1.48 Theorem. Given $a, b, c \in \mathbb{Z}$ and a, b not both 0, there exists $x, y \in \mathbb{Z}$ that satisfy the equation $ax + by = c$ if and only if $(a, b) \mid c$.

Proof. Let $a, b, c \in \mathbb{Z}$ and a, b not both 0 be given. Suppose there exists $x, y \in \mathbb{Z}$ that satisfy the equation $ax + by = c$. Let $d = (a, b)$ where $a = da'$ and $b = db'$ for some $a', b' \in \mathbb{Z}$. Thus,

$$\begin{aligned} c &= ax + by \\ &= da'x + db'y \\ &= d(a'x + b'y). \end{aligned}$$

Thus, $(a, b) \mid c$.

Now suppose $(a, b) \mid c$. Let $d = (a, b)$. By Theorem 1.40, there exists $s, t \in \mathbb{Z}$ such that $as + bt = d$. We have that $d \mid c$ such that $(as + bt) \mid c$. By definition, for some $k \in \mathbb{Z}$,

$$\begin{aligned} c &= (as + bt)k \\ &= ask + btk. \end{aligned}$$

By CPI, let $sk = x$ and $tk = y$ for some $x, y \in \mathbb{Z}$. Thus, $ax + by = c$.

Since both directions of the biconditional statement are true, there exists $x, y \in \mathbb{Z}$ that satisfy the equation $ax + by = c$ if and only if $(a, b) \mid c$. $\square$