University of Illinois
ECE 310

Profs. Bresler & Liang

Midterm Exam

7:00-8:30pm, Thursday, October 12, 2017

Name Key

UIN ________________

Section: 10:00 AM 3:00 PM

Score __________

Problem	Pts.	Score
1	24	
2	5	
3	10	
4	6	
5	20	
6	6	
7	16	
8	5	
9	8	
Total	100	

Instructions

- You may not use any books, calculators, or notes other than one handwritten two-sided sheets of 8.5" x 11" paper.
- Show all your work to receive full credit for your answers.
- When you are asked to "calculate", "determine", or "find", this means providing closed-form expressions (i.e., without summation or integration signs).
- Neatness counts. If we are unable to read your work, we cannot grade it.
- Turn in your entire booklet once you are finished. No extra booklet or papers will be considered.

(24 Pts.)

1. For each of the systems with input $x[n]$ and output $y[n]$ shown in the table, indicate by "yes" or "no" whether the properties indicated apply to the system.

+2 each

	Linear	SI	Causal	Stable
$y[n] = \frac{1}{x[n+1]}$	No	Yes	No	No
$y[n] = \frac{1}{2}y[n-1] + x[n]$	Yes	Yes	Yes / No	Yes / No
$y[n] = x[n]\sin^2(x[n])$	No	Yes	Yes	Yes

if not causal, then not stable.

(5 Pts.)

2. Given that the z-transform of $x[n] = 5^n u[n]$ is $X(z) = \frac{z}{z-5}$, the DTFT of $x[n]$ is given by (select one only)

(a) $\frac{e^{j\omega}}{e^{j\omega}-5}$

(b) $\frac{e^{-j\omega}}{e^{-j\omega}-5}$

(c) $\frac{1}{1-5e^{-j\omega}}$

+5 (d) Does not exist

$x[n]$ is causal, so the ROC is $|z|>5$

ROC does not contain the unit circle, so we cannot substitute $z=e^{j\omega}$

↳ DTFT does not exist.

(10 Pts.)

3. A causal LSI system has transfer function $H(z) = \frac{z}{z^2+j}$. Determine whether the following bounded input $x[n]$ will produce an unbounded output $y[n]$.

(a) $\delta[n]$ — No poles — Y N — +2

(b) $\sin(n\frac{\pi}{4})u[n]$ — Poles at $e^{j\pi/4}$, $e^{-j\pi/4}$ — Y N — +2

(c) $(j)^n u[n]$ — Pole at j — Y N — +2

(d) $(-e^{j\frac{\pi}{4}})^n u[n]$ — Pole at $-e^{j\pi/4}$ — Y N — +2

(e) $e^{jn\frac{5\pi}{4}} u[n]$ — Pole at $e^{j5\pi/4}$ — Y N — +2

$$H(z) = \frac{z}{z^2+j} = \frac{1}{(z-e^{-j\pi/4})(z-e^{j3\pi/4})}$$

Need a pole at $e^{-j\pi/4}$ or $e^{j3\pi/4}$ to produce an unbounded output.

(6 Pts.)

4. Calculate the inverse z-transform of $X(z) = 1 + 3z^{-5} + 5z^{-100}$, ROC, $|z| > 0$.

$$1 \longleftrightarrow \delta[n]$$

$$3z^{-5} \longleftrightarrow 3\delta[n-5]$$

$$5z^{-100} \longleftrightarrow 5\delta[n-100]$$

$$\boxed{x[n] = \delta[n] + 3\delta[n-5] + 5\delta[n-100]}$$

+2 +2 +2

(20 Pts.)

5. The input $x[n] = 2^n\,(u[n] - 3u[n-1])$ to an unknown **causal** LSI system produces output $y[n] = (3^n - 2^n)u[n]$. Use the z-transform approach to determine the unit pulse response $h[n]$ of the system.

$$x[n] = 2^n u[n] - 3\cdot 2^n u[n-1] = 2^n u[n] - 6\cdot 2^{n-1} u[n-1]$$

$$\hookrightarrow X(z) = \frac{z}{z-2} - \frac{6}{z-2} = \frac{z-6}{z-2}, \ |z|>2$$ +4

$$y[n] = 3^n u[n] - 2^n u[n] \rightarrow Y(z) = \frac{z}{z-3} - \frac{z}{z-2} = \frac{z}{(z-2)(z-3)}, \ |z|>3$$ +4

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z(z-2)}{(z-2)(z-3)(z-6)} = \frac{z}{(z-3)(z-6)}, \ |z|>6$$ +4

Approach 1: $\frac{H(z)}{z} = \frac{1}{(z-3)(z-6)} = -\frac{1}{3}\left(\frac{z}{z-3}\right) + \frac{1}{3}\left(\frac{z}{z-6}\right)$ after partial fractions and factoring the z back in

$$\rightarrow \boxed{h[n] = -\frac{1}{3}(3)^n u[n] + \frac{1}{3}(6)^n u[n]}$$

Approach 2: $H(z) = \frac{z}{(z-3)(z-6)} = -z^{-1}\left(\frac{z}{z-3}\right) + 2z^{-1}\left(\frac{z}{z-6}\right)$ after direct application of partial fractions +8

$$\rightarrow \boxed{h[n] = 2(6)^{n-1}u[n-1] - (3)^{n-1}u[n-1]}$$

(6 Pts.)

6. Compute the convolution $x[n] * h[n]$ for $x[n] = 3^{-n}u[n]$ and $h[n] = \{\underset{\uparrow}{0}, 1, 2\}$.

$$h[n] = \delta[n-1] + 2\delta[n-2]$$

$$x[n] * h[n] = x[n] * (\delta[n-1] + 2\delta[n-2])$$

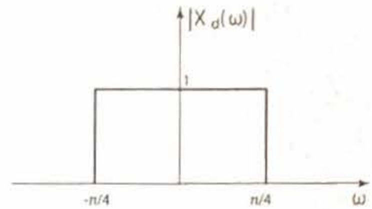
$$= x[n-1] + 2x[n-2]$$

$$= \boxed{3^{-(n-1)}u[n-1] + 2\cdot 3^{-(n-2)}u[n-2]}$$

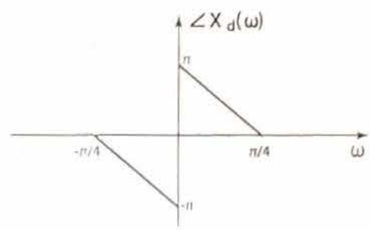
+2 +2 +2 (no third term)

(16 Pts.)

7. Let $X_d(\omega)$ be defined as in the figure below.



$$= \begin{cases} 1, & |\omega| < \pi/4 \\ 0, & \text{else} \end{cases}$$ +2



$$= \begin{cases} \pi - 4\omega, & 0 < \omega < \pi/4 \\ -\pi - 4\omega, & -\pi/4 < \omega < 0 \\ 0, & \text{else} \end{cases}$$ +4

Determine $\mathrm{DTFT}^{-1}\{X_d(\omega)\}$. Your answer should not contain any complex numbers.

$$x[n] = \mathrm{DTFT}^{-1}\{X_d(\omega)\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_d(\omega)e^{j\omega n}d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi/4}^{0} e^{j(-\pi-4\omega)}e^{j\omega n}d\omega + \frac{1}{2\pi}\int_{0}^{\pi/4} e^{j(\pi-4\omega)}e^{j\omega n}d\omega$$ +5

$$= \frac{-1}{2\pi}\int_{-\pi/4}^{\pi/4} e^{-j4\omega}e^{j\omega n}d\omega \quad (\text{factor out } e^{-j\pi} = e^{j\pi} = -1)$$

$$= \frac{-1}{2\pi}\left[\frac{1}{-j(4-n)}e^{-j\omega(4-n)}\right]\Big|_{-\pi/4}^{\pi/4}$$

$$= \frac{1}{2\pi j(4-n)}\left[e^{-jn\frac{\pi}{4}} - e^{jn\frac{\pi}{4}}\right]$$

$$= \boxed{\frac{-1}{\pi(4-n)}\sin\left(\frac{\pi}{4}n\right)} = \boxed{-\frac{1}{4}\operatorname{sinc}\left(\frac{\pi}{4}(n-4)\right)}$$

+5

(5 Pts.)

8. The frequency response $H_d(\omega)$ of a digital filter is defined as below:

$$|H_d(\omega)| = \begin{cases} 0, & -\pi < \omega < 0 \\ \frac{1}{4}, & 0 \leq \omega < \frac{3\pi}{4} \\ 1, & \frac{3\pi}{4} \leq \omega \leq \pi \end{cases} \qquad \angle H_d(\omega) = \begin{cases} 2(-\frac{\pi}{2} - \omega), & -\pi < \omega \leq -\frac{\pi}{2} \\ 0, & -\frac{\pi}{2} < \omega < \frac{\pi}{2} \\ 2(-\frac{\pi}{2} + \omega), & \frac{\pi}{2} \leq \omega \leq \pi \end{cases}$$

Determine the system's output $y[n]$ for input

$$x[n] = \cos\left(\frac{2\pi}{3}n\right) + \cos\left(\frac{3\pi}{2}n\right) + (-1)^n$$

Is the system real? $|H_d(\omega)| =$

No conjugate symmetry, so the system is <u>not</u> real. Need to break cosines into complex exponentials.

$$x[n] = \frac{1}{2}e^{j\frac{2\pi}{3}n} + \frac{1}{2}e^{-j\frac{2\pi}{3}n} + \frac{1}{2}e^{j\frac{3\pi}{2}n} + \frac{1}{2}e^{-j\frac{3\pi}{2}n} + e^{j\pi n} \quad \text{as } (-1)^n = e^{j\pi n}$$

$$= \frac{1}{2}e^{j\frac{2\pi}{3}n} + \frac{1}{2}e^{-j\frac{2\pi}{3}n} + \frac{1}{2}e^{-j\frac{\pi}{2}n} + \frac{1}{2}e^{j\frac{\pi}{2}n} + e^{j\pi n} \quad \text{as } \cos\left(\frac{3\pi}{2}n\right) = \cos\left(-\frac{\pi}{2}n\right) = \cos\left(\frac{\pi}{2}n\right)$$

$$\rightarrow y(n) = \frac{1}{2}e^{j\frac{2\pi}{3}n}H_d\left(\frac{2\pi}{3}\right) + \frac{1}{2}e^{-j\frac{2\pi}{3}n}H_d\left(-\frac{2\pi}{3}\right) + \frac{1}{2}e^{-j\frac{\pi}{2}n}H_d\left(-\frac{\pi}{2}\right) + \frac{1}{2}e^{j\frac{\pi}{2}n}H_d\left(\frac{\pi}{2}\right) + e^{j\pi n}H_d(\pi)$$

- $|H_d(\frac{2\pi}{3})| = \frac{1}{4}$, $\measuredangle H_d(\frac{2\pi}{3}) = 2(-\frac{\pi}{2} + \frac{2\pi}{3}) = \frac{\pi}{3}$
- $|H_d(-\frac{2\pi}{3})| = 0$
- $|H_d(-\frac{\pi}{2})| = 0$
- $|H_d(\frac{\pi}{2})| = \frac{1}{4}$, $\measuredangle H_d(\frac{\pi}{2}) = 0$
- $|H_d(\pi)| = 1$, $\measuredangle H_d(\pi) = 2(-\frac{\pi}{2} + \pi) = \pi$

So $y(n) = \frac{1}{2}e^{j\frac{2\pi}{3}n}\left(\frac{1}{4}e^{j\frac{\pi}{3}}\right) + \frac{1}{2}e^{j\frac{\pi}{2}n}\left(\frac{1}{4}\right) + (-1)^n e^{j\pi}$

$$= \boxed{\frac{1}{8}e^{j(\frac{2\pi}{3} + \frac{\pi}{3})} + \frac{1}{8}e^{j\frac{\pi}{2}n} - (-1)^n}$$

(+2) (+2) (+1)

(8 Pts.)

9. The output, $Y_d(\omega)$, in the frequency domain of an LSI system is related to the input $X_d(\omega)$ to the system by

$$Y_d(\omega) = X_d(\omega)e^{-j2.5\omega}$$

A student claimed that the system's unit pulse response is: $h[n] = \delta[n - 2.5]$. Is it correct? If not, derive the correct unit pulse response for the system.

(+2) The student is <u>not</u> correct – 2.5 is not an integer, so $\delta[n-2.5]$ does not exist in DT.

(+2)

$$\hookrightarrow h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j2.5\omega} e^{j\omega n}\, d\omega.$$

$$= \frac{1}{2\pi} \frac{1}{j(n-2.5)} e^{j\omega(n-2.5)} \Big|_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi j(n-2.5)}\left[e^{j\pi(n-2.5)} - e^{-j\pi(n-2.5)}\right]$$

$$= \frac{\sin(\pi(n-2.5))}{\pi(n-2.5)} = \boxed{\mathrm{sinc}(\pi(n-2.5))}$$ (+4)