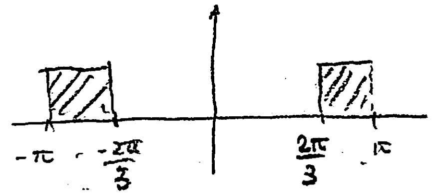
Ex 2: Design HPF, length - 62, $\omega_c = \frac{2\pi}{3}$

① Ideal response:



② $M = 61 \Rightarrow$ Type IV because Type II has $A(\pi) = 0$

③ Type IV has response $D(\omega) = A(\omega) \cdot e^{j\left(\frac{\pi}{2} - \omega\frac{M}{2}\right)} = A(\omega) \cdot j \cdot e^{-j\omega M/2}$

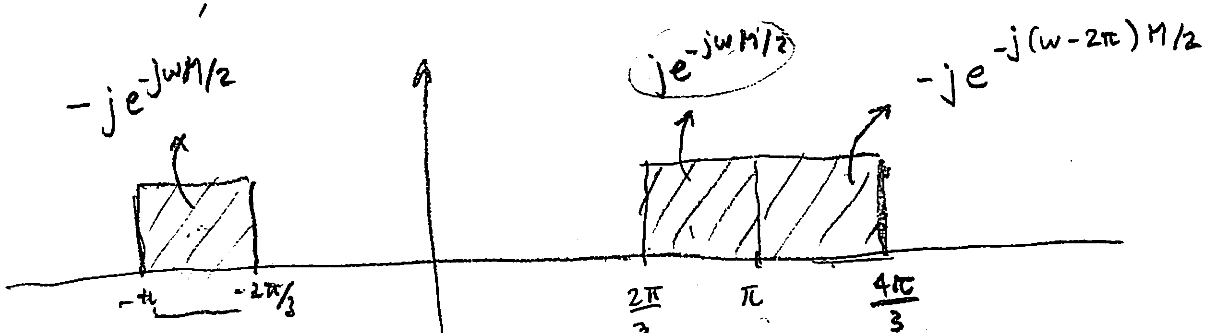
$$\Rightarrow D(\omega) = \begin{cases} je^{-j\omega M/2} & \frac{2\pi}{3} \le \omega \le \pi \\ 0, & -\frac{2\pi}{3} \le \omega \le \frac{2\pi}{3} \\ -je^{-j\omega M/2} & -\pi \le \omega \le -\frac{2\pi}{3} \end{cases}$$

For $d[n]$ to be real, we need $D(\omega) = D^*(-\omega)$

For $-\pi \le \omega \le -2\pi/3$:

$$D(\omega) = \left(j e^{-j(-\omega)M/2}\right)^* = -je^{-j\omega M/2}$$

Before $DTFT^{-1}$, let's use periodicity of $D(\omega)$:



$$= -je^{-j\omega M/2} e^{j\pi M}$$
$$= je^{-j\omega M/2}$$

④ $d[n] = DTFT^{-1}\{D(\omega)\} = \frac{1}{2\pi}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} je^{-j\omega M/2} e^{j\omega n}\, d\omega = (-1)^n \frac{\sin\left(\frac{\pi}{3}(n - M/2)\right)}{\pi(n - M/2)}$

⑤ $h[n] = d[n] \cdot w[n]$

Lecture 22 . IIR Filter Design (via the bilinear transformation) (BLT)

- Goal: design digital IIR filter $H(z) = \frac{B(z)}{A(z)}$ according to some specs (e.g. LPF, $\omega_c = 0.8\pi$, ...)

- Idea: convert a practical analog filter $H_L(s) = \frac{B_L(s)}{A_L(s)}$ to $H(z)$

- Recall: Laplace transform of $h(t)$: $H_L(s) = \int_{-\infty}^{\infty} h(t)\, e^{-st}\, dt$

(equivalent of z-transform in CT. $CTFT\{h(t)\} = H_L(s)\big|_{s=j\Omega}$)

BLT: $H_L(s) = \frac{B_L(s)}{A_L(s)} \xrightarrow[\boxed{s = \alpha \frac{1-z^{-1}}{1+z^{-1}}}]{} H(z) = \frac{B(z)}{A(z)}$

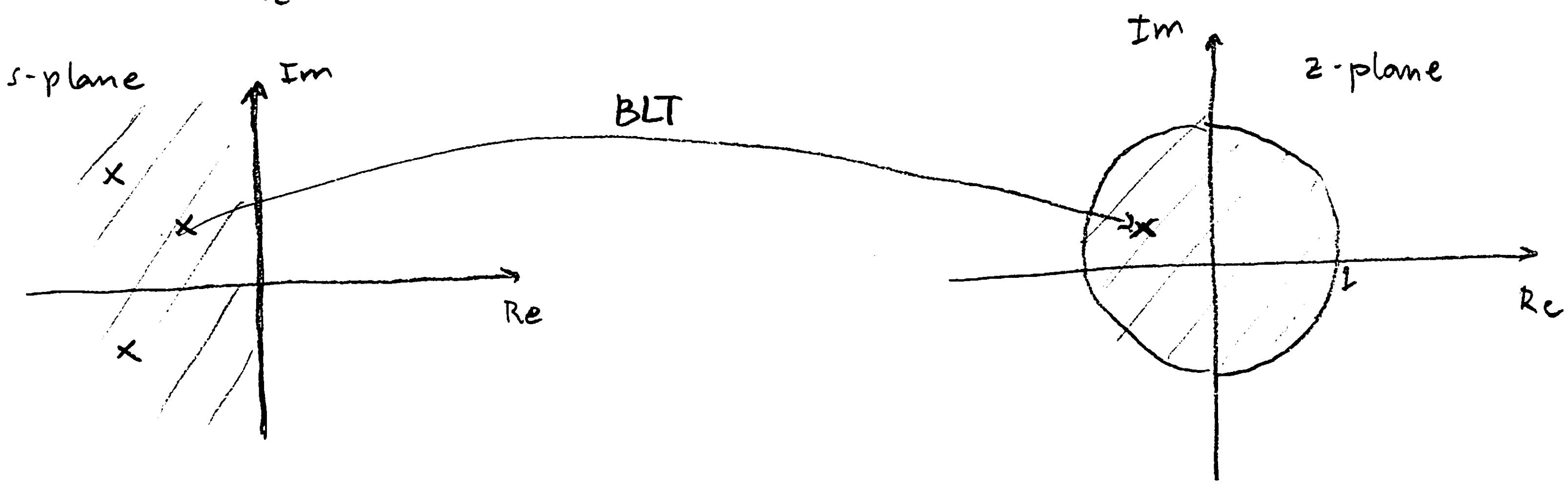
$\alpha \rightarrow$ real, >0, control parameter

Ex. $H_L(s) = \frac{2+s}{3+s}$

$\xrightarrow[(\alpha=1)]{BLT}$ $H(z) = \dfrac{2 + \frac{1-z^{-1}}{1+z^{-1}}}{3 + \frac{1-z^{-1}}{1+z^{-1}}} = \dfrac{\frac{3+z^{-1}}{1+z^{-1}}}{\frac{4+2z^{-1}}{1+z^{-1}}} = \dfrac{3+z^{-1}}{4+2z^{-1}} = \dfrac{B(z)}{A(z)}$

Why do we use the BLT? stability.

$$H_L(s) = \frac{B_L(s)}{A_L(s)} \qquad\qquad H(z) = \frac{B(z)}{A(z)}$$



(if causal) stable $\Leftrightarrow$ all poles are in left plane

Let's check if BLT does that.

First, we notice that $s = \alpha \cdot \frac{1-z^{-1}}{1+z^{-1}} \Leftrightarrow s(1+z^{-1}) = \alpha(1-z^{-1}) \Leftrightarrow (\alpha+s)z^{-1} = \alpha - s$

$$\Leftrightarrow z = \frac{\alpha+s}{\alpha-s}$$

Let $s_0 = \sigma_0 + j\Omega_0 \Rightarrow z_0 = \frac{\alpha+s_0}{\alpha-s_0} = \frac{\alpha+\sigma_0+j\Omega_0}{\alpha-\sigma_0-j\Omega_0}$

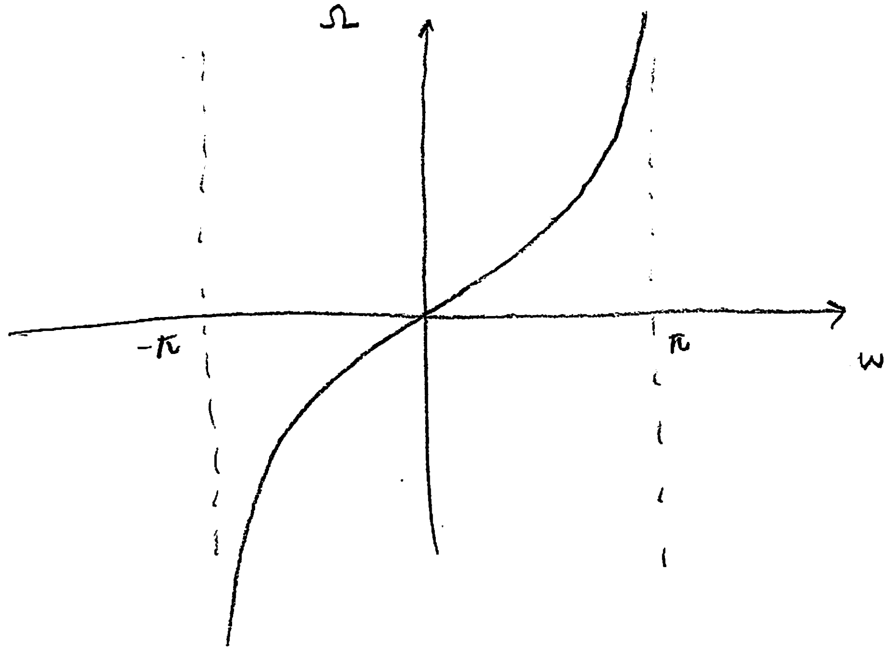
$$|z_0| = \frac{|\alpha+\sigma_0+j\Omega_0|}{|\alpha-\sigma_0-j\Omega_0|} = \frac{\sqrt{(\alpha+\sigma_0)^2+\Omega_0^2}}{\sqrt{(\alpha-\sigma_0)^2+\Omega_0^2}} \Longrightarrow \begin{cases} \sigma_0 = 0 \Rightarrow |z_0| = 1 \\ \sigma_0 < 0 \Rightarrow |z_0| < 1 \\ \sigma_0 > 0 \Rightarrow |z_0| > 1 \end{cases}$$

We also want to know where a CT freq Ω maps to in DT

Recall that $H_d(w) = H(z)\Big|_{z=e^{jw}}$, $H_c(\Omega) = H_L(s)\Big|_{s=j\Omega}$

$$s = \alpha \frac{1-z^{-1}}{1+z^{-1}} \Rightarrow j\Omega = \alpha \frac{1-e^{-jw}}{1+e^{-jw}} = \frac{\alpha \cancel{e^{-jw/2}}\left(e^{jw/2} - e^{-jw/2}\right)}{\cancel{e^{-jw/2}}\left(e^{jw/2} + e^{jw/2}\right)}$$

$$= \alpha \; j \frac{\sin(w/2)}{\cos(w/2)} \Rightarrow \Omega = \alpha \tan\left(\frac{w}{2}\right)$$



squeezing $\Omega \in (-\infty, \infty)$
into $w \in (-\pi, \pi)$

Now we can map $H_L(s)$ from a known CT filter to DT

- We will consider three main types of filters
  - Butterworth, Chebyshev (type I/II), Elliptic

Ex: Butterworth (order n): $H_L(s)$ is obtained by taking the left-plane poles of $\dfrac{1}{1+(-s^2)^n}$