

## From Circular Back to Linear Convolution

$$(f \circledast g)[n] = \sum_{m=0}^{N-1} f[m]g[\langle n - m \rangle_N]$$

Where  $f[n]$  and  $g[n]$  are  $N$  long. Linear convolution is different from circular convolution. However, if we zero-pad  $f$  and  $g$  to at least length  $N+N-1$ , we can use circular convolution to calculate linear convolution.

### Example

Let  $\{x_n\}_{n=0}^3 = \{1, 2, 3\}$  and  $\{h_n\}_{n=0}^3 = \{1, 1, 0\}$ . Compute  $z[n]$ , the linear convolution of  $x[n]$  and  $h[n]$  using circular convolution.

### Solution

To compute the cyclic convolution, we flip one signal and shift it to the right, but we're now assuming that the flipped signal is **periodic**. So we can use a modified version of the table method, shifting the signal over and determining where the terms overlap:

$ 1 \ 2 \ 3 $	
No shift: <b>0 1  1 0 1  1</b>	$\rightarrow z[0] = 4$
Shift by 1: <b>1 0  1 1 0  1</b>	$\rightarrow z[1] = 3$
Shift by 2: <b>1 1  0 1 1  0</b>	$\rightarrow z[2] = 5$

(The bold part is the flipped  $h_n$ ). Because we've reached the length of the original signal, we can stop. Therefore:

$x_n \circledast h_n = \{4, 3, 5\}$

Now if we zero-pad both signals to length  $3 + 2 - 1 = 4$  (note that  $h_n$  has a trailing zero) :

$$\{x_n\}_{n=0}^3 = \{1, 2, 3, 0\} \quad \text{and} \quad \{h_n\}_{n=0}^3 = \{1, 1, 0, 0\}$$

and evaluate this new circular convolution, we will see it matches the result of the linear convolution of  $\{1, 2, 3\}$  and  $\{1, 1, 0\}$ .

## The Fast Fourier Transform (FFT)

**DFT:**  $X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi k n}{N}}$

- $N$  complex multiplications and additions for each  $k$  value
- $N$   $k$  values in total.
- The DFT has a complexity of  $O(N^2)$

**FFT (radix-2):** reduces the computation complexity to  $O(N \log_2 N)$ .

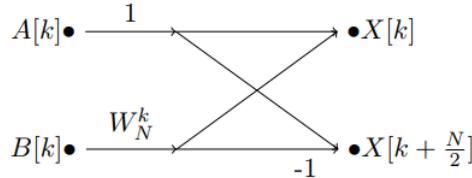
$$\begin{aligned} X[k] &= \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi k n}{N}} \\ &= \sum_{n_{\text{even}}} x[n]e^{-j\frac{2\pi k n}{N}} + \sum_{n_{\text{odd}}} x[n]e^{-j\frac{2\pi k n}{N}} \end{aligned}$$

The even and odd parts can be rewritten and treated as  $N/2$  point DFTs respectively and can be further broken down into its even and odd parts. This is why the length of  $x[n]$  has to be  $2^k$  (powers of 2).

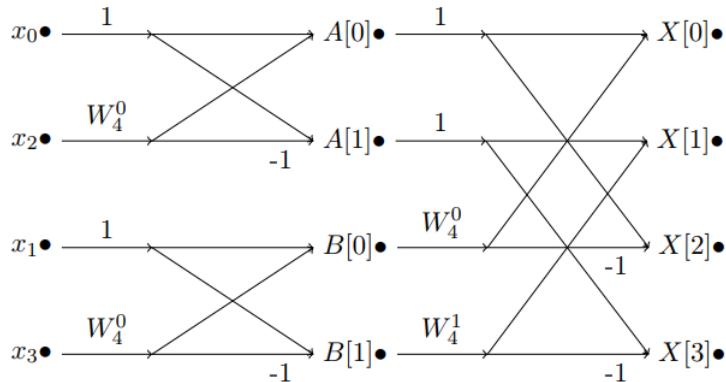
The end result of the FFT is:

$$\begin{aligned} X[k] &= A[k] + W_N^k B[k] \\ X\left[k + \frac{N}{2}\right] &= A[k] - W_N^k B[k] \end{aligned}$$

If interested, the derivation is on page 440 of your textbook.



### Example of a 4-point DFT



## Convolution via FFT

Convolution has a computation complexity of  $O(N^2)$ , but if we use FFT, we can improve it to  $O(N \log N)$ .  
**HINT:** We know from the DFT properties, if  $x_n$  and  $h_n$  are of length  $N$ , then:

$$\text{DFT}\{x_n\} \cdot \text{DFT}\{h_n\} \leftrightarrow x_n \circledast h_n$$

### steps:

If we are given  $x_n$  and  $h_n$  of different lengths  $N$  and  $M$ :

- We can zero-pad  $x_n$  and  $h_n$  to length  $N + M - 1$ . Now, the circular convolution will have the same result as a linear convolution.
- We can zero-pad  $x_n$  and  $h_n$  further to length  $2^{\text{ceil}(\log_2(N+M-1))}$  to enable FFT on  $x_n$  and  $h_n$
- We calculate linear convolution by:  $\text{FFT}^{-1}\{\text{FFT}\{x_n\} \cdot \text{FFT}\{h_n\}\}$