

Lecture 11

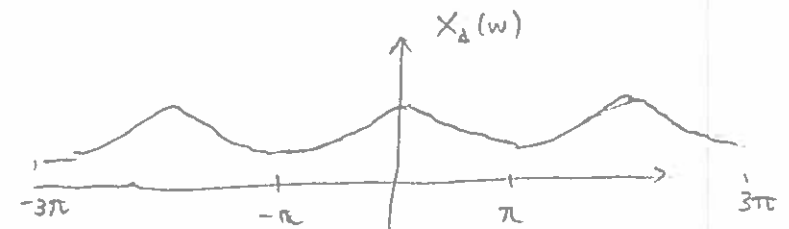
DTFT

$$\{x[n]\} \xleftrightarrow{\text{DTFT}}$$

$$X_d(\omega) \quad \left(\text{equivalently, } X(e^{j\omega}) \right)$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_d(\omega) e^{j\omega n} d\omega$$

$$X_d(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$



(*) $X_d(\omega)$ is 2π -periodic. So we can write $x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_d(\omega) e^{j\omega n} d\omega$ for any interval of length 2π

Ex: DTFT of $x[n] = 1$ for all n ? Given: $1 \xleftrightarrow{\text{DTFT}} 2\pi \delta(\omega)$?

$$\text{Let } \underline{Y_d(\omega) = \delta(\omega)}. \text{ Then } y[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \overbrace{Y_d(\omega)}^{\delta(\omega)} e^{j\omega n} d\omega = \frac{1}{2\pi} e^{j0n} = \frac{1}{2\pi} \cdot \frac{1}{2\pi} \xleftrightarrow{\text{DTFT}} \delta(\omega)$$

Looks right!

↳ not 2π -periodic! Not a valid DTFT.

Correct DTFT pair:

$$1 \xleftrightarrow{\text{DTFT}} 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2k\pi)$$

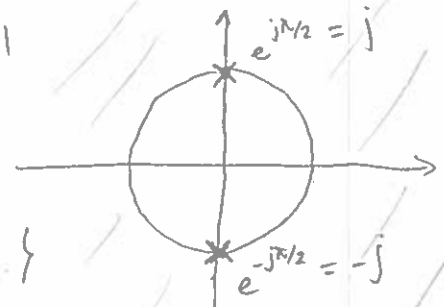
Connection between DTFT and the z -transform

$$X_d(\omega) = X(z) \Big|_{z=e^{j\omega}} \quad \text{If ROC of } z\text{-transform contains unit circle}$$

What if it doesn't? DTFT may still exist for some values of ω if you have poles on the unit circle

$$\text{Ex: } x[n] = \cos\left(\frac{\pi}{2}n\right) u[n] = \frac{e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi}{2}n}}{2} u[n] = \frac{1}{2} e^{j\frac{\pi}{2}n} u[n] + \frac{1}{2} e^{-j\frac{\pi}{2}n} u[n]$$

z-transform: $X(z) = \frac{1}{2} \frac{1}{1-jz^{-1}} + \frac{1}{2} \frac{1}{1+jz^{-1}}$, ROC: $|z| > 1$



To compute DFT, we will use properties. $x[n] = \{1, 0, -1, 0, 1, \dots\}$

$$x[n] + x[n-2] = \delta[n]$$

DTFT \downarrow

$$X_d(w) + e^{-j2w} X_d(w) = 1$$

$$\Rightarrow X_d(\omega)(1 + e^{-j2\omega}) = 1$$

$$\Rightarrow X_L(\omega) = \frac{1}{1 + e^{-j2\omega}} \quad \text{if } e^{-j2\omega} \neq -1 \iff \omega \neq \frac{\pi}{2} + k\pi, \quad k=0,1,2,\dots$$

Back to LTI systems



time: impulse response $h[n]$

$$y[n] = x[n] * h[n]$$

z-domain: transfer function $H(z)$

$$Y(z) = X(z) H(z)$$

DTFT (frequency): frequency domain response $H_d(\omega)$

$$Y_d(\omega) = X_d(\omega) H_d(\omega)$$

Response to a sinusoidal input:



$$y[n] = (x * h)[n]$$

$$= \sum_{k=-\infty}^{\infty} h[k] x[n-k] = \sum_{k=-\infty}^{\infty} h[k] e^{j\omega_0(n-k)}$$

$$= \sum_{k=-\infty}^{\infty} h[k] e^{j\omega_0 n} e^{-j\omega_0 k}$$

$$= e^{j\omega_0 n} \underbrace{\sum_{k=-\infty}^{\infty} h[k] e^{-j\omega_0 k}}_{H_d(\omega_0)}$$

$$\Rightarrow y[n] = \underbrace{H_d(\omega_0)}_{\text{freq. response}} \cdot \underbrace{e^{j\omega_0 n}}_{x[n]}$$

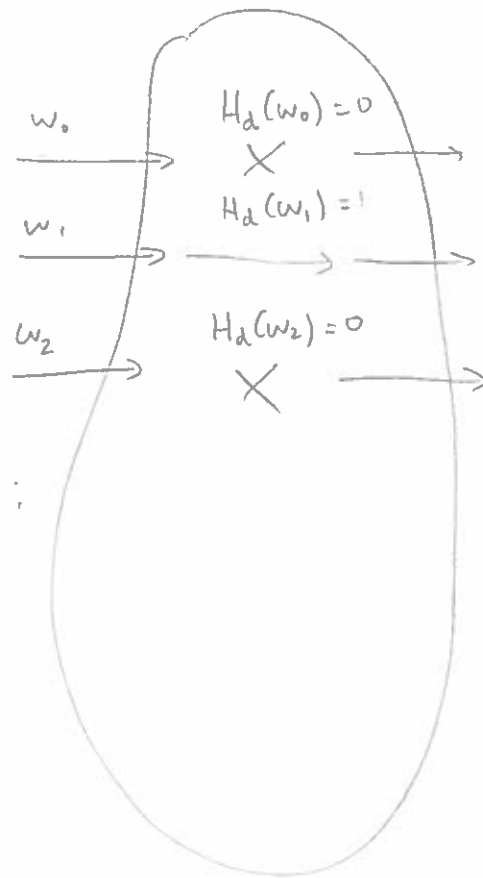


$x[n]$ is essentially a
"sum" of complex sinusoids

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_d(\omega) e^{j\omega n} d\omega$$

||

$$\sum_k \left(\frac{X_d(\omega_k)}{2\pi} \right) \cdot e^{j\omega_k n}$$



$$y[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_d(\omega) H_d(\omega) e^{j\omega n} d\omega$$

system is acting
as a filter!

What if input is a real sinusoid?

Given: $\cos(\omega_0 n)$ $\xrightarrow{H_d(\omega)}$ $H_d(\omega_0) \cos(\omega_0 n)$?

First, let's consider a signal $x[n] = a[n] + j b[n]$ for all n .

$$x[n] = a[n] + j b[n] \xrightarrow{h} y[n]$$

Suppose $h[n]$ is real-valued.

$$y[n] = (a[n] + j b[n]) * h[n] = \underbrace{(a[n] * h[n])}_{\text{real-valued}} + j \underbrace{(b[n] * h[n])}_{\text{real-valued}}$$

System processes real and imaginary parts independently

$$\begin{aligned} x[n] &= (A e^{j\phi}) e^{j\omega_0 n} \xrightarrow{h} y[n] = (A e^{j\phi}) H_d(\omega_0) \cdot e^{j\omega_0 n} \\ &= A e^{j(\omega_0 n + \phi)} \\ &= \underbrace{A \cos(\omega_0 n + \phi)}_{\text{real part}} + j \underbrace{A \sin(\omega_0 n + \phi)}_{\text{imaginary part}} \\ &= A e^{j\phi} |H_d(\omega_0)| e^{j\angle H_d(\omega_0)} e^{j\omega_0 n} \\ &= A |H_d(\omega_0)| e^{j(\omega_0 n + \phi + \angle H_d(\omega_0))} \\ &= \underbrace{A |H_d(\omega_0)| \cos(\omega_0 n + \phi + \angle H_d(\omega_0))}_{\text{real part}} \\ &\quad + j \underbrace{A |H_d(\omega_0)| \sin(\omega_0 n + \phi + \angle H_d(\omega_0))}_{\text{imaginary part}} \end{aligned}$$

In conclusion:

$$A \cos(\omega_0 n + \phi) \xrightarrow{h} \underbrace{A |H_d(\omega_0)|}_{\text{magnitude response}} \underbrace{\cos(\omega_0 n + \phi + \angle H_d(\omega_0))}_{\text{phase response}}$$

Ex. $x[n] = 1 + 2 \cos(\pi n) \rightarrow \boxed{h} \rightarrow y[n] = ?$

$$h[n] = \left(\frac{1}{2}\right)^n u[n]$$

\mathbb{Z} \downarrow

$$H(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}$$

\downarrow

$$H_d(\omega) = \frac{1}{1 - \frac{1}{2}e^{-j\omega}}$$

$$H_d(0) = \frac{1}{1 - \frac{1}{2}e^{-j0}} = \frac{1}{1 - \frac{1}{2}} = 2$$



$1 = 1 \cdot \cos(0 \cdot n)$, so we need $H_d(0)$ and $H_d(\pi)$

$$H_d(0) = 2, \quad H_d(\pi) = 2/3 \quad \left(|H_d(0)| = 2, \angle H_d(0) = 0, |H_d(\pi)| = 2/3, \angle H_d(\pi) = 0 \right)$$

$$\Rightarrow y[n] = 1 \cdot |H_d(0)| + 2 \cdot \overbrace{|H_d(\pi)|}^{2/3} \cos(\pi n + 0) = 2 + \frac{4}{3} \cos(\pi n)$$