2.8 Lemma. Let p and $q_1, q_2, ..., q_n$ all be primes and let k be a natural number such that $pk = q_1 q_2 ... q_n$. Then $p = q_i$ for some i.

Proof. Let p and $q_1, q_2, ..., q_n$ all be primes and let k be a natural number such that $pk = q_1 q_2 ... q_n$ be given. Consider the base case where $n = 1$ such that

$$pk = q_1.$$

Since p is prime, $k = 1$ by primality. Thus, $p = q_1$. By induction, suppose $pk = q_1 q_2 ... q_b$ with $1 < n < b$ such that $p = q_i$. We want to show when $pk = q_1 q_2 ... q_b q_{b+1}$, $p = q_i$. Since $pk = q_1 q_2 ... q_b q_{b+1}$, we can rewrite this as

$$p | q_1 q_2 ... q_b q_{b+1}.$$

Let $a = q_1 q_2 ... q_b$ for some $a \in \mathbb{Z}$ such that $p | a q_{b+1}$. Suppose $(p, a) = 1$. By Theorem 1.41, since $p | a q_{b+1}$ and $(p, a) = 1$, $p | q_{b+1}$. By definition, $pt = q_{b+1}$ for some $t \in \mathbb{Z}$. Since p is prime, $t = 1$ by primality. Thus, $p = q_i$. Since our base case and inductive hypothesis is true, the Lemma is true. $\square$