A.19 Theorem. For every natural number n, $1^2 + 2^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

Proof. Let $P(n)$ be the statement $1^2 + 2^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$. We consider the base case where $n = 1$.

For $P(1)$,

$$\begin{aligned} 1^2 &= \frac{1(1+1)(2(1)+1)}{6}, \\ 1 &= \frac{1(2)(2+1)}{6} \\ &= \frac{2(3)}{6} \\ &= \frac{6}{6} \\ &= 1. \end{aligned}$$

Since the base case is true, we will prove by induction. Suppose now, $1^2 + 2^2 + 2^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$ for some natural number k. We want to show $1^2 + 2^2 + 2^2 + \cdots + k^2 + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$. It follows,

$$\begin{aligned} 1^2 + 2^2 + 2^2 + \cdots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\ &= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} \\ &= \frac{(k+1)(2k^2 + k + 6k + 6)}{6} \\ &= \frac{(k+1)(2k^2 + 7k + 6)}{6} \\ &= \frac{(k+1)(k+2)(2k+3)}{6}. \end{aligned}$$

Since $P(k+1)$ is true, given $P(k)$ is true, and the base case of $P(n=1)$ is true, $1^2 + 2^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ is true by induction. $\square$