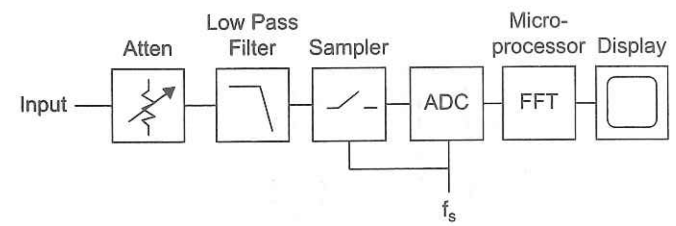

Figure 4-4 The simplified block diagram of the fast Fourier transform spectrum analyzer.

The FFT analyzer accomplishes the same thing that the bank-of-filters analyzer does, but without the need for many band-pass filters. Instead, the FFT analyzer uses digital signal processing to implement the equivalent of many individual filters. When considering the operation of the FFT analyzer, it is appropriate to think in terms of a bank of parallel filters, each filtering a portion of the frequency spectrum. A typical FFT spectrum analyzer is shown in Figure 4-5.

Conceptually, the FFT approach is simple and straightforward: digitize the signal and compute the spectrum. In practice, some effects must be accounted for to make the measurement meaningful.

4.4 Sampled Waveform

In a sampled system, the time domain waveform (Figure 4-6a) is effectively multiplied by the sample function (Figure 4-6b) to produce the sampled waveform (Figure 4-6c).

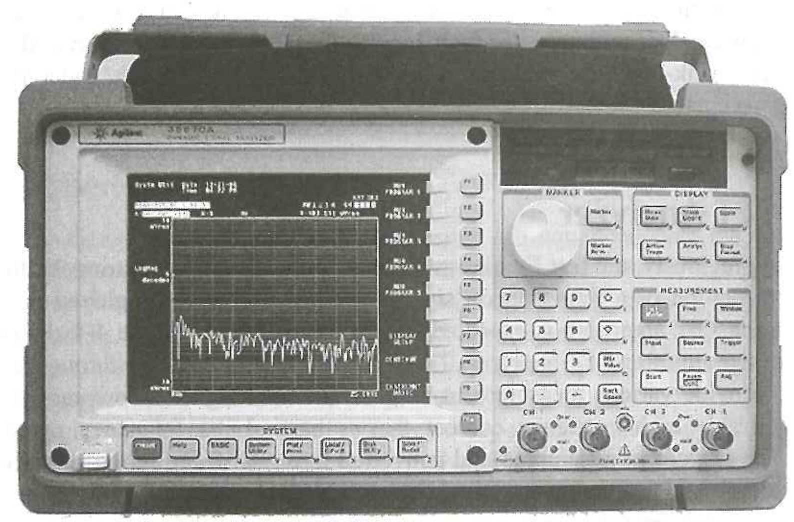

Figure 4-5 A typical four-channel FFT-based spectrum analyzer. (© Keysight Technologies, Inc. Reproduced with Permission, Courtesy of Keysight Technologies, Inc.)

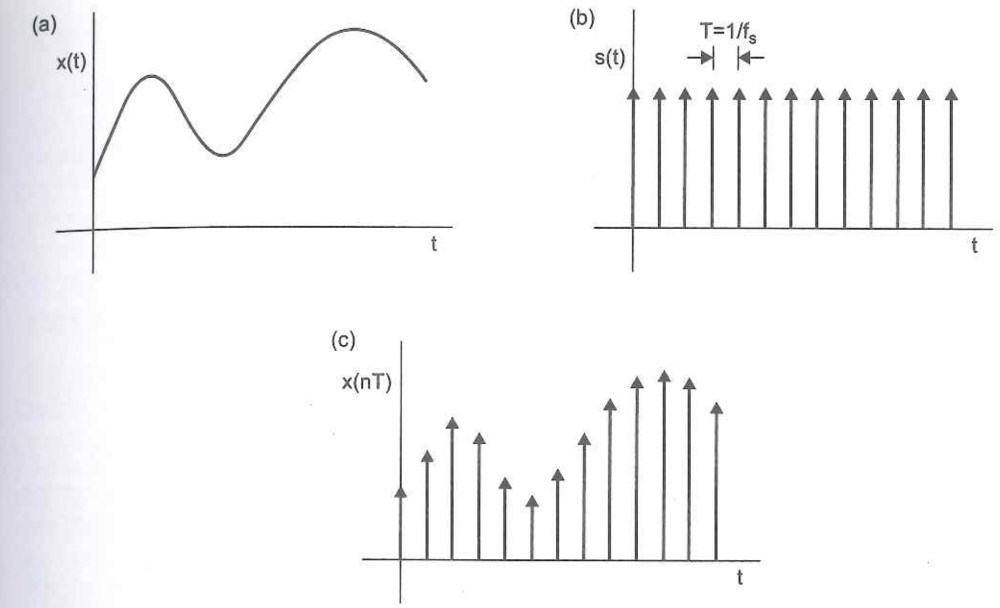

Figure 4-6 (a) A time domain waveform. (b) The sampling function. (c) The sampled waveform.

The sampling function is shown as a series of impulse functions, spaced at $T = 1/f_s$, where f_s is the sample rate of the system.

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \tag{4-3}$$

When these impulse functions are multiplied with the original waveform, they produce a new series of impulse functions with each one weighted according to the original waveform.

$$x(nT) = \sum_{n=-\infty}^{\infty} x(t)\,\delta(t - nT) \tag{4-4}$$

The sampled analog waveform is converted into a sequence of digital numbers using an ADC. The output of the ADC is an array or record of numbers representing the sampled waveform. The sampled and digitized version of the waveform still retains the shape and information content of the unsampled waveform, if the sample rate is sufficiently high.

4.5 Sampling Theorem

The waveform must be sampled often enough to produce a digitized time record that faithfully represents the original waveform. The *sampling theorem* states that a baseband signal must be sampled at a rate greater than twice the highest frequency present in the signal. The minimum acceptable sample rate is called the *Nyquist rate.* Thus,

$$f_s > 2f_{max} \tag{4-5}$$

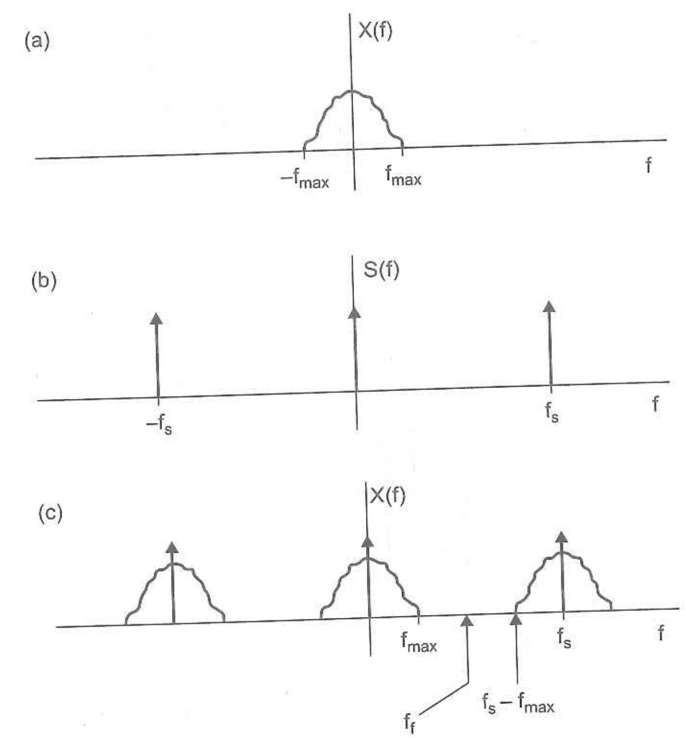

Figure 4-7 (a) The spectrum of the unsampled waveform. (b) The spectrum of the sampling function. (c) The spectrum of the sampled waveform.

where

f_s = sample rate
f_{max} = highest frequency of interest

Figure 4-7a shows the frequency spectrum, $X(f)$, of a signal, $x(t)$, with a maximum frequency of f_{max}. The frequency spectrum of the sampling function, as given by Table 3-1, is an infinite number of impulse functions spaced every f_s in frequency (Figure 4-7b). The spectrum of the sampled waveform can be derived by convolving[2] $X(f)$ with $S(f)$, which results in the original spectrum $X(f)$ appearing centered around each impulse function of $S(f)$ (Figure 4-7c).

This type of spectrum is always found in sampled systems—the baseband signal is repeated at integer multiples of the sample frequency. Notice that the spectrum between 0 and f_s is symmetrical about $f_s/2$, which is also called the *folding frequency*, f_f. The original signal can be recovered by applying a low-pass filter with a cutoff frequency of f_f, as long as the frequency content centered around f_s does not encroach on the baseband signal. Stated mathematically, the following condition must be met:

$$f_s - f_{max} > f_f \tag{4-6}$$

[2] For a discussion of the fine points of convolution, see McGillem and Cooper (1974).

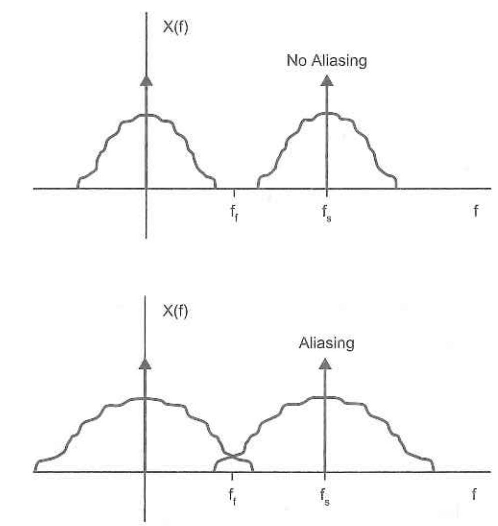

Figure 4-8 Aliasing occurs when the sample rate is not high enough.

which is just a restatement of the sampling theorem since

$$f_s - f_{max} > f_s/2 \tag{4-7}$$

$$f_s/2 > f_{max} \tag{4-8}$$

$$f_s > 2f_{max} \tag{4-9}$$

Figure 4-8 shows the spectra of two sampled signals: one where the sampling theorem is met and another that violates the sampling theorem. Notice that when the sampling theorem is violated unwanted frequency components show up below f_f. This phenomenon is known as *aliasing,* since these undesirable frequency components appear under the alias of another (baseband) frequency.

To prevent aliasing in an FFT analyzer, two conditions must be met:

1. The input signal must be band limited. In other words, there must exist an f_{max} above which no frequency components are present.[3] This is usually accomplished by inserting a low pass filter, commonly known as an anti-alias filter, in the signal path. (This is the low-pass filter shown in Figure 4-4.)
2. The input signal must be sampled at a rate that satisfies the sampling theorem.

The sampling frequency required by the sampling theorem is the minimum theoretical value that can reconstruct the signal properly. In practice, it is necessary to use a sampling frequency somewhat higher than this value. Figure 4-9 shows the frequency response of a

[3] In practice, frequency components above f_{max} are allowed to exist but must be sufficiently attenuated so that they do not affect the measurement.

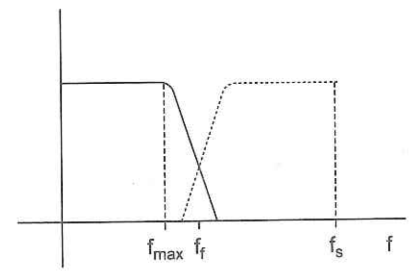

Figure 4-9 The response of the anti-alias filter requires that the sample rate be somewhat higher than the sampling theorem states.

practical low-pass filter. The filter will have a finite slope above its cutoff frequency, f_{max}. The mirrored response of the filter above the folding frequency is also shown. The overlap between the filter response and its mirrored response represent the region where aliasing can occur. The system is designed so that the folding frequency (and the sampling frequency) are large enough that the anti-alias response has room to roll off. Thus, f_{max}, the highest frequency that the analyzer will measure, must be significantly less than f_f. For practical filter implementations, f_s is typically 2.5 times f_{max}.

As shown, aliasing can be explained in the frequency domain, but it is also helpful to consider it briefly in the time domain. Figure 4-10 shows a set of sample points that fit two different waveforms. One of the waveforms has a frequency that violates the sampling theorem; the other does not. (The higher-frequency waveform violates the sampling theorem, of course.) Unless an anti-alias filter removes the unwanted alias frequency, the two sampled sine waves will be indistinguishable when processed digitally.

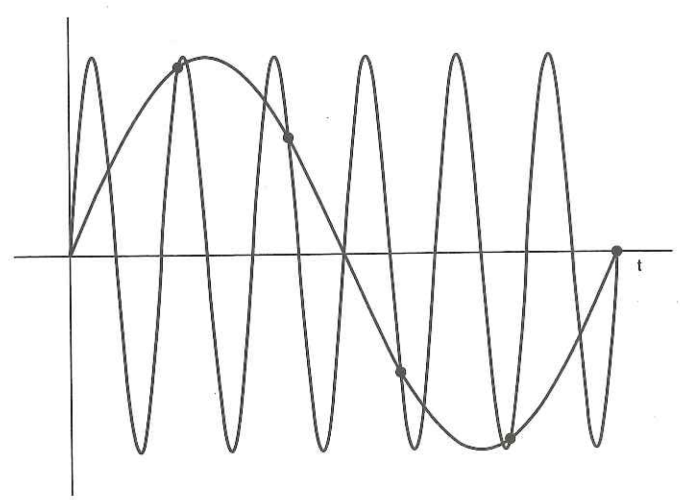

Figure 4-10 Aliasing in the time domain.

4.6 FFT Properties

The FFT is a record-oriented algorithm. A time record, N samples long, is the input, and the frequency spectrum, N samples long, is the output. Recall from Chapter 3 that N is often restricted to being a power of 2 to simplify the FFT computation. A typical record length for an FFT analyzer is 1024 sample points. The frequency spectrum produced by the FFT is symmetrical about the folding frequency. Thus, the first half of the output record is redundant with the second half, and the sample points numbered 0 to $N/2$ are retained. This implies that the effective length of the output record is $(N/2) + 1$. These are complex points (real + j imaginary) containing both magnitude and phase information.

Practically speaking, the output of the FFT is $(N/2) + 1$ points, extending from 0 Hz to f_f. Not all of these points are usually displayed though since the anti-alias filter begins to roll off before f_f. A common configuration is 1024 samples in the time record, producing 513 unique complex frequency domain points, with 401 of these actually displayed.

The $N/2$ (or so) frequency domain points are often referred to as *bins* and are usually numbered from 0 to $N/2$ (e.g., 0 to 512 for $N = 1024$). These bins are equivalent to the individual filter/detector outputs in the bank-of-filters analyzer. Bin 0 contains the DC level present in the input signal and is also known as the *DC bin*. The bins are spaced equally in frequency, with the frequency step, f_{step} being the reciprocal of the time record length.[4]

$$f_{step} = 1/\text{length of time record} \tag{4-10}$$

The length of the time record can be determined from the sample rate and the number of sample points in the time record.

$$f_{step} = f_s/\text{N} \tag{4-11}$$

The frequency associated with each bin is given by

$$f_n = nf_s/N \tag{4-12}$$

where

n = the bin number

The frequency of the last bin, containing the maximum frequency out of the FFT, is $f_s/2$. Therefore, the frequency range of an FFT is 0 Hz to $f_s/2$. (Note that this frequency is intentionally *not* called f_{max}, which is reserved for the upper-frequency limit of the instrument and which may not be the same as the last FFT bin.)

Suppose one cycle of a sine wave fits exactly into one time record, as shown in Figure 4-11. This sine wave will show up in bin 1 of the FFT output. If the frequency of the sine wave is doubled, then two sine waves will fit into one time record and their energy will appear in bin 2. Tripling the original sine wave frequency will cause a frequency domain response in bin 3, and so forth.

[4] The term *frequency step* does not mean that some frequencies will be missed by the FFT. The output of the FFT is equivalent to the bank-of-filters analyzer, with contiguous band-pass filters centered at each bin.