ECE 310: Problem Set 8

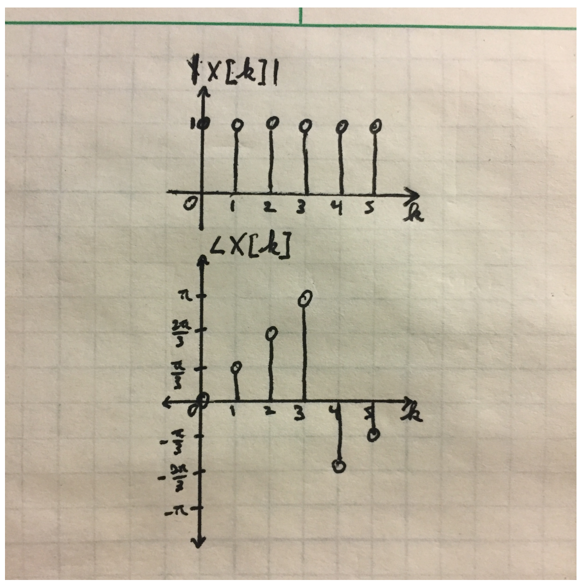
Due: 5pm, Friday October 26, 2018

1. For each of the following finite length sequences, determine the DFT $X[k]$. Sketch the magnitude and phase for parts (a) and (b). (30pts: 8,8,7,7pts)

   Note: The following DFT's are valid only in the range $0 \leq k \leq N-1$.

   (a) $x[n] = \delta(n-5)$, $0 \leq n \leq 5$

$$X[k] = \sum_{n=0}^{5} \delta(n-5) e^{-j\frac{2\pi}{6}kn} = e^{-j\frac{5\pi}{3}k}$$



   (b) $x[n] = \{0, 0, 0, 1, 1, 1\}$

$$\begin{aligned}
X[k] &= \sum_{n=3}^{5} e^{-j\frac{2\pi}{6}kn} \\
&= e^{-j\pi k} \sum_{n=0}^{2} e^{-j\frac{\pi}{3}kn} \\
&= e^{-j\pi k} \frac{1 - e^{-j\frac{\pi}{3}3k}}{1 - e^{-j\frac{\pi}{3}k}} \\
&= e^{-j\pi k} \frac{e^{-j\frac{\pi}{2}k}\left(e^{j\frac{\pi}{2}k} - e^{-j\frac{\pi}{2}k}\right)}{e^{-j\frac{\pi}{6}k}\left(e^{j\frac{\pi}{6}k} - e^{-j\frac{\pi}{6}k}\right)} \\
&= e^{-j\pi k} e^{-j\frac{\pi}{3}k} \frac{\sin(\frac{\pi}{2}k)}{\sin(\frac{\pi}{6}k)}
\end{aligned}$$

$$= e^{-j\frac{4\pi}{3}k}\frac{\sin(\frac{\pi}{2}k)}{\sin(\frac{\pi}{6}k)}$$

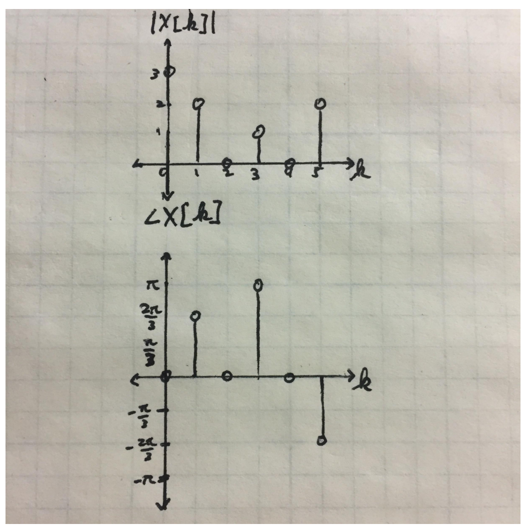
We see that this expression is undefined at $k = 0$. To solve for this case, simply solve the DFT summation for $k = 0$, and we arrive at the following DFT:

$$X[k] = \begin{cases} 3, & k = 0 \\ e^{-j\frac{4\pi}{3}k}\frac{\sin(\frac{\pi}{2}k)}{\sin(\frac{\pi}{6}k)}, & k \neq 0 \end{cases}$$
$$= \left\{3, 2e^{j\frac{2\pi}{3}}, 0, -1, 0, 2e^{-j\frac{2\pi}{3}}\right\}$$

Alternative:

$$\begin{aligned} X[k] &= \sum_{n=3}^{5} e^{-j\frac{2\pi}{6}kn} \\ &= e^{-j\pi k} + e^{-j\frac{4\pi}{3}k} + e^{-j\frac{5\pi}{3}k} \\ &= e^{-j\pi k}\left(1 + e^{-j\frac{\pi}{3}k} + e^{-j\frac{2\pi}{3}k}\right) \\ &= e^{-j\pi k}e^{-j\frac{\pi}{3}k}\left(1 + e^{-j\frac{\pi}{3}k} + e^{j\frac{\pi}{3}k}\right) \\ &= e^{-j\frac{4\pi}{3}k}\left(1 + 2\cos(\frac{\pi}{3}k)\right) \\ &= \left\{3, 2e^{j\frac{2\pi}{3}}, 0, -1, 0, 2e^{-j\frac{2\pi}{3}}\right\} \end{aligned}$$

Note: the first expression gives the Dirichlet (or periodic) sinc, which we should expect upon seeing a rectangular pulse, while the second gives a more straightforward magnitude-phase form, though both are equivalent.



Note: Because the magnitude at $k = 2$ and $k = 4$ is 0, technically the phase can be anything; here we have chosen to represent it as 0.

(c) $x[n] = \cos(\frac{\pi}{4}n)$, $0 \le n \le 7$

$$\begin{aligned}
X[k] &= \sum_{n=0}^{7} \cos(\frac{\pi}{4}n) e^{-j\frac{2\pi}{8}kn} \\
&= \frac{1}{2}\sum_{n=0}^{7} e^{j\frac{\pi}{4}n} e^{-j\frac{\pi}{4}kn} + \frac{1}{2}\sum_{n=0}^{7} e^{-j\frac{\pi}{4}n} e^{-j\frac{\pi}{4}kn} \\
&= \frac{1}{2}\frac{1 - e^{-j2\pi(k-1)}}{1 - e^{-j\frac{\pi}{4}(k-1)}} + \frac{1}{2}\frac{1 - e^{-j2\pi(k+1)}}{1 - e^{-j\frac{\pi}{4}(k+1)}}
\end{aligned}$$

This expression is undefined at $k = 1$ and $k = 7$ but evaluates to 0 everywhere else. To solve for these cases, solve the DFT summation for $k = 1$ and $k = 7$ (it is easiest to use the equation in the second line). We arrive at the following DFT:

$$\begin{aligned}
X[k]|_{k=1} &= \frac{1}{2}\sum_{n=0}^{7} e^{j\frac{\pi}{4}n} e^{-j\frac{\pi}{4}n} + \frac{1}{2}\sum_{n=0}^{7} e^{-j\frac{\pi}{4}n} e^{-j\frac{\pi}{4}n} \\
&= \frac{1}{2}\sum_{n=0}^{7} 1 + \frac{1}{2}\sum_{n=0}^{7} e^{-j\frac{\pi}{2}n} \\
&= \frac{1}{2}\cdot 8 + 0 = 4 \\
X[k]|_{k=7} &= \frac{1}{2}\sum_{n=0}^{7} e^{j\frac{\pi}{4}n} e^{-j\frac{7\pi}{4}n} + \frac{1}{2}\sum_{n=0}^{7} e^{-j\frac{\pi}{4}n} e^{-j\frac{7\pi}{4}n} \\
&= \frac{1}{2}\sum_{n=0}^{7} e^{-j\frac{3\pi}{2}n} + \frac{1}{2}\sum_{n=0}^{7} 1 \\
&= 0 + \frac{1}{2}\cdot 8 = 4 \\
X[k] &= 4\delta[k-1] + 4\delta[k-7]
\end{aligned}$$

(d) $x[n] = \{1, 0, 1, 0, 1, 0, 1\}$, $0 \le n \le 6$.

$$\begin{aligned}
X[k] &= \sum_{n=0}^{6} x[n] e^{-j\frac{2\pi}{7}kn} \\
&= \sum_{n=0}^{3} e^{-j\frac{4\pi}{7}kn} \\
&= \frac{1 - e^{-j\frac{4\pi}{7}4k}}{1 - e^{-j\frac{4\pi}{7}k}} \\
&= \frac{e^{-j\frac{8\pi}{7}k}(e^{j\frac{8\pi}{7}k} - e^{-j\frac{8\pi}{7}k})}{e^{-j\frac{2\pi}{7}k}(e^{j\frac{2\pi}{7}k} - e^{-j\frac{2\pi}{7}k})} \\
&= e^{-j\frac{6\pi}{7}k}\frac{\sin(\frac{8\pi}{7}k)}{\sin(\frac{2\pi}{7}k)}
\end{aligned}$$

This expression is undefined at $k = 0$. To solve for this case, simply solve the DFT summation at $k = 0$, and we arrive at the following DFT:

$$X[k] = \begin{cases} 4\,, & k = 0 \\ e^{-j\frac{6\pi}{7}k}\frac{\sin(\frac{8\pi}{7}k)}{\sin(\frac{2\pi}{7}k)}\,, & k \neq 0 \end{cases}$$

Alternative:

$$\begin{aligned}
X[k] &= \sum_{n=0}^{6} x[n]e^{-j\frac{2\pi}{7}kn} \\
&= e^{-j\frac{2\pi}{7}k\cdot 0} + e^{-j\frac{2\pi}{7}k\cdot 6} + e^{-j\frac{2\pi}{7}k\cdot 2} + e^{-j\frac{2\pi}{7}k\cdot 4} \\
&= e^{-j\frac{6\pi}{7}k}\left(e^{j\frac{6\pi}{7}k} + e^{-j\frac{6\pi}{7}k}\right) + e^{-j\frac{6\pi}{7}k}\left(e^{j\frac{2\pi}{7}k} + e^{-j\frac{2\pi}{7}k}\right) \\
&= 2e^{-j\frac{6\pi}{7}k}\left(\cos(\frac{6\pi}{7}k) + \cos(\frac{2\pi}{7}k)\right)
\end{aligned}$$

2. Let $X[k] = \left\{1, e^{-j\frac{\pi}{2}}, 0, e^{j\frac{\pi}{2}}\right\}$ be the DFT of $\left\{x[n]\right\}_{n=0}^{3}$. (15pts, 5pts each)

   (a) Show that $x[n]$ is real valued.
   A real sequence will produce a DFT with the following symmetry:

$$X^*[k] = X[\langle -k\rangle_N]$$

   We confirm that this symmetry holds across $X[k]$.

$$\begin{aligned}
X^*[0] = 1 &= X[0] = X[\langle 0\rangle_N] \\
X^*[1] = e^{j\frac{\pi}{2}} &= X[3] = X[\langle -1\rangle_N] \\
X^*[2] = 0 &= X[2] = X[\langle -2\rangle_N]
\end{aligned}$$

   Note: a modulo operator on a negative operand behaves as follows: $\langle -k\rangle_N = \langle N - k\rangle_N$

   (b) Compute the IDFT of $X[k]$ to find $x[n]$.

$$\begin{aligned}
x[n] &= \frac{1}{4}\sum_{k=0}^{3} X[k]e^{j\frac{2\pi}{4}kn} \\
&= \frac{1}{4}\left(1 + e^{j\frac{\pi}{2}(n-1)} + 0 + e^{j\frac{\pi}{2}(3n+1)}\right) \\
&= \frac{1}{4}\left(1 + e^{j\pi n}2\cos(\frac{\pi}{2}n + \frac{\pi}{2})\right) \\
&= \frac{1}{4}\left(1 - (-1)^n 2\sin(\frac{\pi}{2}n)\right) \\
&= \frac{1}{4}\left(1 + 2\sin(\frac{\pi}{2}n)\right) \\
&= \left\{\frac{1}{4}, \frac{3}{4}, \frac{1}{4}, -\frac{1}{4}\right\}
\end{aligned}$$

(c) Use DFT properties to find the IDFT of the sequence $Y[k] = \{1, -1, 0, -1\}$.

First, note that $Y[k]$ may be expressed in terms of $X[k]$ as follows.

$$Y[k] = X[k]e^{-j\frac{\pi}{2}k}$$

We see that this frequency modulation allows us to apply the time-shift property:

$$x[\langle n-m\rangle_N] \overset{DFT}{\longleftrightarrow} W_N^{km}X[k]$$

By inspection, we find that $m = 1$, which corresponds to a right circular shift by one.

$$\begin{aligned} y[n] &= \frac{1}{4}\Big(1 + 2\sin(\frac{\pi}{2}\langle n-1\rangle_4)\Big) \\ &= \Big\{-\frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{1}{4}\Big\} \end{aligned}$$

3. Let $X[k]$ be the 8-point DFT of the sequence $x[n] = \{1, -1, 2, 3, -3, 0, 0, 0\}$. Let $y[n]$ be a finite length sequence whose DFT is $Y[k] = X[k]e^{-j\frac{2\pi}{6}kn_0}$, where $n_0 = 3$. Determine the sequence $y[n]$. (10pts)

Use the property of time shift and frequency modulation. Though $n_0 = 3$, we know that 3 *cannot* be the shift factor because this would correspond to a 6-point DFT where $N = 6$; thus, we must find a form where the given frequency modulation corresponds to W_N^{km} for our given value of N.

$$W_N^{km} = e^{-j\frac{2\pi}{8}km} = e^{-j\frac{\pi}{4}km} = e^{-j\pi k}$$

By inspection, we find that $m = 4$, which corresponds to a right circular shift by four.

$$y[n] = \{-3, 0, 0, 0, 1, -1, 2, 3\}$$

4. Consider the real finite length sequence $\{x[n]\}_{n=0}^{4} = \{0, 1, 2, 3, 4\}$, and its DFT $\{X[k]\}_{k=0}^{4}$. (15pts: 7,8pts)

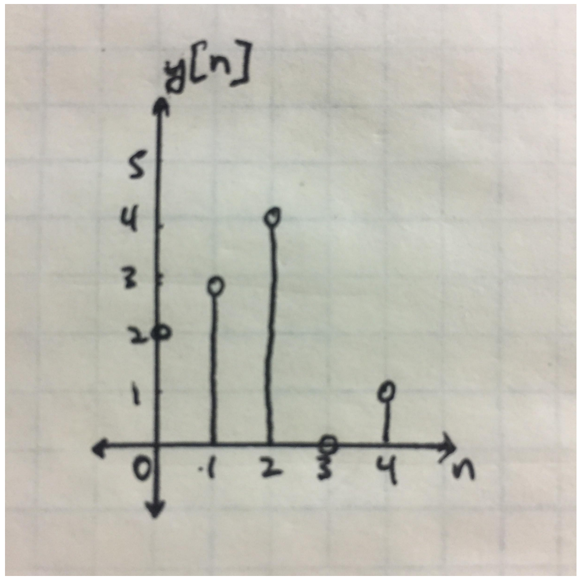
(a) Let $y[n]$ be a finite length sequence whose DFT is $Y[k] = e^{j4\pi k/5}X[k]$. Sketch $y[n]$.

Use the property of time shift and frequency modulation.

$$W_N^{km} = e^{-j\frac{2\pi}{5}km} = e^{j\frac{4\pi}{5}k}$$

By inspection, we find that $m = -2$, which corresponds to a left circular shift by 2.

$$y[n] = \{2, 3, 4, 0, 1\}$$



(b) Let $w[n]$ be a finite length sequence whose DFT is $W[k] = \mathrm{Im}\{X[k]\}$. Sketch $w[n]$.

For a complex number $z = a + bj$, we have the following definition, where z^* denotes the complex conjugate of z.

$$\mathrm{Im}\{z\} := \frac{z - z^*}{2j}$$

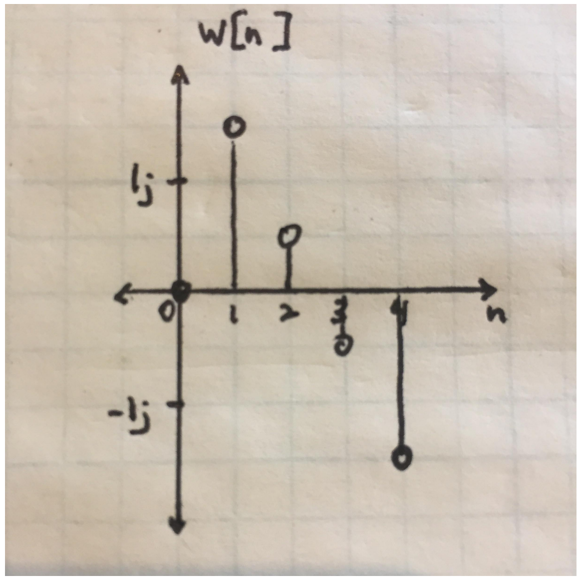
This allows us to rewrite $W[k]$ as follows. Recall that because $x[n]$ is real, we have the symmetry $X[k] = X^*[\langle -k \rangle_N]$ and $X^*[k] = X[\langle -k \rangle_N]$.

$$\begin{aligned} W[k] &= \frac{X[k] - X^*[k]}{2j} \\ &= \frac{X[k] - X[\langle -k \rangle_N]}{2j} \end{aligned}$$

Lastly, we can use this form of $W[k]$ to solve for $w[n]$ in relation to $x[n]$.

$$\begin{aligned} w[n] &= \frac{x[n] - x[\langle -n \rangle_N]}{2j} w[0] = 0 \\ w[1] &= \frac{1-4}{2j} = j\frac{3}{2} \\ w[2] &= \frac{2-3}{2} = j\frac{1}{2} \\ w[3] &= -w[2] = -j\frac{1}{2} \\ w[4] &= -w[1] = -j\frac{3}{2} \end{aligned}$$

Note: this results in a circularly-odd sequence, which confirms our understanding that any sequence can be decomposed into a circularly-even and -odd sequence, with the even and odd components contributing to the real and imaginary componets, respectively, of the DFT.



5. A real-time continuous signal $x_c(t)$ is bandlimited to frequencies below 5 kHz. The signal $x_c(t)$ is sampled with a sampling rate, of 10,000 Hz to produce a sequence $x[n] = x_c(nT)$. Let $X[k]$ be the 1000-point DFT of $x[n]$. To what continuous-time frequencies do the indices $k = 150$ and $k = 800$ correspond? (10pts, 5pts each)

   First, find the corresponding discrete time frequencies by multiplying by $2\pi/N$.

$$150 \cdot \frac{2\pi}{1000} = \frac{3\pi}{10}\text{rads}$$
$$800 \cdot \frac{2\pi}{1000} = \frac{8\pi}{5} - 2\pi = -\frac{2\pi}{5}\text{rads}$$

   Note that we must subtract 2π from the second frequency so that it will be in the principle range of $[-\pi, \pi]$. Next, convert from discrete time to continuous time frequencies by dividing by $2\pi T$.

$$\frac{3\pi}{10} \cdot \frac{10000}{2\pi} = 1500 \text{ Hz}$$
$$-\frac{2\pi}{5} \cdot \frac{10000}{2\pi} = -2000 \text{ Hz}$$

6. Let X[k] denote the 48-point DFT of $x[n]$, $0 \leq n \leq 47$. The sequence $y[n]$ is obtained by zero-padding $x[n]$ to length 256. Determine k_0 such that $Y[48] = X[k_0]$. (4pts)

   We look for k_0 such that:

$$\frac{2\pi}{256}48 = \frac{2\pi}{48}k_0$$

   Solving for k_0, we obtain $k_0 = 9$.

7. Let $X[k]$, $0 \leq k \leq 20$ and $X_d(\omega)$ respectively be the 21-point DFT and DTFT of a *real-valued* sequence $\{x[n]\}_{n=0}^{7}$ that is zero-padded to length 21. Determine all the **correct** relationships and justify your answer. (16 pts, 4pts each)

   (a) $X[19] = X_d(-\frac{4\pi}{21})$

   True. $X[19]$ corresponds to frequency $\frac{2\pi(19)}{21} = \frac{38\pi}{21} - 2\pi = -\frac{4\pi}{21}$.

   (b) $X[2] = X_d^*(-\frac{4\pi}{21})$

   True. $X[2]$ corresponds to frequency $\frac{2\pi(2)}{21} = \frac{4\pi}{21}$. Because $x[n]$ is real, conjugate symmetry holds, so $X_d(\frac{4\pi}{21}) = X_d^*(-\frac{4\pi}{21})$.

   (c) $X[12] = X_d(-\frac{4\pi}{21})$

   False. $X[12]$ corresponds to frequency $\frac{2\pi(12)}{21} = \frac{24\pi}{21} - 2\pi = -\frac{18\pi}{21}$, which we cannot relate to $-\frac{4\pi}{21}$.

   (d) $X[4] = X_d^*(-\frac{4\pi}{21})$

   False. $X[4]$ corresponds to frequency $\frac{2\pi(4)}{21} = \frac{8\pi}{21}$, which we cannot relate to $-\frac{4\pi}{21}$.