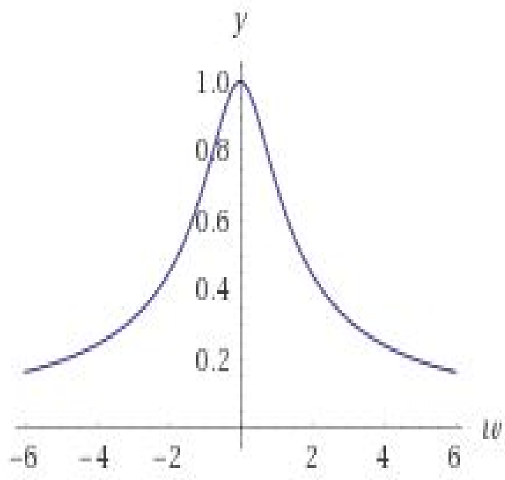
University of Illinois **Fall 2018**

ECE210 / ECE211 - Homework 12 - Solutions

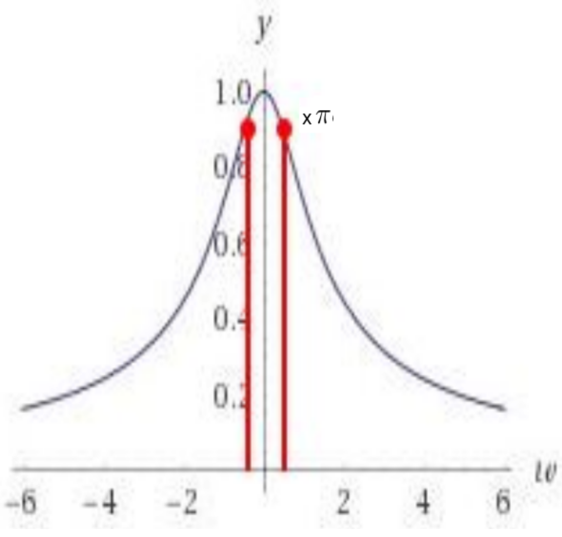
Due : Wednesday, November 28 at 6 p.m.

1. Consider a poor quality LPF with frequency response $H(\omega) = \frac{1}{1+j\omega}$:

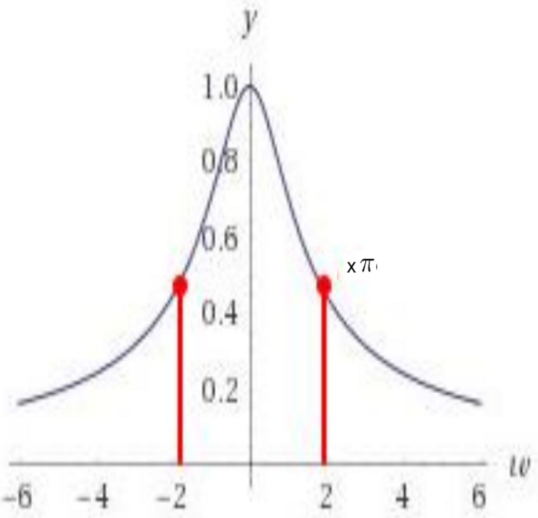
   (a) Plot or sketch $|H(\omega)|$.



   (b) Plot the response to $\cos(t/2)$ on the same plot as $|H(\omega)|$.
   $cos(t/2) \rightarrow \pi\delta(\omega - \frac{1}{2}) + \pi\delta((\omega + \frac{1}{2}))$
   $H(1/2) = \frac{1}{1+j\frac{1}{2}} = 0.8 - 0.4j$
   $\therefore |H(1/2)| = |H(-1/2)| = 0.894$
   $Y(\omega) = H(1/2)\pi\delta(\omega - \frac{1}{2}) + H(-1/2)\pi\delta((\omega + \frac{1}{2})) = 0.894\pi\delta(\omega - \frac{1}{2}) + 0.894\pi\delta((\omega + \frac{1}{2}))$



   (c) Plot the response to $\cos(2t)$ on the same plot as $|H(\omega)|$.
   $cos(2t) \rightarrow \pi\delta(\omega - 2) + \pi\delta((\omega + 2))$
   $H(2) = \frac{1}{1+j2} = 0.2 - 0.4j$
   $\therefore |H(2)| = |H(-2)| = 0.447$
   $Y(\omega) = H(2)\pi\delta(\omega - 2) + H(-2)\pi\delta((\omega + 2)) = 0.447\pi\delta(\omega - 2) + 0.447\pi\delta((\omega + 2))$



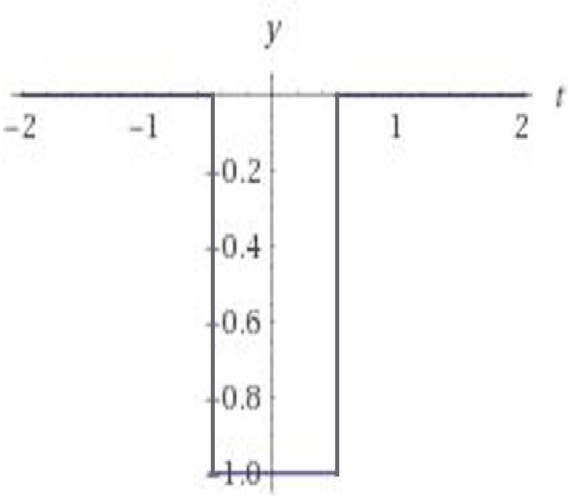
(d) In what sense are the responses in parts (b) and (c), samples of $H(\omega)$?
$cos(\omega_o t)$ samples $H(\omega)$ at frequency $\omega_o [rad/sec]$.

(e) Is $H(\omega)$ a reconstruction from samples? Why?
Yes, because the sum of all samples will make up $H(\omega)$.

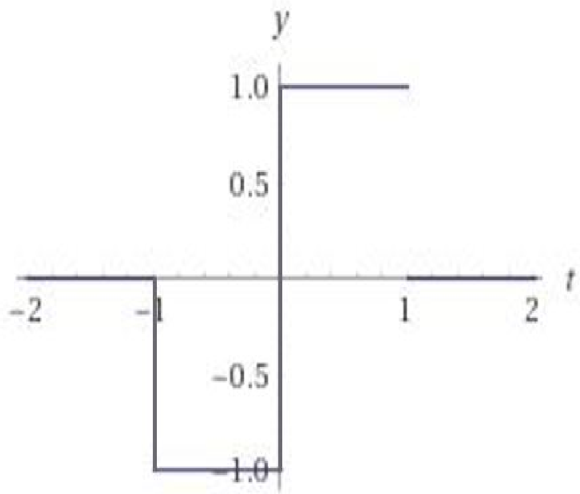
(f) How is $H(\omega)$ related to both the Fourier transform and the Fourier series of the output?
Fourier series decomposes the signal whereas the Fourier transform transforms the signal to frequency domain. Therefore, the Fourier transform of the Fourier series of an input signal will get scaled by $|H(\omega)|$ to get the output.

2. A system is described by an impulse response $h(t) = \delta\left(t - \frac{1}{2}\right) - \delta\left(t + \frac{1}{2}\right)$. Sketch the system response $y(t) = h(t) * f(t)$ to the following inputs:

(a) $f(t) = u(t)$
$y(t) = [\delta(t - \frac{1}{2}) - \delta(t + \frac{1}{2})] * u(t) = u(t - \frac{1}{2}) - u(t + \frac{1}{2}) = -rect(t)$



(b) $f(t) = \text{rect}(t)$
$y(t) = [\delta(t - \frac{1}{2}) - \delta(t + \frac{1}{2})] * rect(t) = rect(t - \frac{1}{2}) - rect(t + \frac{1}{2})$



3. Determine the Fourier transform of the following signals-simplify the results as much as you can. For parts (a), (b) and (c), sketch the magnitude and phase of the result.

   (a) $f(t) = 5\cos(3t) + 2\sin(10t)$.
   $f(t) = 5cos(3t) + 2sin(10t) \leftrightarrow F(\omega) = 5\pi[\delta(\omega-3) + \delta(\omega+3)] + j2\pi[\delta(\omega+10) - \delta(\omega-10)]$

   (b) $x(t) = 2\cos^2(2t)$.
   $x(t) = 2cos^2(2t) = 1 + cos(4t) \leftrightarrow X(\omega) = 2\pi\delta(\omega) + \pi[\delta(\omega-4) + \delta(\omega+4)]$

   (c) $y(t) = e^{-t}u(t) * \cos(4t)$.
   $y(t) = e^{-t}u(t) * cos(4t) \leftrightarrow Y(\omega) = \frac{\pi}{1+j\omega}[\delta(\omega-4) + \delta(\omega+4)] = \frac{\pi}{1+j4}\delta(\omega-4) + \frac{\pi}{1-j4}\delta(\omega+4)$

   (d) $z(t) = (1 + \cos(2t))e^{-t}u(t)$
   $z(t) = (1 + cos(2t))e^{-t}u(t) \leftrightarrow Z(\omega) = \frac{1}{1+j\omega} + \frac{1+j\omega}{(1+j\omega)^2+4}$

4. Determine the inverse Fourier transforms of the following:

   (a) $F(\omega) = j[\delta(\omega-2) - \delta(\omega+2)] + 8\delta(\omega-4)$.
   $F(\omega) = j[\delta(\omega-2) - \delta(\omega+2)] + 8\delta(\omega-4) \leftrightarrow f(t) = -\frac{1}{\pi}sin(2t) + \frac{4}{\pi}e^{j4t}$

   (b) $A(\omega) = 3\pi\cos(2\omega)$.
   $A(\omega) = 3\pi cos(2\omega) \leftrightarrow a(t) = \frac{3\pi}{2}(\delta(t+2) + \delta(t-2))$

   (c) $B(\omega) = \sum_{n=-\infty}^{\infty} \frac{3}{1+n^2}\delta(\omega-n)$
   $B(\omega) = \sum \frac{3}{1+n^2}\delta(\omega-n) \leftrightarrow \frac{3}{2\pi}\sum \frac{e^{jnt}}{1+n^2}$

   (d) $C(\omega) = \frac{8}{2+j\omega} + 4\pi\delta(\omega)$
   $C(\omega) = \frac{8}{2+j\omega} + 4\pi\delta(\omega) \leftrightarrow 8e^{-2t}u(t) + 2$

5. Consider the following zero-state input-output relations for a variety of systems. In each case, determine whether the system is zero-state linear, time invariant, and causal. You must show work to prove each property.

   (a) $y(t) = f(t-1) + f(-t-1)$
      i. Let $f(t) = f_1(t) + f_2(t)$
      Let $f_1(t) \rightarrow y_1(t)$ and $f_2(t) \rightarrow y_2(t)$
      $f_1(t-1) + f_2(t-1) + f_1(-t-1) + f_2(-t-1) = y_1(t) + y_2(t) = y(t)$
      $\therefore$ Zero-state linear
      ii. $f((t-t_o)-1) + f(-(t-t_o)-1) \neq f(t-1-t_o) + f(-t-1-t_o)$
      $\therefore$ Time variant
      iii. Depends on future inputs $-t+1$
      $\therefore$ Non-causal

(b) $y(t) = f(2t)$.

i. Let $f(t) = f_1(t) + f_2(t)$
Let $f_1(t) \rightarrow y_1(t)$ and $f_2(t) \rightarrow y_2(t)$
$f_1(2t) + f_2(2t) = y_1(t) + y_2(t) = y(t)$
$\therefore$ Zero-state linear

ii. $f(2(t - t_o)) \neq f(2t - t_o)$
$\therefore$ Time variant

iii. $t = 1 : f(2)$
$\therefore$ Non-causal

(c) $y(t) = 5f(t) * u(t)$.

i. Convolution is a linear operator $\therefore$ Zero-state linear

ii. $f(t - \tau) \rightarrow \int_{-\infty}^{t-\tau} 5f(\tau)d\tau = y(t - \tau)$
$\therefore$ Time invariant

iii. Depends only on past and current values
$\therefore$ Causal

(d) $y(t) = \delta(t - 4) * f(t) - \int_{-\infty}^{t+2} f^2(\tau)d\tau$.

i. $f^2(\tau)$ is not linear $\therefore$ Zero-state nonlinear

ii. $f((t - t_o) - 4) - \int_{-\infty}^{(t-t_o)+2} f^2(\tau)d\tau = f(t - t_o - 4) - \int_{-\infty}^{t+2} f^2(x - t_o)dx$
$\therefore$ Time invariant

iii. Depends on future values $t + 2$
$\therefore$Non-causal

(e) $y(t) = \int_{-\infty}^{t-2} f(\tau)d\tau$. (Hint: is it possible to write this as a convolution?)

i. Convolution is a linear operator $\therefore$ Zero-state linear

ii. Time invariant

iii. Causal

6. Find the impulse responses $h(t)$ of the LTI systems having the following unit-step responses.

(a) $g(t) = 5u(2t - 5)$.
$\frac{d}{dt}[5u(2t - 5)] = 10\delta(2t - 5)$

(b) $g(t) = t^3u(t)$.
$\frac{d}{dt}[t^3u(t)] = 3t^2u(t) + t^3\delta(t) = 3t^2u(t)$

(c) $g(t) = (2 - e^{-t})u(t - 5)$.
$\frac{d}{dt}[(2 - e^{-t})u(t - 5)] = e^{-t}u(t - 5) + (2 - e^{-5})\delta(t - 5)$

7. Determine the minimum sampling frequencies ω_s needed to sample the following analog signals without causing aliasing error.
$[k = 1000]$

(a) Arbitrary signal $f(t)$ with bandwidth 20 kHz.
$(2\pi)(2)(20k) = 80\pi k\ [rad/sec]$

(b) $f_1(t) = \text{sinc}(4000\pi t)$.
$(2)(4k\pi) = 8\pi k[rad/sec]$

(c) $f_2(t) = \text{sinc}^2(4000\pi t)$. Compare this sampling frequency to the one in part (b).
$(2)(4k\pi + 4k\pi) = 16\pi k[rad/sec]$
This is double that of part (b) because the convolution, in the frequency domain, doubles the width.

(d) $f_3(t) = \text{sinc}(4000\pi t)\cos(12000\pi t)$. Compare this sampling frequency to the one in part (b).
$(2)(12k\pi + 4k\pi) = 32\pi k[rad/sec]$
This is larger than that of part (b) because the modulating cosine moves the spectrum up in the frequency domain.