ECE 310: Problem Set 3

Due:5pm, Friday September 21, 2018

Note: this solution set exclusively makes use of the unilateral (right-sided) z-transform. However, both bilateral and unilateral z-transforms may be used and will arrive at the same solution as all given signals are right sided, and shifting always occurs to the right.

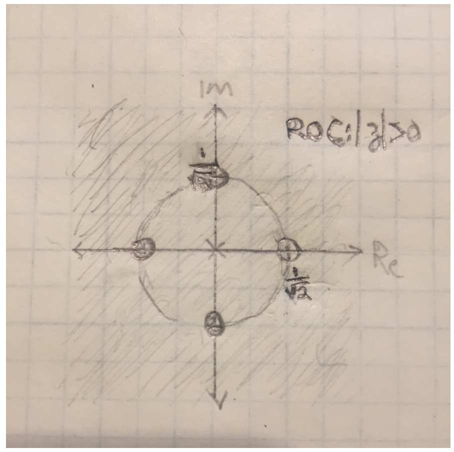
Problem 1 (20pts)

(a) $x[n] = \frac{1}{2}\delta[n-4] - 2\delta[n]$ (6pts)

$$\begin{aligned} X(z) &= \sum_{n=0}^{\infty} x[n]z^{-n} \\ &= \sum_{n=0}^{\infty} (\frac{1}{2}\delta[n-4] - 2\delta[n])z^{-n} \\ &= \boxed{-\frac{1}{2}z^{-4} - 2, \quad \text{ROC: } z \neq 0} \\ &= \boxed{\frac{-4z^4+1}{2z^4}, \quad \text{ROC: } z \neq 0} \end{aligned}$$

Both boxed answers are acceptable.

The poles are located at the origin with multiplicity four, and the zeros are on four equipartitions of the circle with radius $1/\sqrt{2}$ in the complex plane, $z = (1/\sqrt{2})\exp(j2\pi/n)$ for $n \in [0, 3]$. Alternative (recommended): use the given z-transform pair $\delta[n-k] \leftrightarrow z^{-k}$ and the property of linearity.



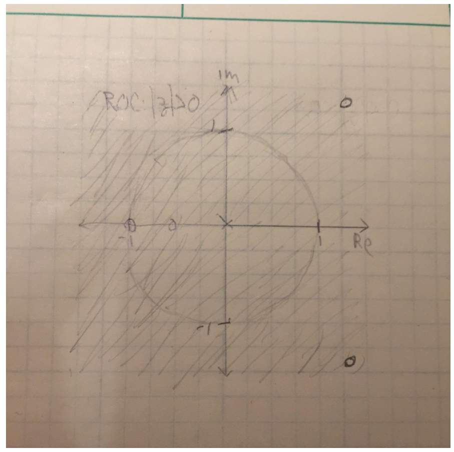
(b) $x[n] = \begin{cases} [1,-1,0,4,2], & 0 \le n \le 4 \\ 0, & otherwise \end{cases}$ (6pts)

Rewrite $x[n]$ as $x[n] = \delta[n] - \delta[n-1] + 4\delta[n-3] + 2\delta[n-2]$

$$\begin{aligned} X(z) &= \sum_{n=0}^{\infty} x[n] z^{-n} \\ &= \sum_{n=0}^{\infty} (\delta[n] - \delta[n-1] + 4\delta[n-3] + 2\delta[n-2]) z^{-n} \\ &= \boxed{1 - z^{-1} + 4z^{-3} + 2z^{-4}, \quad \text{ROC: } z \neq 0} \\ &= \boxed{\frac{z^4 - z^3 + 4z + 2}{z^4}, \quad \text{ROC: } z \neq 0} \end{aligned}$$

Both boxed answers are acceptable.

The poles are located at the origin with multiplicity four, and the zeros are located at the solution to the equation $z^4 - z^3 + 4z + 2 = 0$: $z = -1$, $z = -.5\overline{74}$, $z \approx 1.2874 \pm 1.35i$. This fourth order polynomial can be solved using Wolfram Alpha, MATLAB, orr any solver. Alternative (recommended): use the given z-transform pair $\delta[n-k] \leftrightarrow z^{-k}$ and the property of linearity.



(c) $x[n] = \left(\frac{1}{4}\right)^n nu[n] + \left(\frac{1}{3}\right)^n u[n]$ (8pts)

Use the given z-transforms $na^n u[n] \leftrightarrow (az^{-1})/(1-az^{-1})^2$ and $a^n u[n] \leftrightarrow 1/(1-az^{-1})$ and the property of linearity. Define $x_1[n] = \left(\frac{1}{4}\right)^n nu[n]$ and $x_2[n] = \left(\frac{1}{3}\right)^n u[n]$, such that $x[n] = x_1[n] + x_2[n]$. Let $X_1(z) \leftrightarrow x_1[n]$ and $X_2(z) \leftrightarrow x_2[n]$

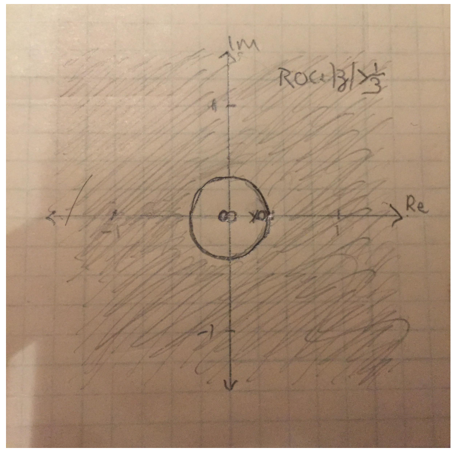
$$\begin{aligned} X(z) &= X_1(z) + X_2(z) \\ &= \frac{\frac{1}{4}z^{-1}}{\left(1-\frac{1}{4}z^{-1}\right)^2} + \frac{1}{1-\frac{1}{3}z^{-1}} \\ &= \boxed{\frac{z^3 - \frac{1}{4}z^2 - \frac{1}{48}z}{(z-\frac{1}{3})(z-\frac{1}{4})^2}, \quad \text{ROC: } |z| > 1/3} \end{aligned}$$

Also acceptable:

$$\boxed{\frac{1-\frac{1}{4}z^{-1} - \frac{1}{48}z^{-2}}{\left(1-\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)^2}, \quad \text{ROC: } |z| > 1/3}$$

Poles are located at $z = 1/3$ with multiplicity one and $z = 1/4$ at multiplicity two. Zeros are located at $z = 0$ and $z = (1/8) \pm \sqrt{(7/3)}/8$.

Alternative (not recommended): Solve analytically using the definition of the z-transform as with (a) and (b).



Problem 2 (20 pts)

$$x[n] \leftrightarrow X(z) = \frac{1}{1-\frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}$$

(a) $y[n] = x[n-2]$ (5pts)

Use the property of time shifting, $x[n-k] \leftrightarrow z^{-k}$.

$$\boxed{Y(z) = \frac{z^{-2}}{1-\frac{1}{2}z^{-1}}, \quad \text{ROC: } |z| > \frac{1}{2}}$$

(b) $y[n] = x[n] * x[n-1]$ (5pts)

Use the properties of time shifting and convolution of sequences, $x_1[n] * x_2[n] \leftrightarrow X_1(z)X_2(z)$.
Let $x_1[n] = x[n], X_1(z) = X(z), \quad |z| > 1/2$
Let $x_2[n] = x[n-1], X_2(z) = z^{-1}X(z), \quad |z| > 1/2$.

$$\begin{aligned} Y(z) = X_1(z)X_2(z) &= z^{-1}X^2(z) \\ &= z^{-1}\left(\frac{1}{1-\frac{1}{2}z^{-1}}\right)^2 \\ &= \boxed{\frac{z}{(z-\frac{1}{2})^2}, \quad \text{ROC: } |z| > 1/2} \end{aligned}$$

Also accepted:

$$\boxed{\frac{z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)^2}, \quad \text{ROC: } |z| > 1/2}$$

(c) $y[n] = nx[n]$ (5pts)

Use the property of differentiation, $nx[n] \leftrightarrow -z\frac{dX(z)}{dz}$

$$\begin{aligned}
Y(z) &= -z\frac{dXz}{dz} \\
&= -z\frac{d\left(1/(1-\frac{1}{2}z^{-1})\right)}{dz} \\
&= -z\left(\frac{-\frac{1}{2}z^{-2}}{(1-\frac{1}{2}z^{-1})^2}\right) \\
&= \frac{2z}{4z^2-4z+1} \\
&= \boxed{\frac{2z}{(2z-1)^2}, \quad \text{ROC: } |z| > 1/2}
\end{aligned}$$

Also accepted:

$$\boxed{\frac{\frac{1}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})^2}, \quad \text{ROC: } |z| > 1/2}$$

(d) $y[n] = \left(\frac{3}{2}\right)^n x[n]$ (5pts)

Use the property of scaling, $a^n x[n] \leftrightarrow X(a^{-1}z), \quad \text{ROC: } |a|R_x$.

$$\begin{aligned}
a &= 3/2 \\
\Rightarrow Y(z) &= X(\frac{2}{3}z) \\
&= \frac{1}{1-\frac{1}{2}\left(\frac{2}{3}z\right)^{-1}} \\
&= \frac{1}{1-\frac{3}{4}z^{-1}}, \quad \text{ROC: } |z| > \frac{1}{2}\cdot\frac{3}{2} \\
&= \boxed{\frac{1}{1-\frac{3}{4}z^{-1}}, \quad \text{ROC: } |z| > \frac{3}{4}}
\end{aligned}$$

Problem 3 (20 pts)

$$x[n] \leftrightarrow X(z) = 1/(1-0.6z^{-1}), \quad |z| > 0.6$$

Begin by solving for x[n] using the inverse z-transform. Use the given z-transform pair $a^n u[n] \leftrightarrow 1/(1-az^{-1})$ and note that $a = 0.6$. This gives us

$$x[n] = 0.6^n u[n]$$

(a) $Y(z) = X^*(3z^*)$ $(7pts)$

Use the properties of scaling and complex conjugation, $x^*[n] \leftrightarrow X^*(z^*)$. Since we have determined that $x[n]$ is real, the second property has no effect.

$$\begin{aligned}
Y(z) &= X^*(3z^*) \\
Y(z) \leftrightarrow y[n] &= \left(\frac{1}{3}\right)^n x^*[n] \\
&= \left(\frac{1}{3}\right)^n x[n] \\
&= \left(\frac{1}{3}\right)^n 0.6^n u[n] \\
&= \boxed{0.2^n u[n]}
\end{aligned}$$

(b) $Y(z) = X^2(z)$ (7*pts*)

Use the properties of convolution of series.

$$\begin{aligned}
Y(z) &= X(z)X(z) \\
Y(z) \leftrightarrow y[n] &= x[n] * x[n] \\
&= 0.6^n u[n] * 0.6^n u[n] \\
&= \sum_{k=-\infty}^{\infty} (0.6)^k u[k] (0.6)^{(n-k)} u[n-k] \\
&= \sum_{k=0}^{n} (0.6)^{(k+n-k)} \\
&= 0.6^n \sum_{k=0}^{n} 1 \\
&= \boxed{(n+1)0.6^n u[n]}
\end{aligned}$$

Alternative (recommended): Use the given z-transform pair $na^n u[n] \leftrightarrow (az^{-1})/(1-az^{-1})^2$ and the property of time shifting.

$$\begin{aligned}
Y(z) = \left(\frac{1}{1-0.6z^{-1}}\right)^2 &= \frac{1}{(1-0.6z^{-1})^2} \\
&= 0.6^{-1} z \frac{0.6z^{-1}}{(1-0.6z^{-1})^2} \\
&= 0.6^{-1}(n+1)0.6^{(n+1)} u[n] \\
&= \boxed{(n+1)0.6^n u[n]}
\end{aligned}$$

(c) $Y(z) = -dX(z)/dz$ (6*pts*)

Use the properties of derivatives and time shifting.

$$
\begin{aligned}
Y(z) &= z^{-1} \cdot -z dX(z)/dz \\
Y(z) \leftrightarrow y[n] &= (n-1)x[n-1] \\
&= \boxed{(n-1)(0.6)^{n-1}u[n-1]}
\end{aligned}
$$

Problem 4 (20pts)

(a) $x[n] = \left(\frac{1}{4}\right)^{(n-1)} \sin\left(\frac{n\pi}{4} + \frac{\pi}{8}\right) u[n-1]$ (10pts)

Use the time-shift property.

$$
\begin{aligned}
x[n] &= \left(\frac{1}{4}\right)^{(n-1)} \sin(\frac{n\pi}{4} + \frac{\pi}{8})u[n-1] \\
&= \left(\frac{1}{4}\right)^{(n-1)} \sin(\frac{(n-1)\pi}{4} + \frac{\pi}{4} + \frac{\pi}{8})u[n-1] \\
&= \left(\left(\frac{1}{4}\right)^{(n-1)} u[n-1]\right) \sin(\frac{(n-1)\pi}{4} + \frac{3\pi}{8}) \\
&= \left(\left(\frac{1}{4}\right)^{(n-1)} u[n-1]\right) \left(\sin(\frac{(n-1)\pi}{4})\cos(\frac{3\pi}{8}) + \cos(\frac{(n-1)\pi}{4})\sin(\frac{3\pi}{8})\right)
\end{aligned}
$$

Now use the given z-transforms

$$
(r^n \cos\omega_0 n)u[n] \leftrightarrow \frac{1 - (r\cos\omega_0)z^{-1}}{1 - 2(r\cos\omega_0)z^{-1} + r^2 z^{-2}}
$$

and

$$
(r^n \sin\omega_0 n)u[n] \leftrightarrow \frac{(r\sin\omega_0)z^{-1}}{1 - 2(r\cos\omega_0)z^{-1} + r^2 z^{-2}}
$$

and the time shifting property, which yields

$$
\begin{aligned}
X(z) &= \cos(\frac{3\pi}{8})\frac{(\frac{1}{4}\sin(\frac{\pi}{4}))z^{-2}}{1 - 2(\frac{1}{4}\cos(\frac{\pi}{4}))z^{-1} + \frac{1}{4}^2 z^{-2}} + \sin(\frac{3\pi}{8})\frac{z^{-1} - (\frac{1}{4}\cos(\frac{\pi}{4}))z^{-2}}{1 - 2(\frac{1}{4}\cos(\frac{\pi}{4}))z^{-1} + \frac{1}{4}^2 z^{-2}} \\
&= \frac{\cos(\frac{3\pi}{8})(\frac{1}{4}\sin(\frac{\pi}{4}))z^{-2} + \sin(\frac{3\pi}{8})\left(z^{-1} - (\frac{1}{4}\cos(\frac{\pi}{4}))z^{-2}\right)}{1 - \frac{1}{2}\cos(\frac{\pi}{4})z^{-1} + \frac{1}{4}^2 z^{-2}} \\
&= \frac{\sin(\frac{3\pi}{8})z^{-1} + \frac{1}{4}\sin(\frac{\pi}{4})\left(\cos(\frac{3\pi}{8}) - \sin(\frac{3\pi}{8})\right)z^{-2}}{1 - \frac{1}{2}\cos(\frac{\pi}{4})z^{-1} + \frac{1}{4}^2 z^{-2}} \\
&= \boxed{\frac{\sin(\frac{3\pi}{8})z^{-1} + \frac{\sqrt{2}}{4}\sin(\frac{\pi}{4})\sin(\frac{3\pi}{8})z^{-2}}{1 - \frac{1}{2}\cos(\frac{\pi}{4})z^{-1} + \frac{1}{4}^2 z^{-2}}, \quad \text{ROC: } |z| > \frac{1}{4}}
\end{aligned}
$$

(b) $x[n] = n^2 \left(\frac{1}{2}\right)^n u[n]$ (10pts)

Use the property of differentiation.

$$\begin{aligned}
x[n] &= n^2 \left(\frac{1}{2}\right)^n u[n] \\
&= n \cdot \left(n \left(\frac{1}{2}\right)^n u[n]\right) \\
&= n x_1[n] \\
x[n] \leftrightarrow X(z) &= -z \frac{dX_1(z)}{dz} \\
&= -z \cdot d\left(\frac{\frac{1}{2}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)^2}\right)/dz \\
&= -z \cdot \frac{\left(-\frac{1}{2}z^{-2}\right)\left(1+\frac{1}{2}z^{-1}\right)}{\left(1-\frac{1}{2}z^{-1}\right)^3} \\
&= -z \cdot \frac{-\frac{1}{2}z - \frac{1}{4}}{\left(z-\frac{1}{2}\right)^3} \\
&= \boxed{\frac{\frac{1}{2}z^2 + \frac{1}{4}z}{\left(z-\frac{1}{2}\right)^3}, \quad \text{ROC: } |z| > \frac{1}{2}}
\end{aligned}$$

Also accepted:

$$\boxed{\frac{\frac{1}{2}z^{-1} + \frac{1}{4}z^{-2}}{\left(1-\frac{1}{2}z^{-1}\right)^3}, \quad \text{ROC: } |z| > \frac{1}{2}}$$

Problem 5 (20pts)

(a) $X(z) = 2 + 3z^{-2} + z^{-4}, \quad |z| > 0$ (6pts)

Use the given z-transform pair $\delta[n-k] \leftrightarrow z^{-k}$ and the property of linearity.

$$\boxed{X(z) \leftrightarrow x[n] = 2\delta[n] + 3\delta[n-2] + \delta[n-4]}$$

(b) $X(z) = \frac{1}{\left(1-\frac{1}{2}z^{-1}\right)} + \frac{2}{\left(1-\frac{1}{3}z^{-1}\right)}, \quad |z| > \frac{1}{2}$ (6pts)

Use the given z-transform pair $a^n u[n] \leftrightarrow 1/(1-az^{-1})$ and the property of linearity. Let $X_1(z) = \frac{1}{\left(1-\frac{1}{2}z^{-1}\right)}$ and $X_2(z) = \frac{2}{\left(1-\frac{1}{3}z^{-1}\right)}$.

$$\begin{aligned}
X(z) &= X_1(z) + X_2(z) \\
X(z) &\leftrightarrow x[n] \\
x[n] &= x_1[n] + x_2[n] \\
&= \boxed{\left(\frac{1}{2}\right)^n u[n] + 2\left(\frac{1}{3}\right)^n u[n]}
\end{aligned}$$

(c) $X(z) = \frac{1}{(1-z^{-1})\left(1-\frac{1}{2}z^{-1}\right)^2}, \quad |z| > 1$ (8pts)

First use partial fraction decomposition to reduce X(z) into a sum of known z-transform pairs.

$$
\begin{aligned}
X(z) &= \frac{1}{\left(1-z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)^2} \\
&= \frac{A}{\left(1-z^{-1}\right)} + \frac{B}{\left(1-\frac{1}{2}z^{-1}\right)} + \frac{C}{\left(1-\frac{1}{2}z^{-1}\right)^2}
\end{aligned}
$$

Use the cover-up method to solve for A and C.

$$
\begin{aligned}
A &= \left.\frac{1}{\left(1-\frac{1}{2}z^{-1}\right)^2}\right|_{z=1} = \frac{1}{\left(1-\frac{1}{2}\right)^2} = 4 \\
C &= \left.\frac{1}{\left(1-z^{-1}\right)}\right|_{z=1/2} = \frac{1}{(1-2)} = -1
\end{aligned}
$$

We can now choose any value of z that is not a pole to solve for B, since this equivalence holds for all values of z. We will choose $z = 2$.

$$
\begin{aligned}
\frac{4}{\left(1-\frac{1}{2}\right)} + \frac{B}{\left(1-\frac{1}{4}\right)} - \frac{1}{\left(1-\frac{1}{4}\right)^2} &= \frac{1}{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)^2} \\
8 + \frac{4}{3}B - \frac{16}{9} &= \frac{32}{9} \\
B &= -2
\end{aligned}
$$

This gives

$$
\begin{aligned}
X(z) &= \frac{4}{\left(1-z^{-1}\right)} - \frac{2}{\left(1-\frac{1}{2}z^{-1}\right)} - \frac{1}{\left(1-\frac{1}{2}z^{-1}\right)^2} \\
&= 4\frac{1}{\left(1-z^{-1}\right)} - 2\frac{1}{\left(1-\frac{1}{2}z^{-1}\right)} - 2z\frac{\frac{1}{2}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)^2}
\end{aligned}
$$

Now make use of the given z-transform pairs $na^n u[n] \leftrightarrow (az^{-1})/(1-az^{-1})^2$ and $a^n u[n] \leftrightarrow 1/(1-az^{-1})$, while also noting that the third term is time-shifted. This gives

$$
\begin{aligned}
X(z) \leftrightarrow x[n] &= 4u[n] - 2\left(\frac{1}{2}\right)^n u[n] - 2(n+1)\left(\frac{1}{2}\right)^{n+1} u[n+1] \\
&= \boxed{4u[n] - 2\left(\frac{1}{2}\right)^n u[n] - (n+1)\left(\frac{1}{2}\right)^n u[n]}
\end{aligned}
$$