Lemma. Let $n_1, n_2, ..., n_k$ be natural numbers. If $(n_i, n_j) = 1$ such that $i \neq j$ and $1 \leq i, j, \leq k$, then $(n_1 n_2 ... n_{k-1}, n_k) = 1$.

Proof. Let $k = 3$ be our base case. By Theorem 2.29, $(n_1 n_2, n_3) = 1$. Suppose all k is true where $1 \leq k \leq t$. We want to show $(n_1 n_2 ... n_t, n_{t+1}) = 1$. Since we know up to t is true, by Theorem 2.29, $(n_1 n_2 ... n_t, n_{t+1}) = 1$. Thus, if $(n_i, n_j) = 1$ such that $i \neq j$ and $1 \leq i, j, \leq k$, then $(n_1 n_2 ... n_{k-1}, n_k) = 1$. $\square$

3.29 Theorem. (Chinese Remainder Theorem). Suppose $n_1, n_2, ..., n_L$ are positive integers that are pairwise relatively prime, that is, $(n_i, n_j) = 1$ for $i \neq j$, $1 \leq i, j \leq L$. Then the system of L congruences

$$\begin{aligned} x &\equiv a_1 \pmod{n_1} \\ x &\equiv a_2 \pmod{n_2} \\ &\vdots \\ x &\equiv a_L \pmod{n_L} \end{aligned}$$

has a unique solution modulo the product $n_1 n_2 ... n_L$.

Proof. Let $L = 2$. Consider this the base case. Thus,

$$\begin{aligned} x &\equiv a_1 \pmod{n_1} \\ x &\equiv a_2 \pmod{n_2}. \end{aligned}$$

Since $(n_1, n_2) = 1$, by Theorem 3.28, $x \equiv x' \pmod{n_1 n_2}$. Thus, the base case is true. Suppose this is true for all L where $1 \leq L \leq K$. By induction, we want to show

$$\begin{aligned} x &\equiv a_1 \pmod{n_1} \\ x &\equiv a_2 \pmod{n_2} \\ &\vdots \\ x &\equiv a_L \pmod{n_K} \\ x &\equiv a_L \pmod{n_{K+1}} \end{aligned}$$

also has a unique solution modulo the product $n_1n_2...n_Kn_{K+1}$. Thus, the system of congruences is

$$\begin{aligned}
x &\equiv a_1 \pmod{n_1} \\
x &\equiv a_2 \pmod{n_2} \\
&\vdots \\
x &\equiv a_K \pmod{n_K} \\
x &\equiv a_{K+1} \pmod{n_{K+1}}.
\end{aligned}$$

By our induction hypothesis and Theorem 3.28, we know up to K is
$x \equiv x' \pmod{n_1n_2...n_K}$. Thus,

$$\begin{aligned}
x &\equiv x' \pmod{n_1n_2...n_K} \\
x &\equiv a_{K+1} \pmod{n_{K+1}}.
\end{aligned}$$

By Theorem 3.28, since $(n_1n_2...n_K, n_{K+1}) = 1$ by the Lemma and Theorem 2.29, and solution x satisfies

$$x \equiv x'' \pmod{n_1n_2...n_Kn_{K+1}},$$

for $x'' \in \mathbb{Z}$. Thus, the system of L congruences has a unique solution modulo the product $n_1n_2...n_L$. $\square$