ECE310: Quiz#2 (3pm Section E) Fall 2018 Solutions

1. (6 pts) Consider the system specified by the following input-output relation:

$$y[n] = \cos(x[n])$$

Determine if it is: (i) linear or non-linear, (ii) time-invariant or time-varying. **Justify your answers with proofs or counter-examples.**

Solution: To test linearity, we first input $x_1[n]$ and $x_2[n]$ to get outputs $y_1[n]$ and $y_2[n]$:

$$\begin{aligned} x_1[n] &\to y_1[n] = \cos(x_1[n]) \\ x_2[n] &\to y_2[n] = \cos(x_2[n]) \end{aligned}$$

Now, we input $x_3[n] = \alpha x_1[n] + \beta x_2[n]$; if the system is linear, then the output is $y_3[n] = \alpha y_1[n] + \beta y_2[n]$. However, doing so gives:

$$\begin{aligned} x_3[n] \to y_3[n] &= \cos(x_3[n]) \\ &= \cos(\alpha x_1[n] + \beta x_2[n]) \\ &= \cos(\alpha x_1[n])\cos(\beta x_2[n]) - \sin(\alpha x_1[n])\sin(\beta x_2[n]) \\ &\neq \alpha y_1[n] + \beta y_2[n] = \alpha\cos(x_1[n]) + \beta\cos(x_2[n]) \end{aligned}$$

Therefore, the system is **not linear.**

To test time-invariance, we input $x_1[n] = x[n-n_0]$, and check whether $y_1[n] = y[n-n_0]$. This means that shifting the input results in the same shift in the output. Doing so gives:

$$\begin{aligned} x_1[n] = x[n-n_0] \to y_1[n] &= \cos(x_1[n]) \\ &= \cos(x[n-n_0]) \\ &= y[n-n_0] \end{aligned}$$

Therefore, the system is **time invariant.**

2. (4 pts) Compute the convolution $x[n] * h[n]$ for $x[n] = (-2)^n u[n-1]$ and $h[n] = (n-1)(u[n] - u[n-3])$.

Solution, Approach 1: Use the definition of convolution:

$$
\begin{aligned}
x[n] * h[n] &= \sum_{k=-\infty}^{\infty} x[n-k]h[k] \\
&= \sum_{k=-\infty}^{\infty} (-2)^{n-k} u[n-k-1](k-1)(u[k] - u[k-3]) \\
&= \sum_{k=0}^{2} (-2)^{n-k} u[n-k-1](k-1) \\
&= (-2)^{n-0} u[n-0-1](-1) + (-2)^{n-2} u[n-2-1](1) \\
&= \boxed{(-2)^n \left[\frac{1}{4} u[n-3] - u[n-1]\right]}
\end{aligned}
$$

Solution, Approach 2: Use the fact that $h[n]$ can be represented as a sum of impulse functions. Since $u[n] - u[n-3]$ is only nonzero for $n \in \{0, 1, 2\}$, we can write

$$
\begin{aligned}
h[n] &= (n-1)(\delta[n] + \delta[n-1] + \delta[n-2]) \\
&= (0-1)\delta[n] + (1-1)\delta[n-1] + (2-1)\delta[n-2] \\
&= -\delta[n] + \delta[n-2]
\end{aligned}
$$

Using the fact that $x[n] * \alpha\delta[n-n_0] = \alpha x[n-n_0]$, we find the convolution result to be

$$
\begin{aligned}
x[n] * h[n] &= -x[n] + x[n-2] \\
&= \boxed{-(-2)^n u[n-1] + (-2)^{n-2} u[n-3]}
\end{aligned}
$$

It can be verified that the two expressions are equivalent.