## ECE 310: Quiz #2 (10am Section G) Fall 2018 Solutions

1. Consider the system specified by the following input-output relationship:

$$y[n] = e^{j\pi n^2} x[n]$$

(i) Determine if it is linear or non-linear. (4 pts)

Let us apply two arbitrary input signals, $x_1[n]$ and $x_2[n]$, such that

$$\begin{aligned} y_1[n] &= e^{j\pi n^2} x_1[n] \\ y_2[n] &= e^{j\pi n^2} x_2[n] \end{aligned}$$

Now apply the input signal $x[n] = a_1 x_1[n] + a_2 x_2[n]$ to the given system, with $a_1,\ a_2$ being arbitrary scalars. This gives:

$$\begin{aligned} y[n] &= e^{j\pi n^2} x[n] \\ &= e^{j\pi n^2} (a_1 x_1[n] + a_2 x_2[n]) \\ &= a_1 e^{j\pi n^2} x_1[n] + a_2 e^{j\pi n^2} x_2[n] \\ &= a_1 y_1[n] + a_2 y_2[n] \end{aligned}$$

This system has the properties of additivity and homogeneity. $\boxed{\textbf{Linear}}$

(ii) Determine if it is time-invariant or time varying. (2 pts)

Let us apply an input signal $x_1[n] = x[n]$ such that

$$y_1[n] = e^{j\pi n^2} x_1[n]$$

Now apply a shifted input, $x_2[n] = x_1[n - n_0]$ with output $y_2[n]$ and compare with the shifted output, $y_1[n - n_0]$.

$$\begin{aligned} y_2[n] &= e^{j\pi n^2} x_2[n] \\ &= e^{j\pi n^2} x_1[n - n_0] \\ y_1[n - n_0] &= e^{j\pi (n-n_0)^2} x_1[n - n_0] \\ y_1[n - n_0] &\neq y_2[n] \end{aligned}$$

The output of the time-shifted input is not equal to the time-shifted output. $\boxed{\textbf{Time-variant}}$

Alternative method: provide a counterexample, for example using the unit step response with

$$x_1[n] = u[n]$$

and the time shifted input

$$x_2[n] = u[n - 1]$$

The first gives outputs

$$\begin{aligned} y_1[-1] &= 0 \\ y_1[0] &= 1 \\ y_1[1] &= e^{j\pi} \\ y_1[2] &= e^{j4\pi} \end{aligned}$$

The second gives outputs

$$\begin{aligned}
y_2[0] &= 0 \\
y_2[1] &= e^{j\pi} \\
y_2[2] &= e^{j4\pi} \\
y_2[3] &= e^{j9\pi}
\end{aligned}$$

Clearly, $y_2[n] \neq y_1[n-1]$.

2. Compute the convolution $x[n] * h[n]$ for $x[n] = \left(\frac{1}{2}\right)^n u[n-2]$ and $h[n] = (n^2+1)(u[n] - u[n-2])$. (4 pts)

$$\begin{aligned}
x[n] * h[n] &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\
&= \sum_{k=-\infty}^{\infty} (k^2+1)(u[k] - u[k-2]) \cdot \left(\frac{1}{2}\right)^{n-k} u[(n-k)-2)] \\
&= \sum_{k=0}^{1} (k^2+1) \left(\frac{1}{2}\right)^{n-k} u[n-k-2] \\
&= \left[(0+1)\left(\frac{1}{2}\right)^{n-0} u[n-2]\right] + \left[(1+1)\left(\frac{1}{2}\right)^{n-1} u[n-3]\right] \\
&= \boxed{\left(\frac{1}{2}\right)^{n} u[n-2] + \left(\frac{1}{2}\right)^{n-2} u[n-3]}
\end{aligned}$$