1.40 Theorem. For any integers a and b not both 0, there are integers x and y such that $ax + by = (a, b)$.

Proof. Let $d = (a, b)$ and $ax + by = k$ for $k \in \mathbb{N}$. Since $d|a$ and $d|b$, $d|k$. Thus, $d \leq k$. Let $S = \{\text{all } c \text{ that can be written as } ax + by \mid c \in \mathbb{N}\}$. Letting $x = a$ and $b = y$, we find that $a^2 + b^2$ equals a natural number. Thus, the set is non-empty. By the WOANN, there exists a smallest element, call it k. Suppose k does not divide a. By TDA,

$$\begin{aligned} a &= kq + r, \\ r &= a - kq \text{ with } 0 < r < k. \end{aligned}$$

Substituting $k = ax + by$ into r,

$$\begin{aligned} r &= a - q(ax + by) \\ &= a - aqx + bqy \\ &= a(1 - qx) + b(qy). \end{aligned}$$

Notice now that r can be written as $ax' + by'$ which contradicts k being the smallest that can be expressed in that form. Thus, $k|a$. Without loss of generality, the same argument can be such that $k|b$. Thus, $k = (a, b) = d$.

Gathering our info, we have $(a, b) \leq k$ and $(a, b) = k$. Since k cannot be greater than AND equal to (a, b), it must be that $k = (a, b)$. Thus, $ax + by = (a, b)$. $\square$