1.43 Theorem. Let $a, b, n \in \mathbb{Z}$. If $(a, n) = 1$ and $(b, n) = 1$, then $(ab, n) = 1$.

Proof. Let $a, b, n \in \mathbb{Z}$ be given such that $(a, n) = 1$ and $(b, n) = 1$. By Theorem 1.38, since $(a, n) = 1$ and $(b, n) = 1$, there exists $x, y, s, t \in \mathbb{Z}$ such that $ax+ny = 1$ and $bs+nt = 1$, respectively. Multiplying the two equations together,

$$\begin{aligned} 1 &= (ax + ny)(bs + nt) \\ &= axbs + axnt + nybs + n^2yt \\ &= ab(xs) + n(axt + ybs + nyt). \end{aligned}$$

By CPI, $xy = q$ and $axt+ybs+nyt = r$ for some $q, r \in \mathbb{Z}$. Thus, $abq+nr = 1$. By Theorem 1.39, $(ab, n) = 1$. $\square$