A.20 Theorem. For every natural number $n > 3$, $2^n < n!$.

Proof. Let $P(n)$ be the statement $2^n < n!$ for$n > 3$. We consider the base case where $n = 4$.

For $P(4)$,

$$\begin{aligned} 2^4 &< 4!, \\ 16 &< 24. \end{aligned}$$

Since the base case is true, we will prove by induction. Suppose now, $2^k < k!$ for $k > 3$. We want to show $2^{k+1} < (k+1)!$. Multiplying both sides by $k+1$,

$$(k+1)2^k < (k+1)k!.$$

Observing the right side of (1) and remembering algebra, for any natural number z, $z! = z(z-1)!$. Substituting z for $k+1$,

$$\begin{aligned} z(z-1)! &= (k+1)(k+1-1)! \\ &= (k+1)k! \\ &= (k+1)!. \end{aligned}$$

Now we find (1) to be

$$(k+1)2^k < (k+1)!. \tag{1}$$

Suppose we compare the left side of (2) to the left side of the hypothesis. Assuming $k \geq 4$, we can say $2 < (k+1)$ is true. Factoring, we find $2^{k+1} = (2)2^k$. Because $2 < (k+1)$, we can say

$$\begin{aligned} 2^{k+1} &< (k+1)2^k < (k+1)!, \\ 2^{k+1} &< (k+1)!. \end{aligned}$$

Since $P(k+1)$ is true, given $P(k)$ is true, and the base case of $P(n=4)$ is true, $2^n < n!$ for every natural number $n > 3$ is true by induction. $\square$