Lecture 12

Recap

$$\{x[n]\} \longrightarrow \overset{\text{LTI}}{\boxed{S}} \longrightarrow \{y[n]\}$$

impulse response $h[n] \overset{\text{DTFT}}{\longleftrightarrow} H_d(w)$ frequency response

$$x[n] = e^{jw_0 n} \text{ --- } \boxed{H} \text{ --- } y[n] = H_d(w_0) \cdot e^{jw_0 n}$$

If h is real valued,

$$x[n] = \cos(w_0 n + \phi) \text{ --- } \boxed{H} \text{ --- } y[n] = |H_d(w_0)| \cos(w_0 n + \phi + \angle H_d(w_0))$$

If h is complex-valued,

$$H_d(w_0) = |H_d(w_0)| e^{j\angle H_d(w_0)}$$

$$x[n] = \cos(w_0 n + \phi) = \frac{1}{2} e^{j(w_0 n + \phi)} + \frac{1}{2} e^{\overbrace{-j(w_0 n + \phi)}^{j((-w_0 n - \phi))}} \text{ --- } \boxed{H} \text{ --- } y[n] = \frac{H_d(w_0)}{2} e^{j(w_0 n + \phi)} + \frac{H_d(-w_0)}{2} e^{j(-w_0 n - \phi)}$$

$$= \frac{|H_d(w_0)|}{2} e^{j(w_0 n + \phi + \angle H_d(w_0))}$$

$$+ \frac{|H_d(-w_0)|}{2} e^{j(-w_0 n - \phi + \angle H_d(-w_0))}$$

Ex: Causal LTI system described by $y[n] - \frac{1}{2} y[n-2] = x[n]$. (real-valued impulse response)

What is the output for input $x[n] = 5 \cdot \cos\left(\frac{\pi}{4}(n+1)\right)$?

We need to compute $H_d(\pi/4)$.

$$Y_d(w) - \frac{1}{2} e^{-j(2w)} Y_d(w) = X_d(w) \Rightarrow Y_d(w)\left(1 - \frac{1}{2} e^{-j2w}\right) = X_d(w)$$

$$\Rightarrow H_d(w) = \frac{Y_d(w)}{X_d(w)} = \frac{1}{1 - \frac{1}{2} e^{-j2w}}$$

$$H_d\left(\frac{\pi}{4}\right) = \frac{1}{1 - \frac{1}{2} e^{-j\pi/2}} = \frac{1}{1 + j/2} \cdot \frac{1 - j/2}{1 - j/2} = \frac{1 - j/2}{1 + 1/4} = \frac{4}{5} - \frac{2j}{5} \Rightarrow$$

$$\left|H_d\left(\frac{\pi}{4}\right)\right| = \sqrt{\left(\frac{4}{5}\right)^2 + \left(\frac{2}{5}\right)^2} = \frac{2\sqrt{5}}{5}$$

$$\angle H_d\left(\frac{\pi}{4}\right) = \operatorname{atan2}\left(\frac{-2}{5}, \frac{4}{5}\right) = -0.46$$

$$y[n] = 5 \left| H_d\left(\frac{\pi}{4}\right) \right| \cos\left(\frac{\pi}{4}(n+1) + \angle H_d\left(\frac{\pi}{4}\right)\right)$$

$$= 2\sqrt{5} \cos\left(\frac{\pi}{4}(n+1) - 0.46\right)$$

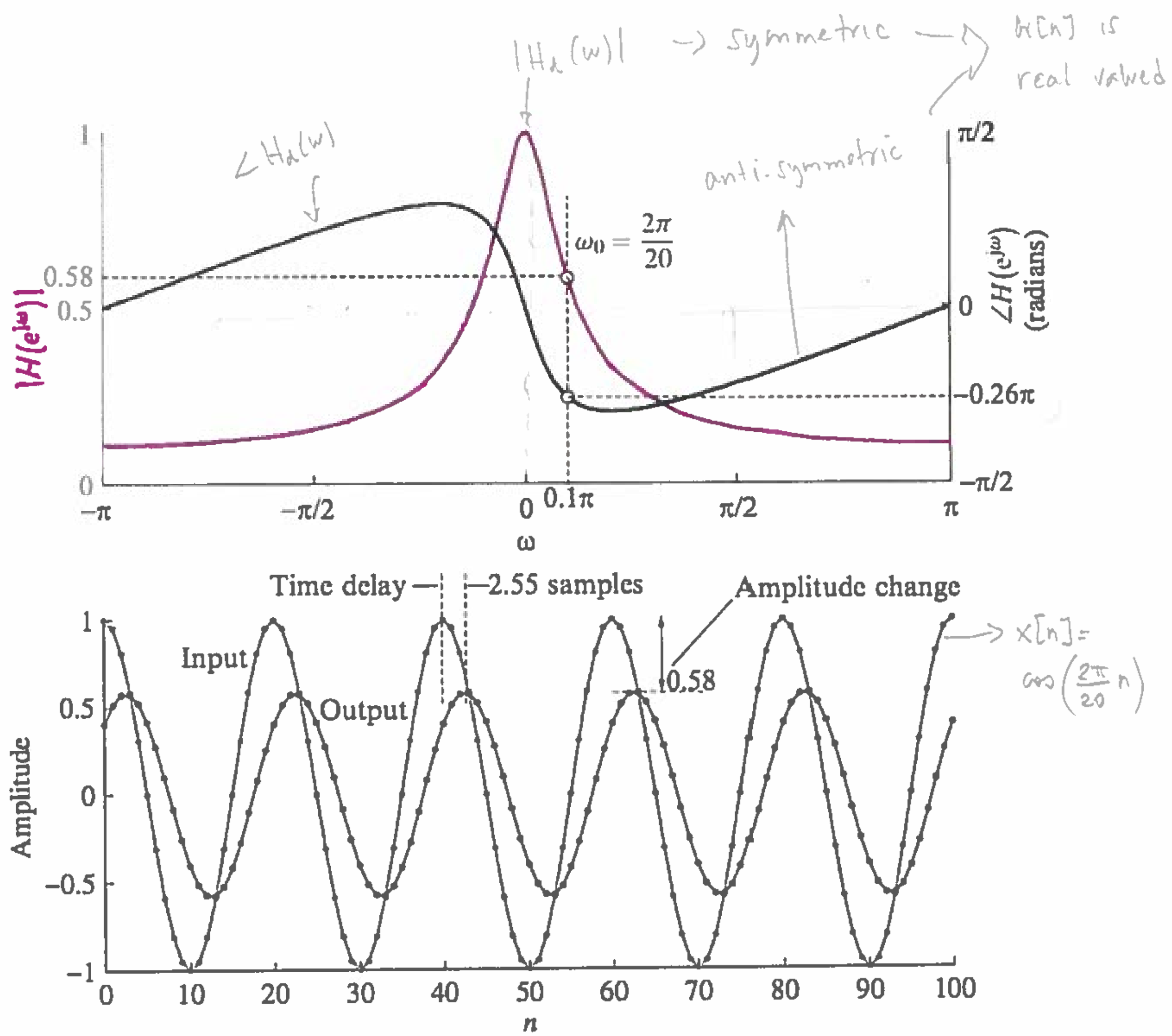


Figure 5.1 Magnitude and phase response functions and input–output signals for the LTI system defined by (5.15). The higher frequency suffers more attenuation than the lower frequency (lowpass filter).

$$y[n] = \underbrace{\left|H_d\left(\tfrac{\pi}{10}\right)\right|}_{0.58} \cos\left(\tfrac{\pi}{10}n + \underbrace{\angle H_d\left(\tfrac{\pi}{10}\right)}_{-0.26\pi}\right)$$

If $x[n]$ is not a sinusoid

$$x[n] \longrightarrow \boxed{H} \longrightarrow y[n] ?$$

$$\underbrace{x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_d(w)\, e^{jwn}\, dw}_{} \longrightarrow y[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_d(w)\, H_d(w)\, e^{jwn}\, dw$$

$x[n]$ can be represented as
a "sum" of complex sinusoids

$$x[n] = e^{j\omega_0 n} \longrightarrow \boxed{H} \longrightarrow y[n] = H_d(\omega_0)\, e^{j\omega_0 n}$$

$$-\infty < n < \infty$$

What if signal $x[n] = 0$ for $n < 0$?

$$x[n] = e^{j\omega_0 n} u[n] \longrightarrow \boxed{H} \longrightarrow y[n] = ?$$

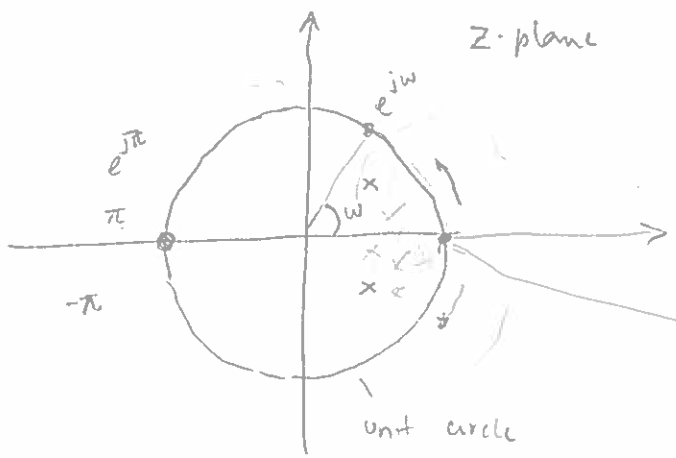
$$y[n] = \sum_{k=-\infty}^{\infty} x[n-k]\, h[k] = \sum_{k=-\infty}^{\infty} h[k]\, e^{j\omega_0 (n-k)} u[n-k] = e^{j\omega_0 n} \sum_{k=-\infty}^{\infty} h[k]\, e^{-j\omega_0 k} u[n-k]$$

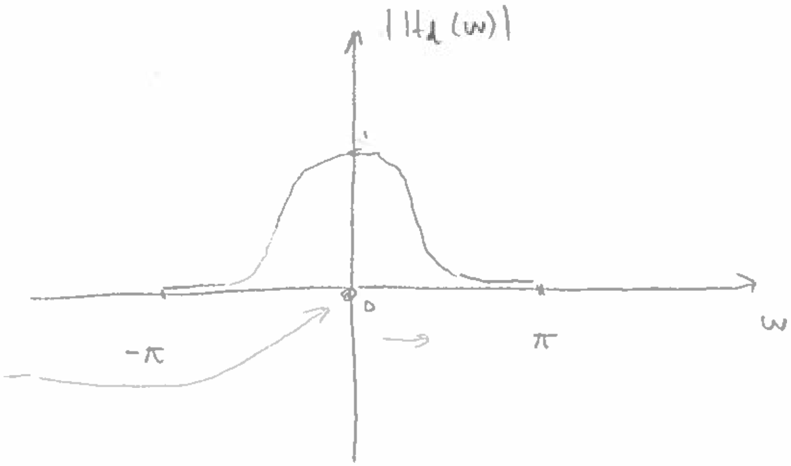
$$= e^{j\omega_0 n} \underbrace{\sum_{k=-\infty}^{n} h[k]\, e^{-j\omega_0 k}}_{\neq H_d(\omega_0)}$$

$$H_d(\omega_0) = \sum_{k=-\infty}^{\infty} h[k]\, e^{-j\omega_0 k} = \lim_{n \to \infty} \sum_{k=-\infty}^{n} h[k]\, e^{-j\omega_0 k}$$

For large n, $\quad y[n] \approx H_d(\omega_0)\, e^{j\omega_0 n}$

Let's try to design a low pass filter





$$H(z) = \frac{1+z^{-1}}{(1-p_1 z^{-1})(1-p_2 z^{-1})}$$

poles: $\underbrace{0.5+0.5j}_{p_1}$, $\underbrace{0.5-0.5j}_{p_2}$

(Next: MATLAB Demo of this filter)