4.2 Theorem. Let $a, n \in \mathbb{N}$ with $(a, n) = 1$. Then $(a^j, n) = 1$ for any $j \in \mathbb{N}$.

Proof. Let $j = 2$ be the base case such that $(a^2, n) = 1$. Since $(a, n) = 1$, by Theorem 1.38, there exists $x, y \in \mathbb{Z}$ such that $ax + ny = 1$. Multiplying by $ax + ny$,

$$\begin{aligned} ax + ny &= (ax + ny)(ax + ny) \\ &= a^2x^2 + 2axny + n^2y^2 \\ &= a^2x^2 + n(2axy + ny^2) \\ &= a^2x' + ny' \end{aligned}$$

where, by CPI, integers $x' = x^2$ and $y' = 2axy + ny^2$. Looking at the left hand side, we know that $ax + ny = 1$. Thus,

$$1 = a^2x' + ny'.$$

By Theorem 1.39, $(a^2, n) = 1$. Thus, the base case is true. Suppose our assumption is true for all j where $1 \leq j \leq k$. By induction, we want to show $(a^{k+1}, n) = 1$ is also true. By our assumption, $(a^k, n) = 1$. By Theorem 1.38, there exists $t, u \in \mathbb{Z}$ such that $a^k t + nu = 1$. Multiplying by $ax + ny$,

$$\begin{aligned} ax + ny &= (a^k t + nu)(ax + ny) \\ &= a^{k+1}tx + a^k tny + axnu + n^2uy \\ &= a^{k+1}tx + n(a^k ty + axu + nuy) \\ &= a^{k+1}t' + nu' \end{aligned}$$

where, by CPI, integers $t' = tx$ and $u' = a^k ty + axu + nuy$. Looking at the left hand side, we know that $ax + ny = 1$. Thus,

$$1 = a^{k+1}t' + nu'.$$

By Theorem 1.39, $(a^{k+1}, n) = 1$. Thus, $(a^j, n) = 1$ for any $j \in \mathbb{N}$. $\square$