2.12 Theorem Let a and b be natural numbers greater than 1 and let $p_1^{r_1}p_2^{r_2}...p_m^{r_m}$ be the unique prime factorization of a and let $q_1^{t_1}q_2^{t_2}...q_s^{t_s}$ be the unique prime factorization of b. Then $a|b$ if and only if for all $i \leq m$ there exists a $j \leq s$ such that $p_i = q_j$ and $r_i \leq t_j$.

Proof. Let a and b be natural numbers greater than 1 and let $p_1^{r_1}p_2^{r_2}...p_m^{r_m}$ be the unique prime factorization of a and let $q_1^{t_1}q_2^{t_2}...q_s^{t_s}$ be the unique prime factorization of b be given such that $a|b$ if and only if for all $i \leq m$ there exists a $j \leq s$ such that $p_i = q_j$ and $r_i \leq t_j$.

Suppose $a|b$. By the FTA, write $a = p_1^{r_1}p_2^{r_2}...p_m^{r_m}$ and $b = q_1^{t_1}q_2^{t_2}...q_s^{t_s}$. We will show two things. First, for any $1 \leq i \leq m$ there is a corresponding value $1 \leq j \leq s$ such that $p_i = q_j$. Second, for such i and j, $r_i \leq t_j$. By definition, $b = an$ for some $n \in \mathbb{Z}$ such that

$$q_1^{t_1}q_2^{t_2}...q_s^{t_s} = p_1^{r_1}p_2^{r_2}...p_m^{r_m}(n).$$

By the FTA, write $n = d_1^{v_1}d_2^{v_2}...d_g^{v_g}$. Equivalently, we have

$$\begin{aligned} q_1^{t_1}q_2^{t_2}...q_s^{t_s} &= (p_1^{r_1}p_2^{r_2}...p_m^{r_m})(d_1^{v_1}d_2^{v_2}...d_g^{v_g}), \\ &= p_1^{r_1}d_1^{v_1}p_2^{r_2}d_2^{v_2}...p_m^{r_m}d_g^{v_g}. \end{aligned}$$

Without loss of generality, we can combine all common primes on the right hand side and keep the $p_i^{r_i}$ naming convention such that

$$q_1^{t_1}q_2^{t_2}...q_s^{t_s} = p_1^{r_1}p_2^{r_2}...p_m^{r_m}.$$

By the uniqueness part of the FTA, the prime factorization of b must equal the right hand side. Thus, for any $1 \leq i \leq m$ there is a corresponding value $1 \leq j \leq s$ such that $p_i = q_j$, and secondly, for such i and j, $r_i \leq t_j$ (their exponents).

Suppose that without loss of generality that $a = p_1^{r_1} \cdots p_m^{r_m}$ and $b = p_1^{t_1} \cdots p_m^{t_m} \cdots p_s^{t_s}$ where $r_i \leq t_i$ for all $1 \leq i \leq m$. We will show $a|b$. Let $k \in \mathbb{Z}$ with a UPF of $p_1^{v_1}p_2^{v_2}...p_g^{v_g}$ such that when multiplied to a,

$$\begin{aligned} ak &= (p_1^{r_1}p_2^{r_2}...p_m^{r_m})(p_1^{v_1}p_2^{v_2}...p_g^{v_g}) \\ &= p_1^{r_1}p_1^{v_1}p_2^{r_2}p_2^{v_2}...p_m^{r_m}p_g^{v_g}. \end{aligned}$$

Letting $r_i + v_i = t_i$, $v_i = t_i - r_i$ where $v_i \geq 0$ because $r_i \leq t_i$, we now combine common factors. Thus,

$$an = p_1^{t_1} p_2^{t_2} ... p_s^{t_s}.$$

Let the $p_1^{t_1} p_2^{t_2} ... p_s^{t_s} = b$. Thus, $a|b$. $\square$