CHAPTER 3

Fourier Theory

The most common way of representing signals is in the time domain. Another representation of a signal is via the frequency domain, which is inherent in spectrum measurements. In the frequency domain, the signal is described in terms of its frequency content, plotting the amount of power present at each frequency. A complete frequency domain representation includes both the magnitude and phase of the signal. The frequency domain is related to the time domain by a body of knowledge generally known as Fourier theory, named for Jean Baptiste Joseph Fourier (1768–1830). This includes the series representation known as the *Fourier series* and the transform techniques known as the *Fourier transform*. Discrete (digitized) signals can be transformed into the frequency domain using the *discrete Fourier transform* (DFT).

3.1 Periodicity

A signal or function is *periodic* if it meets the following criterion:

$$x(t) = x(t + T) \quad \text{for all } t \tag{3-1}$$

where

T = period of the function

In other words, a periodic function can be shifted in time by exactly one period and the resulting new function will look the same as the original one. A periodic function of time repeats itself every T seconds (Figure 3-1).

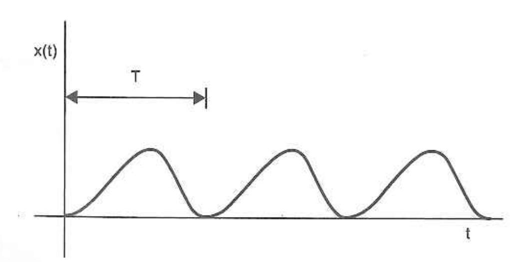

Figure 3-1 A periodic signal repeats every T seconds.

3.2 Fourier Series

Most periodic signals can be represented by a series expansion of sines and cosines. There are some mathematical limitations on the represented signal, but physically realizable signals meet these constraints.[1]

The Fourier series representation of a periodic function has the form[2]

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos 2\pi n f_0 t + b_n \sin 2\pi n f_0 t) \tag{3-2}$$

where

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t) \cos 2\pi n f_0 t \, dt \tag{3-3}$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t) \sin 2\pi n f_0 t \, dt \tag{3-4}$$

where

f_0 = fundamental frequency in hertz
T = period of the signal

T and f_0 are related by

$$f_0 = \frac{1}{T} \tag{3-5}$$

The frequency in rad/sec (ω_0) is

$$\omega_0 = 2\pi f_0 \tag{3-6}$$

Using the Fourier series, a periodic signal can be expanded into a summation of sines and cosines. The weighting of these sines and cosines are given by the a_n and b_n coefficients. These coefficients are found by integrating (over one period) the function multiplied by the sine or cosine associated with that coefficient. The sine and cosine terms are all harmonically related to the fundamental frequency, ω_0. The $a_0/2$ term is simply the average (DC) value of the waveform and can often be found by inspection.

It may be inconvenient to work with separate sine and cosine terms, so the two terms can be combined into one sinusoid with an appropriate magnitude and phase angle.

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\sqrt{a_n^2 + b_n^2}\cos(2\pi n f_0 t + \theta_n) \tag{3-7}$$

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[1] We will take a less than rigorous mathematical approach.

[2] The reader should be aware that there are several different ways of defining the Fourier series, with subtle differences in formulation.

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where

$$\theta_n = \tan^{-1}(-b_n/a_n)$$

Alternatively, a_n and b_n can be combined into a complex coefficient that gives the complex form of the Fourier series. Instead of sines and cosines, a complex exponential is used:

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_0 t} \tag{3-8}$$

where

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-j2\pi n f_0 t} \, dt \tag{3-9}$$

The two Fourier series representations are related by

$$c_n = (a_n - jb_n)/2 \tag{3-10}$$

The complex coefficient can also be expressed in magnitude/phase format:

$$c_n = |c_n|\angle\theta_n \tag{3-11}$$

Note that the complex form of the Fourier series is usually shown with n ranging from negative infinity to positive infinity whereas the original form restricts n to positive values. The complex form is chosen in anticipation of the Fourier transform, which includes negative frequencies. The factor of 2 that appears in equation (3-10) accounts for the presence of twice as many terms (both positive and negative) in the complex form. Frequency domain representations that include only positive frequencies are called *single sided*; those that include both positive and negative frequencies are called *double sided.*

3.3 Fourier Series of a Square Wave

As an example of the significance and utility of the Fourier series, the coefficients of a square wave will be determined. In addition, the square wave is a common signal in electrical systems (Figure 3-2a).

$$\begin{aligned} a_n &= \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos(2\pi n f_0 t)\, dt \\ &= \frac{2}{T}\int_{-T/2}^{0} (-1)\cos(2\pi n f_0 t)\, dt + \frac{2}{T}\int_{0}^{T/2} (1)\cos(2\pi n f_0 t)\, dt \\ &= \frac{2}{T}\left[-\frac{1}{2\pi n f_0}\sin(2\pi n f_0 t)\Big|_{-T/2}^{0} + \frac{1}{2\pi n f_0}\sin(2\pi n f_0 t)\Big|_{0}^{T/2}\right] \\ &= 0 \end{aligned}$$

$$
\begin{aligned}
b_n &= \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin(2\pi n f_0 t)\,dt \\
&= \frac{2}{T}\int_{-T/2}^{0} (-1)\sin(2\pi n f_0 t)\,dt + \frac{2}{T}\int_{0}^{T/2} (1)\sin(2\pi n f_0 t)\,dt \\
&= \frac{2}{T}\left[\frac{1}{2\pi n f_0}\cos(2\pi n f_0 t)\Big|_{-T/2}^{0} - \frac{1}{2\pi n f_0}\cos(2\pi n f_0 t)\Big|_{0}^{T/2}\right] \\
&= \frac{1}{n\pi}(2 - 2\cos n\pi) \\
&= \frac{4}{n\pi} \quad \text{for } n \text{ odd} \\
&= 0 \quad \text{for } n \text{ even}
\end{aligned}
$$

The Fourier series for the square wave is

$$
x(t) = \frac{4}{\pi}\sin(2\pi f_0 t) + \frac{4}{3\pi}\sin(6\pi f_0 t) + \frac{4}{5\pi}\sin(10\pi f_0 t) + \cdots \tag{3-12}
$$

Therefore, the ideal square wave has only odd harmonics. With the particular phase chosen for the square wave, the a_n (cosine) terms are all zero, while the odd b_n (sine) terms remain nonzero. If the phase of the square wave were changed relative to $t = 0$, a_n could be nonzero but only for the odd harmonics. Similarly, at just the right phase b_n could become zero.

The square wave and its harmonics can be examined graphically, which helps show their relationship. Figure 3-2b shows the first three harmonics of the square wave. Figures 3-2c–f show a square wave constructed from a finite number of its harmonics. Note how the harmonics tend to fill in the square wave as each additional harmonic is added to the plot. It takes an infinite number of harmonics to produce a perfect square wave, but in practice the

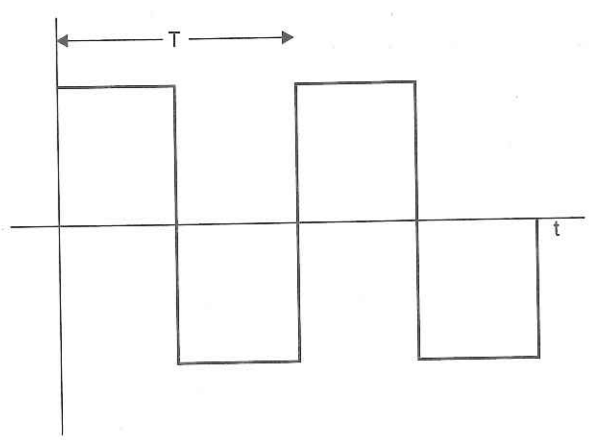


Figure 3-2a The square wave.

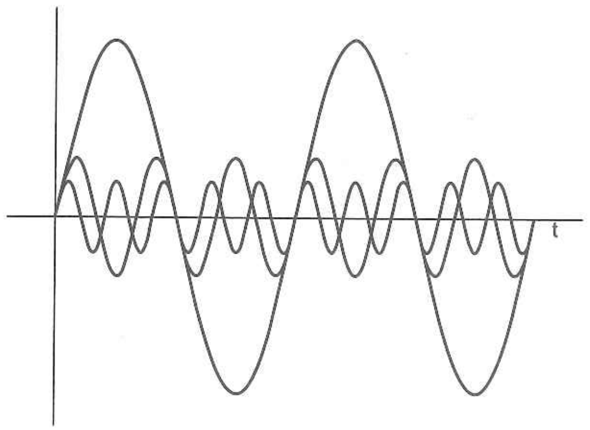


Figure 3-2b The fundamental, third harmonic, and fifth harmonic of the square wave.

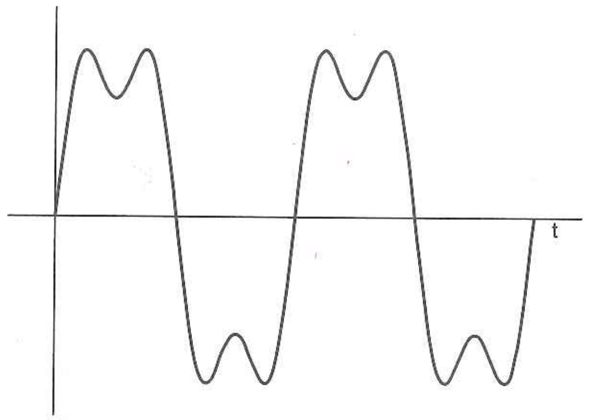


Figure 3-2c The square wave with only the fundamental and third harmonic included.

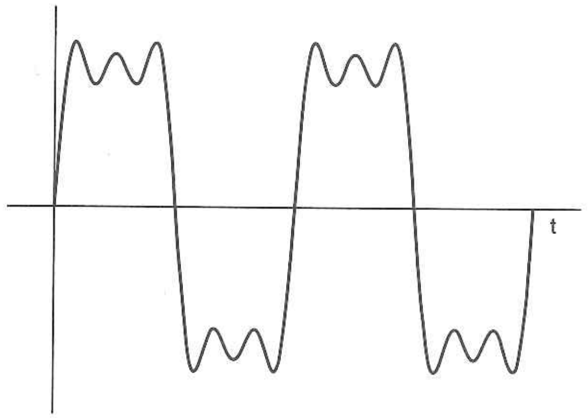
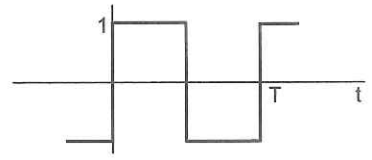
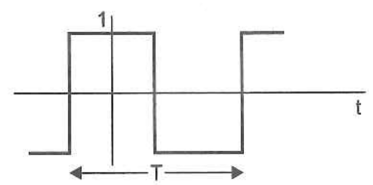
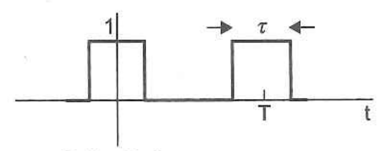
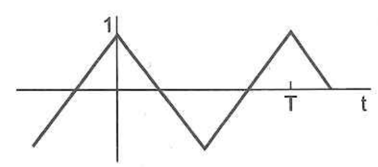
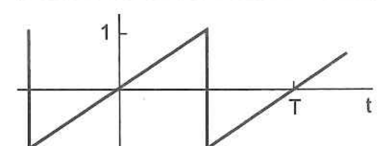
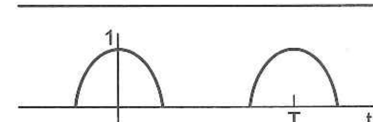


Figure 3-2d The square wave with up to the fifth harmonic included.

3.4 Fourier Series of Other Waveforms

The Fourier series representation of other periodic waveforms can be determined using the techniques given. For convenience, the Fourier series representations of some common waveforms are tabulated in Table 3-1.

Table 3-1 Fourier Series of Waveforms

 Square Wave (Odd)	$\frac{4}{\pi}\sum_{\substack{n=1\\ \text{odd}}}^{\infty}\frac{1}{n}\sin\left(\frac{2\pi n t}{T}\right)$
 Square Wave (Even)	$\frac{4}{\pi}\sum_{\substack{n=1\\ \text{odd}}}^{\infty}\frac{(-1)^{(n-1)/2}}{n}\cos\left(\frac{2\pi n t}{T}\right)$
 Pulse Train	$\frac{\tau}{T}+\frac{2\tau}{T}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{\pi n\tau}{T}\right)}{\frac{\pi n\tau}{T}}\cos\left(\frac{2\pi n t}{T}\right)$
 Triangle Wave	$\frac{8}{\pi^2}\sum_{\substack{n=1\\ \text{odd}}}^{\infty}\frac{1}{n^2}\cos\left(\frac{2\pi n t}{T}\right)$
 Sawtooth Wave	$\frac{2}{\pi}\sum_{\substack{n=1\\ \text{odd}}}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{2\pi n t}{T}\right)$
 Half-Wave Cosine	$\frac{1}{\pi}+\frac{1}{2}\cos\left(\frac{2\pi t}{T}\right)-\frac{2}{\pi}\sum_{\substack{n=2\\ \text{odd}}}^{\infty}\frac{(-1)^{n/2}}{n^2-1}\cos\left(\frac{2\pi n t}{T}\right)$

Example 3.1

Determine the amplitude and frequency of the fundamental of the waveform shown in Figure 3-4. If the signal is a voltage present across 50 Ω, what is the power level in dBm of the fundamental? Determine the amplitude of the second harmonic and express it in decibels relative to the fundamental.

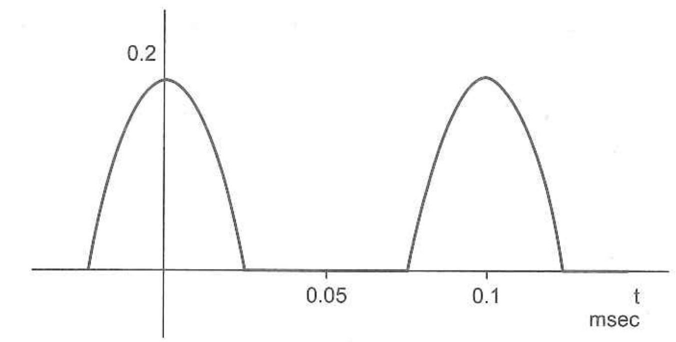


Figure 3-4 The half-wave rectified sine wave is a periodic signal.

From Table 3-1, the first few terms of the Fourier series of the half cosine wave are

$$x(t)=\frac{1}{\pi}+\frac{1}{2}\cos(2\pi t/T)+\frac{2}{3\pi}\cos(4\pi t/T)$$

The waveform shown in Figure 3-4 has a peak voltage of 0.2 V, so the Fourier series is multiplied by 0.2.

$$x(t)=0.2\left[\frac{1}{\pi}+\frac{1}{2}\cos(2\pi t/T)+\frac{2}{3\pi}\cos(4\pi t/T)\right]$$

The frequency of the fundamental is $1/T = 1/(0.1\text{ msec}) = 10$ kHz.

The amplitude of the fundamental is $0.2(1/2) = 0.1$ V zero-to-peak. Converting this value to RMS gives $0.707 \times 0.1 = 0.0707$ V. Using equation (2-12), the amplitude in dBm (50 Ω) is $20\log(0.0707/0.223) = -9.98$ dBm.

The amplitude of the second harmonic is $0.2(2/3\pi) = 0.0424$ V zero-to-peak, or 0.030 V RMS. Expressed as decibels relative to the fundamental, the second harmonic is $20\log(0.030/0.0707) = -7.45$ dB.

3.5 Fourier Transform

Although the Fourier series representation of a signal is very powerful, it is limited to periodic signals. Signals that are not periodic may be represented in the frequency domain by the Fourier transform. The Fourier transform of a time domain signal $x(t)$ is

$$X(f)=\int_{-\infty}^{\infty}x(t)\,e^{-j2\pi ft}\,dt \tag{3-13}$$

where

$X(f)$ = frequency domain representation of the signal
$x(t)$ = time domain representation of the signal
f = frequency

The Fourier transform transforms a time domain signal into a continuous frequency domain representation. Recall that the Fourier series representation, by definition, contains only the fundamental frequency and its harmonics. Not only are these discrete frequencies, but they are also harmonically related. The Fourier transform can represent discrete frequencies but is more often used to represent continuous functions in the frequency domain. Thus, a one-time event (e.g., a pulse) in the time domain can also be represented in the frequency domain.

Mathematically, the frequency domain representation is a complex function, containing both magnitude and phase information. Although many spectrum measurements are performed just using the magnitude of the signal, phase is required for a full representation of the signal.

3.6 Fourier Transform of a Pulse

As an example and because it is a common electrical signal, we will determine the Fourier transform of a single pulse (Figure 3-5a).

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\, dt \\ &= \int_{-T/2}^{T/2} e^{-j2\pi ft} dt = \left.\frac{e^{-j2\pi ft}}{-j2\pi f}\right|_{-T/2}^{T/2} \qquad (3\text{-}14) \\ &= \frac{e^{j\pi fT} - e^{-j\pi fT}}{j2\pi f} = T\frac{\sin(\pi Tf)}{\pi Tf} \end{aligned}$$

The frequency domain representation for a pulse is of the form $(\sin x)/x$ (Figure 3-5b). Notice that the function is continuous and extends over the entire frequency axis, both positive and negative. Thus, a perfect pulse occupies an infinite bandwidth. However, the amplitude of the frequency content tends to decrease with increasing frequency, and, in practice, a finite bandwidth can be assumed.

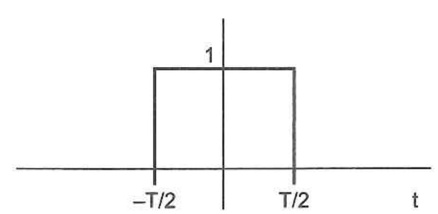


Figure 3-5a A single time domain pulse.

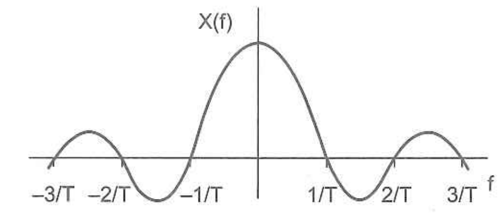


Figure 3-5b The spectrum of a single pulse.

The zero crossings of $X(f)$ are often used as a means of estimating the bandwidth of the pulse. Most of the pulse's energy is in the main lobe, which exists at frequencies below $f = 1/T$. As the width of the time domain pulse is decreased, T becomes smaller. In the frequency domain, as T becomes smaller, the first zero crossing moves out to a higher frequency. Therefore, the narrower the pulse, the wider the bandwidth in the frequency domain. This should make sense intuitively, since a narrower pulse requires higher frequency content to recreate the waveform in the time domain. This is true of signals in general—the faster the voltage changes in the time domain, the wider the bandwidth in the frequency domain.

3.7 Inverse Fourier Transform

The *inverse Fourier transform* converts the frequency domain representation (obtained by the Fourier transform) back into the time domain representation. The inverse transform is given by

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{2\pi ft}\, df \qquad (3\text{-}15)$$

Thus, Fourier theory provides a means of transforming a time domain signal into the frequency and (just as important) provides a means of getting the frequency domain representation back into the time domain.

The time domain and frequency domain representations of a signal are known as *transform pairs*. They are unique in that each time domain representation has only one frequency domain representation and vice versa. A table of common Fourier transform pairs is given in Table 3-2.

3.8 Fourier Transform Relationships

Many mathematical operations in the time domain have a corresponding operation in the frequency domain. These relationships are often used to reduce the difficulty of finding a transform of a particular function. These relationships also lend insight into how the time and frequency domain relate. Table 3-3 is a compilation of commonly used Fourier transform relationships.

Table 3-3 Properties of the Fourier Transform

	$x(t)$	$X(f)$
Magnitude scaling	$Ax(t)$	$AX(f)$
Time scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{f}{a}\right)$
Linearity	$x_1(t) + x_2(t)$	$X_1(f) + X_2(f)$
Time delay	$x(t - t_0)$	$e^{-j2\pi f t_0}X(f)$
Time derivative	$\frac{d^n}{dt^n}x(t)$	$(j2\pi f)^n X(f)$
Modulation	$x(t)\cos(2\pi f_0 t)$	$\frac{1}{2}[X(f-f_0)+X(f+f_0)]$
Complex modulation	$e^{j2\pi f_0 t}x(t)$	$X(f-f_0)$
Multiplication	$x_1(t)\,x_2(t)$	$\int_{-\infty}^{\infty} X_1(\lambda)\,X_2(f-\lambda)\,d\lambda$
Convolution	$\int_{-\infty}^{\infty} x_1(\lambda)x_2(t-\lambda)\,d\lambda$	$X_1(f)\,X_2(f)$
Symmetry	$X(t)$	$x(-f)$

3.9 Discrete Fourier Transform

The Fourier transform is mostly an analysis tool, a powerful means of understanding how signals behave in a system. It is not directly used in a measurement system to produce the frequency domain representation of a signal. The DFT is a discrete version of the Fourier transform. It allows a sampled time domain signal to be transformed into a sampled frequency domain form. Digitizing a real-world signal in the time domain and performing a DFT produces the frequency domain representation of the signal. Thus, the DFT goes beyond being just an analysis tool to being a way to implement the measurement.

We had previously introduced the complex form of the Fourier series. It is rewritten here with a slight change of variable (the period T has become t_p and harmonic number n is replaced by k).

$$c_k = \frac{1}{t_p}\int_{-t_p/2}^{t_p/2} x(t)e^{-j2\pi k f_0 t}\,dt \tag{3-16}$$

Consider the periodic waveform shown in Figure 3-6a. Suppose that a sampled version of one period of this waveform is available (Figure 3-6b). The Fourier series can be applied to this sampled waveform, with the minor change that the time domain waveform is not continuous. This means that $x(t)$ will be replaced by $x(nT)$, where T is the time between

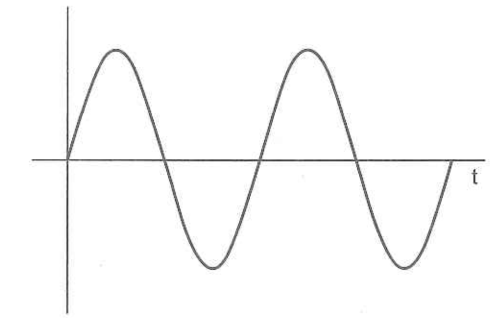


Figure 3-6a A periodic signal to be sampled.

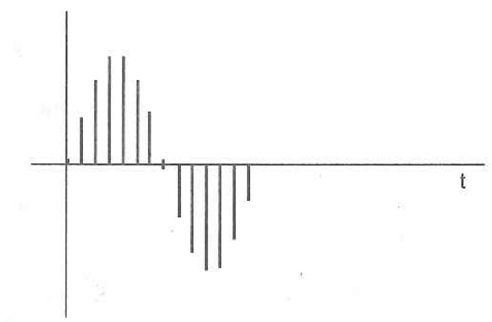


Figure 3-6b The sampled version of one period of the signal.

samples. Also, instead of an integration, a discrete summation of the sampled waveform will be performed with the result multiplied by the time between samples, T.

$$c_k = \frac{T}{t_p}\sum_{n=0}^{N-1} x(nT)e^{-j2\pi k f_0 nT} \tag{3-17}$$

Note that the range of n was chosen to be from 0 to $N-1$, producing N samples. This particular range is not mandatory but is customary for defining the DFT. The fundamental frequency, f_0, is also the spacing between the discrete frequency points. We will rename f_0 as F and attempt to provide consistent notation. Finally, the DFT is usually defined to be N times the complex Fourier series coefficient.[3]

$$X(kF) = Nc_k \tag{3-18}$$

$$X(kF) = \frac{NT}{t_p}\sum_{n=0}^{N-1} x(nT)e^{-j2\pi kFnT} \tag{3-19}$$

[3] This is only a scale factor and does not affect the frequency content of the DFT. In instrumentation use, the DFT must have appropriate scale factors added to properly calibrate the instrument.

Since the number of samples, N, times the sample time, T, equals the period, t_p, the equation simplifies to give the common form of the DFT.

$$X(kF) = \sum_{n=0}^{N-1} x(nT)e^{-j2\pi kFnT} \tag{3-20}$$

where

N = number of samples
F = spacing of the frequency domain samples
T = sample period in the time domain

In instrumentation use, the input to the DFT is a record of time domain data obtained by sampling the signal being analyzed. The sample rate, f_s, is equal to $1/T$. After N time domain samples are collected, the DFT algorithm uses the time domain samples to produce N frequency domain samples, spaced F Hz apart. These N frequency domain samples are not totally independent. The set of samples numbered less than $N/2$ are redundant with the samples numbered above $N/2$. For an N point DFT, only the samples up to and including $N/2$ frequency domain points are normally retained. In general, these points are complex numbers, providing vector information.

Remember that we started the derivation with the Fourier series and not the Fourier transform. As the number of time domain samples, N, increases (and therefore the number of frequency domain samples increases), we can stop considering the DFT as a small set of spectral lines and start thinking of it as a good approximation to the continuous Fourier transform.

The inverse of the DFT, the *inverse discrete Fourier transform* (IDFT), is given by

$$x(nT) = \frac{1}{N}\sum_{k=0}^{N-1} X(kF)\, e^{j2\pi FTkn} \tag{3-21}$$

The IDFT provides a means for converting the discrete frequency domain information back into a discrete time domain waveform. As one might imagine, the DFT and IDFT have properties that are very similar to their continuous counterparts.

3.10 Limitations of the DFT

The DFT is only an approximation to the Fourier transform. It differs from the continuous Fourier transform in several important ways.

Obviously, due to the quantized nature of the DFT, it is valid at only certain frequencies. The frequency resolution of the DFT can be increased by using a larger number of samples.

The theory behind the DFT implicitly assumes that the waveform was periodic. Whether this is the case or not, the mathematics of the DFT will treat the sampled waveform as if it repeats. This causes a phenomenon known as leakage, which is an important limitation of the DFT, but one that can be minimized by proper use of time domain windowing. Leakage is discussed further in Chapter 4.

Since the DFT is performed with digital arithmetic, it is subject to the limitations imposed by the particular algorithm chosen. In particular, finite arithmetic effects due to the number of bits used can limit the dynamic range and noise performance of the DFT.

3.11 Fast Fourier Transform

The *fast Fourier transform* (FFT) is a very quick and efficient algorithm for implementing a DFT. The original basis for the FFT was developed by J. W. Cooley and J. W. Tukey in 1965. Although it is often implied that there is just one FFT, in reality an entire class of algorithms are commonly referred to as the FFT. An FFT algorithm gains a significant speed advantage over the DFT by carefully selecting and organizing intermediate results. Ignoring finite arithmetic effects, the results are the same whether an FFT or a DFT is used.

The number of computations required for a DFT is on the order of N^2, where N is the number of samples, or record length. The FFT, on the other hand, requires $N \log_2 N$ computations ($\log_2$ indicates the base 2 logarithm). The most common FFT algorithms require N to be a power of 2. A typical record length in a spectrum analyzer might be 2^{10}, or 1024. This means a DFT would require over 1 million computations, whereas an FFT would require only 10,240 computations. Assuming all computations take the same amount of time, the FFT could be computed in less than 1% of the DFT computation time. Clearly, this is a substantial time savings and explains why the FFT dominates in instrumentation use.

Examining the details of how and why an FFT is implemented is beyond the scope of this book. For our purposes, we will consider the FFT to be simply an efficient implementation of a DFT. For more information, see Oppenheim and Schafer (1975).

3.12 Relating Theory to Measurements

When the instrument user attempts to relate Fourier theory to an actual measurement, some notable differences will appear. The major differences are summarized here:

1. The spectrum analyzer normally shows a one-sided spectrum, whereas the Fourier transform and perhaps the Fourier series (depending on which form is used) show a two-sided spectrum.
2. The frequency resolution (resolution bandwidth) of the spectrum analyzer determines the width and shape of discrete spectral lines. Ideally, the lines are infinitely thin, but they appear with a finite width due to the resolution bandwidth of the analyzer.
3. Other distortion and noise effects generated internal to the spectrum analyzer will affect the measurement. For example, the noise floor of the analyzer may obscure low-level frequency components or distortion products may appear as additional spectral lines.

In particular, it can be a problem relating the amplitude predicted by Fourier theory to the measured amplitude. In an attempt to reconcile theory and measurement, let us consider a simple, but instructive case: the cosine. We apply both the Fourier series and the Fourier transform to this signal and then compare the results with a practical spectrum measurement.

Consider the time domain waveform

$$v(t) = V_0 \cos 2\pi f_0 t \tag{3-22}$$

The RMS value, as measured by an RMS-reading voltmeter, would be 0.707 times the zero-to-peak value. This value should agree with the spectrum analyzer measurement:

$$V_{RMS} = 0.707\, V_0 \tag{3-23}$$

The Fourier series for this voltage waveform can easily be found by inspection.

$$v(t) = a_1 \cos 2\pi f_0 t \tag{3-24}$$

where

$$a_1 = V_0$$

This implies a single spectral line at f_0, with a zero-to-peak amplitude of V_0.

From Table 3-2, the Fourier transform of the waveform is

$$V(f) = \frac{V_0}{2}[\delta(f - f_0) + \delta(f + f_0)] \tag{3-25}$$

Since the Fourier transform is a two-sided representation, with both positive and negative frequencies, the frequency domain representation indicates two impulse functions: one at $+f_0$ and the other at $-f_0$. The amplitudes of each of these impulse functions is $V_0/2$. This amplitude is doubled to convert the double-sided amplitude to the equivalent single-sided amplitude. Thus, the zero-to-peak amplitude equal to V_0 agrees with the Fourier series analysis and, if multiplied by 0.707 to obtain the RMS value, agrees with the voltmeter reading and a spectrum analyzer reading.

3.13 Finite Measurement Time

The discussion of the Fourier series and the Fourier transform both involved integrals that cover all time, that is, from $-\infty$ to $+\infty$. Therefore, to ascertain correctly the frequency domain representation of a signal, the time domain function must be known for all time. For theoretical analysis, this does not present a problem, but real-world measurements occur in a finite time. Normally, the spectrum analyzer user simply performs the measurement over some convenient time interval and assumes that the time interval chosen adequately represents the signal. Mathematically speaking, the signal is assumed to be *stationary*.[4]

The characteristics of many signals are constant over time in which case such an assumption is justified. By definition, a periodic signal repeats over and over again for all time, producing a constant spectrum. Some other signals change quite rapidly and should not be assumed to have constant spectrums. As an example consider a radio transmitter. If the modulating signal is a person's voice, the spectrum of the signal will change quickly and unpredictably as the radio operator speaks different words. A measurement taken at any particular time will not represent the signal over all time. However, if a constant audio tone modulates the radio signal, the spectrum is constant.

When measuring a signal's spectrum, we should consider the possibility that the signal's spectral content may be varying. If this variation is slow compared with the duration of the measurement, it is not of concern. However, if the signal varies fast enough, the spectrum analyzer may not produce the desired result. In particular, traditional swept spectrum analyzers can have very slow sweep rates, depending on the measurement setup. These analyzers may not be suitable for measuring fast changing signals. FFT-based analyzers and real-time spectrum analyzers acquire the measured signal much faster and are much more effective in capturing changes in spectral content. This will be explored further in Chapters 4 and 5.

[4] A signal is stationary if its statistical nature does not change with time, which implies that its spectrum is constant. For a more rigorous discussion, see Oppenheim and Willsky (1996).

Bibliography

Agilent Technologies. "Fundamentals of Signal Analysis," Application Note 243, Publication Number 5952-8898E, June 2000.

Brigham, E. Oran. *The Fast Fourier Transform and Its Applications*. Englewood Cliffs, NJ: Prentice Hall, Inc., 1988.

Cooley, J. W., and Tukey, J. W., "An Algorithm for the Machine Calculation of Complex Fourier Series," *Math. Computation*, Vol. 19 (1965).

Hayt, William H. Jr., Jack E. Kemmerly, and Steven M. Durbin. *Engineering Circuit Analysis*, 7th ed. New York: McGraw-Hill Book Company, 2007.

McGillem, Clare D., and George R. Cooper. *Continuous and Discrete Signal and System Analysis*. New York: Holt, Rinehart and Winston, Inc., 1974.

Oppenheim, Alan V., and Alan S. Willsky. *Signals and Systems*, 2d ed. Upper Saddle River, NJ: Prentice Hall, Inc., 1996.

Oppenheim, Alan V., and Ronald W. Schafer. *Digital Signal Processing*. Englewood Cliffs, NJ: Prentice-Hall, Inc., 1975.

Ramirez, Robert W. *The FFT, Fundamentals and Concepts*. Englewood Cliffs, NJ: Prentice Hall, Inc., 1985.

Schwartz, Mischa. *Information, Transmission, Modulation, and Noise*, 3rd ed. New York: McGraw-Hill Book Company, 1980.

Ziemer, Rodger E., William H. Tranter, and D. Ronald Fannin. *Signals and Systems: Continuous and Discrete*, 4th ed. Upper Saddle River, NJ: Prentice Hall, Inc., 1998.