1.53 Theorem. Let $a, b, c \in \mathbb{Z}$ with a and b not both 0. If $x = x_0$, $y = y_0$ is an integer solution to the equation $ax + by = c$ (that is, $ax_0 + by_0 = c$) then for every $k \in \mathbb{Z}$, the numbers

$$x = x_0 + \frac{kb}{(a,b)} \text{ and } y = y_0 - \frac{ka}{(a,b)}$$

are integers that also satisfy the linear Diophantine equation $ax + by = c$. Moreover, every solution to the linear Diophantine equation $ax + by = c$ is of this form.

Proof. Let $a, b, c \in \mathbb{Z}$ with a and b not both 0 be given. Let $x = x_0$, $y = y_0$ be an integer solution to the equation $ax + by = c$. Thus,

$$\begin{aligned} c &= a\left[x_0 + \frac{kb}{(a,b)}\right] + b\left[y_0 - \frac{ka}{(a,b)}\right] \\ &= ax_0 + \frac{kab}{(a,b)} + by_0 - \frac{kab}{(a,b)} \\ &= ax_0 + by_0. \end{aligned}$$

Thus, $x = x_0$, $y = y_0$ is an integer solution to the equation $ax + by = c$.

Moreover, let $m, n \in \mathbb{Z}$ such that $m = x - x_0$ and $n = y - y_0$. Notice $x = x_0 + m$ and $y = y_0 + n$. Substituting,

$$\begin{aligned} c &= a(x_0 + m) + b(y_0 + n) \\ &= ax_0 + by_0 + am + bn. \end{aligned}$$

Recalling $ax_0 + by_0 = c$ and substituting,

$$\begin{aligned} c &= c + am + bn, \\ 0 &= am + bn. \end{aligned}$$

Thus, $bn = -am$. Letting $d = (a, b)$ such that $d|a$ and $d|b$. By definition, $a = dA$ and $b = dB$ for $A, B \in \mathbb{Z}$. Substituting,

$$dBn = -dAm,$$
$$Bn = -Am.$$

Thus, $B| - Am$ and we know $d = 1$, and without loss of generality, $B \neq 0$. By Theorem 1.41, $B|m$ such that $m = Bk$ for $k \in \mathbb{Z}$. By substitution,

$$Bn = -AkB,$$
$$n = -Ak.$$

Collecting ourselves, $c = a(x_0 + m) + b(y_0 + n)$, $m = Bk$, $n = -Ak$, $A = \frac{a}{d}$, $B = \frac{b}{d}$, and $d = (a, b)$. Substituting in what we know,

$$\begin{aligned} c &= a(x_0 + Bk) + b(y_0 + (-Ak)) \\ &= a\left[x_0 + \frac{b}{d}k\right] + b\left[y_0 - \frac{a}{d}k\right] \\ &= a\left[x_0 + \frac{bk}{(a,b)}\right] + b\left[y_0 - \frac{ak}{(a,b)}\right]. \end{aligned}$$

Thus, every solution to the linear Diophantine equation $ax + by = c$ is of this form. $\square$