APPH E3300 Applied Electromagnetism

Michael Cai

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Abstract

Vector Analysis (Chapter 1) II. Electrostatics (Chapter 2) III. Solution Techniques for Electrostatic Problems Laplace's Equation (3.1) Method of Images (3.2) Multipole Expansions (3.4) IV. Electric Fields in Matter (Chapter 4) Polarization (4.1-4.3) Dielectric Materials (4.4) V. Application Examples of Electrostatics VI. Magnetostatics (Chapter 5) VII. Magnetic Fields in Matter (Chapter 6) Magnetization (6.1-6.4.1) Ferromagnetism (6.4.2) VIII. Application Examples of Magnetostatics

Chapter 1

Overview of EM

Lecture 1

Review of Class Syllabus Jan 22 10:10

Chapter 2

Overview of EM

Lecture 2

Remark. Differential Calculus: Gradient, Divergence

Jan 27 10:10

Definition 2.0.1. Del operator

$$\nabla = i_x \frac{\partial}{\partial x} + i_y \frac{\partial}{\partial y} + i_z \frac{\partial}{\partial z}$$

The del operator alone is meaningless, however when used with scalar or vector fields, it gives rise to interesting concepts

Definition 2.0.2. We define Divergence as

$$\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \sum_i^n \frac{\partial f(\vec{A})}{\partial A_i} \vec{e_i}$$

For a given divergence, we define the flux of a vector field. We define a surface with area dx, dy with normal vector $\hat{n}$. The flux differential is defined to be

$$d\phi = \hat{n} \cdot \vec{A} \mathrm{d}x \mathrm{d}y$$

For a large surface, the flux is defined to be

$$\phi = \int d\phi = \int \vec{A} \cdot \hat{n} ds$$

From here we get Gauss's divergence theorem (Green's Theorem)

$$\oint \vec{A} \cdot \hat{n} ds = \int_V (\nabla \cdot \vec{A}) dV$$

Given a uniform vector field, the flux is 0. Next we define the curl of the vector with is

$$\nabla \times \vec{A}$$

which defined how much the vector "swirls" around points. THe circulation of a field is denoted as

$$C = \oint \vec{A} \cdot d\vec{l}$$

We will get the between the curl and the circulation to be Stoke's Theorem

$$\oint \vec{A} \cdot d\vec{l} = \int_s (\nabla \times A) \cdot \vec{n} dS$$

Remark. Second Order Derivatives: scalar fields and vector fields

Gradient $\vec{\nabla} f$ which is vector. We can apply the del operator again with the dot product or the cross product which are two ways. For the divergence $\vec{\nabla} \cdot \vec{A}$ is applying the del operator again $\vec{\nabla}(\nabla A)$. For the curl, we have two combinations that are the same as the gradient. Thus, there are 5 second order derivatives...

Definition 2.0.3.

$$\nabla \cdot (\nabla f) = \sum_i^n \frac{\partial^2}{\partial x_i^2} f = \nabla^2 f$$

which is also defined as the laplacian

Definition 2.0.4.

$$\nabla \times (\nabla f) = (\frac{\partial^2}{\partial yz} - \frac{\partial^2}{\partial z, y})\hat{x} + (\frac{\partial^2}{\partial z, x} - \frac{\partial^2}{\partial x, z})\hat{y} + (\frac{\partial^2}{\partial x, y} - \frac{\partial^2}{\partial y, x})\hat{z} = 0$$

Definition 2.0.5.

$$\nabla(\nabla \cdot A)$$

Definition 2.0.6.

$$\nabla \cdot (\nabla \times A) = 0$$

Definition 2.0.7.

$$\nabla \times (\nabla \times A) = \nabla(\nabla \cdot A) - \nabla^2 A$$

Remark. Curvilinear Coordinates:

There exists curvilinear coordinates which are useful in projection in orthoganal directions in 3D.

Example. The use of spherical and cylindrical coordinates are important for point charges, conducting wires, calculating the flux for certain special objects and planes, and other important objects. Spherical coordinates are useful in certain descriptions such as that of a point charge.

Definition 2.0.8. Spherical Coordinates:
Suppose we had a point p in spherical coordinates with coordinates (r, ϕ, θ) where the ranges are going to be

$$0 \leq r \leq \infty$$

$$0 \leq \phi \leq 2\pi$$

$$0 \leq \theta \leq \pi$$

where ϕ represents the movement in the x-y plane and θ represents rotation on the z-y plane. The unit vectors are going to follow in the same planes. The conversion is as follows for cartesian to spherical

$$x = r\cos\phi\sin\theta, y = r\sin\phi\sin\theta, z = r\cos\theta$$

The infintisemal displacement will be

$$d\hat{l} = dr\hat{r} + rd\theta\hat{\theta} + r\sin\phi d\phi\hat{\phi}$$

For a small volume, we have

$$dV = r^2\sin\theta dr d\phi d\theta, (dxdydz)$$

Definition 2.0.9. Cylindrical Coordinates (r, ϕ, z)

$$0 \leq r \leq \infty, 0 \leq \phi \leq 2\pi, -\infty \leq z \leq \infty$$

intnfintisemal displacement:

$$d\hat{l} = dr\hat{r} + dz\hat{z} + rd\phi\hat{\phi}$$

Volume:

$$dV = rdrd\phi dz$$

Lecture 3

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Remark. Dirac delta function:
Consider a vector field $\vec{V} = \frac{1}{r^2}\hat{r}$. The divergence of the field will result to 0 which is surprising

$$\vec{\nabla} \cdot \vec{v} = \vec{\nabla} \cdot \left(\frac{1}{r^2}\hat{r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{1}{r^2}\right) = 0$$

For Gauss's Theorem ,we have that

$$\int \vec{\nabla} \cdot \vec{v} = \oint \vec{v} \cdot \hat{n} dS = 4\pi$$

This 4π value holds for any radius, thus there exists a singularity which we can use to define the dirac delta function.

Definition 2.0.10. Properties:

$$\delta(x) = 0, x \neq 0$$

$$\int_{-\infty}^{\infty} \delta(x)\,\mathrm{d}x = 1$$

$$\int_{-\infty}^{\infty} \delta(x) f(x)\,\mathrm{d}x = f(0) \int_{-\infty}^{\infty} \delta(x)\,\mathrm{d}x = f(0)$$

$$\delta(x-a) = 0, x \neq a$$

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)\,\mathrm{d}x = f(a)$$

There are ways to approximate $\delta(x)$ by having a rectangular series ranging from $\pm\frac{1}{2n}$

$$\delta_n(x), n \hookrightarrow \infty$$

Another example is the function

$$\delta_n(x) = \frac{n}{\sqrt{\pi}} e^{-n^2x^2}$$

where we must have

$$\lim_{n\to\infty} \delta_n(x) f(x)\mathrm{d}x = f(0)$$

and the normalization will will still have that

$$\int_{-\infty}^{\infty} \delta(x)\,\mathrm{d}x = 1$$

Definition 2.0.11. 3D Dirac Delta function

$$\delta^3 = 0, \vec{x} \neq 0$$

$$\int_{-\infty}^{\infty} \delta(\vec{r} - \vec{r}_0)\,\mathrm{d}V = 0$$

$$\int_{-\infty}^{\infty} \delta^3(\vec{r} - \vec{r}_0) f(\vec{r})\,\mathrm{d}V = f(\vec{r}_0)$$

The physical case of a delta function is going to have the form

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\hat{r}}{r^2}$$

$$\vec{\nabla}\cdot\vec{E} = \frac{q}{4\pi\epsilon_0}\left(\vec{\nabla}\cdot\frac{\hat{r}}{r^2}\right) = \frac{q}{\epsilon_0}\delta^3(\vec{r})$$

where we later find that $q\delta^3(r)$ to be the charge density. Many times the derivative of the delta function is not well defined. However, when we place it in a integral it is intuitive...

$$\int_{-\infty}^{\infty} \mathrm{d}x f(x)\frac{\mathrm{d}\delta(x)}{\mathrm{d}x} = \frac{-\mathrm{d}f(x)}{\mathrm{d}x}\,|_{x=0}$$

Theorem 2.0.1. Helmholt'z Theorem

Given $\vec{\nabla}\cdot\vec{F} = D$ and $\vec{\nabla}\times\vec{F} = \vec{c}$ where the divergence of the curl is 0. The theorem states that if $D, \vec{c} \hookrightarrow 0, \vec{r} \hookrightarrow \infty$ and we have the condition $\vec{F} \hookrightarrow 0, \vec{r} \hookrightarrow \infty$. We can write $\vec{F}$ as two parts.

$$\vec{F} \equiv -\vec{\nabla}\phi + \vec{\nabla}\times\vec{A}$$

where

$$\vec{\nabla} \cdot \vec{F} = -\nabla^2 \phi = D$$

$$\vec{\nabla} \times \vec{F} = \nabla \times (\vec{\nabla} \vec{A}) = \vec{c}$$

If $\vec{c} = 0$, $\vec{F}$ is irrotational or curless and is written as $\vec{F} = -\vec{\nabla}\phi$ where $\oint \vec{F} \cdot d\vec{l} = 0$ or also independent from the path. Another special case is when $D = 0$, we call the field $\vec{F}$ to be divergenceless or solenoidal where we write $\vec{F} = \vec{\nabla} \times \vec{A} \Rightarrow o \int \vec{F} \cdot d\vec{S} = 0$ which means that it is independent of the surface for any given boundary.

Remark. Electrostatics:

Suppose we had a point charge q_0, q_1 at positions $\vec{r}_0, \vec{r}_1$ where we can find that

$$\vec{F}_{01} \frac{q_0 q_1}{4\pi\epsilon_0} \frac{\hat{r}_{01}}{|\vec{r}_{01}|^2}, \vec{r}_{01} = \vec{r}_0 - \vec{r}_1$$

Suppose we have q_0 fixed and move q_1 in multiple places, we will see that the force

$$\vec{F}_{01}(\vec{r}_0) = q_0 \frac{q_1}{4\pi\epsilon_0} \frac{\hat{r}_0}{|\vec{r}^2|}$$

to be the field generated by charge q_1 . Thus generally we have that

$$\vec{F} = q\vec{E}$$

For a general set of particles we have

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_j \frac{q_j(\vec{r} - \vec{r}_j)}{|\vec{r} - \vec{r}_j|^3}$$

We can then generalize for a continuous charge distribution where we look at an infintesimal distrubtion of $dq = \rho dV$ to be

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\vec{r}_1)(\vec{r} - \vec{r}_1)}{|\vec{r} - \vec{r}_1|^3} dV$$

where the charge density ρ is in units of $\frac{C}{m^3}$. We will later see for surface charges such as a charge sheet, the charge is ditributed on the sheet with density σ , the infinitesimal charge will be $dq = \sigma dA$. For line charges, we can write for a small segment to be $dq = \lambda dl, \lambda = \frac{C}{m}$

Example. Consider an $\vec{E}$ field of a circular loop of charge along the z-axis. We calculate that

$$dE_z = \frac{1}{4\pi\epsilon_0} \cdot \frac{\lambda dl}{R^2 + z^2} \cdot \frac{z}{\sqrt{R^2 + z^2}}$$

$$E_z = \frac{1}{4\pi\epsilon_0} \frac{\lambda}{R^2 + z^2} \cdot \frac{z}{\sqrt{R^2 + z^2}} \int_0^{2\pi} dl$$
$$= \frac{R}{2\epsilon_0} \frac{\lambda}{R^2 + z^2} \cdot \frac{z}{\sqrt{R^2 + z^2}}$$

Example. Another example is calculating the electric field of an infinitely large charge sheet at a point z above the sheet. We can partition the sheet into a circle and see the symmetry that exists for rings. For a single ring we know th result. We can then integrate to get that

$$E_z = \frac{z\sigma}{2\epsilon_0} \int_0^\infty \frac{R}{(R^2 + z^2)^{\frac{3}{2}}} dR$$

$$E_z = \frac{z\sigma}{2\epsilon_0} \frac{1}{\sqrt{R^2+z^2}} \Big|_0^\infty$$

$$E_z = \frac{\sigma}{2\epsilon_0}$$

Remark. Gauss's Law

$$\vec{E}(\vec{r}) = \frac{q}{4\pi\epsilon_0} \frac{\hat{r}}{r^2}$$

$$\oint_S \vec{E}(\vec{r}) \cdot \hat{n} dS = \frac{q}{4\pi\epsilon_0} \frac{\hat{r}}{R^2} 4\pi R^2 = \frac{q}{\epsilon_0}$$

We ge there that the total flux is going to be

$$Flux = \oint_S \vec{E} \cdot \hat{n} dS = \sum_j q_j/\epsilon_0$$

Lecture 4

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Lecture 5

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Remark. Recall the integration technique for a point z away from a ring of charge q.

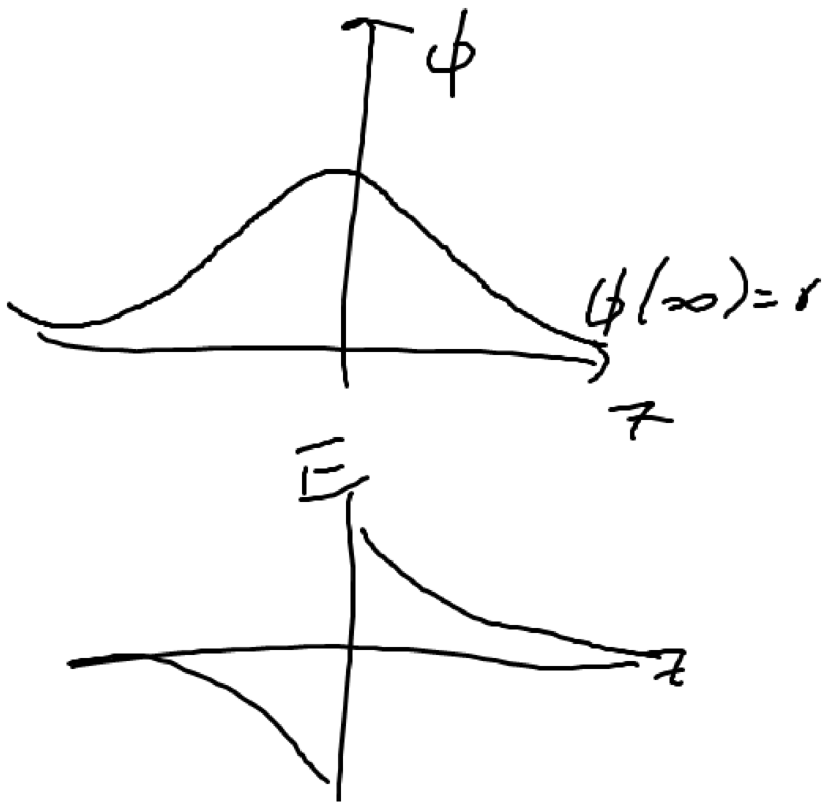

Figure 2.1

This can lead to a generalized conclusion about boundary conditions:

Example. Condisider a box specifically Gauss's Box:

$$\oint \vec{E} \cdot \hat{n} dS = \sigma \frac{S}{\epsilon}$$

If we look at the coordinates perpendicular to the box which we can denote as

$$\left(E_{\perp}^{above} - E_{\perp}^{below}\right) S = \sigma \frac{S}{\epsilon_0}$$

$$\left(E_{\perp}^{above} - E_{\perp}^{below}\right) = \frac{\sigma}{\epsilon_0}$$

This is discontinuous and causes the electric field to jump for the perpendicular component. Now we can look at the parellel component with the surface. If we perform the loop integral over the section is that the electric field

$$\oint \vec{E} \cdot d\vec{l} = 0$$

which comes from the fact that $\nabla \times E = 0$. Thus ew can write that

$$E_{||}^{above} - E_{||}^{below} = 0$$

$$\vec{E}^{above} - \vec{E}^{below} = \frac{\sigma}{\epsilon_0} \hat{n}$$

The potential across the boundary is going to be

$$\phi^{above} - \phi^{below} = \int_a^b \vec{E} \cdot d\vec{l} = 0$$

This tells us that the potential across a charged boundary is still continuous.

Note that we can convert quantities between ϕ , ρ , and E

$$\phi = -\int \vec{E} \cdot d\vec{l}$$

$$\nabla^2 \phi = -\frac{\rho}{\epsilon_0}$$

$$\nabla \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\vec{E} = -\nabla \phi$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r})}{|\vec{r} - \vec{r}'|^2} dr'$$

$$\phi = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r})}{\vec{r} - \vec{r}'} dr'$$

Remark. Electrostatic Energy

The scalar potential is related to the work and energy that we care about. Suppose we wish to move a particle in an electric field from $p_0 \to p_1$ where the particle has charge q. What is the work?

$$W = -\int F \cdot d\vec{r}$$

$$W = -\int q\vec{E} \cdot d\vec{r}$$

$$= q\left[\phi(p) - \phi(p_0)\right]$$

Thus we see that the work is independent of the path and that the field $\vec{E}$ is conservative. If we know the work we can tell the difference to be

$$\phi(p) - \phi(p_0) = \frac{W}{q}$$

From here we can also know the work needed to assemble a particular charge distribution which is the work needed to assemble a distribution of charges q_1, q_2, q_3. We know that $\phi(\infty) = 0$, so we can move the charges one by one. q_1 has no work done on it to move to the correct place. We then move each aprticle one by one. Thus for q_2 , we have

$$W_2 = -\int_{\infty}^{r_{12}} q_2 E_1 d\vec{r} = q_2[\phi(\vec{r}_{12}) - \phi(\infty)] = q_1 \frac{q_2}{4\pi\epsilon|\vec{r_2} - \vec{r_1}|}$$

We can also move the next particle which can be denoted as

$$W_3 = q_3[\phi(\vec{r}_{13}) - 0] + q_3[\phi(\vec{r}_{23}) - 0] = (q_1 + q_2)\frac{q_3}{4\pi\epsilon_0|\vec{r_2} - \vec{r_1}|}$$

$$W_{total} = W_3 + W_2 + W_1$$

we see a structure to calculate the work to be defined to be the work done by 1 to 2, 1 to 3, and 2 to , denoted as

$$W_{total} = W_{12} + W_{13} + W_{23}$$

where

$$W_{ij} = q_i \frac{q_j}{4\pi\epsilon_0|\vec{r}_i - \vec{r}_j|}$$

$$W_{total} = \sum_{i,j,i\neq j,i<j} W_{ij}$$

$$= \frac{1}{8\pi\epsilon_0}\sum_i q_i \sum_j \frac{q_i}{|\vec{r}_i - \vec{r}_j|}$$

$$= \frac{1}{2}\sum_i q_i \phi_j(\vec{r}_i)$$

For a continuous charge distribution we can rewrite this as

$$W = \frac{1}{8\pi\epsilon_0}\int_V \int_{V'} \frac{\rho(\vec{r})\rho(\vec{r}')}{|\vec{r} - (\vec{r})'|} dV dV'$$

$$W = \frac{1}{2}\int_V \rho(\vec{r})\phi(\vec{r})dV$$

The energy can then be expressed in terms of work against the Coloumb interaction. Where in another way we can write

$$W = \frac{1}{2}\int_V \rho(\vec{r})\phi(\vec{r})dV$$

where the charge density is $\epsilon_0(\nabla \cdot \vec{E})$

$$= \frac{1}{2}\int_V e_0(\nabla \cdot \vec{E})\phi dV$$

Notice that

$$\nabla \cdot (\vec{E}\phi) = \nabla\phi \cdot \vec{E} + (\nabla \cdot \vec{E})\phi = -|E|^2 + (\nabla \cdot \vec{E})\phi$$

We can now plug this back and rewrite the work as

$$W = \frac{1}{2}\int_V \epsilon_0 \left[\vec{\nabla} \cdot (\vec{E}\phi) + |\vec{E}|^2\right] dV$$

Suppose that the charge is concetrated at a specific point in space. Then we integrate over the entire charge region. We have

$$\frac{\epsilon_0}{2}\left[\int_S \phi\vec{E}\cdot\hat{n}dS + \int_V |\vec{E}|^2 dV\right]$$

because there are no other charges, we can extend the first term of the surface to be infinitely large.

$$S \to r^2, |\vec{E}| \to \frac{1}{r^2}, \phi \to \frac{1}{r}$$

Thus the first term is proportional to $\frac{1}{r^3}r^2 \to 0$. Thus we can express the term simply as

$$W = \frac{\epsilon_0}{2}\int_V |\vec{E}|^2 dV$$

We can then denote W_e as the energy density associated with the electric field. And this is the other form for expressing the potential energy by calculating the electric field squared. Recall the form we had for the first form and there are some inconsisitencies. For the charge density and potential, we can have negative charge density and a positive potential which means the total energy could be negative. On the other term, the term can neber be negative. The way to reconcile this is integrate over a very small space. If we calculate the work to seemble the point charges, the first equation will blow up.

Lecture 6

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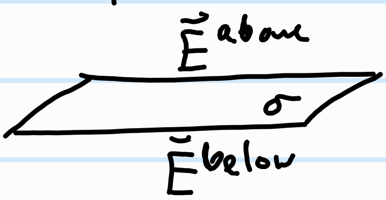
Lecture 6 Applied EM

Review

- boundary conditions

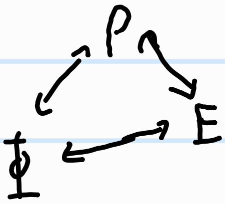


components

$E^\perp$ step $\frac{\sigma}{\varepsilon_0}$

$E^{\parallel}$ continuous

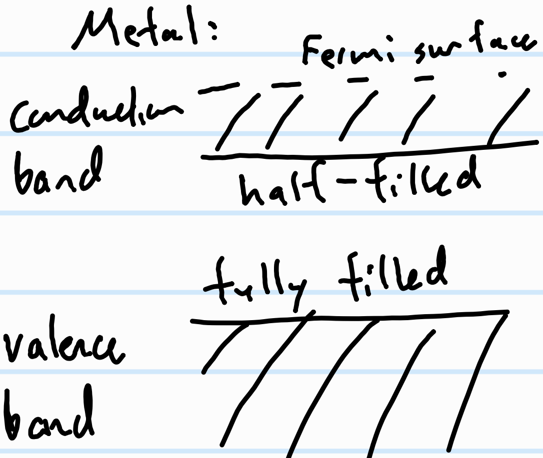
-



- energy $\begin{cases} \frac{1}{2}\int dV\, \rho(\bar{r})\, \Phi(\bar{r}) \\ \int dV\, \frac{1}{2}\varepsilon_0 E^2 \end{cases}$ double count

Conductors

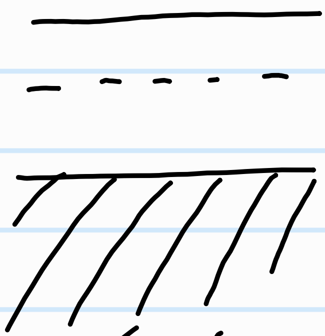
Metal:



Cu.

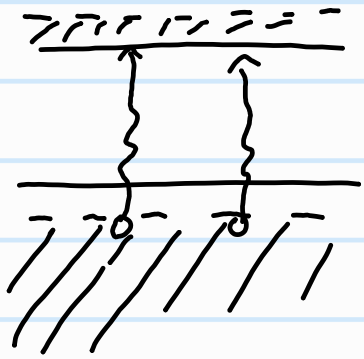
Semi conductors

T=0

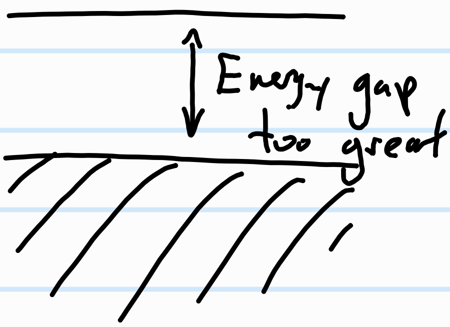


Si. Ge.

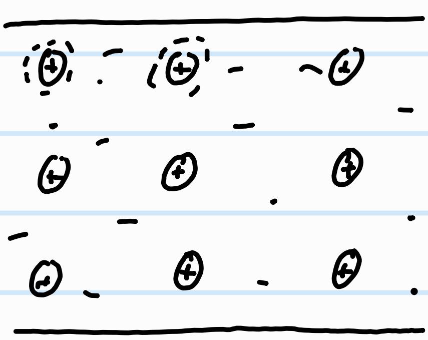
room temp



Insulators

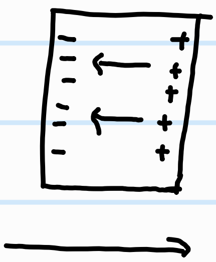


band-gap mentioned



valence band – stays with atom
conduction band electrons float – sea of charge

Ideal conductors in electrostatics



$E_{applied}$

equilibrium implies $\Sigma \bar{E} = 0$ inside cond.
$\bar{E}_{induced} = -\bar{E}_{applied}$
this is the ideal assumption
typical metal settles in ~ps

① $\bar{E}$ inside is $\bar{0}$
② $\nabla \cdot \bar{E} = 0 \rightarrow \rho = 0$ inside conductor
③ all charge on boundary
④ $\Phi = \int \bar{E} \cdot d\bar{s}$ = constant within conductor
⑤ outside of conductor $\bar{E} = \frac{\sigma}{\varepsilon_0}\hat{n}$ $\perp$ to surface
if E parallel to surface, then charge would move on surface

The ideal conductor minimizes total potential energy

eg. solid charged sphere $\rho, R = \frac{3Q^2}{20\pi\varepsilon_0}\frac{1}{R} = W_{vol}$
charged surface $= \frac{Q^2}{8\pi\varepsilon_0}\frac{1}{R} = W_{surf}$ } reduction of W by conductor
0.15 → 0.125 factor

Induced charges

+q

cond. w/ cavity

$\bar{E}_{cavity} = \bar{0}$ (Faraday Cage)

charge in cavity

induced − on inner surface

Gaussian surface through conductor

↳ $\bar{E}=0$, flux $=0 \to$ total $\frac{Q}{\varepsilon_0}=0$

so induced charge $=-q$

conductor is neutral $\to$ outer charge $=+q$

outer σ distribution does not depend on location of cavity/charge

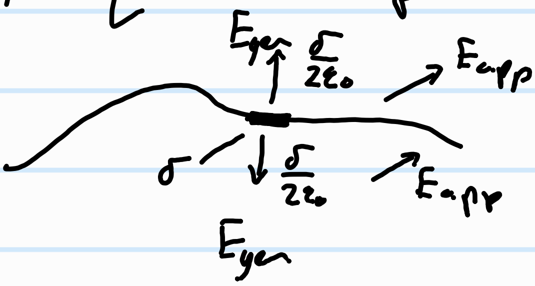
so $\bar{E}_{out} = \frac{q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2} = \frac{\sigma}{\varepsilon_0}\hat{r}$ where $\sigma = \frac{q}{4\pi r^2}$ is constant

sphere acts as point charge @ center

$\bar{F}$ on surface charge

$\bar{F} = q\bar{E} \to$ Force per unit area $\bar{f} = \sigma\bar{E}$

↑ doesn't include field due to itself



$E_{above} = E_{gen,above} + E_{app}$

$E_{below} = E_{gen,below} + E_{app}$

E_{app} = all field not due to local σ

$E_{above} = E_{app} + \frac{\sigma}{2\varepsilon_0}\hat{n}$

$E_{below} = E_{app} - \frac{\sigma}{2\varepsilon_0}\hat{n}$

$$E_{app} = \frac{E_{above} + E_{below}}{2}$$

$\bar{f} = \frac{\sigma}{2}(\bar{E}_{above} + \bar{E}_{below})$ force per area

Conductor: $\bar{E}_{below} = \bar{0}$ $\quad E_{above} = \frac{\sigma}{\varepsilon_0}\hat{n}$

then $\bar{f} = \frac{\sigma^2}{2\varepsilon_0}\hat{n} \to$ always pointed outwards $(\sigma^2 \geq 0)$

f = <u>electrostatic pressure</u>

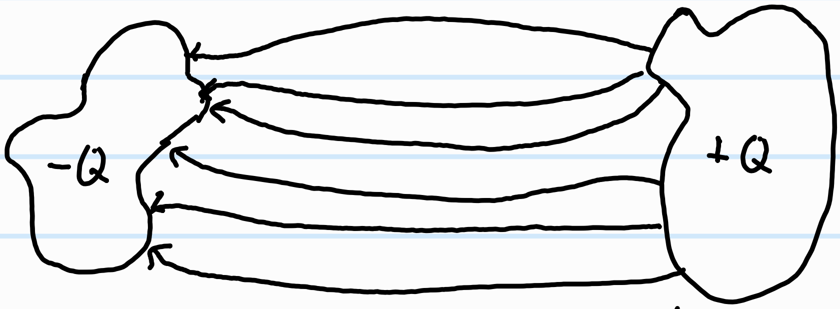
Conclusion: field "felt by" surface $= \frac{\bar{E}_1 + \bar{E}_2}{2}$, pres. $f = \frac{\sigma^2}{2\varepsilon_0}\hat{n} = \frac{\varepsilon_0 E_{out}^2}{2}\hat{n}$

electrostatic pressure depends on magnitude of surface charge density

↳ determined by field outside, charge elsewhere on surface.

Capacitance between 2 conductors

move Q:



potential diff $\Phi = \Phi_+ - \Phi_- = \int_-^+ -\bar{E}\cdot d\bar{s}$

capacitance $C = \frac{Q}{\Phi}$ $\left[\frac{C}{V} = F\right]$

Recall $\Phi = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\bar{r}')}{|\bar{r}-\bar{r}'|}\,dV'$

Φ, ρ superposition $\to$ C is constant

C is a purely geometric quantity of the conductors + dielectric

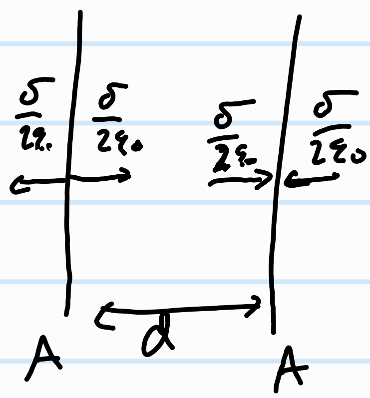
Energy stored in a capacitor

W to move Q $\to \int_0^Q dq$

$dW = dq\,\varphi(q)$ $\quad \varphi(q) = \frac{q}{C}$

$$\int_0^Q dq\,\frac{q}{C} = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}Q\Phi = \frac{1}{2}C\Phi^2$$

Ex. parallel plate cap



Lecture 7

Feb 12 10:10

$$\Delta = q\Delta\phi$$

$$\phi(\vec{r}) = \phi(\vec{r}_0) + \Delta\vec{r} \cdot \nabla\phi \mid_r^= r_0 + \frac{1}{2}(\Delta\vec{r})$$

For the first condition we have

$$E(\vec{r}) = 0$$

For the second condition we have

$$\Delta\phi > 0$$

for all directions

$$\Delta = \Delta y = \Delta z$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} +\right)\phi > 0$$

$$\nabla^2\phi = -\frac{\rho}{\epsilon} > 0 \Rightarrow \rho(\vec{r}) \neq 0$$

If we assume that $\rho = 0$. At least one of the directions should have the derivative

$$\frac{\partial^2\phi}{\partial j^2} < 0$$

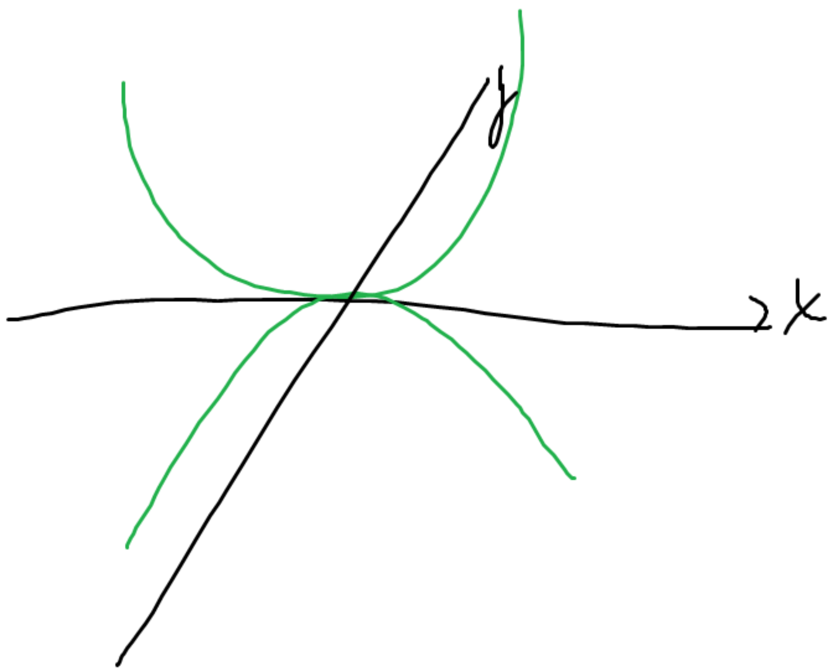


Figure 2.2

Trapped Ions

Example. This is useful for trapped ion quantum computing. There are two ways to hold a ion they use something called a Paul trap which uses a set of electrodes and alternates the electrodes which causes a rotating electric field or (AC field). This creates a quadruple moment which creates a trap in 3D.

The other method is called the **penning trap** where there is a static $\vec{E}$ field and a B field in the z direction.

Electrostatics for atoms, molecules, molecules, and crystals

Example. For the classical pictures of atoms (hydrogen atom), we have the following

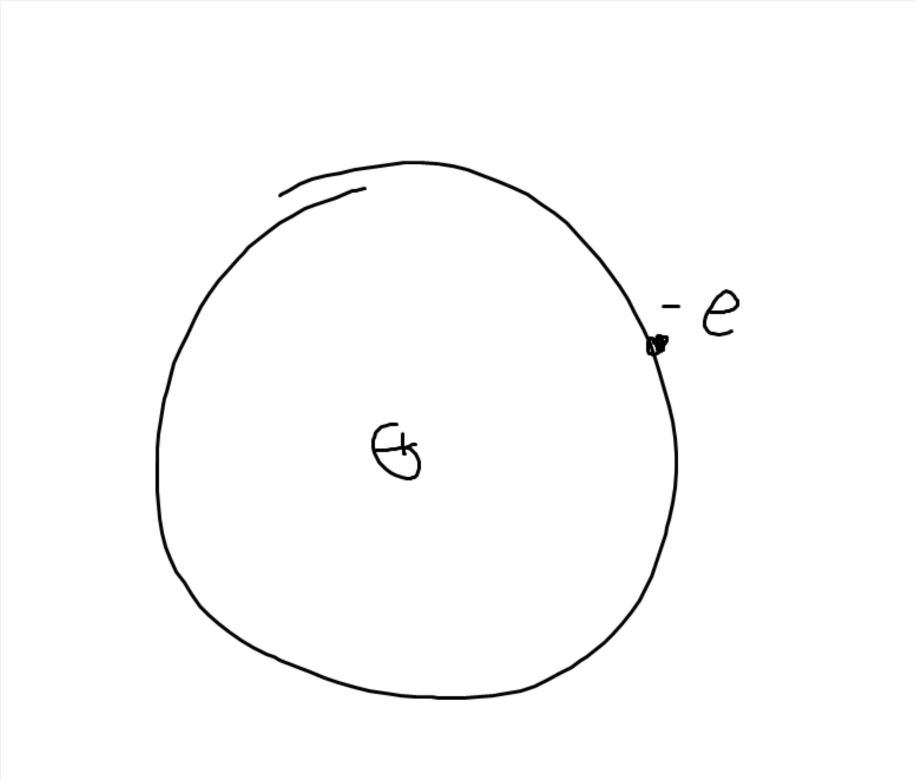


Figure 2.3

$$U = U_k + U_p$$

which is the total energy (kinetic and potential)

$$U_k = \frac{1}{2}mv^2$$

or defined in terms of angular momentum to be

$$\frac{L^2}{2mR^2}$$

The potential energy is going to be

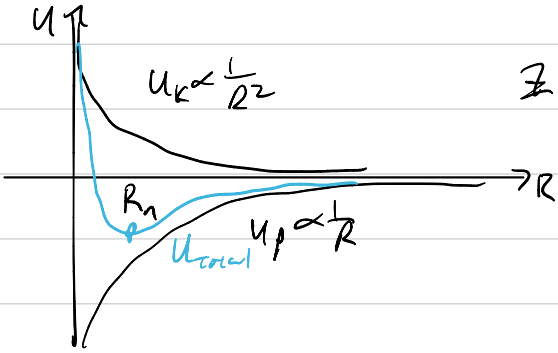
$$U_p = \frac{-q_e q_p}{4\pi\epsilon_0 R}$$

Additionally, Bohr Postulates that

$$L = n\hbar$$

Atomic Example:

For a given n:



Is an equilibrium point

we can take the derivative of the total energy:

$$\frac{dU}{dr} = -\frac{L^2}{mR^3} + \frac{1}{4\pi\varepsilon_0}\frac{e^2}{R^2} = 0$$

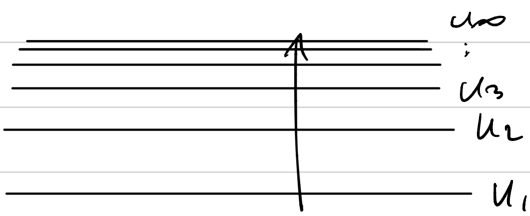
$$= -\frac{1}{R}\left(2U_K + U_p\right) = 0 \Rightarrow U_K = \frac{1}{2}U_p$$

$$R_n = \frac{4\pi\varepsilon_0}{me^2}(n\hbar)^2 \qquad U_n = -\frac{me^4}{32(\pi\varepsilon_0\hbar n)^2}$$

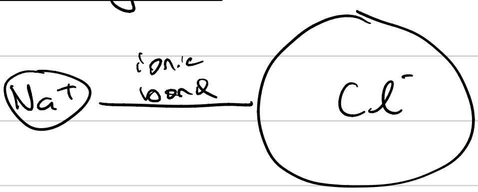
For n=1 ground state:

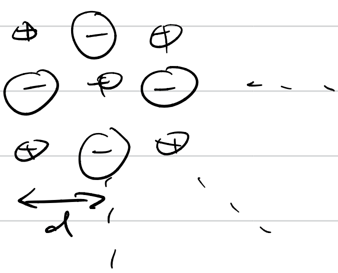
$R_1 = 5.3 \times 10^{-11}$ m $\Rightarrow$ 0.53 Å $= a_0$ "Bohrs Radius"

$U_1 = -2.18 \times 10^{-18}$ J $\rightarrow$ -13.6 eV (Ionization Energy)



Ionic Crystals





face-centered cubic-structure:
like the fcc 3D structure

The distance $d = 2.81 \text{ Å}$

What would be the crystal binding energy ~ "Energy required to separate the crystal into individual ions at infinite distance apart"

2N ions, ~ N molecules

$$U_{total} = \frac{1}{2}\sum_{i=1}^{2N} q_i \phi_i = \frac{1}{8\pi\varepsilon_0}\sum_{\substack{i,j \\ i\neq j}}^{2N} \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|}$$

Average Energy per Molecule

$$\frac{U_{tot}}{N} \sim \frac{-e^2}{4\pi\varepsilon_0}\sum_{j=2}^{2N}\frac{\pm 1}{|\vec{r}_i - \vec{r}_j|}$$

$$= -\frac{e^2}{4\pi\varepsilon_0 R}\underbrace{\sum_{j=2}^{2N}\frac{\pm 1}{|\vec{r}_i - \vec{r}_j|/R}}$$

"Madelung Constant" determined by the crystal structure.

For the crystal to be stable: $M > 0$

NaCl $M = 1.748$

$$\frac{U_{tot}}{N} \simeq -1.43 \times 10^{-18} \text{ J/molecule} = -8.95 \text{ eV/molecule}$$

Experimentally $\Rightarrow$ -8 eV/molecule

Chapter 3: Laplace's Equation

- Motivation

$$\rho \leftrightarrow \phi : \phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} dV'$$

$$\nabla^2\phi = -\frac{\rho}{\varepsilon_0} \text{ (poisson equation)}$$

+boundary condition $\phi \longrightarrow$ then we can easily solve for the potential

$$\left(\phi, \frac{\partial \phi}{\partial n} \text{ at boundary}\right)$$
$\hookrightarrow \hat{n} \perp dl$

If we have no free charges $\Rightarrow$ Poisson Eq $\to$ Laplace's Eq.

where the solutions are harmonic functions

$$\boxed{\nabla^2\phi = 0}$$

Laplace in 1-D

x: $\frac{d^2\phi}{dx^2} = 0$

Lecture 8

Lecture 9

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Lecture 10

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Lecture 11

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Lecture 12

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2.1 Review

Mar 5 10:10

Separation of variables with spherical coordinates will give a special function specifically the spherical harmonics

$$Y_v^m(\theta, \phi)e^{im\phi}P_i^m \cos\theta$$

FOr the cylindrical coordinates we have the Bessel function

$$e^{in\phi}, e^{\pm kz}, \text{ Bessel function}$$

2.2 Cylindrical Coordinates

$$\Phi(r, \phi, z) \approx \sum_{m,n} \left[A_{mn}I_n(k_m r) + B_{mnN_n(l_m r)}\right] e^{\pm in\phi \pm kmz}$$

Example. We have a charged wire where ϕ_0 is the flux inside and ϕ on the surface and $\phi = 0$ for above the surface. We know from here that no charge $\Rightarrow B_{mn} = 0$; symmetry $\Rightarrow n = 0$. Thus we get the equation

$$\Phi(r, \phi, z) = \sum_m A_m J_0(k_m r)e^{-k_m z}$$

From our boundary conditions we have $\Phi(a, \phi, z) = 0$ This type of boundary condition tells the roots and our harmonic is the bessel function. Another boundary condition is

$$\Phi(r, \phi, 0) = \phi_0 \to A_m$$

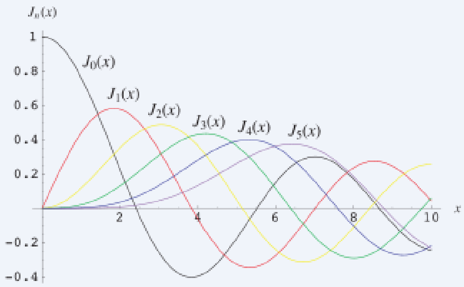

Figure 2.4

The orthoganality condition we get here is simply

$$\int_0^a J_n(k_m r)J_n(k_L r)\,\mathrm{d}r = \frac{a^2}{2}J_{n+1}^2(k_L a)\delta_{mL}$$

2.3 Multipole Expansion

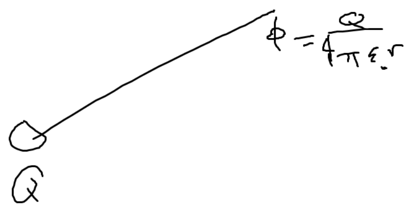

Figure 2.5

figure Suppose we had a dipole here, then we will get the form that the potential to be

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_+ - \frac{q}{r_-}}\right)$$

$$\frac{1}{r\pm} = \frac{1}{r\sqrt{1 \mp \frac{d}{r}\cos\theta}} \rightarrow \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right)$$

$$\frac{1}{r_+} - \frac{1}{r_-} = \frac{d}{r^2}\cos\theta$$

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{qd\cos\theta}{r^2}\infty\frac{1}{r^2}$$

Thus for monopoles the potential falls off as a function of $\frac{1}{r}$ and the dipole will hall of as a function of $\frac{1}{r^2}$. For quadruples we get that the potential falls as a function as $\frac{1}{r^3}$. If we continue with a cube we will have that the potential fall off as a function of $\frac{1}{r^4}$. Thus if we are interested in a far-field approximation we only care about the monopole term, and if 0 then consider the dipole term and continue. This is similar to when expanding the potential term as a function of $\frac{1}{r}$. Thus we can do the following expansion for general charge distributions:

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r'})}{|\vec{r} - \vec{r'}|}dV'$$

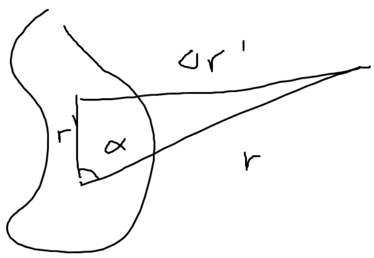

Figure 2.6

Suppose that we want to calculate the potential $\vec{r}$ away we get that

$$(\Delta r')^2 = r^2 + r'^2 - 2rr'\cos\alpha = r^2\left[1 + \left(\frac{r'}{r}\right)^2 - 2\frac{r'}{r}\cos\alpha\right]$$

$$\Delta r' = r\sqrt{1+\epsilon}, \epsilon = \frac{r'}{r}\left(\frac{r'}{r} - 2\cos\alpha\right)$$

For $\frac{r'}{r} \to 0, \epsilon \to 0$

$$\frac{1}{\Delta r'} = \frac{1}{r}(1+\epsilon)^{-\frac{1}{2}} = \frac{1}{r}\left(1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + \dots\right)$$

$$= \frac{1}{r}\left[1 + \frac{r'}{r} + \left(\frac{r'}{r}\right)^2\left(\frac{3\cos^2\alpha - 1}{2}\right) + \dots\right]$$

which are the Legendre polynomials.

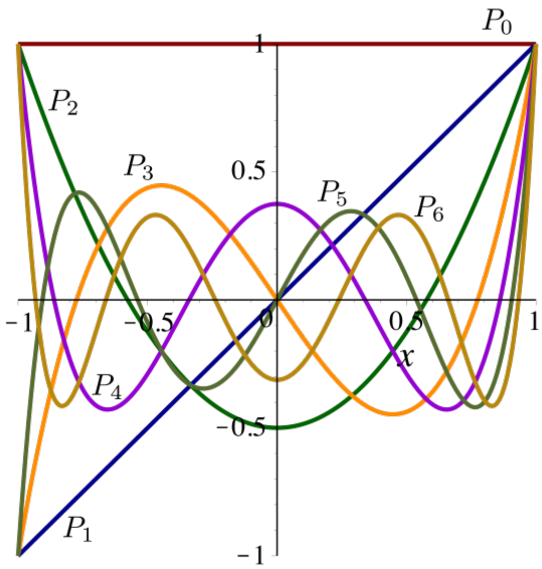

Figure 2.7

$$\frac{1}{\Delta r'} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\alpha)$$

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r}\int\rho(\vec{r'})dV' + \frac{1}{r^2}\int r'\cos\alpha\rho(\vec{r'})dV' + \dots\right]$$

We see that the first term gives the monopole and the second term gives the dipole and the third term will give you the quadrupole. Notice that for the physical dipole we are already making some assumptions. However, the formula above for $\Phi(\vec{r})$ is an exact equation when taking account of higher order terms. Notice the dipole term

$$\Phi_{dip}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int r'\cos\alpha\rho(\vec{r'})dV' \quad r'\cos\alpha = \hat{r}\cdot\vec{r'}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\hat{r}\cdot\int\vec{r'}\rho(\vec{r'})dV'$$

$$\Phi_{dip}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\hat{r}\cdot\vec{p}$$

where the geometry of the charge distribution is

$$\vec{p} = \sum_i^{\infty} q_i\vec{r'_i} = q\vec{d}$$

Remark. For the physical dipole, it gives the dipole term only when $\frac{d}{r} \to 0$. Thus, to construct a mathematical dipole we would need $d$to be very small. We see that for the mathematical dipole we have $\vec{p} = q\vec{d}$. Thus we must have the condition that $d \to 0$ and $q \to \infty$ in order to make $\vec{p}$ finite.

2.4 Dependence on Coordinate Origin

Suppose we placed a charge a distance d on the y-axis from the origin.

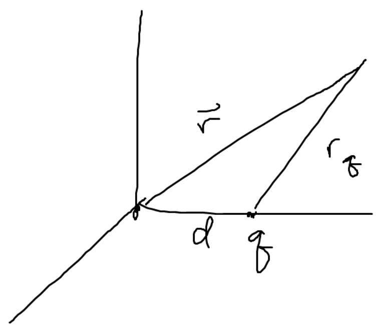

Figure 2.8

$$\Phi = \frac{1}{4\pi\epsilon_0}\frac{q}{r_q}$$

We see that this has a net dipole moment of $\vec{p} = qd\hat{y}$ which would have a nonzero dipole moment. Now suppose we wanted to calculate the dipole in shifted coordinates where

$$\tilde{\vec{p}} = \int \tilde{\vec{r}}' \rho(\vec{r}')dV' = \int (\vec{r}' - \vec{a})\, \rho(\vec{r}')dV'$$

$$= \int \vec{r}' \rho(\vec{r}')dV' - \vec{a}\int \rho(\vec{r}')dV'$$

$$= \vec{p} - \vec{a}Q$$

Thus, we ask where the origin is based on the dipole.

2.5 Electric field of Dipole

$$\Phi_{dip} = \frac{\hat{r}\cdot\vec{p}}{4\pi\epsilon_0 r^2} = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}$$

where the dipole is aligned with the z-axis. We have

$$E_r = -\frac{\partial\Phi_{dip}}{\partial r}, \quad E_\theta = -\frac{1}{r}\frac{\partial\Phi_{dip}}{\partial\theta}, \quad E_\phi = -\frac{1}{r\sin\theta}\frac{\partial\Phi_{dip}}{\partial\phi} = 0$$

$$\vec{E}_{dip} = \frac{p}{4\pi\epsilon_0 r^3}\left(2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right)$$

If we plot this given that we have an ideal small dipole we have

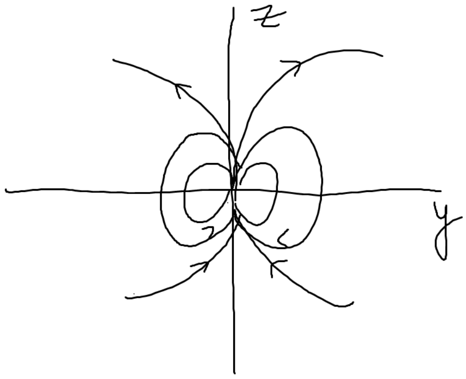

Figure 2.9

If the total charge is nonzero we must find the dipole. If there is something delicate, we can use the quadrupole terms and more which tell us how the field behaves in far-field.

Chapter 3

Electric Fields in Matter

3.1 Polarization

Recall that conductors have positively charged ions fixed to the lattice points while the electrons are freely moving. We can also consider dielectrics. Dielectrics are materials that don't allow current to flow. They are more often called insulators because they are the exact opposite of conductors.

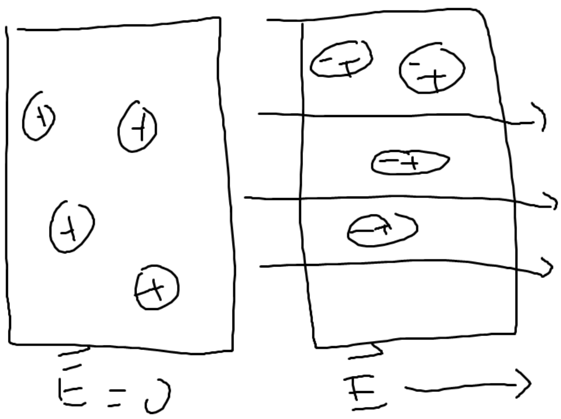

Figure 3.1

This gives rise to a macroscopic polarization that we get to see. For an induced dipole which we see here we get a net dipole moment or a simplified model. We can show that this can be represented in the following:

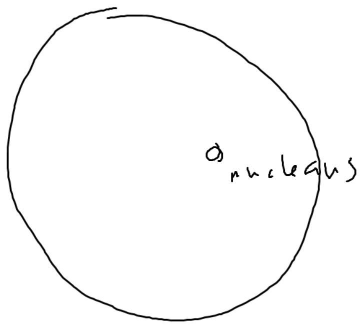

Figure 3.2

When we apply a $\vec{E} \rightarrow$ to the right, we see a shift here but we would like to know the calculation here where the shift balances the force from the external field. Thus, we can look at the force from the electron cloud to the nucleaus would be

$$F_{cloud \rightarrow nucleaus} = Ze\vec{E}_{cloud} = -Ze\vec{E}_{ext}$$

$$E_{cloud} = \frac{\rho r}{3\epsilon_0}\hat{r}$$