1.33 Theorem. Let $a, b, n, r \in \mathbb{Z}$ with a and b not both 0. If $a = bn + r$, then $(a, b) = (b, r)$.

Proof. Let $a, b, n, r \in \mathbb{Z}$ with a and b not both 0 be given such that $a = bn+r$. Since a and b are not both 0, b and r are also both not 0. We want to show $(a, b) = (b, r)$. Let $d_1, d_2 \in \mathbb{Z}$ such that $d_1 = (a, b)$ and $d_2 = (b, r)$.

Since $d_1 = (a, b)$, we can say $d_1 \mid a$ and $d_1 \mid b$, and similarly, $d_2 \mid b$ and $d_2 \mid r$, provided $d_2 = (b, r)$. Observing $a = bn + r$, since $d_2 \mid b$, d_2 divides any multiple of b. Both terms are divisible by d_2, therefore, the sum is also divisible by d_2. Thus, d_2 also divides a.

Since $d_1 = (a, b)$, we can say d_2 is a common factor of a and b. This leads us to

$$d_2 \leq d_1.$$

Observing $a - bn = r$, since $d_1 \mid b$, d_1 divides any multiple of b. Both terms are divisible by d_1, therefore, the sum is also divisible by d_1. Thus, d_1 also divides r.

Since $d_2 = (b, r)$, we can say d_1 is a common factor. This leads us to

$$d_1 \leq d_2.$$

Since d_1 cannot be less than and greater than d_2 at the same time, $(a, b) = (b, r)$. $\square$