Lecture 9

$\infty$ square well

$$|\psi_n\rangle = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$$

$$\langle \psi_m | \psi_n \rangle = \delta_{m,n}$$

$$\hat{H} |\psi_n\rangle = E_n |\psi_n\rangle$$

$$\langle \psi_m | \hat{H} | \psi_n \rangle = \langle \psi_m | E_n | \psi_n \rangle = E_n \delta_{m,n}$$

| $\delta_{m,n} \rightarrow$ identity matrix

$$\hat{H} = \begin{bmatrix} E_1 & & \\ & E_2 & \\ & & \ddots \end{bmatrix}$$ in its own eigenbasis

$H_{m,n} = \langle \psi_m | \hat{H} | \psi_n \rangle$ in $\{\psi_i\}$ basis

Other operators: $\hat{x} |\psi_n\rangle = x \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$

not proportional to $|\psi_n\rangle$

$\hat{p} |\psi_n\rangle \propto \frac{d}{dx} \sin\left(\frac{n\pi x}{a}\right) \propto \cos\left(\frac{n\pi x}{a}\right)$

not proportional to $|\psi_n\rangle$

$$x_{m,n} \triangleq \langle \psi_m | \hat{x} | \psi_n \rangle = \frac{2}{a} \int_0^a \sin\left(\frac{m\pi x}{a}\right) x \sin\left(\frac{n\pi x}{a}\right) dx$$

$$= \frac{1}{a} \int_0^a dx \; x \Big( \underbrace{\cos\left(\frac{(m-n)\pi x}{a}\right)}_{①} - \underbrace{\cos\left(\frac{(m+n)\pi x}{a}\right)}_{②} \Big)$$

use $\int x\cos(\beta x)\,dx = \dfrac{\beta \cancel{\sin(\beta x)} + \beta\cos(\beta x) - \beta}{\beta^2}$

① $\dfrac{a}{(m-n)\pi}\left(\cos\left(\dfrac{(m-n)\pi x}{a}\right) - 1\right)\Big|_0^a$ for $n \neq m$

$\left(\dfrac{a^2}{2} \text{ for } n=m\right)$

??

(Check "a" factors)

② $\dfrac{a}{(n+m)\pi}\left(\cos\left(\dfrac{(m+n)\pi x}{a}\right) - 1\right)\Big|_0^a$

$(m \neq n)$:

$$x_{m,n} = \left(\frac{\cancel{a}}{(m-n)\pi}\left\{\cos((m-n)\pi - 1) - 1\right\} - \frac{\cancel{a}}{(m+n)\pi}\left\{\cos((m+n)\pi - 1) - 1\right\}\right)\frac{1}{\cancel{a}}$$

a?

if $m=n$ $\quad x_{m,n} = \dfrac{a}{2}$

$m+n$ odd $\quad x_{m,n} \neq 0$

$m+n$ even $\quad x_{m,n} = 0$

$$\hat{x} = \begin{bmatrix} a/2 & ? & 0 & ? & 0 \\ ? & a/2 & ? & 0 & ? \\ 0 & ? & a/2 & ? & 0 \\ ? & 0 & ? & \ddots & \end{bmatrix}$$

if $\varphi = \sum_n \alpha_n |\psi_n\rangle$:

$$\langle\varphi|\hat{x}|\varphi\rangle = \sum_{n,m} \alpha_n^* \alpha_m \langle\psi_n|\hat{x}|\psi_m\rangle$$

$$= \sum \alpha_n^* \alpha_m x_{m,n}$$