3.28 Theorem. Let $a, b, m, n \in \mathbb{Z}$ with $m, n > 0$ and $(m, n) = 1$. Then the system

$$\begin{aligned} x &\equiv a \pmod{n} \\ x &\equiv b \pmod{m} \end{aligned}$$

has a solution unique modulo mn.

Proof. Suppose

$$\begin{aligned} x &\equiv a \pmod{n} \\ x &\equiv b \pmod{m} \end{aligned}$$

and

$$\begin{aligned} y &\equiv a \pmod{n} \\ y &\equiv b \pmod{m}. \end{aligned}$$

By Theorem 1.13,

$$\begin{aligned} x - y &\equiv a - a \pmod{n} & \qquad x - y &\equiv b - b \pmod{m}, \\ x - y &\equiv 0 \pmod{n} & \qquad x - y &\equiv 0 \pmod{m}. \end{aligned}$$

Now our system is

$$\begin{aligned} x - y &\equiv 0 \pmod{n} \\ x - y &\equiv 0 \pmod{m}. \end{aligned}$$

This implies, $n|(x-y)$ and $m|(x-y)$. By Theorem 1.42, $mn|(x-y)$, implying $x \equiv y \pmod{mn}$. Since $(m, n) = 1$, we can conclude the system has a solution by Theorem 3.27. Furthermore, the solution is unique modulo mn. $\square$