4.14 Theorem. Let p be a prime and let a be an integer not divisible by p; that is, $(a, p) = 1$. Then

$$a \cdot 2a \cdot 3a \cdots (p-1)a \equiv 1 \cdot 2 \cdot 3 \cdot (p-1) \pmod{p}.$$

Proof. Let $A = \{a, 2a, 3a, ..., (p-1)a\}$ and $N = \{1, 2, 3, ..., p-1\}$. By Theorem 4.13, $ia \equiv j \pmod{p}$ where $1 \leq i, j \leq p-1$. Note that $p|pa$ is equivalent to $pa \equiv 0 \pmod{p}$. By Theorem 1.14, the product of A is congruent to the product of N modulo p. Thus, $a \cdot 2a \cdot 3a \cdots (p-1)a \equiv 1 \cdot 2 \cdot 3 \cdot (p-1) \pmod{p}$. $\square$