Lemma IR For all primes p, $\sqrt{p}$ is irrational.

Proof. Suppose not. That is let $\sqrt{p} \in \mathbb{Q}$. By definition, $\sqrt{p} = \frac{a}{b}$ for $a, b \in \mathbb{Z}$. Suppose $\frac{a}{b}$ is irreducible such that $(a, b) = 1$. Squaring both sides,

$$\begin{aligned} p &= \frac{a^2}{b^2}, \\ pb^2 &= a^2. \end{aligned}$$

Since $p|a^2$, p must be a prime factor of a. Thus $a = pk$ for some $k \in \mathbb{Z}$. Substituting,

$$\begin{aligned} pb^2 &= (pk)^2, \\ b^2 &= pk^2. \end{aligned}$$

Notice that both a and b share the prime factor p. Since primes are greater than 1, this contradictions the assumption that $(a, b) = 1$. Thus, for all primes p, $\sqrt{p}$ is irrational. $\square$

2.19 Theorem. There do not exist natural numbers m and n such that $7m^2 = n^2$.

Proof. Let natural numbers m, n be given. Suppose $7m^2 = n^2$. By the Fundamental Theorem of Arithmetic,

$$7(q_1^{2t_1} q_2^{2t_2} ... q_s^{2t_s}) = p_1^{2r_1} p_2^{2r_2} ... p_m^{2r_m}.$$

By Lemma 2.8,

$$\begin{aligned} 7 &= p_i^{2r_i}, \\ \sqrt{7} &= p_i^{r_i}. \end{aligned}$$

We know that prime numbers are natural numbers, and therefore rational numbers. Yet, by Lemma IR, $\sqrt{7}$ is irrational. Thus, we have a contradiction. Hence, there do not exist natural numbers m and n such that $7m^2 = n^2$. $\square$