

ECE 210 / ECE 211 - Homework 8

Solution

1. Consider the function $f(t) = \Re\{2e^{j3t} + 3e^{-j2t}\}$, where $\Re\{x\}$ denotes the real part of x . Find its period, T , its fundamental frequency, ω_0 , and plot it over at least two periods.

$$\Re\{2e^{j3t} + 3e^{-j2t}\} = 2\cos 3t + 3\cos 2t$$

The period is found as $T = \text{LCM}(T_1, T_2) = \text{LCM}(\frac{2\pi}{3}, \frac{\pi}{2}) = 2\pi$. Then $\omega_0 = 1 \frac{\text{Hz}}{\text{rad}}$

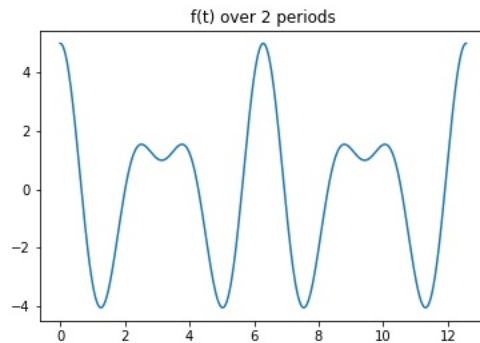


Figure 1: $f(t)$ over 2 periods

2. For each one of the following function of t , indicate whether they are periodic or not. If periodic, indicate its period, and if not periodic indicate why. Assume n is a positive integer.
 $T = \text{LCM}(T_1, T_2)$. If the ratios of all pairs of periods are not a rational number, then the function is not periodic.

(a) $\sin t - \sin\left(\frac{t}{2}\right) - \sin\left(\frac{t}{4}\right)$
 $T = \text{LCM}(2\pi, 4\pi, 8\pi) = 8\pi$

(b) $\sin(2\pi t) + \cos(\sqrt{2}t)$
 $\frac{T_1}{T_2} = \frac{1}{\pi\sqrt{2}} \notin \mathbb{Q}$ so not periodic.

(c) $\sin \frac{\pi t}{2} + \cos(2nt) + \sin \frac{3\pi t}{5} + \sin \frac{et}{2}$
 $\frac{\pi}{e} \notin \mathbb{Q}$ so not periodic.

(d) $\cos(2nt)$
 $T = \frac{\pi}{n}$

(e) $\cos(\pi t) - \cos\left(\frac{\pi}{n}t\right)$
 $T = \text{LCM}(2, 2n) = 2n$

(f) $\sin(nt) \cos(2nt) = \frac{1}{2} \sin(3nt) - \frac{1}{2} \sin(nt)$
 $T = \text{LCM}\left(\frac{2\pi}{3}, 2\pi\right) = 2\pi$

3. What property of RLC circuits allows the use of Fourier series for analysis? Show the property by means of a diagram.

RLC circuits are linear and time-invariant (LTI) systems, allowing their frequency components to be considered independently, meaning the input signal can be decomposed into its Fourier components and treated separately.

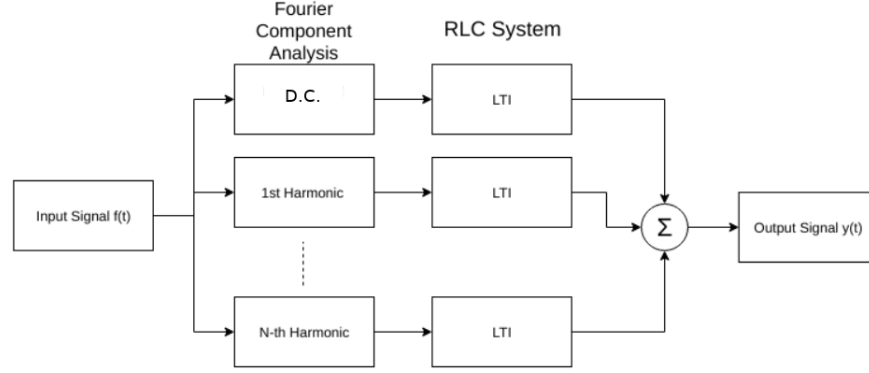


Figure 1: Diagram of Fourier Component Analysis

4. Consider the periodic ramp below, where the angle of the line is $\frac{\pi}{4}$, and the period $T \geq 1$.

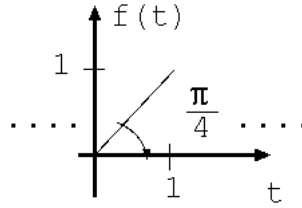


Figure 2: Periodic Ramp

- (a) Its exponential Fourier series coefficients are found as follows:

$$\begin{aligned} F_n &= \frac{1}{T} \int_T f(t) e^{-jn\omega_0 t} dt \\ &= \frac{1}{T} \int_0^1 t e^{-jn\omega_0 t} dt \end{aligned}$$

Using integration by parts with

$$u = t, \quad du = dt, \quad dv = e^{-jn\omega_0 t} dt, \quad v = \frac{e^{-jn\omega_0 t}}{-jn\omega_0}$$

we have

$$\begin{aligned}
F_n &= \frac{1}{T} (uv|_0^1 - \int_0^1 v du) \\
&= \frac{1}{T} \left(\left. \frac{te^{-jn\omega_0 t}}{-jn\omega_0} \right|_0^1 - \int_0^1 \frac{e^{-jn\omega_0 t}}{-jn\omega_0} dt \right) \\
&= -\frac{e^{-jn\omega_0}}{jn\omega_0 T} + \frac{e^{-jn\omega_0} - 1}{n^2 \omega_0^2 T} \\
&= -\frac{e^{-jn \frac{2\pi}{T}}}{jn \frac{2\pi}{T} T} + \frac{e^{-jn \frac{2\pi}{T}} - 1}{n^2 \left(\frac{2\pi}{T}\right)^2 T} \\
&= \frac{je^{-j \frac{2\pi n}{T}}}{2\pi n} + \frac{Te^{-j \frac{2\pi n}{T}} - T}{(2\pi n)^2}
\end{aligned}$$

(b) $\omega_0 = \frac{2\pi}{T} \frac{Hz}{rad}$

(c) $\Delta\omega = \omega_n - \omega_{n-1} = \omega_0 = \frac{2\pi}{T} \frac{Hz}{rad}$. As $\Delta\omega_0 \rightarrow 0$, $T_0 \rightarrow \infty$, meaning the function becomes aperiodic.

5. Consider the following periodic signal $f(t)$ with period $T = 2\pi$ and exponential Fourier series coefficients F_n . The fundamental frequency is $\omega_0 = \frac{2\pi}{T} = 1 \frac{Hz}{rad}$

- (a) Obtain the compact Fourier series coefficients c_n and θ_n for all n and express $f(t)$ in its compact Fourier series form.

The exponential Fourier series form is

$$\begin{aligned}
f(t) &= \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \\
&= \frac{1}{4} (e^{j2t} + e^{-j2t}) + \sum_{n \text{ odd}} \frac{jn}{\pi(4 - n^2)} e^{jnt}
\end{aligned}$$

(b)

$$c_n = 2|F_n| = \begin{cases} \frac{2}{3\pi} & n = 1 \\ \frac{1}{2} & n = 2 \\ \frac{2n}{\pi(n^2 - 4)} & n \text{ odd}, n \neq 1 \\ 0 & \text{else} \end{cases}$$

$$\theta_n = \angle F_n = \begin{cases} \frac{\pi}{2} & n = 1 \\ -\frac{\pi}{2} & n \text{ odd}, n \neq 1 \\ 0 & \text{else} \end{cases}$$

So the compact Fourier series form of the signal is:

$$\begin{aligned}
f(t) &= \sum_{n=0}^{\infty} c_n \cos(n\omega_0 t + \theta_n) \\
&= \frac{2}{3\pi} \cos\left(t + \frac{\pi}{2}\right) + \frac{1}{2} \cos 2t + \sum_{n \text{ odd}, \geq 3} \frac{2n}{\pi(n^2 - 4)} \cos\left(nt - \frac{\pi}{2}\right)
\end{aligned}$$

- (c) Let $f(t)$ be the input to an LTI system with frequency response $H(\omega)$ plotted below. Find the system output, $y(t)$, in compact form.

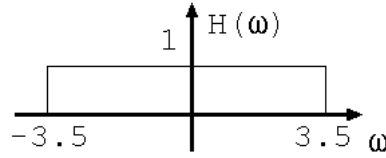


Figure 3:

The filter only passes the sinusoids at $\omega_0, 2\omega_0$ and $3\omega_0$, so

$$y(t) = \frac{2}{3\pi} \cos(t + \frac{\pi}{2}) + \frac{1}{2} \cos 2t + \frac{6}{5\pi} \cos(3t - \frac{\pi}{2})$$

- (d) Obtain the exponential Fourier series coefficients of the periodic signal $g(t)$, with period $T = 2\pi$ s, in terms of the exponential Fourier series coefficients F_n .

$$g(t) = \begin{cases} \sin(2(t + \frac{\pi}{4})) & -\frac{\pi}{2} \leq t < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} \leq t < \frac{3\pi}{2} \end{cases} = -f(t + \frac{\pi}{2})$$

Using the scaling and time shift properties,

$$G_n = -F_n e^{jn\omega_0 t_0} = -F_n e^{\frac{j\pi n}{2}}$$

- (e) Express $g(t)$ in exponential Fourier series form.

$$\begin{aligned} g(t) &= \frac{1}{4}(e^{j2t} + e^{-j2t}) + \sum_{n \text{ odd}} \frac{-jn e^{\frac{j\pi n}{2}}}{\pi(4 - n^2)} e^{jnt} \\ &= \frac{1}{4}(e^{j2t} + e^{-j2t}) + \sum_{n \text{ odd}} \frac{j^{n-1}n}{\pi(4 - n^2)} e^{jnt} \end{aligned}$$

6. Consider a periodic signal $f(t)$ with period $T = 2$ s, given by:

$$g(t) = \begin{cases} \sin(2(t + \frac{\pi}{4})), & -\frac{\pi}{2} \leq t < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} \leq t < \frac{3\pi}{2} \end{cases}$$

The fundamental frequency is $\omega_0 = \pi \frac{\text{Hz}}{\text{rad}}$.

- (a) The function $f(t)$ can be expressed as a Fourier series with exponential coefficients

$$F_n = \frac{e^{1-jn\pi} - 1}{1 + n^2\pi^2}$$

Let $f(t)$ be the input to an LTI system with frequency response $H(\omega)$, given in the plot below. Determine the steady state output, $y_{ss}(t)$ in compact form.

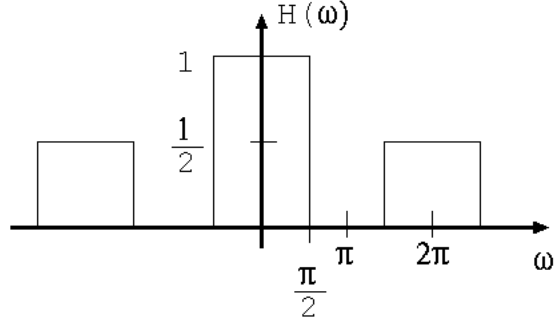


Figure 4: LTI system

The system passes only the sinusoids at 0 and $2\omega_0 = 2\pi$.

$$\begin{aligned}
 y_{ss}(t) &= \frac{c_0}{2} + c_2 \cos(2\pi t + \theta_2) \\
 &= F_0 + 2|F_2| \cos(2\pi t + \angle F_2) \\
 &= e - 1 + \frac{e - 1}{1 + 4\pi^2} \cos 2\pi t \\
 &\approx 1.718 + 0.0424 \cos 2\pi t
 \end{aligned}$$

(b) Consider the periodic signal $g(t)$ with period $T = 2$ s, given by:

$$g(t) = \begin{cases} e^t & -1 \leq t < 0\text{s} \\ e^{-t} & 0 \leq t < 1\text{s} \end{cases}$$

Determine its exponential Fourier coefficients G_n in terms of the exponential Fourier coefficients F_n .

$$g(t) = e^{-1} f(t + 1)$$

Using scaling and time-shift properties,

$$G_n = e^{-1} F_n e^{j\pi n}$$

(c) Express $g(t)$ in exponential Fourier series form.

$$\begin{aligned}
 g(t) &= \sum_{n=-\infty}^{\infty} \frac{e^{1-j\pi n} - 1}{1 + n^2\pi^2} e^{-1} e^{j\pi n} e^{j\pi n t} \\
 &= \sum_{n=-\infty}^{\infty} \frac{1 - e^{j\pi n - 1}}{1 + n^2\pi^2} e^{j\pi n t}
 \end{aligned}$$