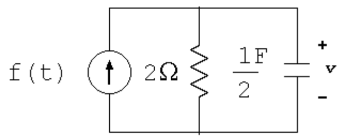
ECE210 / ECE211 - Homework 05

Solution

1. Consider the following circuit with $v\left(0^{-}\right)=2\,V$ and let $f(t)=\cos\left(\sqrt{3}t+\frac{\pi}{3}\right)$ A, For $t>0$, obtain:
   Solution : Equivalent circuit shows below



The ODE for this system is obtained from a KCL at the top node as:

$$f(t)=\frac{v}{2}+i_c=\frac{v}{2}+C\frac{dv}{dt}=\frac{v}{2}+\frac{1}{2}\frac{dv}{dt}\quad\rightarrow\quad 2f(t)=\frac{dv}{dt}+v$$

The particular solution : $v_p(t)=A\cos(\sqrt{3}t)+B\sin(\sqrt{3}t)$

$$2\cos(\sqrt{3}t+\frac{\pi}{3})=\cos(\sqrt{3}t)-\sqrt{3}\sin(\sqrt{3}t)=\frac{dv_p}{dt}+v_p=A\cos(\sqrt{3}t)+B\sin(\sqrt{3}t)-A\sqrt{3}\sin(\sqrt{3}t)+B\sqrt{3}\cos(\sqrt{3}t)$$

Solving : $A=1,\ B=0$
Therefore, the particular solution is $V_p(t)=\cos(\sqrt{3}t)$
The homogeneous solution : $V_h(t)=K\,e^{-t/\tau}$
Here, $\tau=RC=2\Omega\times\frac{1}{2}F=1s$ is the RC time constant.
Fix the initial condition on the total solution, $V(t)=V_h(t)+V_p(t)$, hence

$$V(0^{-})=2=K+1$$
$$K=1$$

The homogenous solution is $V_h(t)=e^{-t}$ V.

(a) the zero-state voltage across the capacitor's terminals, $v_{\mathrm{ZS}}(t)$,
Solution : The zero-state response must satisfy $v_{ZS}(0)=0$

$$V(0^{-})=0=K+1$$
$$K=-1$$

The zero-state solution is $v_{ZS}(t)=-e^{-t}+\cos(\sqrt{3}t)$ V.

(b) the zero-input voltage across the capacitor's terminals, $v_{\mathrm{ZI}}(t)$,
Solution :

$$\frac{dv(t)}{dt}+v(t)=0$$
$$v=Ce^{-t}$$

use initial condtion we find $C=2$

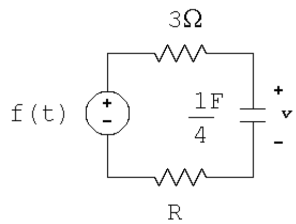
$$v_{ZI}(t)=2e^{-t}\text{ V.}$$

(c) the transient voltage across the capacitor's terminals, $v_{\mathrm{tr}}(t)$,
Solution : $v_{tr}(t)=e^{-t}$ V.

(d) the steady state voltage across the capacitor's terminals, $v_{\mathrm{ss}}(t)$,
Solution : $v_{ss}(t)=\cos(\sqrt{3}t)$ V.

(e) the total voltage across the capacitor's terminals, $v(t)$.
Solution : The total voltage is $V(t) = \cos(\sqrt{3}t) + e^{-t}$ V.
Note that : $v(t) = v_p(t) + v_h(t) = v_{ZI}(t) + v_{ZS}(t) = v_{tr}(t) + v_{ss}(t)$

2. Consider the following circuit with $f(t) = \frac{2}{\sqrt{3}}\cos(\omega t)$ volts and $v(0^-) = v_0$ volts.



It is known that for $t > 0$, $v(t) = Ae^{-t} + B\cos\left(\frac{1}{\sqrt{3}}t\right) + D\sin\left(\frac{1}{\sqrt{3}}t\right)$ volts.

(a) Write the ODE that governs this system for $t > 0$ in terms of R, $v(t)$, and ω.
(b) Find the value of R.
(c) What is the value of ω?
(d) What are the values of B and D?
(e) Identify $v_{tr}(t)$, the transient component of $v(t)$.
(f) Identify $v_{ss}(t)$, the steady-state component of $v(t)$.
(g) What is steady-state phasor V ?

Solution :

(a) $f(t) = v(t) + (3+R)\frac{1}{4}\frac{dv}{dt}$
(b) $\tau = RC = 1s$, hence $R = 1\Omega$.
(c) $w = \frac{1}{\sqrt{3}}$rad/s.
(d) At steady state, the ODE in part (a) must be satisfied

$$\frac{2}{\sqrt{3}}\cos(\frac{1}{\sqrt{3}}t) = B\cos(\frac{1}{\sqrt{3}}t) + D\sin(\frac{1}{\sqrt{3}}t) - \frac{B}{\sqrt{3}}\sin(\frac{1}{\sqrt{3}}t) + \frac{D}{\sqrt{3}}\cos(\frac{1}{\sqrt{3}}t)$$

Solving : $B = \frac{\sqrt{3}}{2}$ and $D = \frac{1}{2}$.
(e) The transient component $v_{tr}(t) = A\,e^{-t}$ V.
We fix the value of A using the initial condition $A + B = v_0$, hence, $A = v_0 - \frac{\sqrt{3}}{2}$
(f) The steady-state component is $v_{ss}(t) = B\cos(\frac{1}{\sqrt{3}}t) + D\sin(\frac{1}{\sqrt{3}}t)$ V, where B and D are solved in part (d)
(g) $V = B + D\,e^{-j\frac{\pi}{2}} = e^{-j\frac{\pi}{6}}$

3. The different parts of this problem are unrelated:

(a) Express $\frac{e^{-j4t} - e^{j4t}}{j4}$ in terms of a cosine function.
Solution :

$$\frac{e^{-j4t} - e^{j4t}}{j4} = -\frac{1}{2}\sin(4t) = -\frac{1}{2}\cos(4t - \frac{\pi}{2})$$

(b) Express $\frac{e^{-j3t} + e^{j3t}}{4}$ in terms of a sine function.
Solution :

$$\frac{e^{-j3t} + e^{j3t}}{4} = \frac{1}{2}\cos(3t) = \frac{1}{2}\sin(3t + \frac{\pi}{2})$$

(c) Express $\text{Re}\{2e^{j\frac{\pi}{3}}e^{-j5t}\}$ in terms of a cosine function.
Solution :

$$Re\{2e^{j\frac{\pi}{3}}e^{-j5t}\} = 2\cos(-5t + \frac{\pi}{3})$$

(d) Determine the phasor F of $f(t) = -2\sin(2t - \frac{\pi}{3})$. Express F in both polar and rectangular coordinates.
Solution :

$$f(t) = -2\cos(2t - \frac{5\pi}{6})$$

the phasor is $F = 2e^{j\frac{\pi}{6}} = \sqrt{3} + j$

(e) Determine the phasor F of $f(t) = \cos(3t - \frac{\pi}{2})$. Express F in both polar and rectangular coordinates.
Solution :

$$F = e^{-j\frac{\pi}{2}} = -j$$

(f) Express the phasor $F = 2 - j2$ in terms of a cosine function $f(t)$ having frequency $\omega = 9\,\frac{\text{rad}}{\text{s}}$.
Solution :

$$F = 2 - j2 = 2\sqrt{2}e^{-j\frac{\pi}{4}}$$
$$f(t) = 2\sqrt{2}\cos(9t - \frac{\pi}{4})$$

(g) Express the phasor $F = 3e^{-j\frac{\pi}{3}}$ in terms of a cosine function $f(t)$ having frequency $\omega = 9\,\frac{\text{rad}}{\text{s}}$.
Solution :

$$f(t) = 3\cos(9t - \frac{\pi}{3})$$

4. Determine the phasor F of the following cosinusoidal functions $f(t)$:

(a) $f(t) = 2\cos(2t + \frac{\pi}{3})$.
Solution :

$$F = 2\,e^{j\frac{\pi}{3}}$$

(b) $f(t) = A\,\sin(wt)$
Solution :

$$F = -A\,j$$

(c) $f(t) = -5\sin(\pi t)$
Solution :

$$F = 5j$$

5. Determine the cosine function $f(t)$ with frequency $w = 2$rad/s, corresponding to the following phasors :

(a) $F = j2$
Solution :

$$f(t) = 2\cos(2t + \frac{\pi}{2})$$

(b) $F = 3\,e^{-j\frac{\pi}{6}}$
Solution :

$$f(t) = 3\cos(2t - \frac{\pi}{6})$$

(c) $F = j2 + 3\,e^{-j\frac{\pi}{6}}$
Solution :

$$F = j2 + 3\left(\frac{\sqrt{3}}{2} - j\frac{1}{2}\right) = \frac{3\sqrt{3}}{2} + \frac{1}{2}$$

$$f(t) = \sqrt{7}\cos\left(2t + \tan^{-1}\left(\frac{1}{3\sqrt{3}}\right)\right)$$

6. Use the phasor method to determine amplitude and phase shift (in rad) of the following signals when written as cosines:

(a) $f(t) = 3\cos(4t) - 4\sin(4t)$
Solution :

$$\begin{aligned} F &= 3 + 4j \\ &= 5\,e^{j\,\tan^{-1}(4/3)} \\ f(t) &= 5\cos\left(4t + \tan^{-1}\left(\frac{4}{3}\right)\right) \end{aligned}$$

(b) $g(t) = 2\left[\cos(wt) + \cos(wt + \frac{\pi}{4})\right]$
Solution :

$$\begin{aligned} G &= 2 + 2e^{j\frac{\pi}{4}} = 2 + 2\left(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right) = \frac{2}{\sqrt{2}}(\sqrt{2}+1) + j\frac{2}{\sqrt{2}} \\ &= 2\sqrt{2+\sqrt{2}}\,e^{j\,\tan^{-1}(1/(\sqrt{2}+1))} \\ g(t) &= 2\sqrt{2+\sqrt{2}}\cos\left(wt + \tan^{-1}\left(\frac{1}{\sqrt{2}+1}\right)\right) \end{aligned}$$