University of Illinois **Fall 2018**

ECE210 / ECE211 - Homework 14
Due date: Wednesday, Dec. 12, 2018

1. For each of the following Laplace transforms $\hat{F}(s)$, determine the inverse Laplace transform $f(t)$.

   (a) $\hat{F}(s) = \frac{s+3}{(s+2)(s+4)}$

   (b) $\hat{F}(s) = \frac{1}{s(s-5)^2}$

   (c) $\hat{F}(s) = \frac{s^2+2s+1}{(s+1)(s+2)}$

   Solution: (a) Partial fraction expansion yields

$$\hat{F}(s) = \frac{s+3}{(s+2)(s+4)} = \frac{1}{2(s+2)} + \frac{1}{2(s+4)}$$

   Thus,

$$f(t) = \frac{e^{-2t}u(t)}{2} + \frac{e^{-4t}u(t)}{2}$$

   (b)

$$\hat{F}(s) = \frac{1}{s(s-5)^2} = \frac{1}{25s} + \frac{1}{5(s-5)^2} - \frac{1}{25(s-5)}$$

   Taking inverse Laplace transform,

$$f(t) = (\frac{1}{25} + \frac{te^{5t}}{5} - \frac{e^{5t}}{25})u(t)$$

   (c)

$$\hat{F}(s) = \frac{s^2+2s+1}{(s+1)(s+2)} = \frac{s+1}{s+2} = 1 - \frac{1}{s+2}$$

$$f(t) = \delta(t) - e^{-2t}u(t)$$

2. Determine the transfer functions $\hat{H}(s)$ and the zero-state response for the LTIC system described by the following ODE:

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y(t) = f(t),$$

   where $f(t) = e^{3t}u(t)$.

   Solution: Taking laplace transform of the equation, we get

$$s^2Y(s) + 3sY(s) + 2Y(s) = F(s)$$

   The transfer function $\hat{H}(s)$ can be obtained as

$$\hat{H}(s) = \frac{Y(s)}{X(s)} = \frac{1}{(s+1)(s+2)}$$

   For $x(t) = e^{3t}u(t)$, $X(s) = \frac{1}{s-3}$. Thus,

$$Y(s) = \frac{1}{(s+1)(s+2)(s-3)} = -\frac{1}{4(s+1)} + \frac{1}{5(s+2)} + \frac{1}{20(s-3)}$$

   Zero state response $y(t)$ is

$$y(t) = \left(-\frac{e^{-t}}{4} + \frac{e^{-2t}}{5} + \frac{e^{3t}}{20}\right)u(t)$$

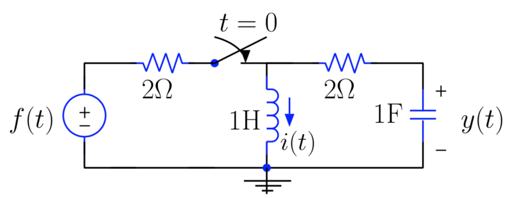
3. Take the Laplace transform of the following ODE to determine $\hat{Y}(s)$ assuming $f(t) = u(t)$, $y(0^-) = 1$, and $y'(0^-) = 0$. Determine $y(t)$ for $t > 0$ by taking the inverse Laplace transform of $\hat{Y}(s)$.

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 4y(t) = \frac{df}{dt} + 2f(t).$$

Solution: Taking laplace transform

$$s^2Y(s) - sy(0^-) - y'(0^-) + 5(sY(s) - y(0^-)) + 4Y(s) = 1 + 2F(s) = 1 + \frac{2}{s}$$
$$Y(s)(s^2 + 5s + 4) = s + \frac{2}{s} + 6$$
$$Y(s) = \frac{s^2 + 6s + 2}{s(s+1)(s+4)} = \frac{1}{2s} + \frac{1}{s+1} - \frac{1}{2(s+4)}$$
$$y(t) = \left(\frac{1}{2} + e^{-t} - \frac{e^{-4t}}{2}\right)u(t)$$

4. Consider the circuit:

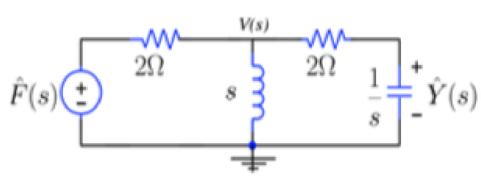


(a) Show that the transfer function of the circuit for $t > 0$ is $\hat{H}(s) = \frac{\hat{Y}(s)}{\hat{F}(s)} = \frac{s}{4s^2+5s+2}$.

(b) What are the characteristic modes of the circuit?

(c) Determine $y(t)$ for $t > 0$ if $f(t) = 1$ V, $y(0^-) = 1$ V, and $i(0^-) = 0$.

Solution: By converting the circuit to the s-domain



$$\hat{Y}(s) = V(s)\frac{1/s}{1/s + 2} = \frac{V(s)}{2s + 1}$$

where

$$V(s) = \frac{s(2s+1)\hat{F}(s)}{4s^2 + 5s + 2}$$
$$\hat{Y}(s) = \frac{\hat{F}(s)s}{4s^2 + 5s + 2}$$
$$\hat{H}(s) = \frac{s}{4s^2 + 5s + 2}$$

(b) The characteristic modes m_1 and m_2 of the circuit correspond to the poles of $\hat{H}(s) : s = \frac{-5 \pm j\sqrt{7}}{8}$. Thus,

$$m_1(t) = \exp\frac{(-5 + j\sqrt{7})t}{8}$$
$$m_2(t) = \exp\frac{(-5 - j\sqrt{7})t}{8}$$

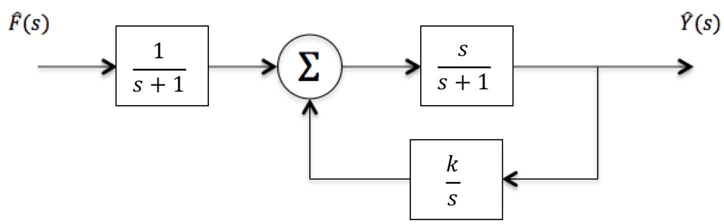
(c) If $f(t) = 1$ V, $F(s) = 1/s$. $i(0^-) = 0$ implies $y'(0^-) = 0$, and $y(0^-) = 1$ is known. Rewriting the laplace transform of the circuit's governing differential equation while including these initial values, we get

$$\hat{Y}(s) = \frac{4s+6}{4s^2+5s+2} = \frac{s+5/8}{(s+5/8)^2+(\sqrt{7}/8)^2} + \frac{7/8}{(s+5/8)^2+(\sqrt{7}/8)^2}$$

Taking inverse laplace transform,

$$y(t) = e^{-5t/8}\left(\cos\left(\frac{\sqrt{7}t}{8}\right) + \sqrt{7}\sin\left(\frac{\sqrt{7}t}{8}\right)\right)u(t)$$

5. (a) Determine the transfer function $\hat{H}(s)$ of the system shown below.



(b) When $k = 2$, determine whether the system is BIBO stable or not.

(c) Which values of k can you have so that the system is BIBO stable?

Solution:

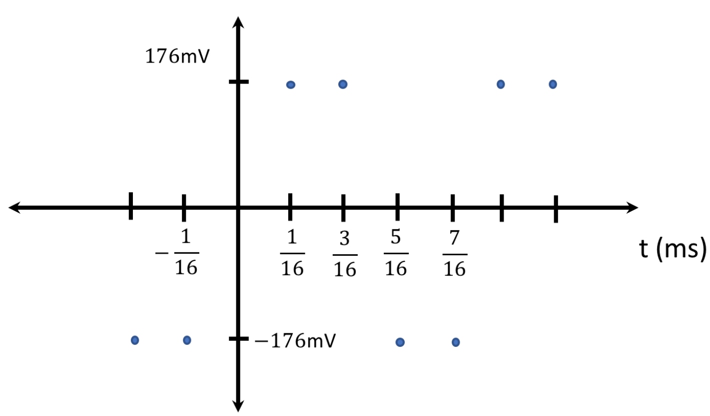
$$\left(\hat{F}(s)\frac{1}{s+1} + \hat{Y}(s)\frac{k}{s}\right)\frac{s}{s+1} = \hat{Y}(s)$$

$$\hat{H}(s) = \frac{s}{(s+1)(s+1-k)}$$

(b) For $k = 2$, the system has a pole in the right half of s-plane. Therefore, the system is unstable.

(c) The system will be BIBO stable if the pole $s = -1+k$ lies in the left half of s-plane, that leads to $-1+k < 0$, or $k < 1$ for stability.

6. Lab 5 is about signal sampling and reconstruction. Let's review a part about what we did there in this question: Part 2.1.6, we had a 2kHz sinusoid wave with amplitude 250 mV. Let's take 4 samples each period, thus our sampling frequency is 8kHz (4 samples per period), what you have in your computer hard drive is shown in the following graph.



(a) Do you think it's possible to get your original 2kHz sine wave back from these dots, as shown in the graph?

(b) Please explain step-by-step how you can reconstruct your original signal from these 4 dots per period by drawing the resulting signal. (those 4 points per period is all you have).

(c) According to Nyquist's sampling principle, as long as you have more than two dots per period, you should be able to get your original sine wave back. If you didn't get a perfect sine wave in part (b), try again.

Solution: (a) Yes it is possible to reconstruct the original sine wave from the points because the sampling rate the greater than the nyquist rate.

(b) The reconstruction can be done by interpolation using time shifted sinc functions

$$y_{rec}(t_{ms}) = \sum_{n=-\infty}^{\infty} y(n) sinc(8\pi(t_{ms} - (2n+1)/16))$$

where t_{ms} represents time in milliseconds, $y(n) = \tilde{y}(n \bmod 4)$ and

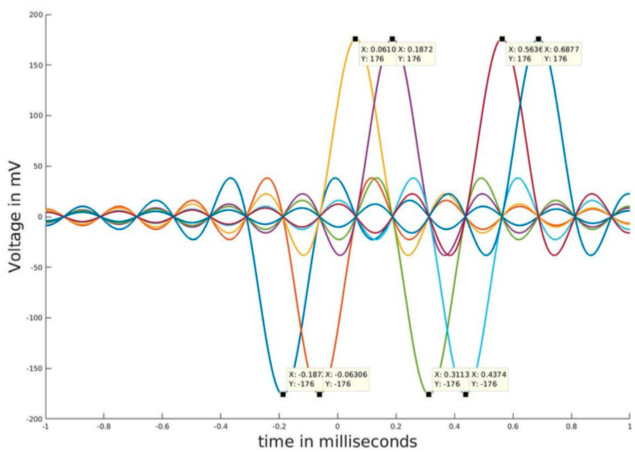
$$\tilde{y}(n) = +176 \text{ mV for } n = 0, 1$$
$$\tilde{y}(n) = -176 \text{ mV for } n = 2, 3$$

y_{rec} could also be written as

$$y_{rec}(t) = \sum_{n=-\infty}^{\infty} y(n) sinc(8000\pi(t - (2n+1)/16000))$$

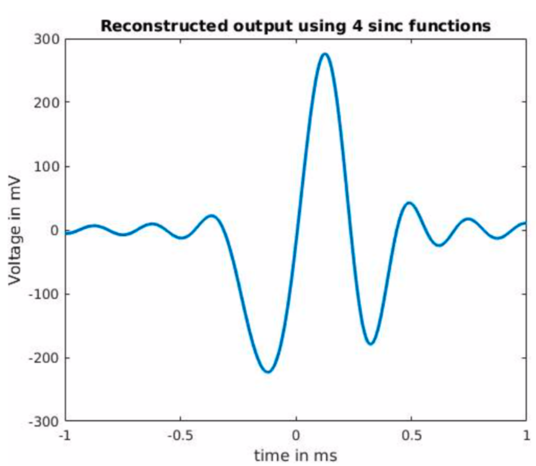
where t is in seconds.

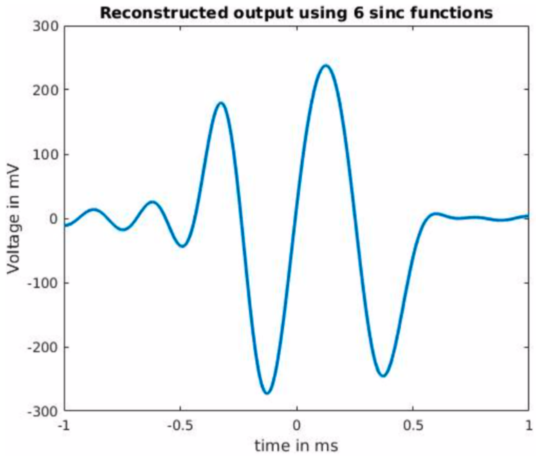
Figure below shows sinc functions associated with the sampled points in $t \in [-3/16, 11/16]$ ms.

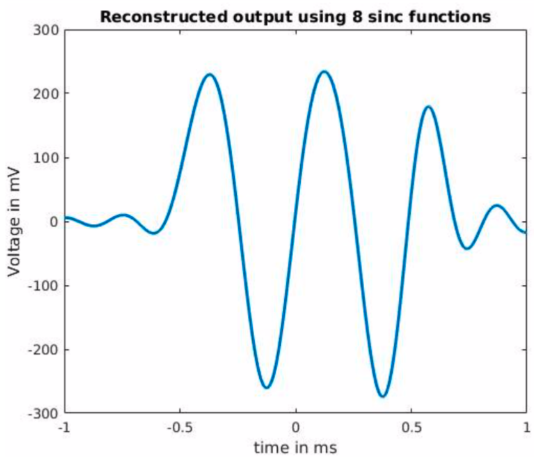


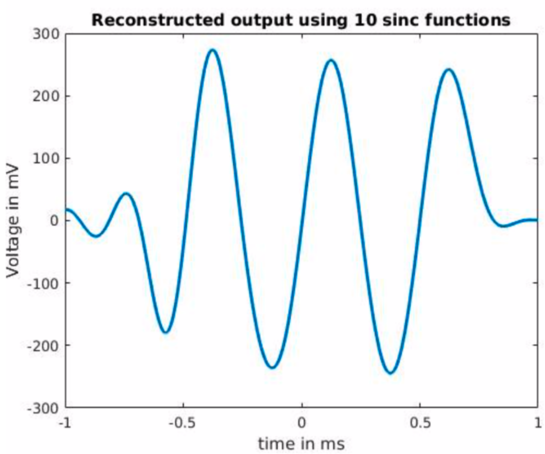
This is just a representative plot showing only 8 sinc functions. In realty, there would be infinitely many such functions associated with each sampled point.

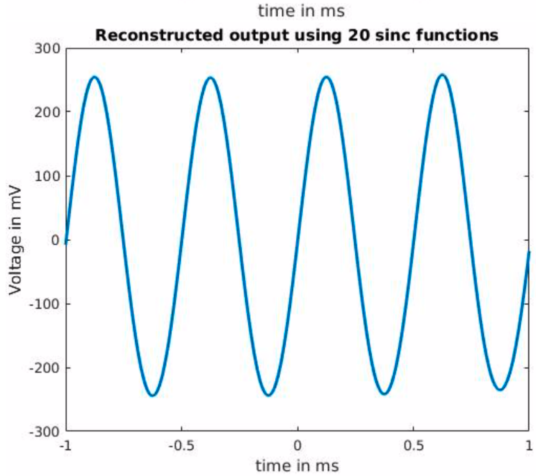
To perform reconstruction, we incrementally add these time shifted sinc functions scaled by the amplitude of each sample. The series of plots below indicate this reconstruction process.

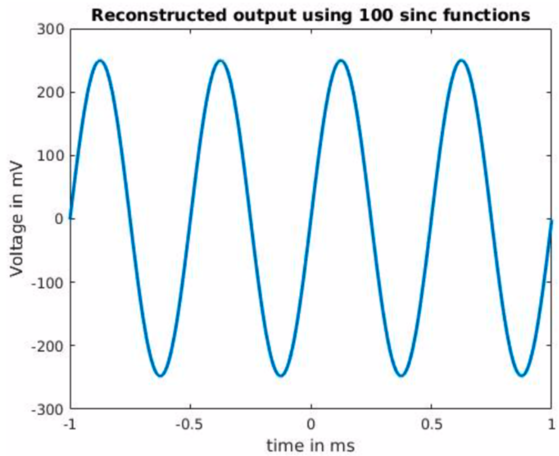



Reconstructed output using 6 sinc functions
Voltage in mV
time in ms


Reconstructed output using 8 sinc functions
Voltage in mV
time in ms


Reconstructed output using 10 sinc functions
Voltage in mV
time in ms


Reconstructed output using 20 sinc functions
Voltage in mV
time in ms


Reconstructed output using 100 sinc functions
Voltage in mV
300
200
100
0
-100
-200
-300
-1
-0.5
0
0.5
1
time in ms