University of Illinois **Fall 2018**

ECE210 / ECE211 - Homework 09
Due date: Wednesday, Oct. 31, 2018

1. Determine the Fourier transform of the following signals, as well as their energy content.

   (a) $x(t) = rect(t-3) + rect(t+3)$

   (b) $x(t) = 4rect(\frac{t}{4})\cos(2\pi f_c t)$

   (c) $x(t) = \frac{1}{1+t^2}$

   (d) $x(t) = t\cos(2\pi f_c t)$

   (e) $x(t) = (3 + 4\cos(2t))e^{5t}u(-t)$

   Solution:

   (a) Using time shift property, we get
   $X(\omega) = \text{sinc}(\frac{\omega}{2})e^{-j3\omega} + \text{sinc}(\frac{\omega}{2})e^{j3\omega} = 2\text{sinc}(\frac{\omega}{2})\cos(3\omega)$
   $W = \int_{-\infty}^{\infty} |x(t)|^2 dt = 1 + 1 = 2$

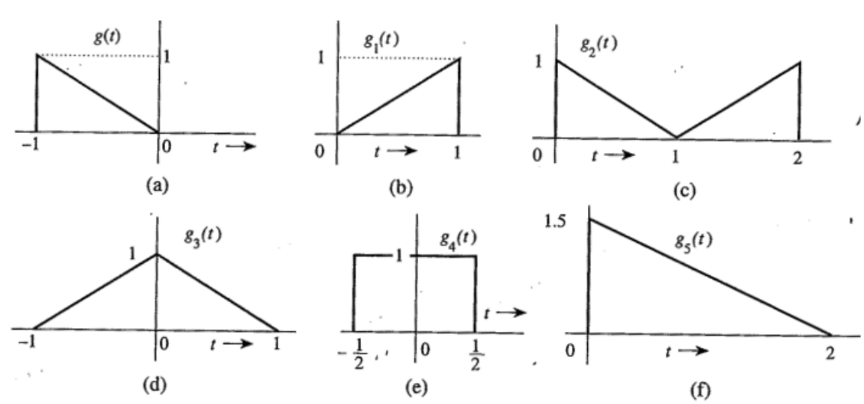
   (b) Using modulation property, we get
   $X(\omega) = 8[\text{sinc}(2(\omega - 2\pi f_c)) + \text{sinc}(2(\omega + 2\pi f_c))]$
   $W = \int_{-\infty}^{\infty} |x(t)|^2 dt = 16\int_{-2}^{2} \cos^2(2\pi f_c t)dt = \sin(8\pi f_c)/(\pi f_c) + 8$

   (c) Using symmetry property, we get:
   $X(\omega) = \pi e^{-|\omega|}$
   $W = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega = \pi\int_0^{\infty} e^{-2\omega} d\omega = \pi/2$

   (d) Using frequency derivative property, we get
   $x(t) = -jt(j\cos(2\pi f_c t)) \quad \longleftrightarrow$
   $X(\omega) = \frac{d}{d\omega}[j\pi(\delta(\omega - 2\pi f_c) + \delta(\omega + 2\pi f_c))] = j\pi(\delta'(\omega - 2\pi f_c) + \delta'(\omega + 2\pi f_c))$
   $W = \int_{-\infty}^{\infty} |x(t)|^2 d\omega = \int_{-\infty}^{\infty} (t\cos(2\pi f_c t))^2 d\omega = \infty$ because the integral over the t^2 diverges.

   (e) Using the Fourier transform table for first term, and the modulation property for second term, we get
   $x(t) = 3e^{5t}u(-t) + (4e^{5t}u(-t))\cos(2t) \quad \longleftrightarrow$
   $X(\omega) = \frac{3}{5-j\omega} + \frac{2}{5-j(\omega-2)} + \frac{2}{5-j(\omega+2)}$
   $W = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{0} (3 + 4\cos(2t))^2 e^{10t} dt = \frac{17709}{3770}$

2. The triangular pulse $g(t)$ in figure (a) below has Fourier transform $G(f) = \frac{1}{(2\pi f)^2}(e^{j2\pi f} - j2\pi f e^{j2\pi f} - 1)$. Use this information, and properties of the Fourier transform to find the Fourier transforms of the signals shown in figures (b), (c), (d), (e), (f) below.



   Solution: First we write down the relation of each function in $(b) - (f)$ to (a), then applying the Fourier transform property such that we don't need to calculate the Fourier transforms of part $(b) - (f)$ explicitly.

(b) $g_1(t) = g(-t) \quad \longleftrightarrow \quad G_1(\omega) = G(-\omega)$

(c) $g_2(t) = g(t-1) + g(-(t-1)) \quad \longleftrightarrow \quad G_2(\omega) = G(\omega)e^{-j\omega} + G(-\omega)e^{-j\omega}$

(d) $g_3(t) = g(t-1) + g(-(t+1)) \quad \longleftrightarrow \quad G_3(\omega) = G(\omega)e^{-j\omega} + G(-\omega)e^{j\omega}$

(e) $g_4(t) = g(t-\frac{1}{2}) + g(-(t+\frac{1}{2})) \quad \longleftrightarrow \quad G_4(\omega) = G(\omega)e^{-j\omega/2} + G(-\omega)e^{j\omega/2}$

(f) $g_5(t) = 1.5\ g(\frac{t-2}{2}) \quad \longleftrightarrow \quad G_5(\omega) = 3\ G(2\omega)e^{-j2\omega}$

3. Let $f(t) = \frac{1}{2\pi} sinc(-\frac{t}{6})[1 - 2\cos(-\frac{t}{6})]$, with Fourier transform $F(\omega) = 6rect(6\omega) - 6rect(6\omega+1) - 6rect(6\omega-1)$. Let $G(\omega) = \frac{1}{2\pi} sinc(\frac{\omega}{6})[1 - 2\cos(\frac{\omega}{6})]$.

   (a) Obtain the inverse Fourier transform of $G(\omega)$, that is, obtain $g(t)$.

   (b) Plot $F(\omega)$ and $g(t)$.

   Solution:

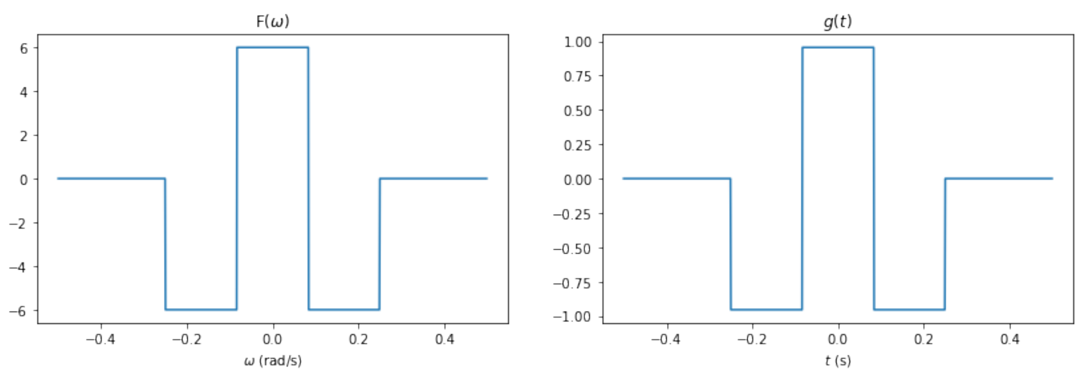
   (a) Using symmetry property of Fourier transform

$$f(t) \leftrightarrow F(\omega)$$
$$F(t) \leftrightarrow 2\pi f(-\omega)$$

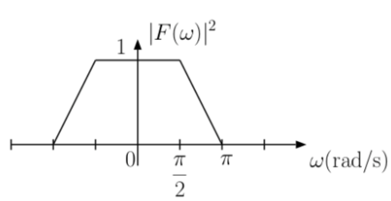
   we find that

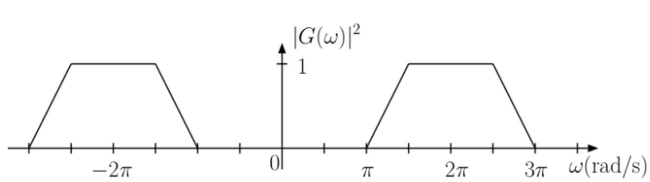
$$\begin{aligned} G(\omega) &= \frac{1}{2\pi} \cdot 2\pi f(-\omega) \\ g(t) &= \frac{1}{2\pi} \cdot F(t) \\ &= \frac{3}{\pi} rect(6t) - \frac{3}{\pi} rect(6t+1) - \frac{3}{\pi} rect(6t-1) \end{aligned}$$

   (b) The $F(\omega)$ and $g(t)$ are plotted as follows



4. Determine the 3-dB bandwidth and the 95%-bandwidth of signals $f(t)$ and $g(t)$ with the following energy spectra:





Solution:

(a) The 3-dB bandwidth is the bandwidth when the energy drops to half of its maximum value. Therefore

i. For $f(t)$: we observe that at $\omega = \frac{3\pi}{4}$ rad/s, the energy $|F(\omega)|^2$ drops to 0.5, therefore the 3-dB bandwidth for $f(t)$ is $\frac{3\pi}{4}$ rad/s.

ii. For $g(t)$: we observe that at $\omega = \frac{5\pi}{4}$ rad/s and $\omega = \frac{11\pi}{4}$ rad/s, the energy $|G(\omega)|^2$ drops to 0.5, therefore the 3-dB bandwidth for $g(t)$ is $\frac{11\pi}{4} - \frac{5}{4} = \frac{3\pi}{2}$ rad/s.

(b) The 95% bandwidth is the bandwidth that contains 95% of total energy. Therefore

i. For $f(t)$ we assume the 95% bandwidth is x such that $\frac{\pi}{2} \le x \le \pi$, then we can find the following relation

$$\frac{1}{2} \cdot (\pi - x)\frac{(\pi - x)}{\pi/2} = \frac{0.05}{2} \cdot \left[1 \cdot \pi + 1 \cdot \frac{\pi}{2}\right]$$

Solving for x gives us the result $\text{BW}_{95\%} = x = 2.533$ rad/s.

ii. For $g(t)$ we use the similar procedure, assuming x_1 and x_2 such that $\pi \le x_1 \le \frac{3\pi}{2}$ and $\frac{5\pi}{2} \le x_2 \le 3\pi$, then we find

$$\begin{aligned}
&x_1 - \pi = 3\pi - x_2 \\
&\frac{1}{2} \cdot (x_1 - \pi)\frac{(x_1 - \pi)}{\pi/2} + \frac{1}{2} \cdot (3\pi - x_2)\frac{(3\pi - x_2)}{\pi/2} = 0.05 \cdot \left[1 \cdot \pi + 1 \cdot \frac{\pi}{2}\right]
\end{aligned}$$

Solving for the equations above, we find $x_1 = 3.75$ rad/s and $x_2 = 8.82$ rad/s, and $\text{BW}_{95\%} = x_2 - x_1 = 5.07$ rad/s.