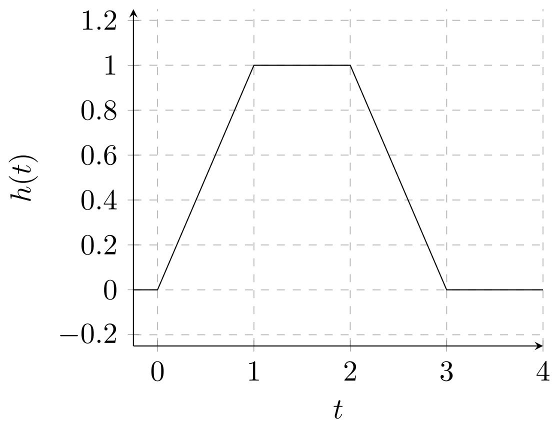
Homework 11 Solutions

ECE 210: Analog Signal Processing

Fall 2018

Problem 1

Let $f(t) = rect(t - \frac{1}{2})$, and $h(t)$ is given as:



And, let $y(t) = f(t) * h(t)$.

(a) Determine $y(t) = f(t) * h(t)$ by direct integration and sketch the result.

(b) Determine the value of t_I, the first instant in time when $y(t)$ is non-zero.

(c) Determine the value of t_F, the last instant in time when $y(t)$ is non-zero.

(d) Determine the values of $y(0)$, $y(1)$, $y(2)$, $y(3)$.

Solution

Part (a) We compute the convolution directly, limiting the bounds of integration to those values of τ over which $f(t - \tau)$ is non-zero.

Since $f(\tau)$ is non-zero over the domain $\tau \in [0, 1]$, $f(t - \tau)$ is non-zero over the domain $\tau \in [t - 1, t]$.

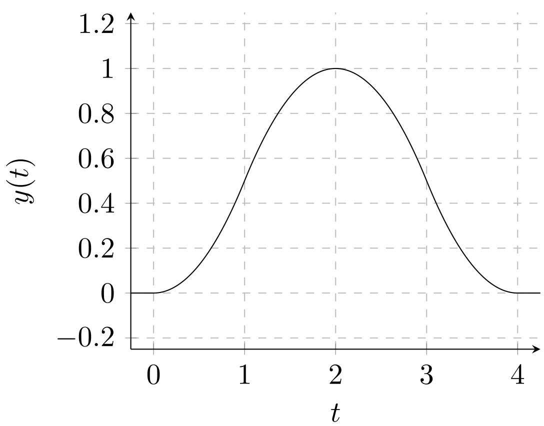
$$
\begin{aligned}
y(t) = \int_{-\infty}^{\infty} h(\tau) \cdot f(t - \tau) d\tau &= \int_{-\infty}^{\infty} h(\tau) \cdot rect(t - \tau - \frac{1}{2}) d\tau \\
&= \int_{t-1}^{t} h(\tau) \cdot 1 \cdot d\tau
\end{aligned}
$$

$y(t)$ is then the area under $f(\tau)$ over the domain $\tau \in [t-1, t]$.

$$y(t) = \begin{cases} \int_0^t \tau d\tau = \frac{1}{2}t^2, & 0 \leq t \leq 1 \\ \int_{t-1}^1 \tau d\tau + \int_1^t 1 \cdot d\tau = \frac{-1}{2}(t^2 - 4t + 2), & 1 \leq t \leq 2 \\ \int_{t-1}^2 1 \cdot d\tau + \int_2^t (3-\tau) d\tau = \frac{-1}{2}(t^2 - 4t + 2), & 2 \leq t \leq 3 \\ \int_{t-1}^3 (3-\tau) d\tau = \frac{1}{2}(t^2 - 8t + 16), & 3 \leq t \leq 4 \\ 0, & Otherwise \end{cases}$$

$y(t)$ is shown below.

$$\boxed{y(t) = \begin{cases} \frac{1}{2}t^2, & 0 \leq t \leq 1 \\ \frac{-1}{2}(t^2 - 4t + 2), & 1 < t \leq 3 \\ \frac{1}{2}(t^2 - 8t + 16), & 3 < t \leq 4 \end{cases}}$$



Part (b) Upon inspection of the graph above, $\boxed{t_I = 0}$.

Part (c) Upon inspection of the graph above, $\boxed{t_F = 4}$.

Part (d) Upon inspection of the graph above, $\boxed{y(0) = 0, \quad y(1) = \frac{1}{2}, \quad y(2) = 1, \quad y(3) = \frac{1}{2}}$.

Problem 2

Let $f(t) = 3u(1+t)$ and $h(t) = e^{-t}u(t)$, and let $y(t) = f(t) * h(t)$. Determine $y(t)$ for all t.

Solution

We compute the convolution by direct integration, limiting the bounds of integration to those values of τ over which $h(\tau)f(t-\tau)$ is non-zero.

$$y(t) = \int_{-\infty}^{\infty} h(\tau)f(t-\tau)d\tau = \int_{-\infty}^{\infty} e^{-\tau} \cdot 3u(\tau)u(t-\tau+1)d\tau$$

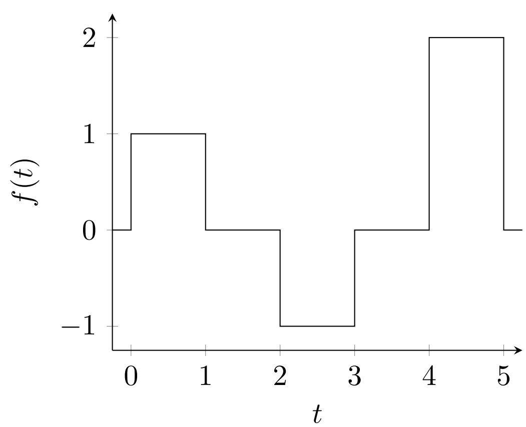
$u(\tau)u(t-\tau+1)$ is non-zero over the domain $\tau \in [0, t+1]$.

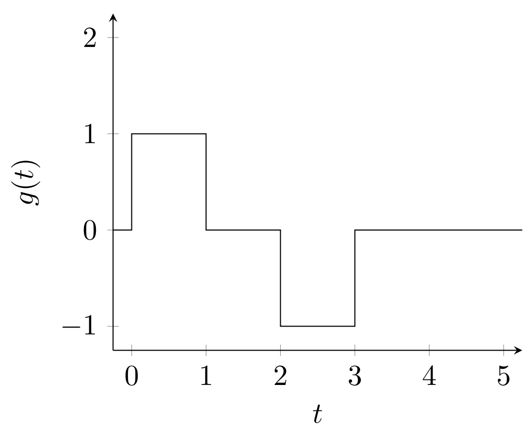
$$\begin{aligned} y(t) &= \int_0^{t+1} 3e^{-\tau} \cdot 1 \cdot d\tau \\ &= -3e^{-\tau}\Big|_0^{t+1}, \quad t \geq -1 \\ &= -3e^{-(t+1)} + 3, \quad t \geq -1 \end{aligned}$$

$$\boxed{y(t) = 3(1 - e^{-(t+1)}), \quad t \geq -1}$$

Problem 3

For the functions shown below:





(a) Determine $x(t) = g(t) * g(t)$ by direct integration and sketch the result.

(b) Determine $y(t) = f(t) * g(t)$ using appropriate properties of convolution and sketch the result.

(c) Determine $z(t) = f(t) * f(t-1)$ using appropriate properties of convolution. Sketch the result.

Solution

To simplify the calculation, let us first calculate the convolution of two shifted rectangle functions.

Let $s(t) = rect(t - \frac{1}{2}) = \begin{cases} 1, & 0 \leq t \leq 1 \\ 0, & Otherwise \end{cases}$.

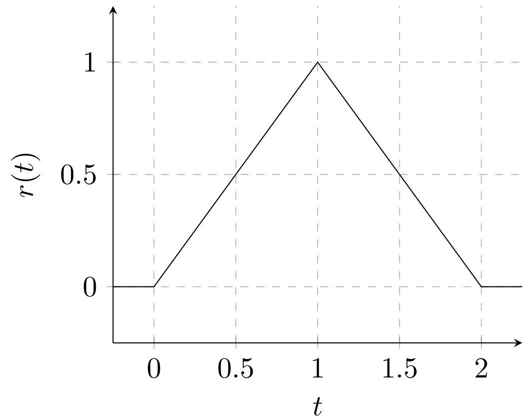
We compute the convolution of $s(t)$ with itself, and we denote this as $r(t)$.

$$r(t) = s(t) * s(t) = \int_{-\infty}^{\infty} s(\tau)s(t-\tau)d\tau$$

$s(\tau)s(t-\tau)$ is non-zero over the domain $\{\tau : 0 \leq \tau \leq 1 \quad and \quad t-1 \leq \tau \leq t\}$.

$$r(t) = \int_{\max(0,t-1)}^{\min(1,t)} 1 \cdot 1 \cdot d\tau = \tau\Big|_{\max(0,t-1)}^{\min(1,t)}$$

$$= \begin{cases} \tau\big|_0^t = t, & 0 \leq t \leq 1 \\ \tau\big|_{t-1}^1 = 2 - t, & 1 < t \leq 2 \end{cases}$$



It can be shown by the same calculation that shifts in the rectangle functions result in a cumulative shift in $r(t)$. The only difference in the calculation is the limits of integration.

$$s(t-a) * s(t-b) = \int_{\max(a,(t-1)-b)}^{\min(1+a,t-b)} 1 \cdot 1 \cdot d\tau = \begin{cases} t - (a+b), & (a+b) \leq t \leq 1 + (a+b) \\ 2 - (t - (a+b)), & 1 + (a+b) < t \leq 2 + (a+b) \end{cases}$$

$$= r(t - (a+b))$$

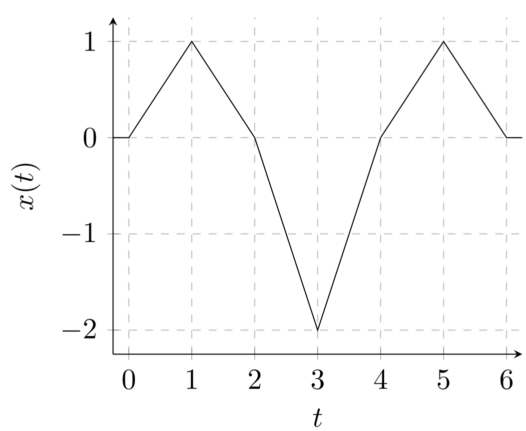
Knowing this, and seeing that $g(t)$ is a linear combination of shifted rectangle functions, we can easily determine $g(t) * g(t)$.

$$g(t) = s(t) - s(t-2)$$

$$\begin{aligned} x(t) = f(t) * g(t) &= \int_{\infty}^{\infty} g(\tau) g(t-\tau) d\tau \\ &= \int_{\infty}^{\infty} \Big(s(\tau) - s(\tau - 2)\Big) \cdot \Big(s(t-\tau) - s(t-\tau-2)\Big) d\tau \\ &= \int_{\infty}^{\infty} \Big(s(\tau)s(t-\tau) - s(t)s(t-\tau-2) - s(\tau-2)s(t-\tau) + s(\tau-2)s(t-\tau-2)\Big) d\tau \\ &= s(t) * s(t) - s(t) * s(t-2) - s(t-2) * s(t) + s(t-2) * s(t-2) \\ &= r(t) - 2r(t-2) + r(t-4) \end{aligned}$$

$x(t)$ is shown below.

$$\boxed{\begin{aligned} & x(t) = r(t) - 2r(t-2) + r(t-4) \\ & where \quad r(t) = \begin{cases} t, & 0 \leq t \leq 1 \\ 2 - t, & 1 < t \leq 2 \end{cases} \end{aligned}}$$

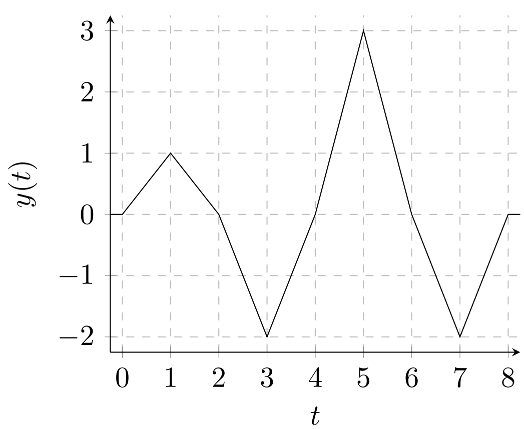


Part (b) Using what we know about $r(t)$ and $s(t)$, we can compute $f(t) * g(t)$ without integration.

$$f(t) = g(t) + 2s(t-4)$$

$$\begin{aligned} y(t) = f(t) * g(t) &= \Big(g(t) + 2s(t-4)\Big) * g(t) = g(t) * g(t) + 2s(t-4) * g(t) \\ &= x(t) + 2s(t-4) * \Big(s(t) - s(t-2)\Big) \\ &= x(t) + 2s(t-4) * s(t) - 2s(t-4) * s(t-2) \\ &= x(t) + 2r(t-4) - 2r(t-6) \\ &= r(t) - 2r(t-2) + 3r(t-4) - 2r(t-6) \end{aligned}$$

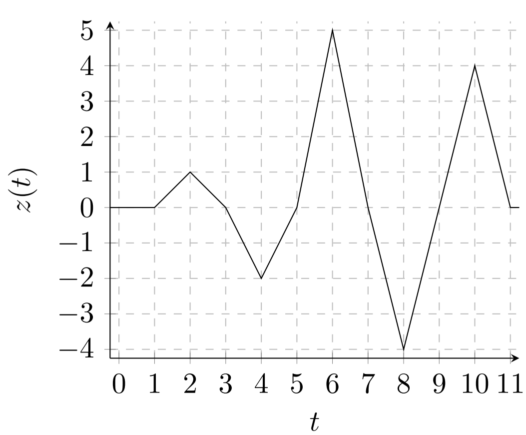
$$\boxed{y(t) = r(t) - 2r(t-2) + 3r(t-4) - 2r(t-6)}$$



Part (c) Similarly, we can compute $f(t) * f(t-1)$ without integration.

$$\begin{aligned} z(t) = f(t) * t(t-1) &= \Big(s(t) - s(t-2) + 2s(t-4)\Big) * \Big(s(t-1) - s(t-3) + 2s(t-5)\Big) \\ &= r(t-1) - r(t-3) + 2r(t-5) - r(t-3) + r(t-5) - 2r(t-7) + 2r(t-5) - 2r(t-7) + 4r(t-9) \\ &= r(t-1) - 2r(t-3) + 5r(t-5) - 4r(t-7) + 4r(t-9) \end{aligned}$$

$$\boxed{\begin{aligned} z(t) = \quad & r(t-1) - 2r(t-3) + 5r(t-5) \\ & - 4r(t-7) + 4r(t-9) \end{aligned}}$$



Problem 4

Given $h(t) = u(t+2)$ and $f(t) = 2\Delta(t+4) = \begin{cases} 2 + 4(t+4), & -\frac{9}{2} \le t \le -4 \\ 2 - 4(t+4), & -4 < t \le \frac{-7}{2} \end{cases}$

(a) Determine $y(t) = h(t) * f(t)$ and sketch the result.

(b) Determine $z(t) = h(t) * \frac{df}{dt}$ using appropriate properties of convolution and sketch the result.

Solution

Part (a) We can simplify the calculation by modeling the time-shifts in $h(t)$ and $f(t)$ as convolutions with Dirac delta functions.

$$y(t) = h(t) * f(t) = u(t+2) * 2\Delta(t+4) = 2 \cdot u(t) * \Delta(t) * \delta(t+6)$$

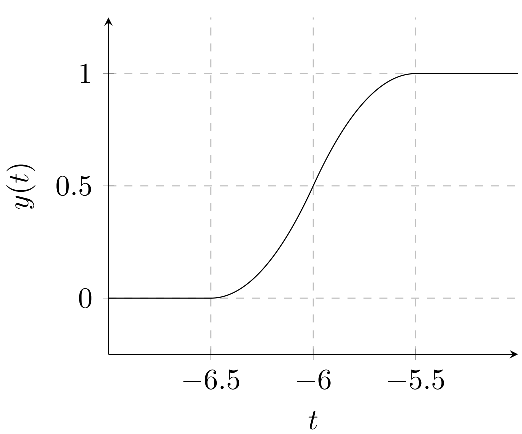
$$\begin{aligned} u(t) * \Delta(t) &= \int_{-\infty}^{\infty} u(\tau)\Delta(t-\tau)d\tau \\ &= \int_{\max(0,t-\frac{1}{2})}^{t+\frac{1}{2}} 1 \cdot \Delta(t-\tau)d\tau \\ &= \int_{-\frac{1}{2}}^{\max(\frac{1}{2},t)} \Delta(\tau')d\tau' \end{aligned}$$

Note that this expression is equal to the cumulative area under $\Delta(t)$.

$$u(t) * \Delta(t) = \begin{cases} 0, & t < -\frac{1}{2} \\ (t+\frac{1}{2})^2, & -\frac{1}{2} \le t < 0 \\ \frac{1}{2} - (t-\frac{1}{2})^2, & 0 \le t \le \frac{1}{2} \\ \frac{1}{2}, & t > \frac{1}{2} \end{cases}$$

$$y(t) = 2 \cdot u(t) * \Delta(t) * \delta(t+6)$$

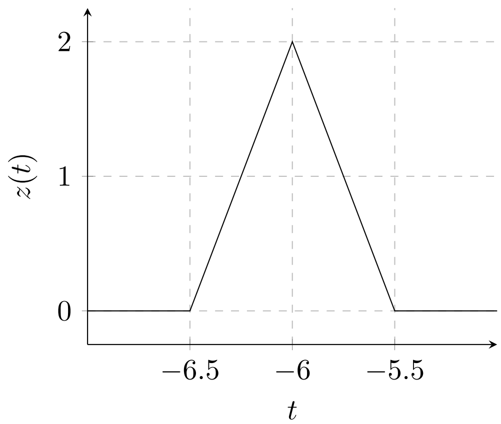
$$\boxed{y(t) = \begin{cases} 0, & t < -\frac{13}{2} \\ 2(t+\frac{13}{2})^2, & -\frac{13}{2} \le t < -6 \\ 1 - 2(t+\frac{11}{2})^2, & -6 \le t \le -\frac{11}{2} \\ 1, & t > -\frac{11}{2} \end{cases}}$$



Part (b) As shown in the calculation above, convolution of a function with a unit step results in the cumulative area under the function. This is equivalent to the integral of the function.

$$
\begin{aligned}
z(t) = h(t) * \frac{df}{dt} &= \delta(t+2) * u(t) * \frac{d}{dt}\Big(2\Delta(t+4)\Big) \\
&= \delta(t+2) * 2\Delta(t+4) \\
&= 2\Delta(t+6)
\end{aligned}
$$

$$\boxed{z(t) = \frac{dy}{dt} = f(t+2) = 2\Delta(t+6)}$$



Note that this is a direct result of the following convolution property:

$$y(t) = f(t) * h(t) \Rightarrow \frac{dy}{dt} = \frac{df}{dt} * h(t) = f(t) * \frac{dh}{dt}$$

Problem 5

Given $f(t) = u(t+1)$ and $g(t) = \frac{1}{2}\Delta(t-2)$, and $q(t) = f(t-2) * g(t)$, determine $q(4)$.

Solution

$$
\begin{aligned}
q(4) &= \int_{-\infty}^{\infty} u(4 - \tau - 2 + 1) g(\tau) d\tau \\
&= \int_{-\infty}^{\infty} u(3 - \tau) g(\tau) d\tau \\
&= \int_{-\infty}^{3} 1 \cdot \frac{1}{2}\Delta(t-2) d\tau \\
&= \frac{1}{2} \int_{1.5}^{2.5} \Delta(t-2) d\tau
\end{aligned}
$$

Note that this expression is equal to half of the area under $\Delta(t-2)$, or $\boxed{\frac{1}{4}}$.

Problem 6

Given $f(t) = u(t+1)$ and $h(t) = t^2 u(2-t)$, and $y(t) = f(t) * h(t)$, determine $y(-4)$

$$\begin{aligned} y(-4) &= \int_{-\infty}^{\infty} u(-4-\tau+1)\tau^2 u(2-\tau) d\tau \\ &= \int_{-\infty}^{-3} \tau^2 d\tau \\ &= \frac{1}{3}\tau^3 \Big|_{-\infty}^{-3} \\ &= \boxed{\infty} \end{aligned}$$

Problem 7

Simplify the following expressions involving the impulse and/or shifted impulse and sketch the results:

(a) $g(t) = \sin(2\pi t)\Big(\frac{du}{dt} + \delta(t+0.5)\Big)$

(b) $a(t) = \int_{-\infty}^{t} \delta(\tau - 1)d\tau + \Delta(\frac{t}{4})\delta(t-2)$

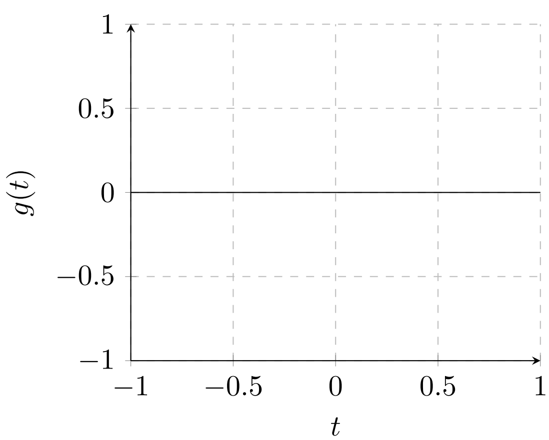
(c) $b(t) = \delta(t-2) * rect(t-3)$

(d) $y(t) = \int_{-\infty}^{2} (\tau^2 - 3)\delta(t-3)d\tau$

Solution

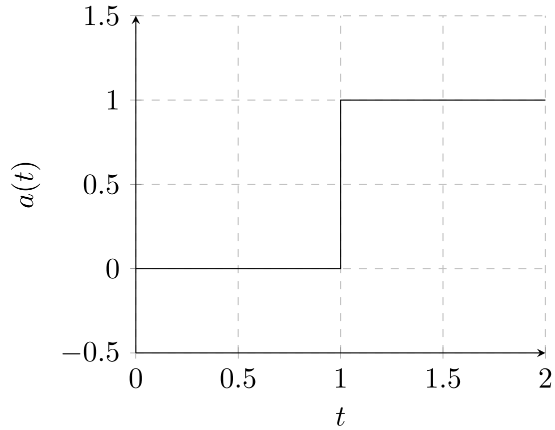
Part (a)

$$\begin{aligned} g(t) &= \sin(2\pi t)\Big(\frac{du}{dt} + \delta(t+0.5)\Big) \\ &= \sin(2\pi t)\Big(\delta(t) + \delta(t+0.5)\Big) \\ &= \underbrace{\sin(2\pi \cdot 0)}_{0}\delta(t) + \underbrace{\sin(2\pi \cdot (-0.5))}_{0}\delta(t-0.5) \\ &= \boxed{0} \end{aligned}$$



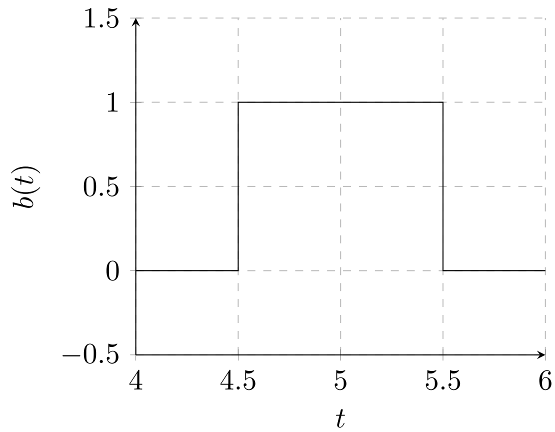
Part (b)

$$
\begin{aligned}
a(t) &= \int_{-\infty}^{t} \delta(\tau-1)d\tau + \Delta(\frac{t}{4})\delta(t-2) \\
&= \int_{-\infty}^{t-1} \delta(\tau)d\tau + \Delta(\frac{2}{4})\delta(t-2) \\
&= u(t-1) + \underbrace{\Delta(\frac{1}{2})}_{0}\delta(t-2) \\
&= \boxed{u(t-1)}
\end{aligned}
$$



Part (c)

$$
\begin{aligned}
b(t) &= \delta(t-2) * rect(t-3) \\
&= rect(t-3-2) \\
&= \boxed{rect(t-5)}
\end{aligned}
$$



Part (d)

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{2} (\tau^2-3)\delta(t-3)d\tau \\
&= \int_{-\infty}^{2} (^2-3)\delta(t-3)d\tau \\
&= 6\underbrace{\int_{-\infty}^{2} \delta(t-3)d\tau}_{0} \\
&= \boxed{0}
\end{aligned}
$$