University of Illinois **Fall 2018**

ECE210 - Homework 13

Due: Wednesday, December 5, 2018 at 6:00 p.m.

1. For each one of the 4 signals $f(t)$ in parts (a), (b), (c), (d), do the following

   i. Obtain its Laplace transform $\hat{F}(s)$.

   ii. Indicate the poles of $\hat{F}(s)$.

   iii. Indicate the ROC of $\hat{F}(s)$.

   (a) $f(t) = u(t-6) - u(t+6)$

   (b) $f(t) = te^{2(t-1)}u(t)$

   (c) $f(t) = (t-1)\,e^{-4t} + \delta(t)$

   (d) $f(t) = e^{2t}\cos(t-1)u(t+1)$.

   Solution:

   (a) $f(t) = u(t-6) - u(t+6)$

   i. Using the Laplace transform definition, we have

$$\hat{F}(s) = \int_{0^-}^{\infty} [u(t-6) - u(t+6)]\, e^{-st}dt = \int_{0^-}^{6} -e^{-st}dt = \frac{-1+e^{-6s}}{s}.$$

   ii. poles:
   Testing if $\hat{F}(s) \to \pm\infty$ as $s \to 0$: $\lim_{s\to 0}\hat{F}(s) = \lim_{s\to 0}\frac{-1+e^{-6s}}{s} = \frac{0}{0}$ (indeterminate). Applying l'Hospital rule we find out that $s = 0$ is not a pole, because $\lim_{s\to 0}\hat{F}(s) \neq \pm\infty$:

$$\lim_{s\to 0}\frac{-1+e^{-6s}}{s} = \lim_{s\to 0}\frac{\frac{d}{ds}\left(-1+e^{-6s}\right)}{\frac{d}{ds}(s)} = \lim_{s\to 0}\frac{-6e^{-6s}}{1} = -6 \neq \infty.$$

   There is a set of poles as $\mathrm{Re}\{s\} \to -\infty$. Therefore, we say that there is a "hidden" pole at $s = -\infty + j\omega$.
   List of poles:

$$s_1 = \{-\infty + j\omega\}.$$

   .

   iii. We recognize the ROC as the region to the right of the rightmost pole : $\sigma = \mathrm{Re}\{s\} > -\infty$.

   (b) $f(t) = te^{2(t-1)}u(t) = e^{-2}te^{2t}u(t)$

   i. Using Laplace transform tables we obtain

$$\hat{F}(s) = \frac{e^{-2}}{(s-2)^2},$$

   ii. list of poles: $s_{1,2} = 2\,$(double pole)

   iii. ROC: $\sigma = \mathrm{Re}\{s\} > 2$.
   This means that this Laplace integral converges only for values of s such that $\mathrm{Re}\{s\} > 2$.

   (c) $f(t) = (t-1)\,e^{-4t} + \delta(t) = te^{-4t} - e^{-4t} + \delta(t)$

   i. Using Laplace transform tables we obtain

$$\hat{F}(s) = \frac{1}{(s+4)^2} - \frac{1}{s+4} + 1 = \frac{s^2+7s+13}{(s+4)^2} = \frac{\left(s - \frac{-7+j\sqrt{55}}{2}\right)\left(s - \frac{-7-j\sqrt{55}}{2}\right)}{(s+4)^2}.$$

   ii. list of poles: $s_{1,2} = \{-4\,(\text{double})\}$

   iii. ROC: $\sigma = \mathrm{Re}\{s\} > -4$.
   This means that this Laplace integral converges only for values of s such that $\mathrm{Re}\{s\} > -4$.

(d) $f(t) = e^{3t}\cos(t-1)u(t-1)$

i. The Laplace transform starts at $t = 0$. Therefore it will be the same as calculating the L.T of $f(t) = e^{2t}\cos(t-1)u(t)$. Using Table 11.1 , we obtain

$$f(t) = e^3 e^{3(t-1)}\cos(t-1)u(t-1) \longleftrightarrow \hat{F}(s) = e^{(3-s)}\frac{s-3}{(s-(3+j))\,(s-(3-j))}.$$

ii. There is a set of poles as $\mathrm{Re}\{s\} \to -\infty$. Therefore, list of poles: $s_{1,2} = \{3+j, 3-j\}$ and $s_3 = \{-\infty + j\omega\}$.

iii. ROC: $\sigma = \mathrm{Re}\{s\} > 3$.
This means that this Laplace integral converges only for values of s such that $\mathrm{Re}\{s\} > 3$.

2. Determine whether the LTIC systems with the following transfer functions are BIBO stable and explain why or why not.

(a) $\hat{H}_1(s) = 2 + \frac{s}{(s-1)(s+2)}$

(b) $\hat{H}_2(s) = \frac{s^2+5s+6}{(s-1+j5)(s-1-j5)}$

(c) $\hat{H}_3(s) = \frac{s^3+1}{(s+2)(s+4)}$

(d) $\hat{H}_4(s) = \frac{1}{s^2+16}$

(e) $\hat{H}_5(s) = \frac{s-2}{s^2-4}$.

Solution:

(a) $\hat{H}_1(s)$ has a pole in the RHP at s = 1, so the system is not BIBO stable.

(b) $\hat{H}_2(s)$ has two conjugate poles at s =1-j5, and s = 1 + j5, both in the RHP, so the system is not BIBO stable.

(c) $\hat{H}_3(s)$ has two poles at s =-2, s = -4 and s = $+\infty$. Because the pole at infinity is not confined to the LHP, the system is not BIBO stable.

(d) $\hat{H}_4(s)$ has two conjugate poles on the imaginary axis at s = j4, and s = -j4. The system is marginally stable, but not BIBO stable.

(e) $\hat{H}_5(s)$ has one pole at s = -2, so the system is BIBO stable. The unstable pole is cancelled with the unstable zeros.

3. For each of the following Laplace transforms $\hat{F}(s)$, determine the inverse Laplace transform $f(t)$.

(a) $\hat{F}(s) = \frac{s+3}{(s+2)(s+4)}$

(b) $\hat{F}(s) = \frac{1}{s(s-5)^2}$

(c) $\hat{F}(s) = \frac{s^2+2s+1}{(s+1)(s+2)}$

Solution:

(a) Expressing as a PFE,
$\hat{F}(s) = \frac{K_1}{(s+2)} + \frac{K_2}{(s+4)}$
Applying the cover-up method, we have
$K_1 = 0.5,\ K_2 = 0.5$, therefore,
$f(t) = \left(\frac{1}{2}e^{-4t} + \frac{1}{2}e^{-2t}\right)u(t)$.

(b) Expressing as a PFE,
$\hat{F}(s) = \frac{1}{s(s-5)^2} = \frac{K_1}{s} + \frac{K_2}{(s-5)^2} + \frac{K_3}{(s-5)}$
Applying the cover-up method, we have
$K_1 = \frac{1}{25},\ K_2 = \frac{1}{5}$, and $K_1 = -\frac{1}{25}$, therefore,
$f(t) = \left(\frac{1}{25} + \frac{1}{5}te^{5t} - \frac{1}{25}e^{5t}\right)u(t)$.

(c) We first simplify the expression by writing
$\hat{F}(s) = \frac{s^2+2s+1}{(s+1)(s+2)} = \frac{(s+1)(s+2)-(s+1)}{(s+1)(s+2)} = 1 - \frac{1}{(s+2)}$
Consequently, $f(t) = \delta(t) - e^{-2t}u(t)$.