

## ECE 210 / ECE 211 - Homework 8

## Solution

1. Consider the function  $f(t) = \Re\{2e^{j3t} + 3e^{-j2t}\}$ , where  $\Re\{x\}$  denotes the real part of  $x$ . Find its period,  $T$ , its fundamental frequency,  $\omega_0$ , and plot it over at least two periods.

$$\Re\{2e^{j3t} + 3e^{-j2t}\} = 2 \cos 3t + 3 \cos 2t$$

The period is found as  $T = \text{LCM}(T_1, T_2) = \text{LCM}(\frac{2\pi}{3}, \frac{\pi}{2}) = 2\pi$ . Then  $\omega_0 = 1 \frac{\text{Hz}}{\text{rad}}$

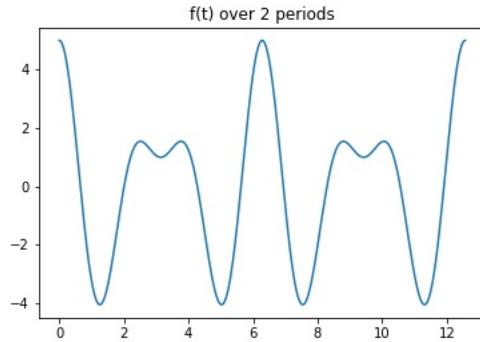


Figure 1:  $f(t)$  over 2 periods

2. For each one of the following function of  $t$ , indicate whether they are periodic or not. If periodic, indicate its period, and if not periodicm indicate why. Assume  $n$  is a positive integer.

$T = \text{LCM}(T_1, T_2)$ . If the ratios of all pairs of periods are not a rational number, then the function is not periodic.

(a)  $\sin t - \sin\left(\frac{t}{2}\right) - \sin\left(\frac{t}{4}\right)$   
 $T = \text{LCM}(2\pi, 4\pi, 8\pi) = 8\pi$

(b)  $\sin(2\pi t) + \cos(\sqrt{2}t)$   
 $\frac{T_1}{T_2} = \frac{1}{\pi\sqrt{2}} \notin \mathbb{Q}$  so not periodic.

(c)  $\sin\frac{\pi t}{2} + \cos(2nt) + \sin\frac{3\pi t}{5} + \sin\frac{et}{2}$   
 $\frac{\pi}{e} \notin \mathbb{Q}$  so not periodic.

(d)  $\cos(2nt)$   
 $T = \frac{\pi}{n}$

(e)  $\cos(\pi t) - \cos\left(\frac{\pi}{n}t\right)$   
 $T = \text{LCM}(2, 2n) = 2n$

(f)  $\sin(nt) \cos(2nt) = \frac{1}{2} \sin(3nt) - \frac{1}{2} \sin(nt)$   
 $T = \text{LCM}\left(\frac{2\pi}{3}, 2\pi\right) = 2\pi$

3. What property of RLC circuits allows the use of Fourier series for analysis? Show the property by means of a diagram.

RLC circuits are linear and time-invariant (LTI) systems, allowing their frequency components to be considered independently, meaning the input signal can be decomposed into its Fourier components and treated separately.

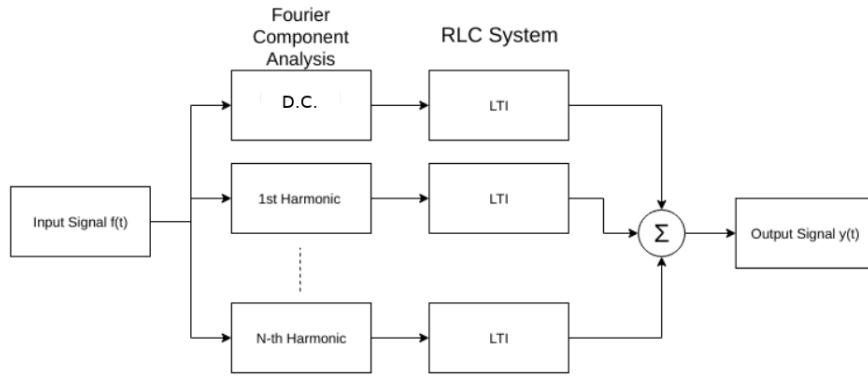


Figure 1: Diagram of Fourier Component Analysis

4. Consider the periodic ramp below, where the angle of the line is  $\frac{\pi}{4}$ , and the period  $T \geq 1$ .

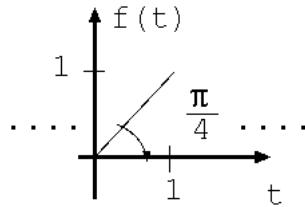


Figure 2: Periodic Ramp

- (a) Its exponential Fourier series coefficients are found as follows:

$$\begin{aligned} F_n &= \frac{1}{T} \int_T f(t) e^{-jn\omega_0 t} dt \\ &= \frac{1}{T} \int_0^1 t e^{-jn\omega_0 t} dt \end{aligned}$$

Using integration by parts with

$$u = t, \quad du = dt, \quad dv = e^{-jn\omega_0 t} dt, \quad v = \frac{e^{-jn\omega_0 t}}{-jn\omega_0}$$

we have

$$\begin{aligned}
F_n &= \frac{1}{T} (uv|_0^1 - \int_0^1 vdu) \\
&= \frac{1}{T} \left( \frac{te^{-jn\omega_0 t}}{-jn\omega_0} \Big|_0^1 - \int_0^1 \frac{e^{-jn\omega_0 t}}{-jn\omega_0} dt \right) \\
&= -\frac{e^{-jn\omega_0}}{jn\omega_0 T} + \frac{e^{-jn\omega_0} - 1}{n^2 \omega_0^2 T} \\
&= -\frac{e^{-jn\frac{2\pi}{T}}}{jn\frac{2\pi}{T} T} + \frac{e^{-jn\frac{2\pi}{T}} - 1}{n^2 (\frac{2\pi}{T})^2 T} \\
&= \frac{je^{-j\frac{2\pi n}{T}}}{2\pi n} + \frac{Te^{-j\frac{2\pi n}{T}} - T}{(2\pi n)^2}
\end{aligned}$$

(b)  $\omega_0 = \frac{2\pi}{T} \frac{\text{Hz}}{\text{rad}}$

(c)  $\Delta\omega = \omega_n - \omega_{n-1} = \omega_0 = \frac{2\pi}{T} \frac{\text{Hz}}{\text{rad}}$ . As  $\Delta\omega_0 \rightarrow 0$ ,  $T_0 \rightarrow \infty$ , meaning the function becomes aperiodic.

5. Consider the following periodic signal  $f(t)$  with period  $T = 2\pi$ s and exponential Fourier series coefficients  $F_n$ . The fundamental frequency is  $\omega_0 = \frac{2\pi}{T} = 1 \frac{\text{Hz}}{\text{rad}}$

- (a) Obtain the compact Fourier series coefficients  $c_n$  and  $\theta_n$  for all  $n$  and express  $f(t)$  in its compact Fourier series form.

The exponential Fourier series form is

$$\begin{aligned}
f(t) &= \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \\
&= \frac{1}{4} (e^{j2t} + e^{-j2t}) + \sum_{n \text{ odd}} \frac{jn}{\pi(4-n^2)} e^{jnt}
\end{aligned}$$

(b)

$$\begin{aligned}
c_n &= 2|F_n| = \begin{cases} \frac{2}{3\pi} & n = 1 \\ \frac{1}{2} & n = 2 \\ \frac{2n}{\pi(n^2-4)} & n \text{ odd}, n \neq 1 \\ 0 & \text{else} \end{cases} \\
\theta_n &= \angle F_n = \begin{cases} \frac{\pi}{2} & n = 1 \\ -\frac{\pi}{2} & n \text{ odd}, n \neq 1 \\ 0 & \text{else} \end{cases}
\end{aligned}$$

So the compact Fourier series form of the signal is:

$$\begin{aligned}
f(t) &= \sum_{n=0}^{\infty} c_n \cos(n\omega_0 t + \theta_n) \\
&= \frac{2}{3\pi} \cos(t + \frac{\pi}{2}) + \frac{1}{2} \cos 2t + \sum_{n \text{ odd}, \geq 3} \frac{2n}{\pi(n^2-4)} \cos(nt - \frac{\pi}{2})
\end{aligned}$$

- (c) Let  $f(t)$  be the input to an LTI system with frequency response  $H(\omega)$  plotted below. Find the system output,  $y(t)$ , in compact form.

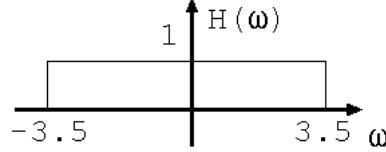


Figure 3:

The filter only passes the sinusoids at  $\omega_0$ ,  $2\omega_0$  and  $3\omega_0$ , so

$$y(t) = \frac{2}{3\pi} \cos(t + \frac{\pi}{2}) + \frac{1}{2} \cos 2t + \frac{6}{5\pi} \cos(3t - \frac{\pi}{2})$$

- (d) Obtain the exponential Fourier series coefficients of the periodic signal  $g(t)$ , with period  $T = 2\pi$ s, in terms of the exponential Fourier series coefficients  $F_n$ .

$$g(t) = \begin{cases} \sin(2(t + \frac{\pi}{4})) & -\frac{\pi}{2} \leq t < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} \leq t < \frac{3\pi}{2} \end{cases} = -f(t + \frac{\pi}{2})$$

Using the scaling and time shift properties,

$$G_n = -F_n e^{jn\omega_0 t_0} = -F_n e^{\frac{j\pi n}{2}}$$

- (e) Express  $g(t)$  in exponential Fourier series form.

$$\begin{aligned} g(t) &= \frac{1}{4}(e^{j2t} + e^{-j2t}) + \sum_{n \text{ odd}} \frac{-jne^{\frac{j\pi n}{2}}}{\pi(4-n^2)} e^{jnt} \\ &= \frac{1}{4}(e^{j2t} + e^{-j2t}) + \sum_{n \text{ odd}} \frac{j^{n-1}n}{\pi(4-n^2)} e^{jnt} \end{aligned}$$

6. Consider a periodic signal  $f(t)$  with period  $T = 2$ s, given by:

$$g(t) = \begin{cases} \sin(2(t + \frac{\pi}{4})), & -\frac{\pi}{2} \leq t < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} \leq t < \frac{3\pi}{2} \end{cases}$$

The fundamental frequency is  $\omega_0 = \pi \frac{Hz}{rad}$ .

- (a) The function  $f(t)$  can be expressed as a Fourier series with exponential coefficients

$$F_n = \frac{e^{1-jn\pi} - 1}{1 + n^2\pi^2}$$

Let  $f(t)$  be the input to an LTI system with frequency response  $H(\omega)$ , given in the plot below. Determine the steady state output,  $y_{ss}(t)$  in compact form.

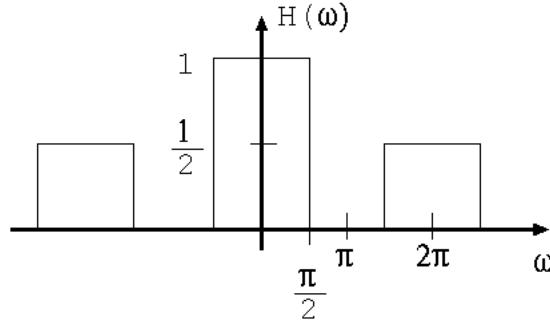


Figure 4: LTI system

The system passes only the sinusoids at 0 and  $2\omega_0 = 2\pi$ .

$$\begin{aligned}
 y_{ss}(t) &= \frac{c_0}{2} + c_2 \cos(2\pi t + \theta_2) \\
 &= F_0 + 2|F_2| \cos(2\pi t + \angle F_2) \\
 &= e - 1 + \frac{e - 1}{1 + 4\pi^2} \cos 2\pi t \\
 &\approx 1.718 + 0.0424 \cos 2\pi t
 \end{aligned}$$

(b) Consider the periodic signal  $g(t)$  with period  $T = 2s$ , given by:

$$g(t) = \begin{cases} e^t & -1 \leq t < 0s \\ e^{-t} & 0 \leq t < 1s \end{cases}$$

Determine its exponential Fourier coefficients  $G_n$  in terms of the exponential Fourier coefficients  $F_n$ .

$$g(t) = e^{-1} f(t+1)$$

Using scaling and time-shift properties,

$$G_n = e^{-1} F_n e^{j\pi n}$$

(c) Express  $g(t)$  in exponential Fourier series form.

$$\begin{aligned}
 g(t) &= \sum_{n=-\infty}^{\infty} \frac{e^{1-j\pi n} - 1}{1 + n^2\pi^2} e^{-1} e^{j\pi n} e^{j\pi nt} \\
 &= \sum_{n=-\infty}^{\infty} \frac{1 - e^{j\pi n-1}}{1 + n^2\pi^2} e^{j\pi nt}
 \end{aligned}$$