DFT Spectral Analysis

Given $x_c(t)$, want to estimate $X_c(\Omega)$ (i.e., frequency contents)

(e.g. $x_c(t) = \sum_{i=1}^{M} A_i \cos(\Omega_i t)$. Find A_i's and Ω_i's)

unknown

$$x_c(t) \xrightarrow[T]{} \overset{y[n]}{\overset{\shortparallel}{\{x[n]\}_{n=0}^{N-1}}} \longrightarrow \boxed{\text{DFT}} \longrightarrow \{Y[k]\}_{k=0}^{N-1} \qquad Y[k] = Y_d\left(\frac{2\pi}{N}\cdot k\right)$$

We observe a "windowed" version of $x[n]$

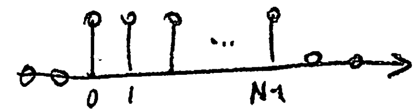
$$y[n] = x[n] \cdot w[n] \quad \text{where} \quad w[n] = \begin{cases} 1 & \text{for } 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases} \quad \text{(rectangular window)}$$

$$Y_d(\omega) = \frac{1}{2\pi} X_d(\omega) \circledast W_d(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_d(\theta)\, W_d(\omega-\theta)\, d\theta \quad \text{(periodic convolution)}$$

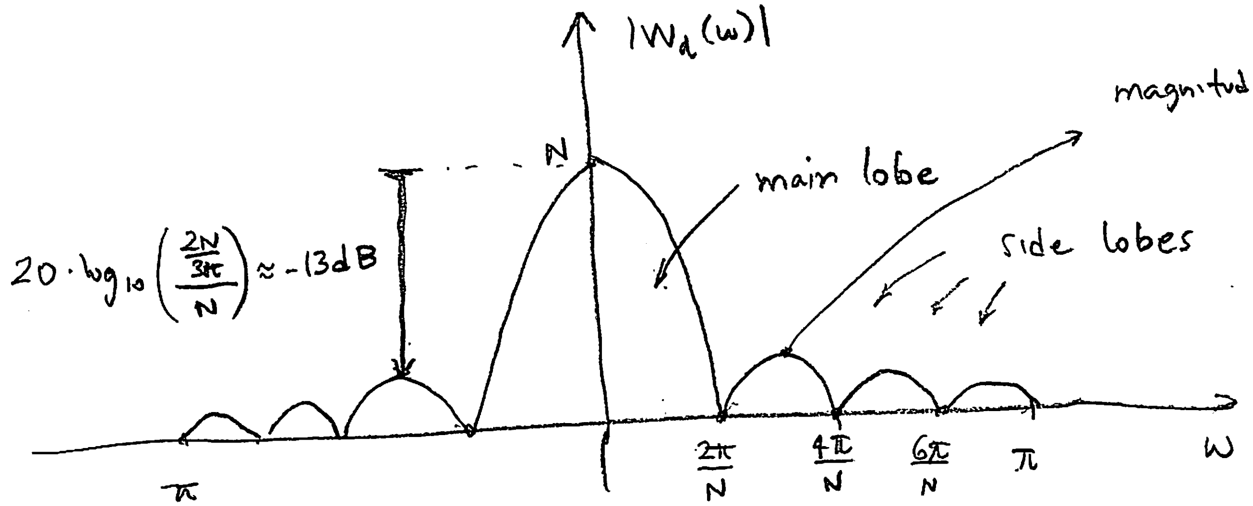
As we will see, convolution with $W_d(\omega)$ will have the effect of "smearing" $X_d(\omega)$

Rectangular window

$$w[n] = \begin{cases} 1 & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$



$$W_d(\omega) = \sum_{n=0}^{N-1} e^{-j\omega n} = \frac{1-e^{-j\omega N}}{1-e^{-j\omega}} = \frac{e^{-j\omega\frac{N}{2}}\left(e^{j\frac{\omega N}{2}} - e^{-j\frac{\omega N}{2}}\right)}{e^{-j\omega/2}\left(e^{j\omega/2} - e^{-j\omega/2}\right)} = e^{-j\omega\frac{(N-1)}{2}} \frac{\sin(\omega N/2)}{\sin(\omega/2)}$$



magnitude of strongest side lobe

$$\left|\frac{\sin\left(\frac{3\pi}{N}\cdot\frac{N}{2}\right)}{\sin\left(\frac{3\pi}{N}\cdot\frac{1}{2}\right)}\right| = \frac{1}{\left|\sin\left(\frac{3\pi}{2N}\right)\right|} \underset{N \text{ large}}{\approx} \frac{2N}{3\pi}$$

- main lobe $\Rightarrow$ smearing (the wider it is, the more smearing)
- side lobes $\Rightarrow$ leaking

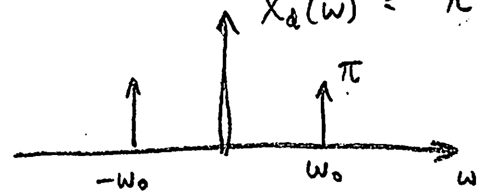
Ex: Spectral analysis of a single sinusoid

$\omega_0 = \Omega_0 T$

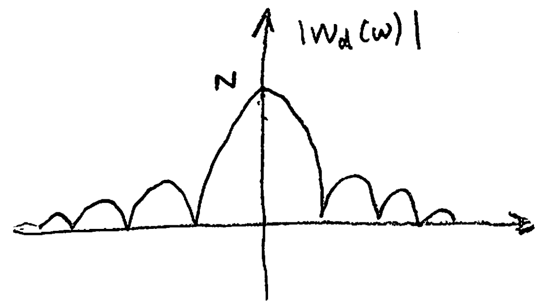
$x_c(t) = \cos(\Omega_0 t)$, $\quad x[n] = \cos(\Omega_0 n T) = \cos(\omega_0 n)$

$X_d(\omega) = \pi\,\delta(\omega-\omega_0) + \pi\,\delta(\omega+\omega_0)$

$x[n] = \cos(\omega_0 n) \quad \overset{DTFT}{\longleftrightarrow}$



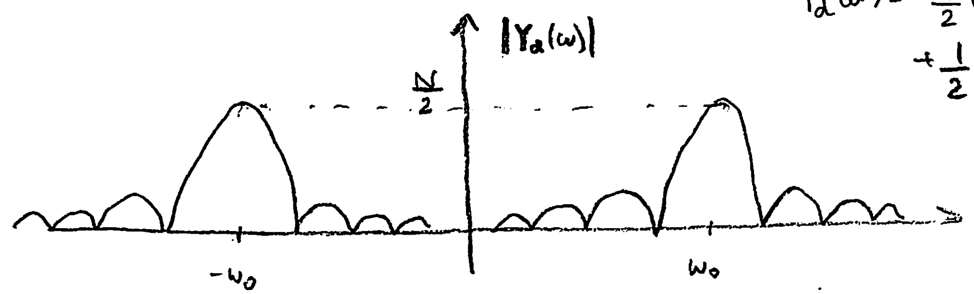
$w[n] \quad \overset{DTFT}{\longleftrightarrow}$



$Y_d(\omega) = \frac{1}{2\pi} X_d(\omega) \circledast W_d(\omega)$

$Y_d(\omega) = \frac{1}{2} W_d(\omega-\omega_0) + \frac{1}{2} W_d(\omega+\omega_0)$

$y[n] = x[n]\cdot w[n] \quad \longleftrightarrow$



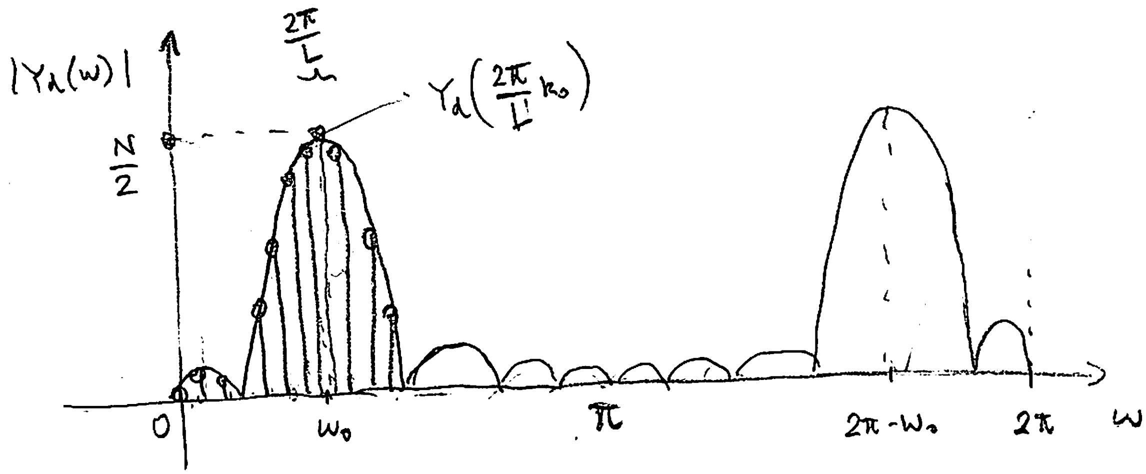
$|Y_d(\omega)| \approx \frac{1}{2}|W_d(\omega+\omega_0)| + \frac{1}{2}|W_d(\omega-\omega_0)|$

- spectral lines of $X_d(\omega)$ were smeared, spread out
- due to side lobes, we have leakage from $\pm\omega_0$ to <u>all</u> frequencies

DFT takes samples of $Y_d(\omega)$

Typically, $y[n] = x[n] \cdot w[n]$ is zero-padded to length L

$$\Rightarrow \quad Y[k] = Y_d\left(\frac{2\pi}{L} k\right)$$



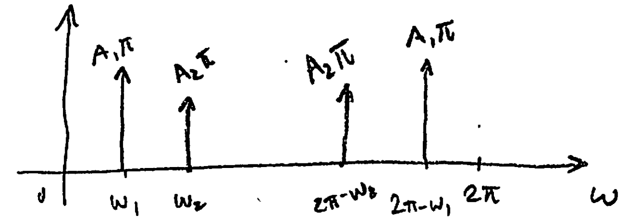
We only sample peak exactly if $\frac{2\pi}{L} k_0 = \omega_0$ for some k_0

Two sinusoids :

$$x_c(t) = A_1 \cos(\Omega_1 t) + A_2 \cos(\Omega_2 t)$$

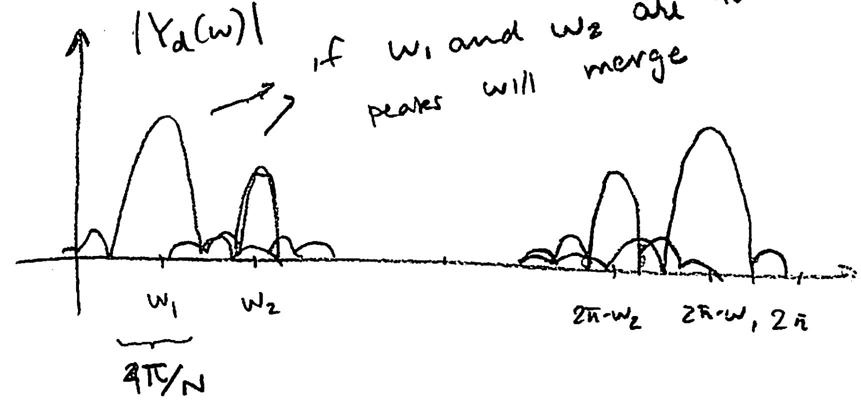
$$x[n] = A_1 \cos(w_1 n) + A_2 \cos(w_2 n)$$

$\xleftrightarrow{\text{DTFT}}$



$$y[n] = x[n] \cdot w[n]$$

$\xleftrightarrow{\text{DTFT}}$



if w_1 and w_2 are too close, peaks will merge

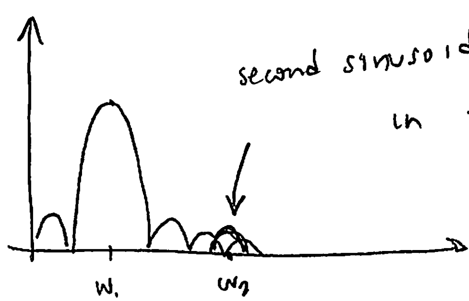
conservative rule : Peaks can be resolved, if the main lobes are disjoint

$$|w_1 - w_2| > \frac{4\pi}{N} \qquad \left(\text{equivalently, } |\Omega_1 - \Omega_2| > \frac{4\pi}{NT}\right)$$

NT → observation window.

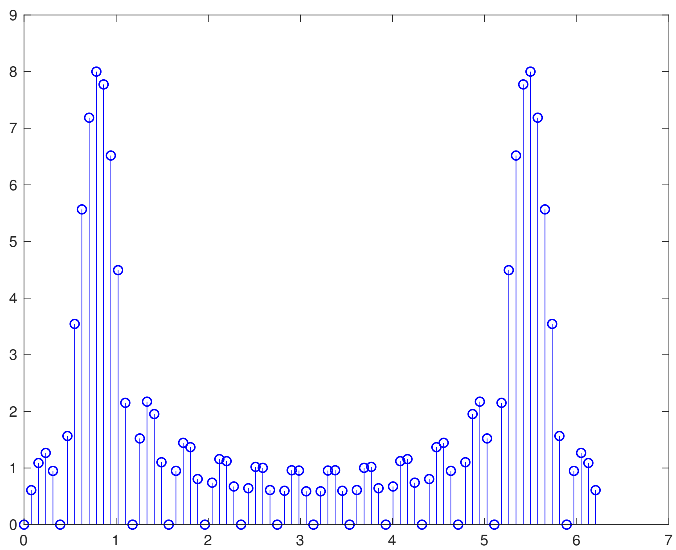
larger NT, better frequency resolution

side lobes reduce amplitude resolution :

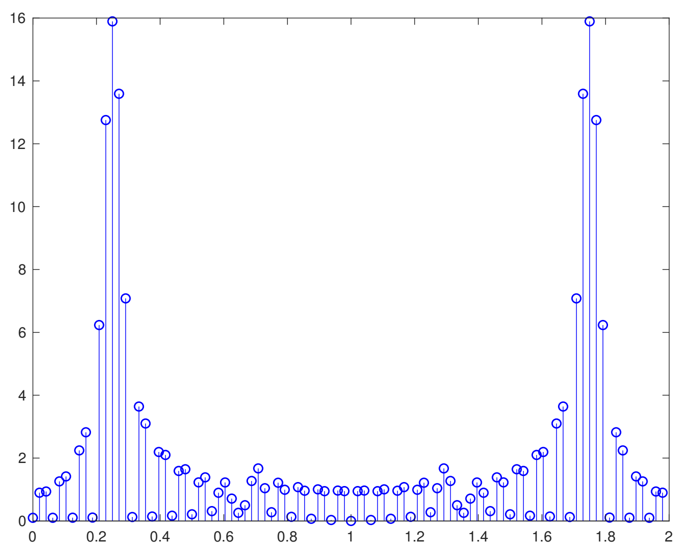


second sinusoid "hidden" in the side lobes → can we use a $W_d(w)$ with smaller side lobes ?

```
% DFT spectral analysis of one sinusoid

N = 16; % window of length N=16
omega_0 = pi/4;
x = cos(omega_0*(1:N));

% Let's do some zero-padding
zpl = 64;
L = N + zpl;
x = [x, zeros(1,zpl)];

X = DFT(x);
w_k = (0:L-1)*2*pi/L;

figure(1);
stem( w_k, abs(X) ,'b');
```

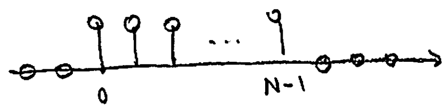


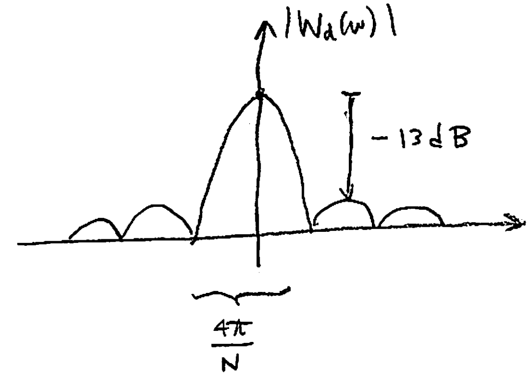
```
% DFT spectral analysis of two sinusoids

N = 32;
omega_0 = pi/4;
omega_1 = 2*pi/3;
x = cos(omega_0*(1:N)) + 0.1*cos(omega_1*(1:N));

% zero-padding
zpl = 64;
L = N + zpl;
x = [x, zeros(1,zpl)];

X = DFT(x);
w_k = ((0:L-1)*2*pi/L) / pi;

figure(2);
stem( w_k, abs(X) ,'b');
```



other window choices

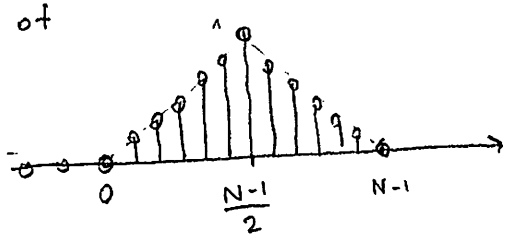
$w_R[n]$

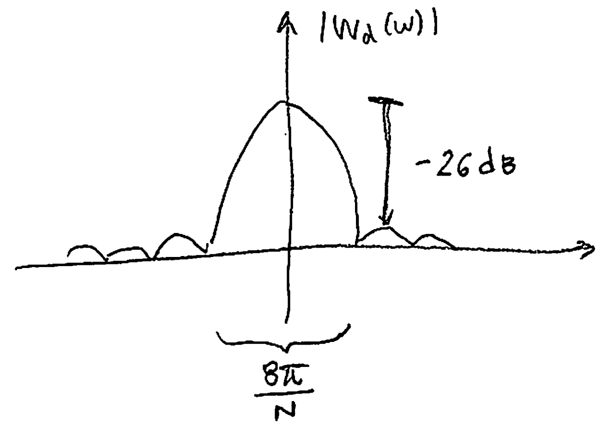




Idea: "square" $W_d(w)$

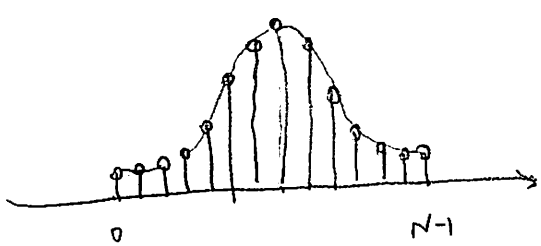
convolution between two $w_R[n]$ of half the length

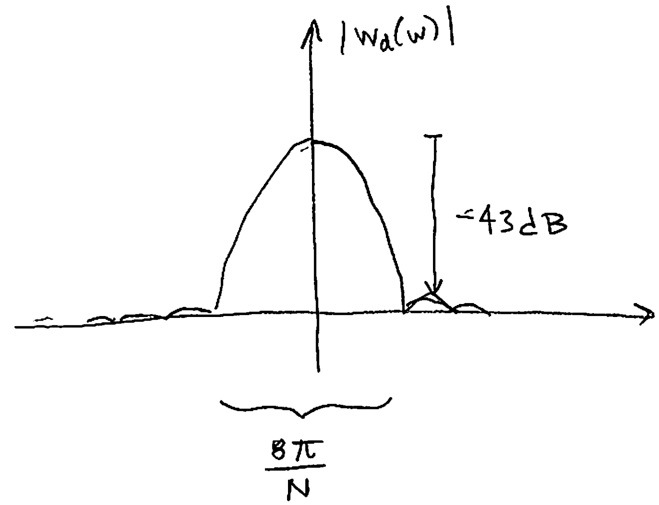




Another approach: raised-cosine

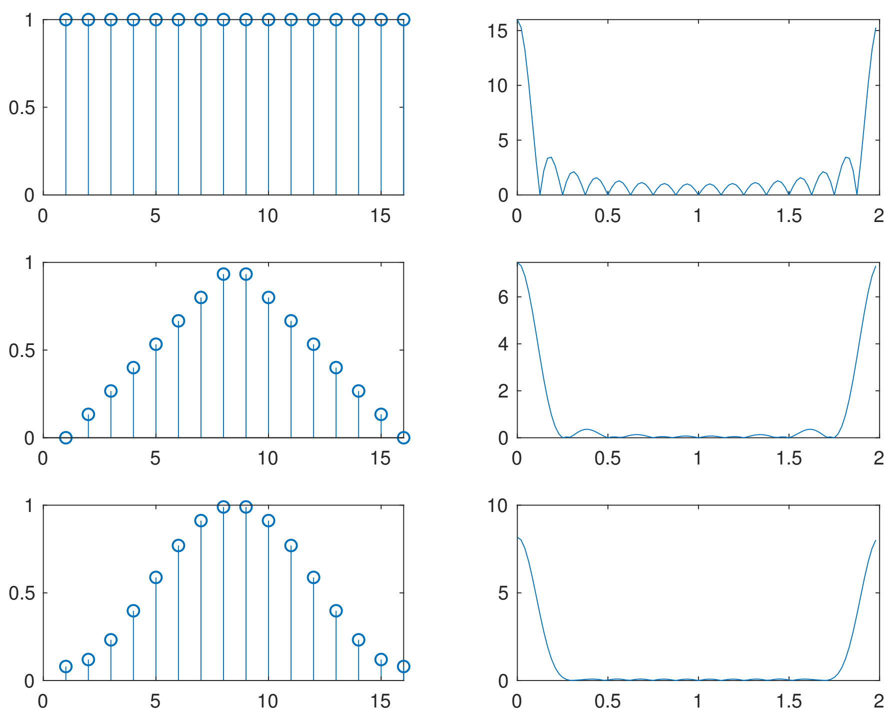
$w[n] = a_0 - a_1 \cos\left(\frac{2\pi}{N-1} n\right)$





Hamming window: $a_0 = 0.54$, $a_1 = 0.46$

```
% Let's look at some window options

N = 16;
zpN = 5*N;
L = N + zpN;

% rectangular window
h0 = [ones(1,N),zeros(1,zpN)];
H0 = DFT(h0);

% triangular window
h1 = [bartlett(N)',zeros(1,zpN)];
H1 = DFT(h1);

% Hamming window (raised cosine)
h2 = [hamming(N)',zeros(1,zpN)];
H2 = DFT(h2);

w_k = ((0:L-1)*2*pi/L) / (pi);

figure(4);
subplot(3,2,1); stem(h0(1:N));
subplot(3,2,2); plot(w_k,abs(H0));
subplot(3,2,3); stem(h1(1:N));
subplot(3,2,4); plot(w_k,abs(H1));
subplot(3,2,5); stem(h2(1:N));
subplot(3,2,6); plot(w_k,abs(H2));
```



```
% DFT spectral analysis of two sinusoids with a different window

N = 32;
omega_0 = pi/4;
omega_1 = 2*pi/3;
x = cos(omega_0*(1:N)) + 0.1*cos(omega_1*(1:N));

x = x .* hamming(N)';

% zero-padding
zpl = 64;
L = N + zpl;
x = [x, zeros(1,zpl)];

X = DFT(x);
w_k = ((0:L-1)*2*pi/L) / pi;

figure(2);
stem( w_k, abs(X) ,'b');
```