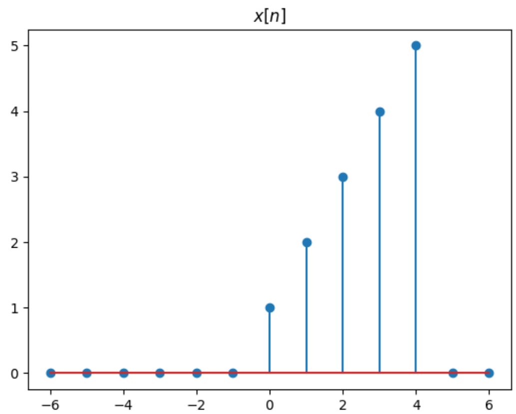
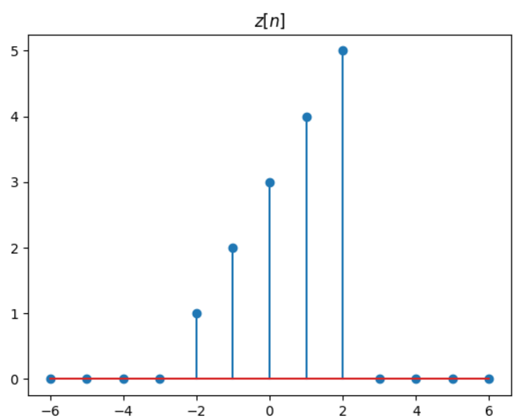
Problem 1: Flipping and shifting

Let $x[n] = \{\cdots, 0, \underset{\uparrow}{1}, 2, 3, 4, 5, 0, \cdots\}$ and $y[n] = x[2-n]$. There are two ways that we can construct $y[n]$ from $x[n]$.



Method 1: Shift $x[n]$ to the left and then flip it.
Assume $z[n]$ is created by shifting $x[n]$ to the left by 2, then

$$z[n] = x[n+2].$$

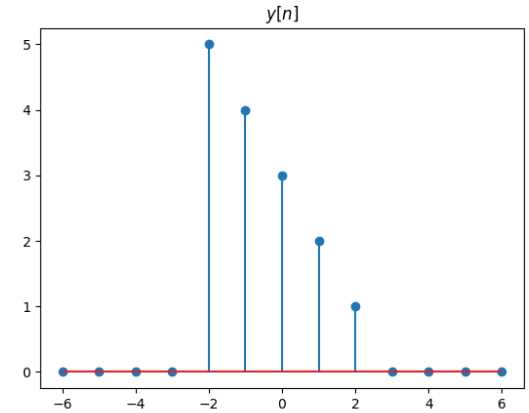


$y[n]$ is the flipped version of $z[n]$, so

$$y[n] = z[-n].$$

Since $z[n] = x[n+2] \Rightarrow z[-n] = x[-n+2]$,

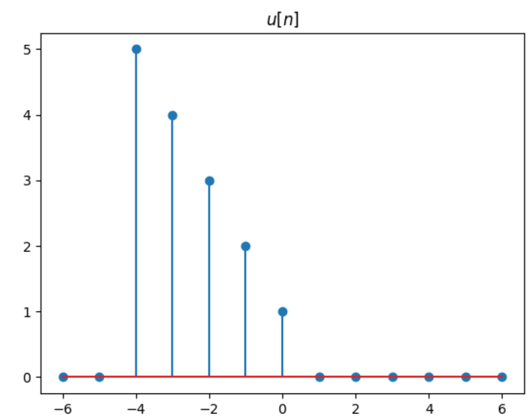
$$y[n] = z[-n] = x[-n+2] = x[2-n].$$



Method 2: Flip $x[n]$ and then shift it to the right.

Assume $z[n]$ is the flipped version of $x[n]$, then

$$z[n] = x[-n].$$



$y[n]$ is created by shifting $z[n]$ to the right by 2 (delaying by 2), so

$$z[n] = z[n-2].$$

Since $z[n] = x[-n] \Rightarrow z[n-2] = z[-(n-2)]$,

$$y[n] = z[n-2] = x[-(n-2)] = x[2-n].$$

Note: Given $y[n] = x[n+k]$,

- If $k < 0$: $y[n]$ is a right-shifted (delayed) version of $x[n]$.
- If $k > 0$: $y[n]$ is a left-shifted version of $x[n]$.

Problem 2: Linearity, shift-invariance, and causality

(a) $y[n] = \max(0, x[n])$

- Non-linear: superposition does not hold
- SI: no operation on indices that makes output shift-variant
- Causal: no access of future indices

(b) $y[n] = y[n-1] + 3^{x[n]}$

- Non-linear: output does not scale with scaled input ($3^{x[n]}$ term)
- SI: no operation on indices that makes output shift-variant
- Causal: no access of future indices

(c) $y[n] = x[|n| - n]$

- Linear: scaled input yields scaled output
- SV: shifted input does not yield shifted output due to access function $|n| - n$
- Not causal: access of future indices

(d) $y[n] = \sum_{m=-\infty}^{n+2} x[m-2]$

- Linear: simply adding previous inputs
- SI: no operation on indices that makes output shift-variant
- Causal: no access to future values / indices

Problem 3: Stability

(a) $y[n] = 3x[n]e^n$
If $|x[n]| < M_0$ then $y[n]$ is not bounded because e^n goes to infinity. Hence, the system is $\boxed{\text{not stable}}$.

(b) $y[n] = x[n] + x[n-1]$
If $|x[n]| \leq M_0$ then the largest output is

$$x[n] + x[n+1] = M_0 + M_0$$

Therefore,

$$|y[n]| \leq 2M_0 < \infty$$

The system is $\boxed{\text{stable}}$ because we have a bound.

(c) $y[n] = \sum_{m=-\infty}^{n} x[m]$
If $|x[m]| < M_0$, there is no bound on $y[n]$ because the output is a summed accumulation of previous inputs. Bound cannot be calculated. Hence, the system is $\boxed{\text{not stable}}$.

Problem 4: Convolution

Consider the finite duration sequences $x[n] = u[n] - u[n-N]$ and $h[n] = n(u[n] - u[n-M]), M \leq N$. Find an analytical expression for the sequence $y[n] = h[n] * x[n]$.

Solution: We can rewrite $x[n]$ and $h[n]$ as

$$x[n] = \begin{cases} 1, & 0 \le n < N \\ 0, & \text{else} \end{cases} \qquad = \{\cdots, 0, \underset{\uparrow}{1}, 1, \cdots, 1, 0, \cdots\}$$

$$h[n] = \begin{cases} n, & 0 \le n < M \\ 0, & \text{else} \end{cases} \qquad = \{\cdots, 0, \underset{\uparrow}{0}, 1, 2, \cdots, M-1, 0, \cdots\}$$

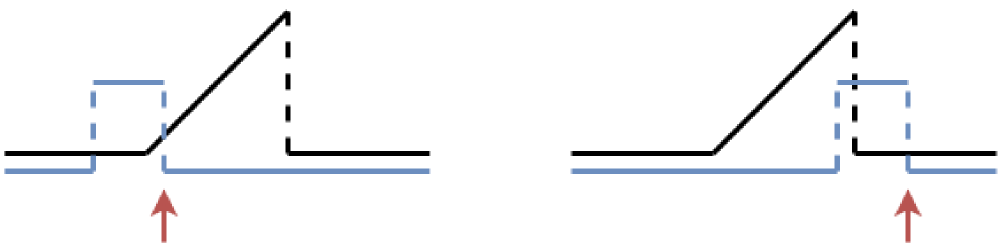
By definition,

$$(x * h)[n] = \sum_{k=0}^{M-1} h[n-k]x[k] = \sum_{k=0}^{M-1} h[k]$$

(Here, $x[n]$ can be seen as a windowing function for $h[n]$).

Question: For which values of n that $(x * h)[n]$ receives non-zero values?

The convolution response only has non-zero values if there is overlap between $h[n]$ and sliding $x[n]$ (or $x[n]$ and sliding $h[n]$). It is more interesting to look at the beginning and the end of this sequence:



The arrow here shows the end of the window. It is clear that the operation does not stop when $n = M-1$ but it continues until $n = (M-1) + N - 1$, where $M - 1$ is the end of $h[n]$, N is the "length" of $x[n]$, and 1 is from overlap condition.

Consider $N = 5, M = 4$, then

$$x[n] = \{\cdots, 0, \underset{\uparrow}{1}, 1, 1, 1, 1, 0, 0, \cdots\}$$

$$h[n] = \{\cdots, 0, \underset{\uparrow}{0}, 1, 2, 3, 0, \cdots\}$$

Unrolling the operation gives us:

n	-4	-3	-2	-1	0	1	2	3	4	5	6	7
					$\downarrow$							
$h[n]$					0	1	2	3				
$x[n]$	1	1	1	1	1							
		1	1	1	1	1						
			1	1	1	1	1					
				1	1	1	1	1				
					1	1	1	1	1			
						1	1	1	1	1		
							1	1	1	1	1	
								1	1	1	1	1

The last non-zero index is $n = (M-1) + N - 1 = 7$. The first non-zero index should usually be 0, however in this case $(x * h)[0] = 0$ because $h[0] = 0$. Therefore, $(x * h)[n] \neq 0$ for $n \in [1, 7], n \in \mathbb{Z}$.