Rotation matrix

$$\underline{P} = x\underline{i} + y\underline{j} + z\underline{k} \Rightarrow P_{xyz}$$
$$= u\underline{i}' + v\underline{j}' + w\underline{k}' \Rightarrow P_{uvw}$$

$$P_{xyz} = \underline{\underline{R}}\, P_{uvw}$$

$\underline{\underline{R}} \longrightarrow$ Rotation matrix (DCM method)

Properties

① $R^{-1} = R^{T}$

② $R^{T}R = I$

③ $\det|R| = 1$

$*$ $\underbrace{\underline{\underline{R}}^{i}}_{\text{coordinates}} = \underline{\underline{C}}^{i}_{m} \underbrace{\underline{\underline{R}}^{m}}_{\text{coordinates}}$

$\Downarrow$

from m to i matrix

{ transformation matrix (almost 4×4)

{ rotation matrix (almost 3×3)

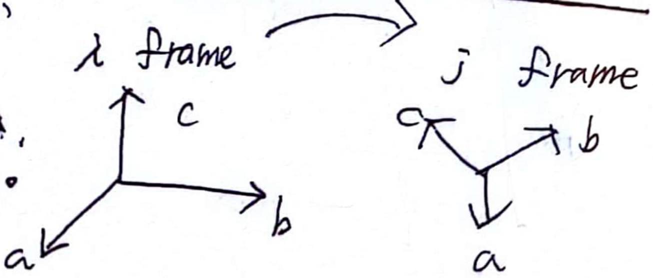
☆ Use DCM for robotics in this note

\# DCM : Direction Cosine Matrix

\# DCM is used for "frame change".

* Generally 'rotation matrix' is used as "value change".

$(x, y) \rightarrow (x', y')$



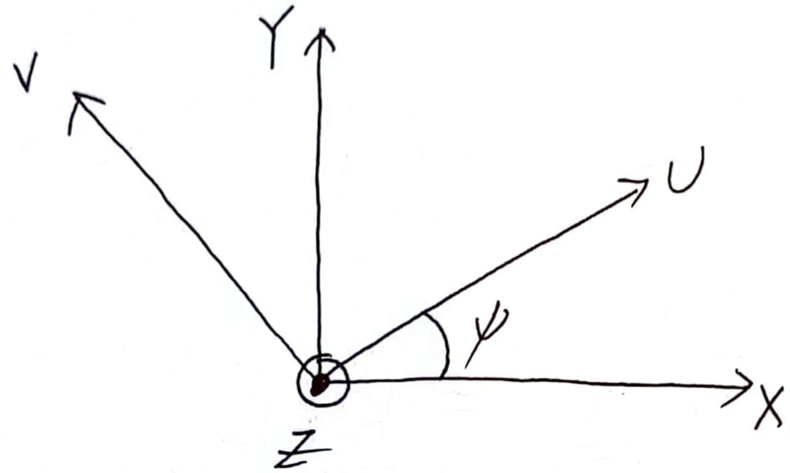
Remember !!

DCM
(Direction Cosine Matrix)

"frame change"

Rotation matrix

"value change"



$$R_Z = R(0,0,\psi) = \begin{bmatrix} \underline{i}_x \cdot \underline{i}_u & \underline{i}_x \cdot \underline{j}_v & \underline{i}_x \cdot \underline{k}_w \\ \underline{j}_y \cdot \underline{i}_u & \underline{j}_y \cdot \underline{j}_v & \underline{j}_y \cdot \underline{k}_w \\ \underline{k}_z \cdot \underline{i}_u & \underline{k}_z \cdot \underline{j}_v & \underline{k}_z \cdot \underline{k}_w \end{bmatrix}$$

$$= \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Similiarly R_y, R_Z could be found.

Z-axis ψ $R_Z = R(\psi)$

$$\begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

y-axis θ $R_y = R(\theta)$

$$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

x-axis ϕ $R_x = R(\phi)$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$$

Remember

z-rotation	y-rotation	x-rotation
$\begin{matrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{matrix}$	$\begin{matrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{matrix}$	$\begin{matrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{matrix}$

Rotation Sequence

$$Z \rightarrow Y \rightarrow X$$

$$R = R_X R_Y R_Z = R(\phi)R(\theta)R(\psi)$$

$$R^{-1} = R_Z R_Y R_X = R(-\psi)R(-\theta)R(-\phi)$$

Transformation Matrix

→ translation Vector ~~matrix~~

$$T = \begin{bmatrix} R_{3\times3} & C_{3\times1} \\ 0_{1\times3} & 1 \end{bmatrix} : 4 \times 4$$

Rotation matrix (DCM) (or Orientation matrix)

→ calibration vector

0 : zero.

C :

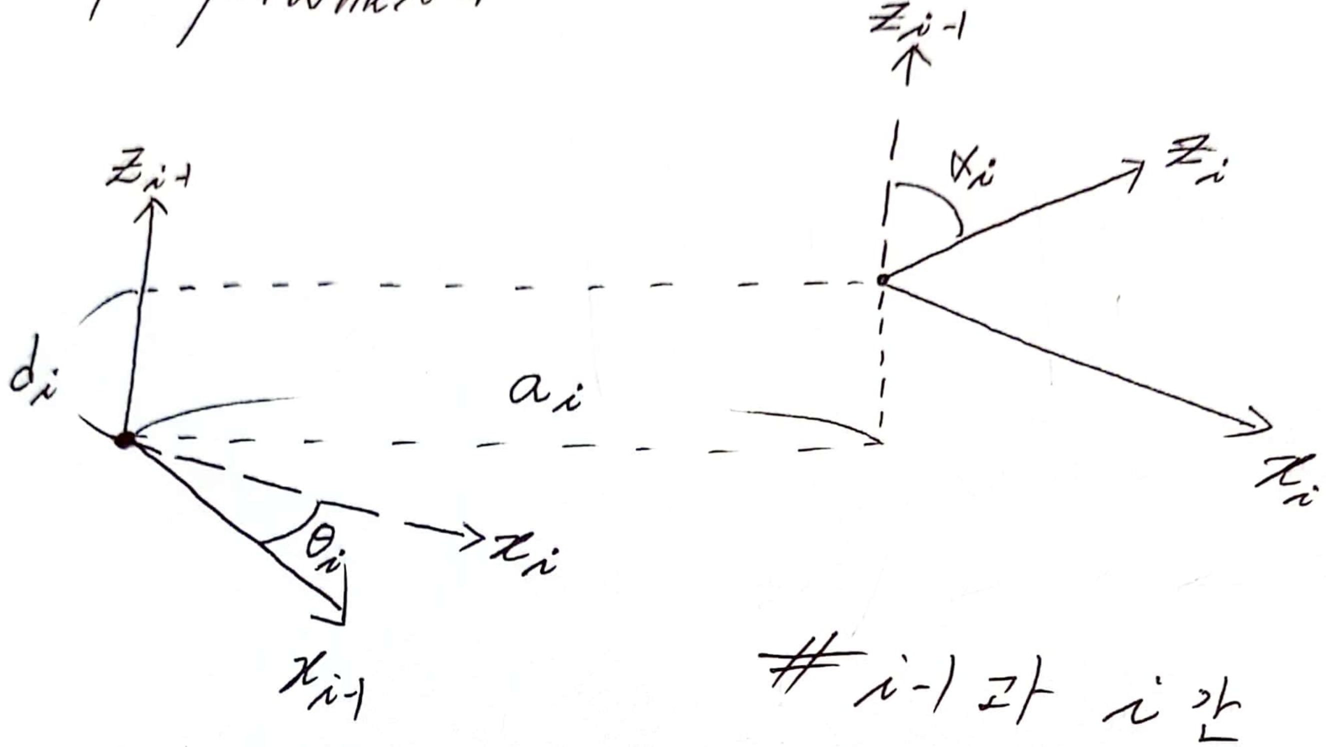
upper side　　front side　　close side

$$* \; T_0^m = \begin{bmatrix} n & s & a & p \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} n & s & a \end{bmatrix}$$

$n \Downarrow$ normal　$s \Downarrow$ sliding　$a \Downarrow$ approach

TBR..

D-H parameter



$i-1$과 i 간 "$i-1$" 기준 물리적 관계

θ_i : x_{i-1}과 x_i 간 각도

α_i : z_{i-1} 과 z_i 간 각도

d_i : x_{i-1} 과 x_i 간 "z_{i-1}" 기준 거리 차이

a_i : z_{i-1}과 z_i 간 "x_i" 기준 거리 차이

* θ_i, α_i 는 모두 각각 두 축이 만드는 평면에서의 각도 관계를 측정

$\theta \rightarrow \alpha \rightarrow d \rightarrow a$ 순으로 해석

로봇 설계 및 해석 순서

① (x_0, y_0, z_0) 설정 / base 축 정의

② 회전축은 모두 "z" 축으로 통일

③ z_{i-1} 과 z_i 가 가지는 화살표의 시작점을 원점으로 설정

④ z_{i-1} 과 z_i 가 모두 수직인 선을 X_i 로 지정

#추가설명 들어갈 수 있음
TBR

⑤ X_{i-1} 과 X_i 간 각도 θ_i

⑥ z_{i-1} 과 z_i 간 각도 α_i

⑦ X_{i-1} 과 X_i 간 거리 d_i ⇒ z_{i-1} 기준

⑧ z_{i-1} 과 z_i 간 거리 a_i ⇒ X_i 기준

"화살" 방향 기준

⑨ D-H parameters 를 이용한 D-H table 작성

??? ⇒ TBR...

joint i	θ_i	α_i	d_i	a_i
i	⋮	⋮	⋮	⋮
⋮				

⑩ Transformation Matrix 작성

$$T_0^m = T_0^1 T_1^2 T_2^3 \cdots T_{m-1}^m$$

Robot kinematics

$$DOF = \lambda(L-A-1) + \sum_{i=1}^{n} f_i$$

(Degree Of Freedom)

$$\lambda = \begin{cases} 6 & (\text{공간 상 움직임}) \\ 3 & (\text{평면 상 움직임}) \end{cases}$$

L = 링크 수

A = 조인트 수 $(=n)$

f_i = i 번째 조인트의 자유도

$$\underline{X}_p = \underline{F}(\underline{q}, l)$$

$\underline{X}_p$: end-effector position

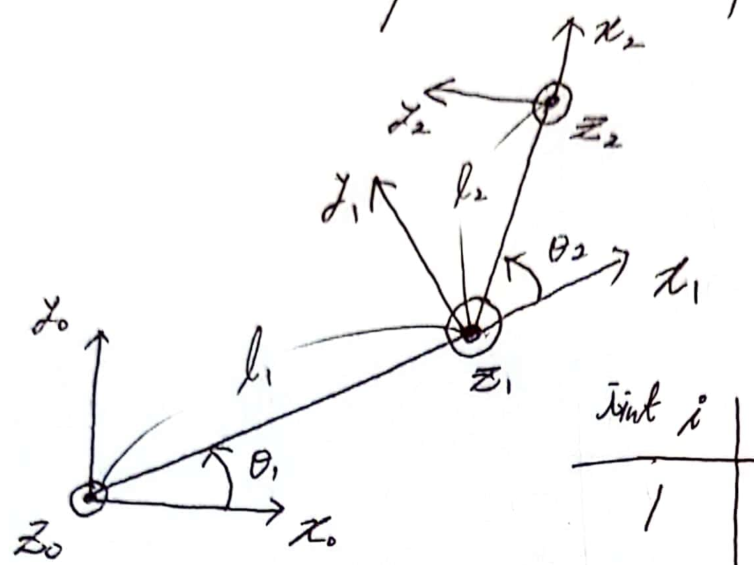
$\underline{q}$: joint angle vector

l : link length

$$* \ \underline{P}_0 = T_0^n \underline{P}_n$$

$\underline{P}_0$: base location vector

$\underline{P}_n$: End Effector location vector ex) 2 link planar manipulator



D-H parameter table

Joint i	θ_i	α_i	d_i	a_i
1	θ_1	0	0	l_1
2	θ_2	0	0	l_2

$$T_0^1 = \begin{bmatrix} C_1 & -S_1 & 0 & l_1 C_1 \\ S_1 & C_1 & 0 & l_2 S_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad T_1^2 = \begin{bmatrix} C_2 & -S_2 & 0 & l_2 C_2 \\ S_2 & C_2 & 0 & l_2 S_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_0^2 = T_0^1 \, T_1^2 = \begin{bmatrix} C_{12} & -S_{12} & 0 & l_1 C_1 + l_2 C_{12} \\ S_{12} & C_{12} & 0 & l_2 S_1 + l_2 S_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

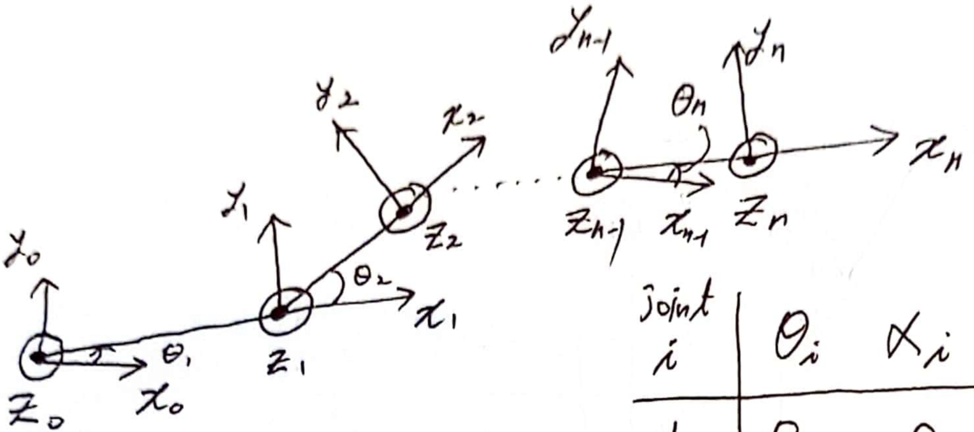
\# $C_1 = \cos\theta_1$ $\quad C_{12} = \cos(\theta_1 + \theta_2)$

$S_1 = \sin\theta_1$ $\quad S_{12} = \sin(\theta_1 + \theta_2)$

End Effector (EE) orientation $\qquad$ EE translation

$$T_0^2 = \begin{bmatrix} R_0^2 & C_0^2 \\ 0_{1\times 3} & 1 \end{bmatrix}$$

ex) n link planar robot



joint i	θ_i	α_i	d_i	a_i
1	θ_1	0	0	l_1
2	θ_2	0	0	l_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	θ_n	0	0	l_n

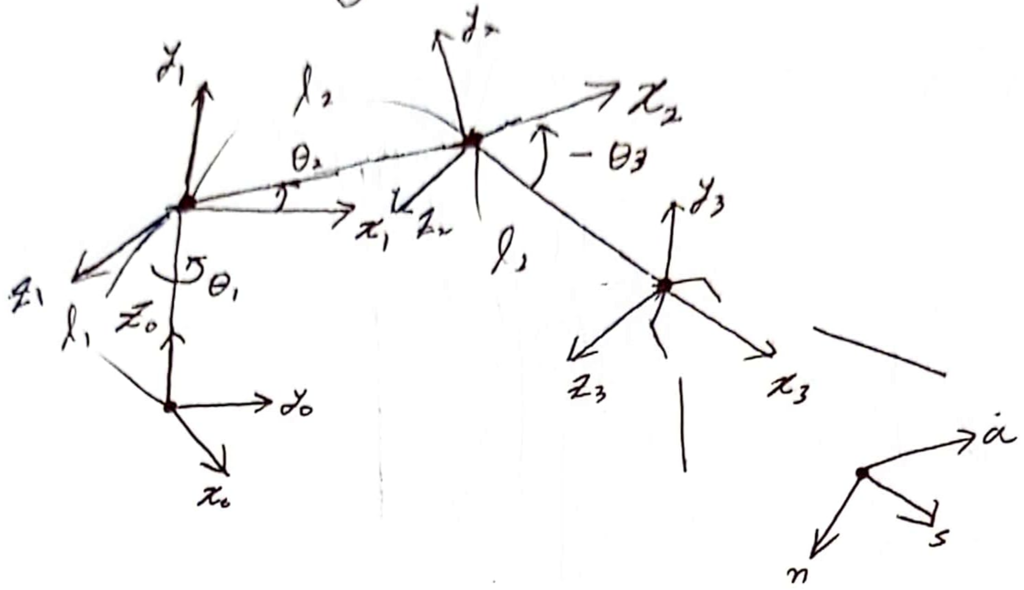
$$T_0^n = T_0^1 T_1^2 T_2^3 \cdots T_{n-1}^n$$

unofficial terminology

$\# \; 1 \to i = 12 \cdots i$

$$T_0^n = \begin{bmatrix} C_{12\cdots n} & -S_{12\cdots n} & 0 & \sum_{i=1}^{n} l_i C_{1\to i} \\ S_{12\cdots n} & C_{1\cdots n} & 0 & \sum_{i=1}^{n} l_i S_{1\to i} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

ex) 3 axis rotary robot ⇒ 지난 수직형 로봇



joint i	θ_i	α_i	d_i	a_i
1	θ_1	90°	l_1	0
2	θ_2	0	0	l_2
3	$-\theta_3$ $(=\theta_3')$	0	0	l_3

※ Rotation sequence

$Z \to Y \to X$

$C = C_x C_y C_z$

$P_f = C P_i = C_x C_y C_z P_i$

$$R_0^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 90^\circ & -\sin 90^\circ \\ 0 & \sin 90^\circ & \cos 90^\circ \end{bmatrix} \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C_1 & 0 & S_1 \\ S_1 & 0 & -C_1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$T_0^1 = \begin{bmatrix} R_0^1 & C_0^1 \\ 0_{1\times3} & 1 \end{bmatrix} = \begin{bmatrix} C_1 & 0 & S_1 & 0 \\ S_1 & 0 & -C_1 & 0 \\ 0 & 1 & 0 & l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_0^3 = T_0^1 T_1^2 T_2^3$$

※ In this note, $-\theta_3$ is represented as θ_3'.

$$= \begin{bmatrix} C_1(C_2C_3 - S_2S_3) & -C_1(C_2C_3 + S_2C_3) & S_1 & d_2C_2C_3 + d_3C_1(C_2C_3 - S_2S_3) \\ S_1(C_2C_3 - S_2S_3) & -S_1(C_2S_3 + S_2C_3) & -C_1 & d_2S_1C_1 + d_3S_1(C_2C_3 - S_2S_3) \\ C_2S_3 + S_2C_3 & C_2C_3 - S_2S_3 & 0 & d_1 + d_2S_2 + d_3(C_2S_3 + S_2C_3) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Jacobian Matrix

$$\boxed{\underline{X}, \dot{\underline{X}}, \ddot{\underline{X}}} \xrightarrow{IK} \boxed{\underline{q}, \dot{\underline{q}}, \ddot{\underline{q}}}$$

$$\underline{X} = \begin{bmatrix} p_x \\ p_y \\ p_z \\ \phi \\ \theta \\ \psi \end{bmatrix} \qquad \underline{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix}$$

$$\underline{X} = \underbrace{\underline{F}}(\underline{q}) \rightarrow \text{Kinematics}$$

#6-DoF robot

$$\frac{d\underline{X}}{dt} = \frac{\partial F(\underline{q})}{\partial \underline{q}} \frac{\partial \underline{q}}{\partial t} = J \dot{\underline{q}}$$

$$J = \begin{bmatrix} \frac{\partial F_1}{\partial q_1} & \cdots & \frac{\partial F_1}{\partial q_n} \\ \vdots & & \vdots \\ \frac{\partial F_n}{\partial q_1} & \cdots & \frac{\partial F_n}{\partial q_n} \end{bmatrix}$$

$$\begin{cases} \dot{\underline{X}} = J\dot{\underline{q}} & \text{--- ①} \\ \ddot{\underline{X}} = \dot{J}\dot{\underline{q}} + J\ddot{\underline{q}} & \text{--- ②} \end{cases}$$

$$\rightarrow \frac{d^2 X}{dt^2} = \frac{\partial J(\underline{q})}{\partial \underline{q}} \frac{\partial \underline{q}}{\partial t} + J(\underline{q}) \frac{\partial^2 \underline{q}}{\partial t^2}$$

* J^{-1} exists and is non-singular

$\det |J| \neq 0$

* $J_{ij} = \frac{\partial F_i}{\partial q_j}$: i location's j angle derivative

From ①,

$$J^{-1}\dot{\underline{X}} = J^{-1}J\dot{\underline{q}}$$

$$J^{-1}\dot{\underline{X}} = \dot{\underline{q}}$$

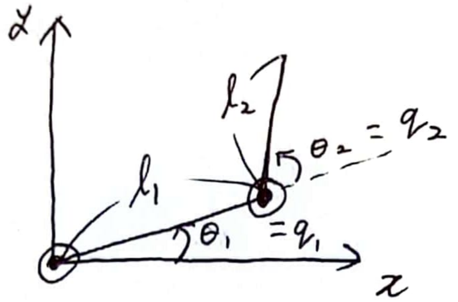
From ②,

$$J\ddot{\underline{q}} = \ddot{\underline{X}} - \dot{J}\dot{\underline{q}}$$

$$\ddot{\underline{q}} = J^{-1}(\ddot{\underline{X}} - \dot{J}\dot{\underline{q}})$$

$$\begin{cases} \underline{\dot{q}} = J^{-1}\underline{\dot{X}} \\ \underline{\ddot{q}} = J^{-1}(\underline{\ddot{X}} - \dot{J}\underline{\dot{q}}) \end{cases}$$

ex) 2 link planar robot



$$\begin{bmatrix} P_x \\ P_y \end{bmatrix} = \begin{bmatrix} l_1 C_1 + l_2 C_{12} \\ l_1 S_1 + l_2 S_{12} \end{bmatrix} = \underline{X}$$

$$\underline{\dot{X}} = J\underline{\dot{q}} = \begin{bmatrix} \frac{\partial P_x}{\partial q_1} & \frac{\partial P_x}{\partial q_2} \\ \frac{\partial P_y}{\partial q_1} & \frac{\partial P_y}{\partial q_2} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} -l_1 S_1 - l_2 S_{12} & -l_2 S_{12} \\ +l_1 C_1 + l_2 C_{12} & l_2 C_{12} \end{bmatrix}}_{= J} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

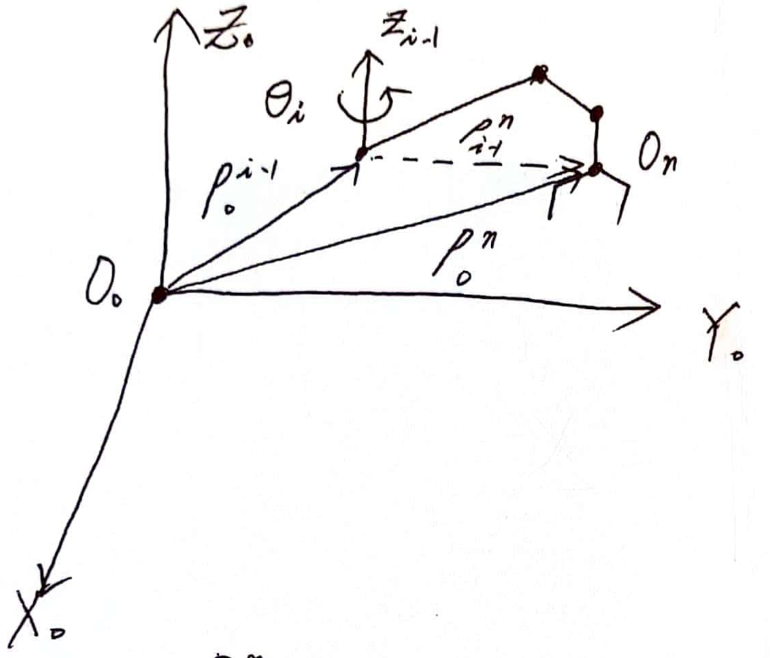
$$|J| = l_1 l_2 S_2$$

We have to avoid this angle kinematically!!

$q_2 = 0$ or π $\Rightarrow$ Singularity!!!

General Jacobian

$$T_0^n = \begin{bmatrix} R_0^n & \underline{P}_0^n \\ 0_{1\times 3} & 1 \end{bmatrix}$$



$$\underline{\dot{X}} = J\underline{\dot{q}}$$

$$\underline{P}_0^n = \underline{P}_0^{n-1} + R_0^{n-1}\underline{P}_{n-1}^n$$

$$\underline{V}_0^n = \underline{V}_0^{n-1} + R_0^{n-1}\underline{V}_{n-1}^n$$

$$\underline{W}_0^n = \underline{W}_0^{n-1} + R_0^{n-1}\underline{W}_{n-1}^n$$

$$\underline{X} = \begin{bmatrix} P_x \\ P_y \\ P_z \\ \phi \\ \theta \\ \psi \end{bmatrix}$$

$$\underline{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix}$$

$$\underline{\dot{X}} = \begin{bmatrix} \dot{P}_x \\ \dot{P}_y \\ \dot{P}_z \\ \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} \underline{V}_0^n \\ \underline{W}_0^n \end{bmatrix} = \begin{bmatrix} J_v \\ J_w \end{bmatrix} \underline{\dot{q}}$$

$$= J_0^n \underline{q}$$

Def. of link's angular velocity (revolute joint)

$$\underline{\omega}_{i-1}^{i} = \dot{q}_i \underline{k} \qquad \underline{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\underline{\omega}_0^n = \omega_0^1 + \sum_{i=1}^{n-1} R_0^i \underline{\omega}_i^{i+1}$$

$$= \dot{q}_1 \underline{k} + R_0^1 \dot{q}_2 \underline{k} + \cdots + R_0^{n-1} \dot{q}_n \underline{k}$$

$$= \dot{q}_1 \underline{z}_0 + \dot{q}_2 \underline{z}_1 + \cdots + \dot{q}_n \underline{z}_{n-1}$$

$$= \sum_{i=1}^{n} \dot{q}_i \underline{z}_{i-1} \qquad (\underline{z}_{i-1} = R_0^{i-1} \underline{k})$$

$$\therefore J_\omega = \left[\frac{\partial \underline{\omega}_0^n}{\partial q_1} \quad \frac{\partial \underline{\omega}_0^n}{\partial q_2} \quad \cdots \quad \frac{\partial \underline{\omega}_0^n}{\partial q_n} \right]$$

$$= [\underline{z}_0 \quad \underline{z}_1 \quad \cdots \quad \underline{z}_{n-1}]$$

$$\boxed{J_\omega = [\underline{z}_0 \quad \underline{z}_1 \quad \underline{z}_2 \cdots \underline{z}_{n-1}]}$$

Def of link's velocity

$$\dot{\underline{p}}_0^n = \frac{\partial \underline{p}_0^n}{\partial q_1}\dot{q}_1 + \frac{\partial \underline{p}_0^n}{\partial q_2}\dot{q}_2 + \cdots + \frac{\partial \underline{p}_0^n}{\partial q_n}\dot{q}_n$$

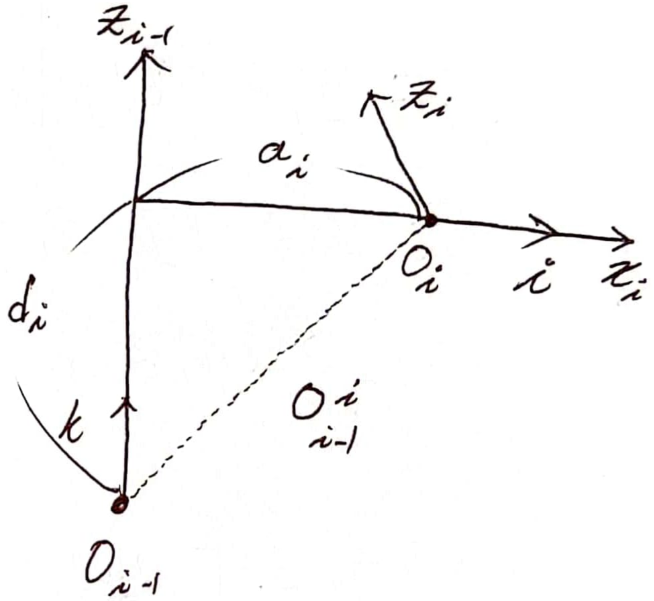
$$= \sum_{i=1}^{n} \frac{\partial \underline{p}_0^n}{\partial q_i}\dot{q}_i$$

※ i joint : sliding joint

$$\underline{d}_{i-1}^i = d_i \underline{k} + R_{i-1}^i a_i \underline{i}$$

$$\dot{\underline{d}}_{i-1}^i = \dot{d}_i \underline{k}$$

(Assume i joint only move)



$$\underline{p}_0^n = \underline{p}_0^{i-1} + R_0^{i-1}\underline{p}_{i-1}^n \qquad \dot{\underline{p}}_0^n = R_0^{i-1}\dot{\underline{p}}_{i-1}^i$$

$$\dot{\underline{p}}_0^n = R_0^{i-1}\dot{d}_i\underline{k} = \dot{d}_i \underline{z}_{i-1}$$

$$\therefore \frac{\partial \underline{p}_0^n}{\partial q_i} = \frac{\partial \underline{p}_0^n}{\partial d_i} = \underline{z}_{i-1} \qquad \left(\dot{d}_i = \frac{d d_i}{d d_i} = 1\right)$$

$$\boxed{\frac{\partial \underline{p}_0^n}{\partial q_i} = \underline{z}_{i-1}}$$

i joint : revolute joint

$$\underline{O}_n - \underline{O}_{i-1} = R_0^{i-1} \underline{p}_{i-1}^n$$

$$\begin{cases} \dot{\underline{p}}_0^n = R_0^{i-1} \dot{\underline{p}}_{i-1}^n \\ \dot{\underline{p}}_{i-1}^n = \dot{q}_i \underline{k} \times \underline{p}_{i-1}^n \end{cases}$$

$\underline{V} = \underline{\omega} \times \underline{r}$

$$\dot{\underline{p}}_0^n = R_0^{i-1} (\dot{q}_i \underline{k} \times \underline{p}_{i-1}^n)$$

$$= \dot{q}_i \underline{z}_{i-1} \times (\underline{O}_n - \underline{O}_{i-1})$$

$\dot{q}_i = \frac{\partial q_i}{\partial q_i} = 1$

$$\boxed{\frac{\partial \underline{p}_0^n}{\partial q_i} = \underline{z}_{i-1} \times (\underline{O}_n - \underline{O}_{i-1})}$$

$$\therefore J_v = \left[\frac{\partial \underline{p}_0^n}{\partial q_1} \ \frac{\partial \underline{p}_0^n}{\partial q_2} \ \cdots \ \frac{\partial \underline{p}_0^n}{\partial q_n} \right]$$

$$= \left[\underline{z}_0 \times (\underline{O}_n - \underline{O}_0) \ \underline{z}_1 \times (\underline{O}_n - \underline{O}_1) \ \cdots \ \underline{z}_{n-1} \times (\underline{O}_n - \underline{O}_{n-1}) \right]$$

$$\boxed{J_v = \left[\underline{z}_0 \times (\underline{O}_n - \underline{O}_0) \ \underline{z}_1 \times (\underline{O}_n - \underline{O}_1) \ \cdots \ \underline{z}_{n-1} \times (\underline{O}_n - \underline{O}_{n-1}) \right]}$$

$$J = \begin{bmatrix} J_v \\ J_w \end{bmatrix} = \begin{bmatrix} \underline{J}_1 & \underline{J}_2 & \cdots & \underline{J}_n \end{bmatrix}$$

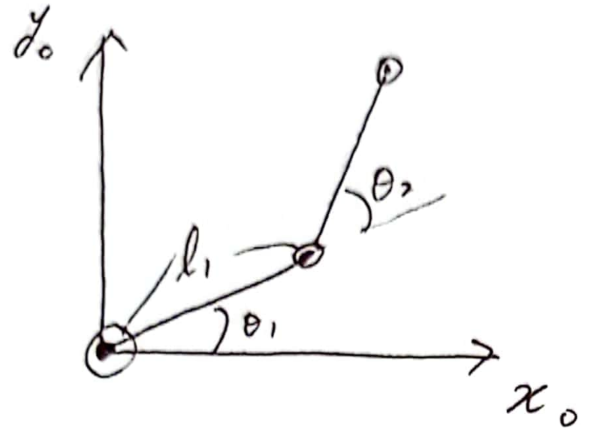
i) revolute joints Jacobian

$$\underline{J}_i = \begin{bmatrix} \underline{z}_{i-1} \times (\underline{O}_n - \underline{O}_{i-1}) \\ \underline{z}_{i-1} \end{bmatrix}$$

ii) sliding joints Jacobian

$$\underline{J}_i = \begin{bmatrix} \underline{z}_{i-1} \\ 0 \end{bmatrix}$$

ex) 2 link planar robot



$$T_0^2 = \begin{bmatrix} C_{12} & -S_{12} & 0 & l_1C_1 + l_2C_{12} \\ S_{12} & C_{12} & 0 & l_1S_1 + l_2S_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\dot{\underline{X}} = J\underline{\dot{q}}$$

$$\underline{J} = \begin{bmatrix} \underline{J}_1 & \underline{J}_2 \end{bmatrix}$$

$$= \begin{bmatrix} \underline{Z}_0 \times (\underline{O}_2 - \underline{O}_0) & \underline{Z}_1 \times (\underline{O}_2 - \underline{O}_1) \\ \underline{Z}_0 & \underline{Z}_1 \end{bmatrix}$$

$$\underline{Z}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \underline{Z}_1 = R_0^1 \underline{Z}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\underline{O}_2 = \begin{bmatrix} l_1C_1 + l_2C_{12} \\ l_1S_1 + l_2S_{12} \\ 0 \end{bmatrix} \qquad \underline{O}_1 = \begin{bmatrix} l_1C_1 \\ l_1S_1 \\ 0 \end{bmatrix} \qquad \underline{O}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{J} = \begin{bmatrix} -l_1S_1 - l_2S_{12} & -l_2S_{12} \\ l_1C_1 + l_2C_{12} & l_2C_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

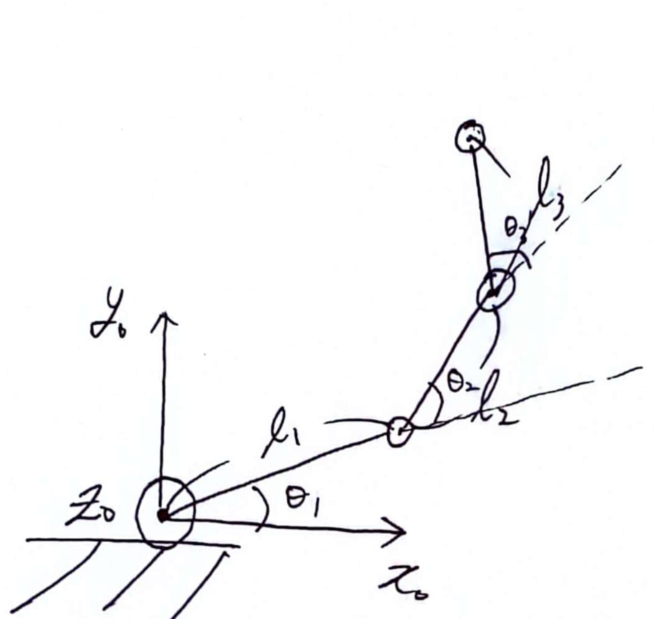
Inverse kinematics

$\Rightarrow$ To find desired angle!!

Ex) 3 link planar robot

$$T_0^3 = \begin{bmatrix} C_{123} & -S_{123} & 0 & l_1C_1 + l_2C_{12} + l_3C_{123} \\ S_{123} & C_{123} & 0 & l_1S_1 + l_2S_{12} + l_3S_{123} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = \begin{bmatrix} l_1C_1 + l_2C_{12} + l_3C_{123} \\ l_1S_1 + l_2S_{12} + l_3S_{123} \\ 0 \end{bmatrix}$$



$$\phi = \theta_1 + \theta_2 + \theta_3 \quad (\text{\# almost } \phi \text{ is const. (defined)})$$

$$\left.\begin{aligned} p_x' &= p_x - l_3 C_{123} = p_x - l_3 C_\phi = l_1 C_1 + l_2 C_{12} \\ p_y' &= p_x - l_3 S_{123} = p_y - l_3 S_\phi = l_1 S_1 + l_2 S_{12} \end{aligned}\right\} \boxed{1}$$

$$p_x'^2 + p_y'^2 = l_1^2 + l_2^2 + 2 l_1 l_2 C_2$$

$$\left\{\begin{aligned} C_2 &= \frac{p_x'^2 + p_y'^2 - l_1^2 - l_2^2}{2 l_1 l_2} \\ S_2 &= \pm\sqrt{1 - C_2^2} \end{aligned}\right. \qquad \therefore \theta_2 = \tan^{-1}\left(\frac{S_2}{C_2}\right)$$

From $\boxed{1}$,

$$p_x' = (l_1 + l_2 C_2) C_1 - l_2 S_2 S_1$$

$$p_y' = (l_1 + l_2 C_2) S_1 + l_2 S_2 C_1$$

$$\underbrace{\begin{bmatrix} -l_2 S_2 & l_1 + l_2 C_2 \\ l_1 + l_2 C_2 & l_2 S_2 \end{bmatrix}}_{=A} \begin{bmatrix} S_1 \\ C_1 \end{bmatrix} = \begin{bmatrix} p_x' \\ p_y' \end{bmatrix}$$

$$\det A = -p_x'^2 - p_y'^2$$

$$\therefore \begin{bmatrix} S_1 \\ C_1 \end{bmatrix} = \frac{1}{-(P_x'^2 + P_y'^2)} \begin{bmatrix} l_2 S_2 & -(l_1 + l_2 C_2) \\ l_1 + l_2 C_2 & -l_2 S_2 \end{bmatrix} \begin{bmatrix} P_x' \\ P_y' \end{bmatrix}$$

$$\begin{cases} C_1 = \dfrac{(l_1 + l_2 C_2) P_x' + l_2 S_2 P_y'}{P_x'^2 + P_y'^2} \\ S_1 = \dfrac{(l_1 + l_2 C_2) P_y' - l_2 S_2 P_x'}{P_x'^2 + P_y'^2} \end{cases} \qquad \therefore \theta_1 = \tan^{-1}\left(\frac{S_1}{C_1}\right)$$

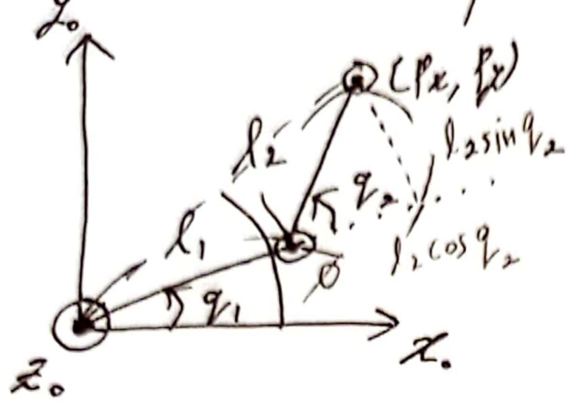
$$\phi = \theta_1 + \theta_2 + \theta_3$$

$$\theta_1 = \tan^{-1}\left(\frac{S_1}{C_1}\right) \quad C_1 = \frac{(l_1 + l_2 C_2) P_x' + l_2 S_2 P_y'}{P_x'^2 + P_y'^2}, \quad S_1 = \frac{(l_1 + l_2 C_2) P_y' - l_2 S_2 P_x'}{P_x'^2 + P_y'^2}$$

$$\theta_2 = \tan^{-1}\left(\frac{S_2}{C_2}\right) \quad C_2 = \frac{P_x'^2 + P_y'^2 - l_1^2 - l_2^2}{2 l_1 l_2}, \quad S_2 = \pm\sqrt{1 - C_2^2}$$

$$\theta_3 = \phi - \theta_1 - \theta_2$$

Ex) 2 link planar robot circle trajectory



$$T_0^2 = \begin{bmatrix} C_{12} & -S_{12} & 0 & l_1C_1+l_2C_{12} \\ S_{12} & C_{12} & 0 & l_1S_1+l_2S_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\underline{P} = \begin{bmatrix} l_1C_1 + l_2C_{12} \\ l_1S_1 + l_2S_{12} \\ 0 \end{bmatrix} = \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix}$$

$$P_x^2 + P_y^2 = l_1^2 + l_2^2 + 2l_1l_2C_2$$

$$C_2 = \frac{P_x^2+P_y^2-l_1^2-l_2^2}{2l_1l_2}$$

$$S_2 = \pm\sqrt{1-C_2^2}$$

$$\begin{cases} \phi = q_1 + q_2 \\ \phi = \tan^{-1}\left(\frac{P_y}{P_x}\right) \\ r = \sqrt{P_x^2+P_y^2} \end{cases}$$

$$\therefore \begin{cases} C_2 = \frac{r^2-l_1^2-l_2^2}{2l_1l_2} \\ S_2 = \pm\sqrt{1-C_2^2} \end{cases}$$

$$q_2 = \tan^{-1}\left(\frac{S_2}{C_2}\right)$$

$$q_1 = \phi - q_2$$

$$= \tan^{-1}\left(\frac{P_y}{P_x}\right) - \tan^{-1}\left(\frac{l_2S_2}{l_1+l_2C_2}\right)$$

$$\therefore \left\{ \underline{q} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} \tan^{-1}\left(\frac{P_y}{P_x}\right) - \tan^{-1}\left(\frac{l_2S_2}{l_1+l_2C_2}\right) \\ \tan^{-1}\left(\frac{S_2}{C_2}\right) \end{bmatrix} \right\}$$