A.18 Theorem. For every natural number n, $1+2+2^2+\cdots+2^n = 2^{n+1}-1$.

Proof. Let $P(n)$ be the statement $1+2+2^2+\cdots+2^n = 2^{n+1}-1$. We consider the base case where $n=0$.

For $P(0)$,

$$\begin{aligned} 2^0 &= 2^{0+1}-1, \\ 1 &= 2-1 \\ &= 1. \end{aligned}$$

Since the base case is true, we will prove by induction. Suppose now, $1+2+2^2+\cdots+2^k = 2^{k+1}-1$ for some natural number k. We want to show $1+2+2^2+\cdots+2^k+2^{k+1} = 2^{k+2}-1$. It follows,

$$\begin{aligned} 1+2+2^2+\cdots+2^k+2^{k+1} &= 2^{k+1}-1+2^{k+1} \\ &= 2(2^{k+1})-1 \\ &= 2^{k+2}-1. \end{aligned}$$

Since $P(k+1)$ is true, given $P(k)$ is true, and the base case of $P(n=0)$ is true, $1+2+2^2+\cdots+2^n = 2^{n+1}-1$ is true by induction. $\square$