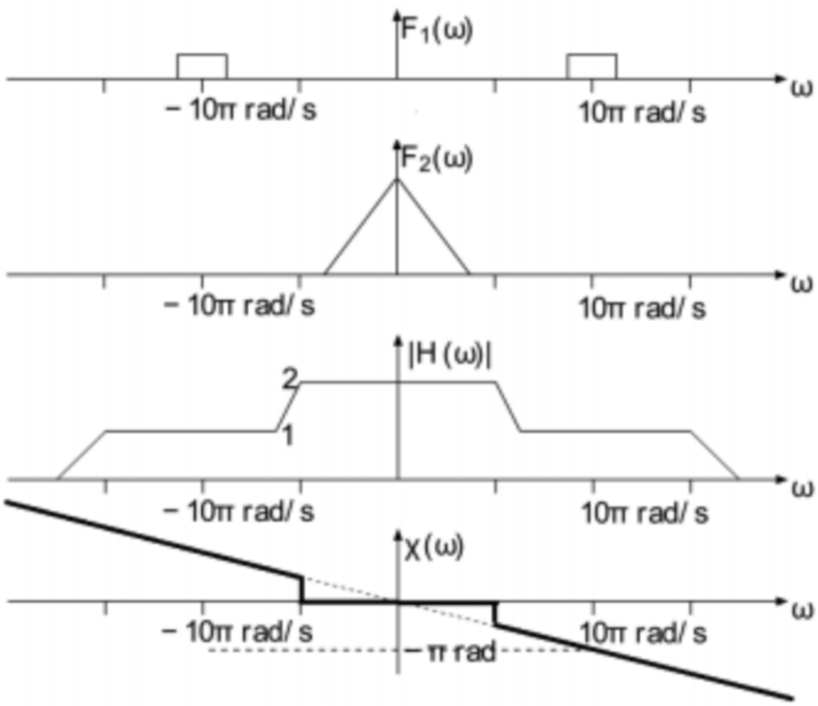
ECE210 / ECE211 - Homework 10

Solution

1. Let $f(t) = f_1(t) + f_2(t)$ such that $f_1(t) \longleftrightarrow F_1(w)$ and $f_2(t) \longleftrightarrow F_2(w)$, and let $H(w) = |H(w)|\, e^{jX(w)}$. The functions $F_1(w)$, $F_2(w)$, $H(w)$ and $X(w)$ are given graphically below. The signal $f(t)$ is the input to an LTI system with a frequency response $H(w)$. Express the output $y(t)$ of the system as a superposition of scaled and/or shifted versions of $f_1(t)$ and $f_2(t)$.



Solution:

For the region where $F_2(w) \neq 0$, we have

$$H(w) = 2$$

Therefore,

$$Y_2(w) = 2F_2(w) \longleftrightarrow y_2(t) = 2f_2(t)$$

Also, for the region where $F_1(w) \neq 0$, we notice a phase that is changing linearly with slope $-\frac{1}{10}$. Hence,

$$H(w) = e^{-j\frac{1}{10}w}$$

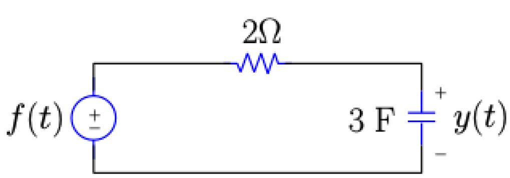
Consequently,

$$Y_1(w) = e^{-j\frac{1}{10}w}F_1(w) \longleftrightarrow y_1(t) = f_1(t - \frac{1}{10})$$

Finally, adding the two results, we obtain

$$y(t) = f_1(t - \frac{1}{10}) + 2f_2(t)$$

2. Consider the circuit shown below.



(a) Consider an arbitrary input $f(t)$ and determine the response, $y(t)$, in the form of an inverse Fourier transform.
(b) Evaluate $y(t)$ for the case $f(t) = e^{-\frac{t}{6}}u(t)$

Solution:

(a) Using voltage division, we have

$$Y(w) = F(w)\frac{\frac{1}{3jw}}{2+\frac{1}{3jw}} = F(w)\frac{1}{6jw+1}$$

Applying the inverse Fourier transform,

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(w)\frac{1}{6jw+1}e^{jwt}dw$$

(b) For the input $f(t) = e^{-\frac{t}{6}}u(t)$ the Fourier transform pair

$$e^{-at}u(t) \longleftrightarrow \frac{1}{a+jw}, a > 0$$

yields

$$F(w) = \frac{1}{\frac{1}{6}+jw}$$

Use the result from (a),

$$Y(w) = F(w)\frac{1}{6jw+1} = \frac{1}{\frac{1}{6}+jw}\frac{\frac{1}{6}}{\frac{1}{6}+jw} = \frac{1}{6}\frac{1}{(\frac{1}{6}+jw)^2}$$

Using the inverse transform pair:

$$te^{-at}u(t) \longleftrightarrow \frac{1}{(a+jw)^2}$$

we have

$$y(t) = \frac{1}{6}te^{-\frac{t}{6}}u(t)$$

3. Given that $f(t)e^{\pm jw_0t} \longleftrightarrow F(w \pm w_0)$, determine the Fourier transform of $g(t) = f(t)\sin(w_0t)$ in terms of scaled and/or shifted versions of $F(w)$

   Solution :

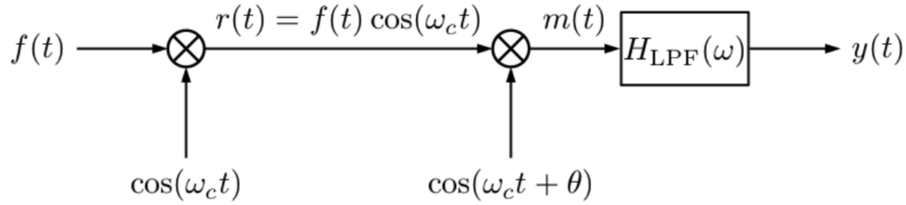
$$f(t)\sin(w_0t) = f(t)\frac{1}{2j}\left[e^{jw_0t} - e^{-jw_0t}\right] = \frac{-j}{2}f(t)\left[e^{jw_0t} - e^{-jw_0t}\right]$$

   Using the frequency-shift property

$$f(t)e^{jw_0t} \longleftrightarrow F(w-w_0)$$

   together with the modulation property $f(t)\sin(w_0t) \longleftrightarrow -\frac{j}{2}F(w-w_0) + \frac{j}{2}F(w+w_0)$

4. A signal $f(t)$ is band limited to the interval $w \in [-\Omega, \Omega]$ and modulated by a cosine carrier of frequency $w_c > \Omega$. The resulting modulated signal $r(t)$ is then coherently demodulated with a mismatched carrier signal $\cos(w_ct+\theta)$, and filtered with an ideal low pass filter $H_{LPF}(w) = rect(\frac{w}{2\Omega})$ as shown in the figure below.



(a) Find an expression for $y(t)$ in terms of $f(t)$ and θ
(b) For what values of θ is the amplitude of $y(t)$ smallest and largest ?
(c) Consider what would happen when θ is slowly time varying. If you were to play $y(t)$ on a loudspeaker, what qualitative effect with this have on the signal you hear ?
Solution :

(a) $m(t) = f(t)\cos(w_c t)\cos(w_c t + \theta) = \frac{f(t)}{2}\left[\cos(2w_c t + \theta) + \cos(\theta)\right] = \frac{f(t)}{2}\left[\frac{e^{j(2w_c t+\theta)} + e^{-j(2w_c t+\theta)}}{2} + \cos(\theta)\right]$
$m(t) = \frac{e^{j\theta}}{4} f(t) e^{j2w_c t} + \frac{e^{-j\theta}}{4} f(t) e^{-j2w_c t} + \frac{\cos(\theta)}{2} f(t)$
Using the frequency shift property we have:

$$\frac{e^{j\theta}}{4} f(t) e^{j2w_c t} + \frac{e^{-j\theta}}{4} f(t) e^{-j2w_c t} \leftrightarrow \frac{e^{j\theta}}{4} F(w - 2w_c) + \frac{e^{-j\theta}}{4} F(w + 2w_c)$$

Therefore, $M(w)$ is :

$$M(w) = \frac{1}{4}\left[F(w - 2w_c)e^{j\theta} + F(w + 2w_c)e^{-j\theta}\right] + \frac{\cos(\theta)}{2} F(w)$$

After the low pass filter, the first part of the above equation is filtered out. Hence

$$Y(w) = \frac{\cos(\theta)}{2} F(w) \longleftrightarrow \frac{\cos(\theta)}{2} f(t)$$

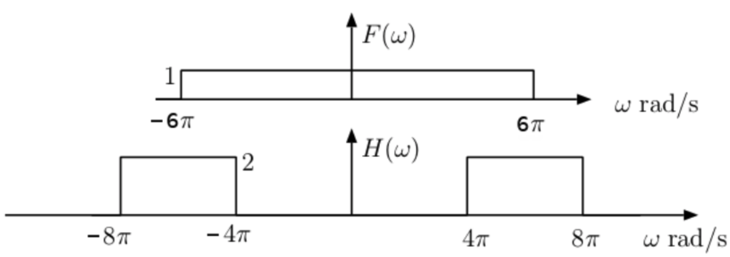
(b) The amplitude is maximized at $\theta = n\pi$, and minimized when $\theta = \frac{2n+1}{2}\pi$, for integer n

(c) The variation of θ acts like an amplitude modulation since $y(t) \approx \frac{\cos(\theta)}{2} f(t)$.

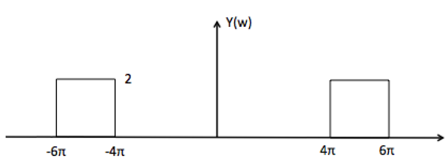
5. A linear system with frequency response $H(w)$ is excited with an input

$$f(t) \longleftrightarrow F(w)$$

$H(w)$ and $F(w)$ are plotted below:

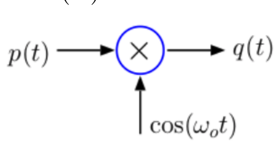


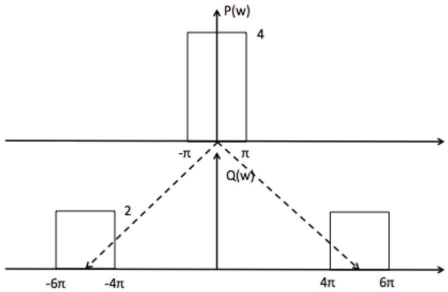
(a) Sketch the Fourier transform $Y(w)$ of the system output $y(t)$ and calculate the energy W_y of $y(t)$.



$$W_y = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |Y(w)|^2\, dw = 8$$

(b) It is observed that output $q(t)$ of the following system equals $y(t)$ determine in part (a).
Sketch $P(w)$ and determine w_0





w_0 is 5π

(c) Express $y(t)$ in terms of $f(t)$
$P(w)$ is amplitude and frequency scaled version of $F(w)$

$$P(w) = 4F(6w)$$

and $y(t)$ is $p(t)$ modulated with $\cos(5\pi t)$

$$y(t) = p(t)\cos(5\pi t)$$

Using time scaled property

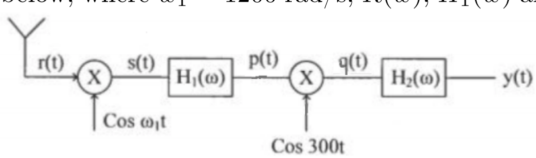
$$f(st) \longleftrightarrow \frac{1}{|s|}F(\frac{w}{s})$$

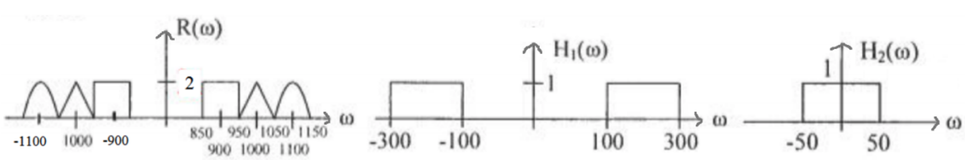
We get $p(t) = 4 \times \frac{1}{6}f(\frac{1}{6}t)$

$$y(t) = \frac{2}{3}f(\frac{1}{6}t)\cos(5\pi t)$$

Solution :

6. Consider the system below, where $w_1 = 1200$ rad/s, $R(w)$, $H_1(w)$ and $H(w)$ are plotted below the system figure.

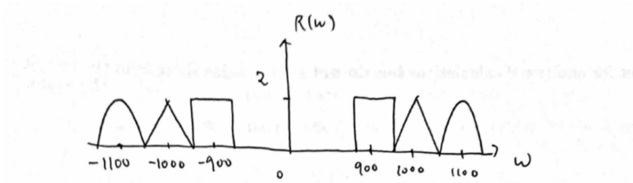


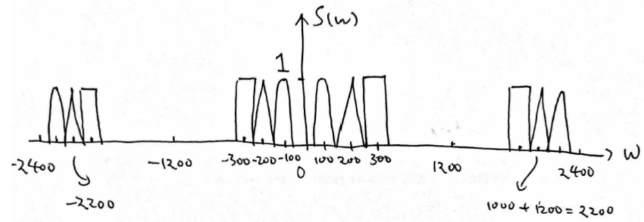


(a) Sketch the spectra $S(w)$, $P(w)$, $Q(w)$, and $Y(w)$.
(b) Calculate the energy of $y(t)$

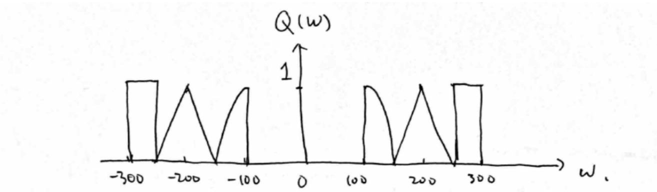
Solution

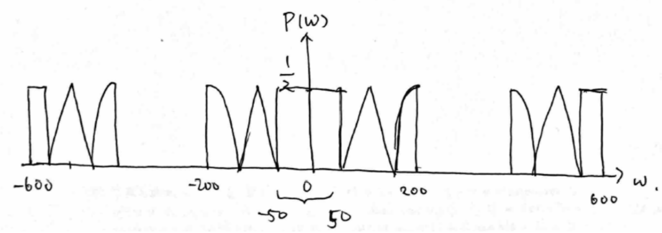
(a) The first modulation displaces the original signal by 1200 rad/s in the frequency domain, creating four copies.



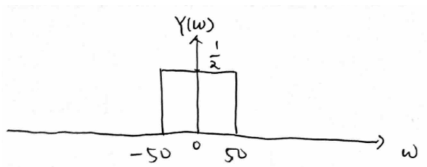


Then the signal is passed through the ideal band pass filter, chopping everything outside of $w \in \pm[100, 300]$ rad/s





Finally, the signal is passed through the ideal low pass filter.



(b) From the sketch in part (a), the energy in the output is $0.5^2 \times 100/2\pi = \frac{25}{2\pi}$J