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Parcial 2 señales y sistemas

(1) Encontrar la expresión del espectro de Fourier [forma exponencial y trigonométrica] para la señal $x(t) = |6\sin(3t + \pi/4)|^2$ con $t \in [-\pi, \pi]$.

solución

$$x(t) = |6\sin(3t+\pi/4)|^2 = 36\sin^2(3t+\pi/4)$$

$$36\left(\mathrm{sen}(3t)\cos(\pi/4) + \mathrm{sen}(\pi/4)\cos(3t)\right)^2$$

$$36\left(\frac{\sqrt{2}}{2}\cdot\left(\mathrm{sen}(3t) + \cos(3t)\right)\right)^2$$

$$36\left(\frac{1}{2}\left(\cancel{\mathrm{sen}^2(3t)} + 2\,\mathrm{sen}(3t)\cos(3t) + \cancel{\cos^2(3t)}\right)\right)$$

$$18\left(1 + 2\,\mathrm{sen}(3t)\cos(3t)\right)$$

$$18 + 18\,\mathrm{sen}(6t)$$

Al graficar la señal $x(t)$ en el cuaderno de colab, se determina que la señal es de tipo <u>IMPAR</u>.

• Expresión del espectro de Fourier → forma trigonométrica.

$$\tilde{x}(t) = a_0 + \sum_{n=-N}^{N} a_n\cos(n\omega_0 t) + b_n\,\mathrm{sen}(n\omega_0 t)$$

$$a_0 = C_0 = \frac{1}{T_f - T_i}\int_{ti}^{tf} x(t)\,dt$$

$$a_0 = \frac{1}{\pi-(-\pi)}\int_{-\pi}^{\pi}(18 + 18\,\mathrm{sen}(6t))\,dt = \frac{18}{2\pi}\int_{-\pi}^{\pi}(1+\mathrm{sen}(6t))\,dt$$

$$= \frac{9}{\pi}\left(\int_{-\pi}^{\pi}dt + \int_{-\pi}^{\pi}\mathrm{sen}(6t)\,dt\right) = \frac{9}{\pi}\left(t\Big|_{-\pi}^{\pi} + \frac{(-\cos(6t))}{6}\Big|_{-\pi}^{\pi}\right)$$

$$= \frac{9}{\cancel{\pi}}\left(2\cancel{\pi} - \left(\frac{\overset{1}{\cancel{\cos(6\pi)}}}{6} - \frac{\overset{1}{\cancel{\cos(-6\pi)}}}{6}\right)\right) = \boxed{18 = a_0 = C_0}$$

$$a_n = \frac{2}{T_f - T_i}\int_{ti}^{tf} \underbrace{x(t)\cos(n\omega_0 t)}_{\text{impar}\cdot\text{par}=\text{impar}}\,dt$$ → como $x(t)$ es impar y $\cos(\theta)$ es par, entonces: $a_n = 0$

$$a_b = \frac{2}{T_f - T_i}\int_{ti}^{tf} x(t)\,\mathrm{sen}(n\omega_0 t)\,dt$$

$$a_b = \frac{2}{2\pi}\int_{-\pi}^{\pi}(18 + 18\,\mathrm{sen}(6t))\,\mathrm{sen}(n\omega_0 t)$$

$$b_n = \frac{18}{\pi}\int_{-\pi}^{\pi}(1+\operatorname{sen}(6t))\operatorname{sen}(n\omega_0 t)\,dt = \frac{18}{\pi}\int_{-\pi}^{\pi}(\operatorname{sen}(n\omega_0 t)+\operatorname{sen}(6t)\operatorname{sen}(n\omega_0 t))\,dt$$

$$b_n = \frac{18}{\pi}\left(\int_{-\pi}^{\pi}\operatorname{sen}(n\omega_0 t)\,dt + \int_{-\pi}^{\pi}\frac{\cos(6t-n\omega_0 t)}{2} - \frac{\cos(6t+n\omega_0 t)}{2}\,dt\right.$$

$$= \frac{18}{\pi}\left(-\frac{\cos(nt)}{n} + \frac{\operatorname{sen}(6t-nt)}{2(6-n)} - \frac{\operatorname{sen}(6t+nt)}{2(6+n)}\right)\Bigg|_{-\pi}^{\pi}$$

$\omega_0 = \frac{2\pi}{T}$

$T = 2\pi$

$\omega_0 = 1 \ \frac{rad}{s}$

$$= \frac{18}{\pi}\left(-\frac{1}{n}\underbrace{\cancel{(\cos(n\pi)-\cos(-n\pi))}}_{0} + \frac{\operatorname{sen}(\pi(6-n))}{2(6-n)} - \frac{\operatorname{sen}(\pi(6+n))}{2(6+n)} - \right.$$

$$-\left(\frac{\operatorname{sen}(-\pi(6-n))}{2(6-n)} - \frac{\operatorname{sen}(-\pi(6+n))}{2(6+n)}\right)$$

$$b_n = \frac{18}{\pi}\left(\frac{\operatorname{sen}(\pi(6-n)) - \operatorname{sen}(-\pi(6-n))}{2(6-n)} + \frac{\operatorname{sen}(-\pi(6+n)) - \operatorname{sen}(\pi(6+n))}{2(6+n)}\right)$$

Calculamos el límite para las indeterminaciones en +6 y -6

$$\frac{18}{\pi}\lim_{n\to+6}\frac{\operatorname{sen}(\pi(6-n)) - \operatorname{sen}(-\pi(6-n))}{2(6-n)} = \frac{0}{0}$$ → Aplicando L'hopital

$$\frac{18}{\pi}\lim_{n\to+6}\frac{\cos(\pi(6-n))(-\pi) - \cos(-\pi(6-n))(\pi)}{-2} =$$

$$\frac{18}{\pi}\left(\frac{\cancel{\cos(\pi(6-6))}^{1}(-\pi) - \cancel{\cos(-\pi(6-6))}^{1}(\pi)}{-2}\right) = \left(\frac{18}{\pi}\right)\left(\frac{-\pi-\pi}{-2}\right) = \frac{-\cancel{2}\cancel{\pi}}{\cancel{-2}}\left(\frac{18}{\cancel{\pi}}\right) = 18$$

$$\frac{18}{\pi}\lim_{n\to-6}\frac{\operatorname{sen}(-\pi(6+n)) - \operatorname{sen}(\pi(6+n))}{2(6+n)} = \frac{0}{0}$$ → Aplicando L'hopital

$$\frac{18}{\pi}\lim_{n\to-6}\frac{\cos(-\pi(6+n))(-\pi) - \cos(\pi(6+n))(\pi)}{2} =$$

$$\frac{18}{\pi}\left(\frac{\cancel{\cos(-\pi(6-6))}^{1}(-\pi) - \cancel{\cos(\pi(6-6))}^{1}(\pi)}{2}\right) = \left(\frac{18}{\pi}\right)\left(\frac{-\pi-\pi}{2}\right) = \left(\frac{18}{\cancel{\pi}}\right)\left(\frac{-\cancel{2}\cancel{\pi}}{\cancel{2}}\right) = -18$$

$$C_n = \frac{a_n - jb_n}{2} \Rightarrow C_6 = \frac{-j18}{2} = -j9$$ → Armónico +6

$$C_{-6} = \frac{-j(-18)}{2} = j9$$ → Armónico -6

$$C_0 = a_0 = 18$$ → Armónico 0

$$C_n = \begin{cases} -j9 & n=6 \\ j9 & n=-6 \\ 18 & n=0 \\ 0 & \forall n / \{-6,0,6\} \end{cases}$$

• Expresión del espectro de Fourier → forma exponencial

$$C_n = \frac{1}{T_f - T_i}\int_{t_i}^{t_f} x(t)\,e^{-jnw_0 t}\,dt \qquad T = \pi - (-\pi) = 2\pi$$

$$C_n = \frac{18}{2\pi}\int_{-\pi}^{\pi}(1+\operatorname{sen}(6t))\,e^{-jnw_0 t}\,dt \qquad w_0 = \frac{2\pi}{T}$$

$$w_0 = \frac{2\pi}{2\pi} = 1\,\frac{rad}{s}$$

$$C_n = \frac{9}{\pi}\left(\int_{-\pi}^{\pi} e^{-jnw_0 t} + \int_{-\pi}^{\pi}\operatorname{sen}(6t)\cdot e^{-jnw_0 t}\,dt\right)$$

Aplicando integral por partes y linealidad

$$C_n = \frac{9}{\pi}\left(\frac{e^{-jnt}}{-jn} - \frac{e^{-jnt}(jn\operatorname{sen}(6t) + 6\cos(6t))}{j^2n^2+36}\right)\Bigg|_{-\pi}^{\pi}$$

$$C_n = \frac{9}{\pi}\left(\frac{e^{-jn\pi}}{-jn} - \frac{e^{jn\pi}}{-jn} - \frac{e^{-jn\pi}(jn\cancel{\operatorname{sen}(6\pi)}^{0} + 6\cancel{\cos(6\pi)}^{1})}{-n^2+36} + \frac{e^{jn\pi}(jn\cancel{\operatorname{sen}(-6\pi)}^{0} + 6\cancel{\cos(-6\pi)}^{1})}{-n^2+36}\right)$$

$$C_n = \frac{9}{\pi}\left(\frac{1(2)}{n}\left(\frac{-e^{-jn\pi}+e^{jn\pi}}{2j}\right) - \frac{6e^{-jn\pi}}{-n^2+36} + \frac{6e^{jn\pi}}{-n^2+36}\right)$$

→ Aplicando → $\operatorname{sen}(\theta) = \left(\frac{e^{j\theta}-e^{-j\theta}}{2j}\right)$; $\cos(\theta) = \left(\frac{e^{j\theta}+e^{-j\theta}}{2}\right)$

$$C_n = \frac{9}{\pi}\left(\frac{2}{n}\cdot\operatorname{sen}(n\pi) + \frac{6\,(2j)}{-n^2+36}\left(\frac{-e^{-jn\pi}+e^{jn\pi}}{2j}\right)\right)$$

$$C_n = \frac{9}{\pi}\left(\frac{2}{n}\cdot\operatorname{sen}(n\pi) + \frac{12j}{-n^2+36}\cdot\operatorname{sen}(n\pi)\right)$$

Calculamos el límite en las indeterminaciones 0, +6, -6

$$\frac{9}{\pi}\lim_{n\to 0}\frac{2\operatorname{sen}(n\pi)}{n} = \frac{0}{0} \to \text{Aplicando L'hopital}$$

$$\frac{9}{\pi}\lim_{n\to 0}\frac{2\pi\cos(n\pi)}{1} = \left(\frac{9}{\pi}\right)2\pi\cancel{\cos(0)}^{1} = 18 = C_0 = a_0$$

Ya encontrado de la anterior forma

Armónico 0

$$\frac{9}{\pi}\lim_{n\to +6}\frac{12j}{-n^2+36}\cdot\operatorname{sen}(n\pi) = \frac{0}{0} \to \text{Aplicando L'hopital}$$

$$\frac{9}{\pi}\lim_{n\to +6}\frac{12j\pi\cdot\cos(n\pi)}{-2n} = \left(\frac{9}{\pi}\right)\frac{j12\pi\cancel{\cos(6\pi)}^{1}}{(-2)(6)_2} = \frac{j36}{-4} = \boxed{-j9 = C_{+6}}$$

Armónico +6

$$\frac{9}{\pi}\lim_{n\to -6}\frac{12j}{-n^2+36}\cdot\operatorname{sen}(n\pi) = \frac{0}{0} \to \text{Aplicando L'hopital}$$

$$\frac{9}{\pi}\lim_{n\to -6}\frac{j12\pi\cos(n\pi)}{-2n} = \left(\frac{9}{\pi}\right)\frac{j12\pi\cancel{\cos(-6\pi)}^{1}}{(-2)(-6)} = \boxed{j9 = C_{-6}}$$

Armónico -6

• Cálculo del error relativo

$$\mathcal{E}_r[\%] = \left[1 - \frac{1}{P_x}\sum_{n=-N}^{N}|C_n|^2\right] 100\%$$

$$\omega_0 = \frac{2\pi}{T}$$

$$P_x = \frac{1}{T}\int_{t_i}^{t_f}|x(t)|^2\,dt \qquad T = \pi - (-\pi) = 2\pi \qquad \omega = \frac{2\pi}{2\pi} = 1\,\frac{rad}{s}$$

$$P_x = \frac{1}{2\pi}\int_{-\pi}^{\pi}|18 + 18\,sen(6t)|^2\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} 18^2 + 2(18)(18)sen(6t) + (18\,sen(6t))^2\,dt$$

$$P_x = \frac{18^2}{2\pi}\int_{-\pi}^{\pi} 1 + 2\,sen(6t) + sen^2(6t)\,dt$$

$$P_x = \frac{162}{\pi}\left(t + 2\left(\frac{-\cos(6t)}{6}\right)\Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\frac{1}{2} - \frac{\cos(12t)}{2}\,dt\right)$$

$$P_x = \frac{162}{\pi}\left(\pi - (-\pi) - \frac{1}{3}\left(\underbrace{\cancel{\cos(6\pi)}}_{1} - \underbrace{\cancel{\cos(-6\pi)}}_{1}\right) + \frac{1}{2}\left(t - \frac{sen(12t)}{12}\right)\Big|_{-\pi}^{\pi}\right) \quad 1 - 1 = 0$$

$$P_x = \frac{162}{\pi}\left(2\pi + \frac{1}{2}\left(\pi - (-\pi) - \frac{1}{12}\left(\underbrace{\cancel{sen(12\pi)}}_{0} - \underbrace{\cancel{sen(-12\pi)}}_{0}\right)\right)\right)$$

$$P_x = \frac{162}{\pi}\left(2\pi + \frac{1}{\cancel{2}}(\cancel{2}\pi)\right) \Rightarrow P_x = \frac{162}{\pi}(2\pi + \pi)$$

$$P_x = \left(\frac{162}{\cancel{\pi}}\right)(3\cancel{\pi}) = 486$$

Entonces $\Rightarrow$ $$\mathcal{E}_r[\%] = \left[1 - \frac{|C_{-6}|^2 + |C_0|^2 + |C_6|^2}{486}\right] \cdot 100\%$$

$$\mathcal{E}_r[\%] = \left[1 - \frac{9^2 + 18^2 + 9^2}{486}\right] \cdot 100\%$$

$$\mathcal{E}_r[\%] = [1 - 1] \cdot 100\%$$

$$\mathcal{E}_r[\%] = 0\,\%$$

(2) Sea la señal portadora $c(t) = A_c \cdot \cos(2\pi F_c t)$, con $A_c, F_c \in \mathbb{R}$ y la señal mensaje $m(t) \in \mathbb{R}$. Encuentre el espectro en la frecuencia de la señal modulada en amplitud (AM).

$$y(t) = \left(1 + \frac{m(t)}{A_c}\right) \cdot c(t)$$

Solución

$$Y(\omega) = \mathcal{F}\{y(t)\} = \mathcal{F}\left\{\left(1 + \frac{m(t)}{A_c}\right) \cdot c(t)\right\} = \mathcal{F}\{c(t)\} + \frac{1}{A_c} \mathcal{F}\{m(t)\, c(t)\}$$

Usando tablas de Fourier

$$\rightarrow C(\omega) = \mathcal{F}\{c(t)\} = \mathcal{F}\{A_c \cdot \cos(2\pi F_c t)\} = A_c \mathcal{F}\left\{\frac{e^{j2\pi F_c t} + e^{-j2\pi F_c t}}{2}\right\}$$

$$\bullet\ \mathcal{F}\{e^{\pm j\omega_0 t}\} = 2\pi \delta(\omega \mp \omega_0)$$

$$\rightarrow \frac{A_c}{\cancel{2}} \left(\cancel{2}\pi \delta(\omega - 2\pi F_c) + \cancel{2}\pi \delta(\omega + 2\pi F_c)\right)$$

$$C(\omega) = A_c \cdot \pi \left(\delta(\omega - 2\pi F_c) + \delta(\omega + 2\pi F_c)\right)$$

$$\frac{1}{A_c} \mathcal{F}\{m(t) \cdot c(t)\} = \frac{1}{\cancel{A_c}} \mathcal{F}\{m(t)\, \cancel{A_c} \cdot \cos(2\pi F_c t)\}$$

$$= \mathcal{F}\left\{\frac{m(t)\, e^{j2\pi F_c t} + m(t)\, e^{-j2\pi F_c t}}{2}\right\}$$

$$\bullet\ \mathcal{F}\{X(t) \cdot e^{\pm j\omega_0 t}\} = X(\omega \mp \omega_0)$$

$$\rightarrow \frac{1}{2}\left(M(\omega - 2\pi F_c) + M(\omega + 2\pi F_c)\right)$$

Espectro de la señal modulada:

$$\boxed{Y(\omega) = A_c \cdot \pi \left(\delta(\omega - 2\pi F_c) + \delta(\omega + 2\pi F_c)\right) + \frac{1}{2}\left(M(\omega - 2\pi F_c) + M(\omega + 2\pi F_c)\right)}$$