Recitation 5

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Welfare Maximization

Suppose (throwing an entire course in graduate-level economics into a black box) that a society's preferences over schools, S, and parks, P, are given by the following *welfare function*:

$$W(S,P) = \frac{3}{4}\ln S + \frac{1}{4}\ln P$$

Suppose further that the government has \$100 million to allocate between these two uses of public funds.

Each school costs \$25 million, and each park costs \$5 million.

What is the welfare-maximizing allocation of schools and parks?

The formal problem that we're solving is

$$\underset{S,P}{\text{Max}}\,\{W(S,P)\} \qquad \text{s.t. } 25S + 5P \leq 100; \qquad S,P \geq 0$$

We'll focus on the final part of solving this problem. First notice that, since welfare is increasing in both S and P, the government will spend all \$100 million (in more common terms – society is improved by any additional school or park, so there is no incentive not to spend all of the money).

In math, this means that

$$25S + 5P = 100$$

(Be sure you understand what this equation means!)

With this, we can eliminate one of the two choice variables, say P, by replacing P everywhere with $P = 20 - 5S$. Thus the problem simplifies to:

$$\underset{S}{\text{Max}}\left\{\frac{3}{4}\ln S + \frac{1}{4}\ln(20-5S)\right\}$$

We're trying to maximize the function $W(S) \equiv \frac{3}{4}\ln S + \frac{1}{4}\ln(20-5S)$ with respect to S. What on earth does this function look like?

Can plot with R or Wolfram Alpha, with output in Figure 1:

Our goal is to identify analytically the exact place where this function is maximized – visually, it's in the neighborhood of $S = 3$.

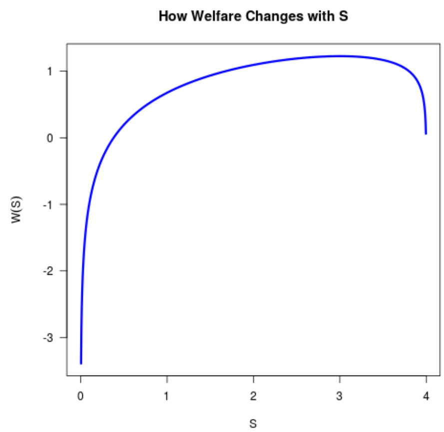

Figure 1: Appearance of Welfare Function at Budget Line

1. What is the analytic/calculus condition that describes the value of S that achieves the maximum of this function?
2. What is the derivative of W with respect to S?
3. What value of S eliminates the derivative (i.e., for which S is $W'(S) = 0$)?
4. *(Complete in your free time)* What is the second derivative of W with respect to S, and what can we learn from it about the critical point discovered in part 3)?

1. $W'(S) = 0$
2. $\frac{3}{4}\frac{1}{S} - \frac{5}{4}\frac{1}{20-5S}$
3. $S = 3$
4. $$W''(S) = -\frac{3}{4}\frac{1}{S^2} - \frac{25}{4}\frac{1}{(20-5S)^2} < 0$$

   The second derivative is always negative, so any critical points correspond to local maxima.

Some Matrix Algebra

Express the following system of equations as a single matrix equation of the form $AX = B$:

$$
\begin{aligned}
2x + y + 2z &= 10 \\
x + y + z &= 6 \\
x + 3y + 2z &= 13
\end{aligned}
$$

$$
A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix}; \qquad B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}
$$

The solution to the matrix equation $AX = B$ is given by $X = A^{-1}B$.

1. Find A^{-1}.
2. Find $A^{-1}B$.

1. $\begin{bmatrix} -1 & 4 & -1 \\ -1 & 2 & 0 \\ 2 & -5 & 1 \end{bmatrix}$
2. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$