**A.21 Theorem.** For every natural number $n$, $1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$.

**Proof.** Let $P(n)$ be the statement $1^3+2^3+3^3+\cdots+n^3 = (1+2+3+\cdots+n)^2$. We consider the base case where $n = 1$.

For $P(1)$,

$$\begin{aligned} 1^3 &= (1)^2, \\ 1 &= 1. \end{aligned}$$

Since the base case is true, we will prove by induction. Suppose now, $1^3 + 2^3 + 3^3 + \cdots + k^3 = (1 + 2 + 3 + \cdots + k)^2$ for some natural number $k$. We want to show $1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 = (1 + 2 + 3 + \cdots + (k+1))^2$. By Theorem A.10, we know the right side to be $\frac{(k+1)^2(k+2)^2}{4}$.

Examining the left side of the equation,

$$\begin{aligned} 1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 &= (1 + 2 + 3 + \cdots + k)^2 + (k+1)^3 \\ &= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 \\ &= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\ &= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\ &= \frac{(k+1)^2[k^2 + 4(k+1)]}{4} \\ &= \frac{(k+1)^2(k^2 + 4k + 4)}{4} \\ &= \frac{(k+1)^2(k+2)^2}{4}. \end{aligned}$$

Since $P(k+1)$ is true, given $P(k)$ is true, and the base case of $1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$ for every natural number $n$ is true by induction. $\square$