```
Clear[Evaluate[Context[] <> "*"]]

(* Here is fake signal data. *)
ω = {0.1325, 0.3275, 0.754, 0.9600, 0.4900};
A = {1.0, 0.001, 0.2, 6.1, 0.1};
ϕ = {2.090, 1.090, 0.000, 1.234, 2.034};
NN = 1024;

proj[u_, v_] := Sum[Conjugate[u[t]] * v[t], {t, 0, NN - 1}] / NN; (* project u onto v *)
easyInt[ωj_, ωk_] := If[ωj == ωk, NN - 1, -i / (2 π (-ωj + ωk)) (1 - Exp[-i 2 π (-ωj + ωk) (NN - 1)])];

(* v are the individual components of the signal.*)
v1[t_] := A[[1]] Exp[-i (2 π ω[[1]] t + ϕ[[1]])];
v2[t_] := A[[2]] Exp[-i (2 π ω[[2]] t + ϕ[[2]])];
v3[t_] := A[[3]] Exp[-i (2 π ω[[3]] t + ϕ[[3]])];
v4[t_] := A[[4]] Exp[-i (2 π ω[[4]] t + ϕ[[4]])];
v5[t_] := A[[5]] Exp[-i (2 π ω[[5]] t + ϕ[[5]])];
signal[t_] := v1[t] + v2[t] + v3[t] + v4[t] + v5[t];


(* In actual implementation, e1 is a complex sinusoidal with frequency component
 determined by first estimating with interpolated FFT, then refined by applying a
 maximizing the projection of u1 onto the signal.  Same applies to e2, e3, ...
  In this mock-up, the frequency is simply copied. *)

(* As said above, assume ω[[1]] was found through a FFT and refinement process. *)
(* en are the frequency component.  u1 are the orthogonalized counterpart. *)
e1[t_] := Exp[-i 2 π ω[[1]] t];
u1[t_] := e1[t];

e2[t_] := Exp[-i 2 π ω[[2]] t];
p21 = proj[u1, e2];
u2[t_] := e2[t] - p21 u1[t];

e3[t_] := Exp[-i 2 π ω[[3]] t];
p31 = proj[u1, e3];
u31[t_] := e3[t] - p31 u1[t];
p32 = proj[u2, u31];
u3[t_] := u31[t] - p32 u2[t];

e4[t_] := Exp[-i 2 π ω[[4]] t];
```

```
p41 = proj[u1, e4];
u41[t_] := e4[t] - p41 u1[t];
p42 = proj[u2, u41];
u42[t_] := u41[t] - p42 u2[t];
p43 = proj[u3, u42];
u4[t_] := u42[t] - p43 u3[t];

e5[t_] := Exp[-i 2 π ω⟦5⟧ t];
p51 = proj[u1, e5];
u51[t_] := e5[t] - p51 u1[t];
p52 = proj[u2, u51];
u52[t_] := u51[t] - p52 u2[t];
p53 = proj[u3, u52];
u53[t_] := u52[t] - p53 u3[t];
p54 = proj[u4, u53];
u5[t_] := u53[t] - p54 u4[t]

Print["1-<u1|u1> = ", 1 - proj[u1, u1]]
Print["1-<u2|u2> = ", 1 - proj[u2, u2]]
Print["1-<u3|u3> = ", 1 - proj[u3, u3]]
Print["1-<u4|u4> = ", 1 - proj[u4, u4]]
Print["1-<u5|u5> = ", 1 - proj[u5, u5]]
Print["<u1|u2> = ", proj[u1, u2]]
Print["<u1|u3> = ", proj[u1, u3]]
Print["<u2|u3> = ", proj[u2, u3]]
Print["<u1|u4> = ", proj[u1, u4]]
Print["<u2|u4> = ", proj[u2, u4]]
Print["<u3|u4> = ", proj[u3, u4]]
Print["<u1|u5> = ", proj[u1, u5]]
Print["<u2|u5> = ", proj[u2, u5]]
Print["<u3|u5> = ", proj[u3, u5]]
Print["<u4|u5> = ", proj[u4, u5]]

(*Plot[{Re[u1[t]],Im[u1[t]],Re[u2[t]],Im[u2[t]],
  Re[u3[t]],Im[u3[t]],Re[u4[t]],Im[u4[t]],Re[u5[t]],Im[u5[t]]},{t,0,1}]*)
```

```
1-<u1|u1> = 0. + 0. i
1-<u2|u2> = 2.05626 × 10^-6 + 0. i
1-<u3|u3> = 1.58154 × 10^-6 + 0. i
1-<u4|u4> = 3.83381 × 10^-6 + 0. i
1-<u5|u5> = 5.57929 × 10^-6 + 0. i
<u1|u2> = -2.92735 × 10^-18 + 2.49366 × 10^-18 i
```

$\langle u1|u3\rangle = -4.06576 \times 10^{-18} - 8.67362 \times 10^{-19}\, i$

$\langle u2|u3\rangle = 8.58792 \times 10^{-10} - 1.25941 \times 10^{-9}\, i$

$\langle u1|u4\rangle = -2.60209 \times 10^{-18} + 2.22261 \times 10^{-18}\, i$

$\langle u2|u4\rangle = 1.83859 \times 10^{-9} - 2.7642 \times 10^{-10}\, i$

$\langle u3|u4\rangle = -3.07444 \times 10^{-10} - 3.31544 \times 10^{-10}\, i$

$\langle u1|u5\rangle = 1.38236 \times 10^{-18} + 5.96311 \times 10^{-18}\, i$

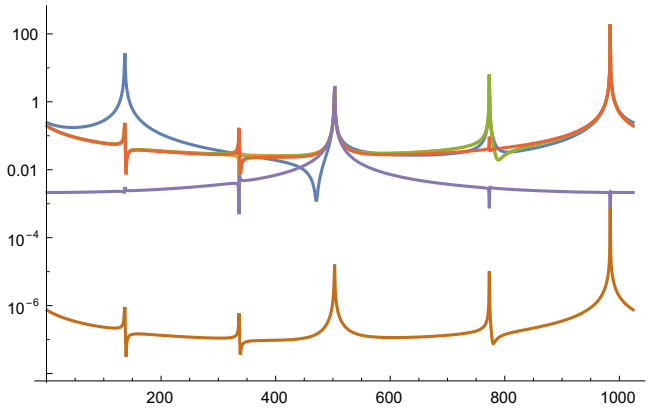
$\langle u2|u5\rangle = 2.86838 \times 10^{-9} - 2.65151 \times 10^{-9}\, i$

$\langle u3|u5\rangle = 1.77363 \times 10^{-9} + 4.0817 \times 10^{-10}\, i$

$\langle u4|u5\rangle = 2.3907 \times 10^{-9} - 1.62472 \times 10^{-9}\, i$

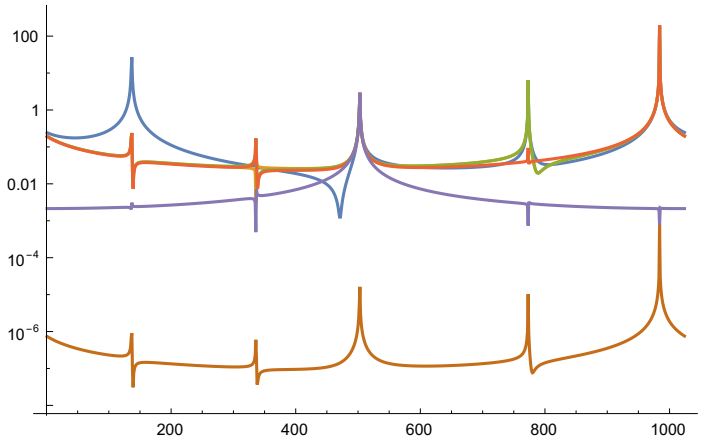
Method 1 : Project each basis component onto the full signal, then subtract.

```
a1 = Conjugate[proj[signal, u1]];
a2 = Conjugate[proj[signal, u2]];
a3 = Conjugate[proj[signal, u3]];
a4 = Conjugate[proj[signal, u4]];
a5 = Conjugate[proj[signal, u5]];
sampledSignal = Table[signal[t], {t, 0, NN - 1}];
sample2 = Table[signal[t] - a1 u1[t], {t, 0, NN - 1}];
sample3 = Table[signal[t] - a1 u1[t] - a2 u2[t], {t, 0, NN - 1}];
sample4 = Table[signal[t] - a1 u1[t] - a2 u2[t] - a3 u3[t], {t, 0, NN - 1}];
sample5 = Table[signal[t] - a1 u1[t] - a2 u2[t] - a3 u3[t] - a4 u4[t], {t, 0, NN - 1}];
sample6 = Table[signal[t] - a1 u1[t] - a2 u2[t] - a3 u3[t] - a4 u4[t] - a5 u5[t], {t, 0, NN - 1}];
ftOrig = Fourier[sampledSignal];
ft2 = Fourier[sample2];
ft3 = Fourier[sample3];
ft4 = Fourier[sample4];
ft5 = Fourier[sample5];
ft6 = Fourier[sample6];
ListLogPlot[{Abs[ftOrig], Abs[ft2], Abs[ft3], Abs[ft4], Abs[ft5], Abs[ft6]},
 PlotRange → {Full, Full}, Joined → True]
```



Method 2 : Project then subtract, project then subtract.

```
a1 = Conjugate[proj[signal, u1]];
a2 = Conjugate[proj[signal[#] - a1 u1[#] &, u2]];
a3 = Conjugate[proj[signal[#] - a1 u1[#] - a2 u2[#] &, u3]];
a4 = Conjugate[proj[signal[#] - a1 u1[#] - a2 u2[#] - a3 u3[#] &, u4]];
a5 = Conjugate[proj[signal[#] - a1 u1[#] - a2 u2[#] - a3 u3[#] - a4 u4[#] &, u5]];
sampledSignal = Table[signal[t], {t, 0, NN - 1}];
sample2 = Table[signal[t] - a1 u1[t], {t, 0, NN - 1}];
sample3 = Table[signal[t] - a1 u1[t] - a2 u2[t], {t, 0, NN - 1}];
sample4 = Table[signal[t] - a1 u1[t] - a2 u2[t] - a3 u3[t], {t, 0, NN - 1}];
sample5 = Table[signal[t] - a1 u1[t] - a2 u2[t] - a3 u3[t] - a4 u4[t], {t, 0, NN - 1}];
sample6 = Table[signal[t] - a1 u1[t] - a2 u2[t] - a3 u3[t] - a4 u4[t] - a5 u5[t], {t, 0, NN - 1}];
ftOrig = Fourier[sampledSignal];
ft2 = Fourier[sample2];
ft3 = Fourier[sample3];
ft4 = Fourier[sample4];
ft5 = Fourier[sample5];
ft6 = Fourier[sample6];
ListLogPlot[{Abs[ftOrig], Abs[ft2], Abs[ft3], Abs[ft4], Abs[ft5], Abs[ft6]},
 PlotRange → {Full, Full}, Joined → True]
```



Cruft

```
ft3 = Fourier[sample3];

sample3 = Table[signal[t] - Conjugate[a1] u1[t] - Conjugate[a2] u2[t], {t, 0, NN - 1}];

Nfreqs = 3;
data = Table[Sum[A[[j]] Exp[-i (2 π ω[[j]] (i - 1) + ϕ[[j]])], {j, 1, Nfreqs}], {i, 1, NN}];


dataOrig = data;
```

```
e1disc = Table[f1[i], {i, 0, NN - 1}];
e2disc = Table[f2[i], {i, 0, NN - 1}];
e3disc = Table[f3[i], {i, 0, NN - 1}];

projection1 = Sum[data[[i]] * e1disc[[i]], {i, 1, NN}];
component1 = Table[projection1 * e1disc[[i]], {i, 1, NN}];
data = data - component1 ;
data1 = data;

Print["Projeciton Amplitude 1: ", projection1]
Print["Check Projection Amplitude 1: ", Sum[data[[i]] * e1disc[[i]], {i, 1, NN}]]

projection2 = Sum[data[[i]] * e2disc[[i]], {i, 1, NN}];
component2 = Table[projection2 * e2disc[[i]], {i, 1, NN}];
data = data - component2 ;
data2 = data;

Print["Projection Amplitude 2: ", projection2]
Print["Check Projection Amplitude 1: ", Sum[data[[i]] * e1disc[[i]], {i, 1, NN}]]
Print["Check Projection Amplitude 2: ", Sum[data[[i]] * e2disc[[i]], {i, 1, NN}]]

projection3 = Sum[data[[i]] * e3disc[[i]], {i, 1, NN}];
component3 = Table[projection3 * e3disc[[i]], {i, 1, NN}];
data = data - component3 ;
data3 = data;

Print["Projection Amplitude 3: ", projection3]
Print["Check Projection Amplitude 1: ", Sum[data[[i]] * e1disc[[i]], {i, 1, NN}]]
Print["Check Projection Amplitude 2: ", Sum[data[[i]] * e2disc[[i]], {i, 1, NN}]]
Print["Check Projection Amplitude 3: ", Sum[data[[i]] * e3disc[[i]], {i, 1, NN}]]

projection1a = Sum[data[[i]] * e1disc[[i]], {i, 1, NN}];
component1a = Table[projection1a * e1disc[[i]], {i, 1, NN}];
data = data - component1a ;
data1a = data;

Print["Projeciton Amplitude 1a: ", projection1a]
Print["Check Projection Amplitude 1: ", Sum[data[[i]] * e1disc[[i]], {i, 1, NN}]]
Print["Check Projection Amplitude 2: ", Sum[data[[i]] * e2disc[[i]], {i, 1, NN}]]
Print["Check Projection Amplitude 3: ", Sum[data[[i]] * e3disc[[i]], {i, 1, NN}]]

projection2a = Sum[data[[i]] * e2disc[[i]], {i, 1, NN}];
component2a = Table[projection2a * e2disc[[i]], {i, 1, NN}];
```

```
data = data - component2a ;
data2a = data;

Print["Projeciton Amplitude 2a: ", projection2a]
Print["Check Projection Amplitude 1: ", Sum[data[[i]] * e1disc[[i]], {i, 1, NN}]]
Print["Check Projection Amplitude 2: ", Sum[data[[i]] * e2disc[[i]], {i, 1, NN}]]
Print["Check Projection Amplitude 3: ", Sum[data[[i]] * e3disc[[i]], {i, 1, NN}]]

projection3a = Sum[data[[i]] * e3disc[[i]], {i, 1, NN}];
component3a = Table[projection3a * e3disc[[i]], {i, 1, NN}];
data = data - component3a ;
data3a = data;

Print["Projeciton Amplitude 3a: ", projection3a]
Print["Check Projection Amplitude 1: ", Sum[data[[i]] * e1disc[[i]], {i, 1, NN}]]
Print["Check Projection Amplitude 2: ", Sum[data[[i]] * e2disc[[i]], {i, 1, NN}]]
Print["Check Projection Amplitude 3: ", Sum[data[[i]] * e3disc[[i]], {i, 1, NN}]]

projection1b = Sum[data[[i]] * e1disc[[i]], {i, 1, NN}];
component1b = Table[projection1b * e1disc[[i]], {i, 1, NN}];
data = data - component1b ;
data1b = data;

Print["Projeciton Amplitude 1b: ", projection1b]
Print["Check Projection Amplitude 1: ", Sum[data[[i]] * e1disc[[i]], {i, 1, NN}]]
Print["Check Projection Amplitude 2: ", Sum[data[[i]] * e2disc[[i]], {i, 1, NN}]]
Print["Check Projection Amplitude 3: ", Sum[data[[i]] * e3disc[[i]], {i, 1, NN}]]

projection2b = Sum[data[[i]] * e2disc[[i]], {i, 1, NN}];
component2b = Table[projection2b * e2disc[[i]], {i, 1, NN}];
data = data - component2b ;
data2b = data;

Print["Projeciton Amplitude 2b: ", projection2b]
Print["Check Projection Amplitude 1: ", Sum[data[[i]] * e1disc[[i]], {i, 1, NN}]]
Print["Check Projection Amplitude 2: ", Sum[data[[i]] * e2disc[[i]], {i, 1, NN}]]
Print["Check Projection Amplitude 3: ", Sum[data[[i]] * e3disc[[i]], {i, 1, NN}]]
```

```
Projeciton Amplitude 1: -15.9203 + 27.8157 i

Check Projection Amplitude 1: 0.0155623 - 0.0271903 i

Projection Amplitude 2: 14.8094 + 28.3898 i

Check Projection Amplitude 1: 0.0111345 - 0.0246082 i
```

Check Projection Amplitude 2: -0.014484 - 0.027766 i

Projection Amplitude 3: 6.40315 - 0.0000175742 i

Check Projection Amplitude 1: 0.00439224 - 0.0195293 i

Check Projection Amplitude 2: -0.0149869 - 0.0285035 i

Check Projection Amplitude 3: $-0.00626319 + 1.71901 \times 10^{-8}\, i$

Projeciton Amplitude 1a: 0.00439224 - 0.0195293 i

Check Projection Amplitude 1: $-4.29349 \times 10^{-6} + 0.0000190903\, i$

Check Projection Amplitude 2: -0.01499 - 0.0285043 i

Check Projection Amplitude 3: -0.00628331 + 0.0000170971 i

Projeciton Amplitude 2a: -0.01499 - 0.0285043 i

Check Projection Amplitude 1: $1.51262 \times 10^{-7} + 0.0000164785\, i$

Check Projection Amplitude 2: 0.0000146606 + 0.000027878 i

Check Projection Amplitude 3: -0.00627885 + 0.0000176093 i

Projeciton Amplitude 3a: -0.00627885 + 0.0000176093 i

Check Projection Amplitude 1: $6.74872 \times 10^{-6} + 0.0000114797\, i$

Check Projection Amplitude 2: 0.0000151557 + 0.0000285997 i

Check Projection Amplitude 3: $6.14162 \times 10^{-6} - 1.72244 \times 10^{-8}\, i$

Projeciton Amplitude 1b: $6.74872 \times 10^{-6} + 0.0000114797\, i$

Check Projection Amplitude 1: $-6.59699 \times 10^{-9} - 1.12216 \times 10^{-8}\, i$

Check Projection Amplitude 2: 0.0000151576 + 0.0000285987 i

Check Projection Amplitude 3: $6.14362 \times 10^{-6} - 3.46651 \times 10^{-8}\, i$

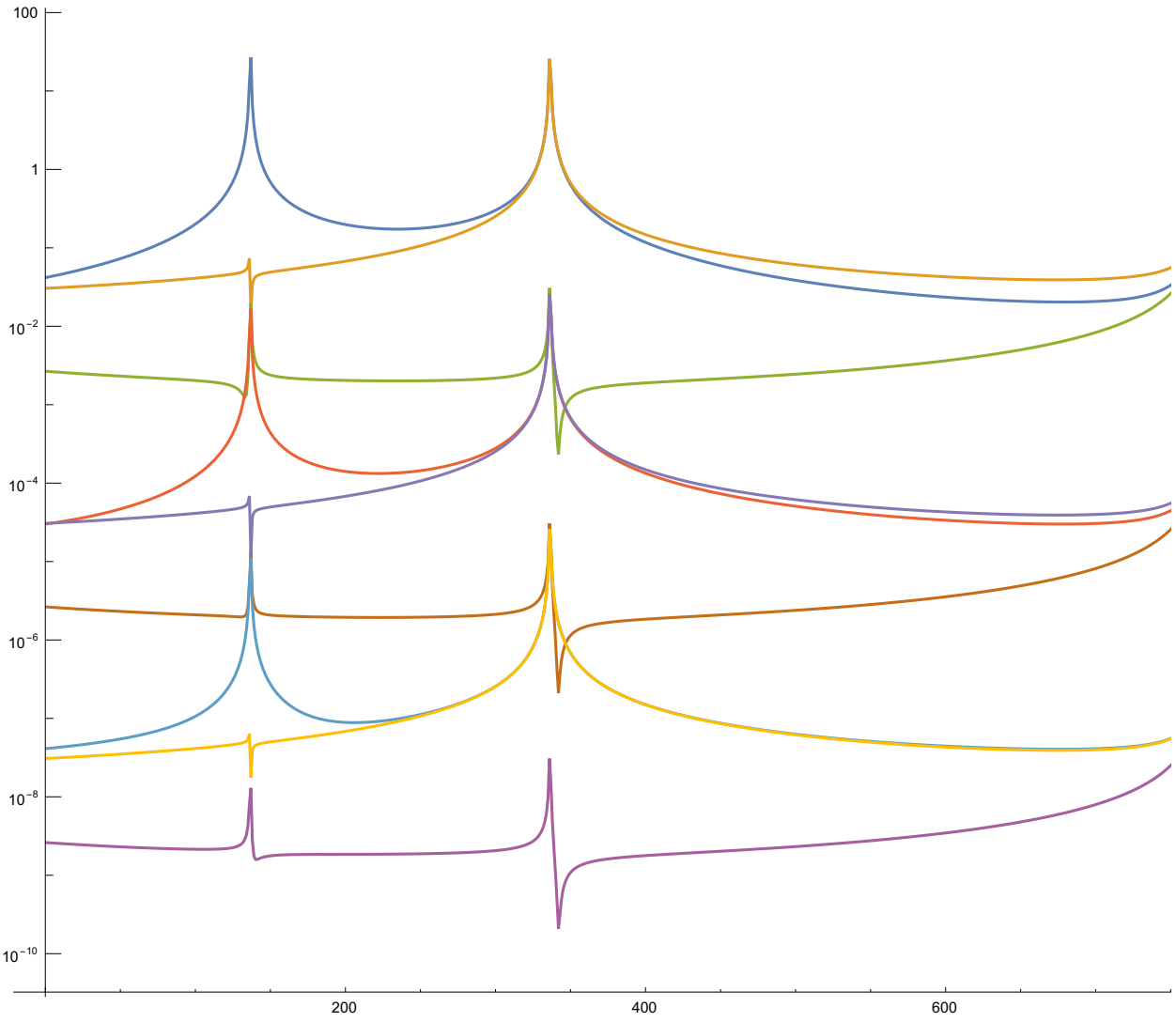
Projeciton Amplitude 2b: 0.0000151576 + 0.0000285987 i

Check Projection Amplitude 1: $-1.10556 \times 10^{-8} - 8.58223 \times 10^{-9}\, i$

Check Projection Amplitude 2: $-1.48246 \times 10^{-8} - 2.79704 \times 10^{-8}\, i$

Check Projection Amplitude 3: $6.13913 \times 10^{-6} - 3.51653 \times 10^{-8}\, i$

```
ftOrig = Fourier[dataOrig];
ft1 = Fourier[data1];
ft2 = Fourier[data2];
ft3 = Fourier[data3];
ft1a = Fourier[data1a];
ft2a = Fourier[data2a];
ft3a = Fourier[data3a];
ft1b = Fourier[data1b];
ft2b = Fourier[data2b];
ListLogPlot[{Abs[ftOrig], Abs[ft1], Abs[ft2], Abs[ft3], Abs[ft1a], Abs[ft2a],
  Abs[ft3a], Abs[ft1b], Abs[ft2b]}, PlotRange → {Full, Full}, Joined → True]
```



```
ωa = 39.408290474 ;
ωb = 10.733638813 ;
ω1 = 39.467277630 ;
list = Flatten[Table[l ωa + k ωb - m ω1, {l, -10, 10}, {k, -10, 10}, {m, -20, 20}]];
-ω1 + 2 ωb
RankedMin[Abs[list], 1]
RankedMin[Abs[list], 2]
ListPlot[list, PlotRange → {Full, {-0.1, 0.1}}]
```

```
-18.
```

```
710.411
```

```
0.
```

```
0.0589872
```