Lecture 7

Recall : $X(z) = \dfrac{B(z)}{A(z)}$ polynomials in z^{-1}

Ex : $X(z) = \dfrac{1+z^{-2}}{4-z^{-2}}$ deg 2, deg 2. Find the inverse z-transform (RoC: $|z| > \frac{1}{2}$)

Trick : Whenever deg B $\geq$ deg A, we can simplify the fraction

poles : $4 - z^{-2} = 0$

$4z^2 - 1 = 0$

$\Rightarrow z^2 = \frac{1}{4} \Rightarrow z = \pm\frac{1}{2}$

$$\frac{5 + (-4 + z^{-2})}{4 - z^{-2}} = \frac{5}{4-z^{-2}} + \frac{-4+z^{-2}}{4-z^{-2}} = \frac{5}{4-z^{-2}} - 1$$

$$\frac{5}{4-z^{-2}} = \frac{5}{4\left(1-\frac{1}{2}z^{-1}\right)\left(1-\left(-\frac{1}{2}\right)z^{-1}\right)} = \boxed{\frac{A_1}{1-\frac{1}{2}z^{-1}} + \frac{A_2}{1-\left(-\frac{1}{2}\right)z^{-1}}}$$

partial fraction expansion

$$A_1\left(1-\left(-\tfrac{1}{2}\right)z^{-1}\right) + A_2\left(1-\tfrac{1}{2}z^{-1}\right) = 5/4$$

set $z = \frac{1}{2}$: $A_1(\overbrace{1-(-1)}^{2}) = 5/4 \Rightarrow A_1 = 5/8$

set $z = -\frac{1}{2}$: $A_2(1-(-1)) = 5/4 \Rightarrow A_2 = 5/8$

$\delta[n-n_0] \overset{Z}{\longleftrightarrow} z^{-n_0}$

$\delta[n] \longleftrightarrow 1$

$$X(z) = \frac{5/8}{1-\frac{1}{2}z^{-1}} + \frac{5/8}{1-\left(-\frac{1}{2}\right)z^{-1}} - 1$$

$$x[n] = \frac{5}{8}\cdot\left(\frac{1}{2}\right)^n u[n] + \frac{5}{8}\left(-\frac{1}{2}\right)^n u[n] - \delta[n]$$

Ex 2: $X(z) = \dfrac{-5z^{-1}}{1-6z^{-1}+9z^{-2}}$ poles: $1-6z^{-1}+9z^{-2} = 0$

$\Rightarrow z^2 - 6z + 9 = 0$, $z = \dfrac{6 \pm \sqrt{\cancel{36-4\cdot 9}}^{\to 0}}{2} = 3$

$= \dfrac{-5z^{-1}}{(1-3z^{-1})^2}$ Partial frac expansion? $\dfrac{A_1}{1-3z^{-1}} + \dfrac{A_2}{1-3z^{-1}}$ X

$\rightarrow$ double pole

Recall: differentiation property of z-transform

$$x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z), \quad \text{ROC } R_x$$

$$n\,x[n] \overset{\mathcal{Z}}{\longleftrightarrow} -z \frac{d}{dz} X(z), \quad \text{ROC } R_x$$

Therefore:

$$a^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-az^{-1}}, \quad |z| > |a|$$

$$n a^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} -z \cdot \frac{d}{dz}(1-az^{-1})^{-1} = -z\cdot(-1)(1-az^{-1})^{-2}\cdot(az^{-2})$$

$$= \frac{az^{-1}}{(1-az^{-1})^2}$$

$$X(z) = -\frac{5}{3}\cdot\frac{3z^{-1}}{(1-3z^{-1})^2}$$

$\mathcal{Z}^{-1} \downarrow$

$$x[n] = -\frac{5}{3}\, n\, 3^n u[n]$$

Aside

z - transform of $x[n]$: $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$

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one-sided z-transform : $X^{+}(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$

If $x[n] = 0$ for $n < 0$, $X(z) = X^{+}(z)$

LTI systems in z - domain

$$\{x[n]\} \longrightarrow \overset{\text{LTI}}{\boxed{S}} \longrightarrow \{y[n]\}$$

If S is LTI, $y[n] = x[n] * h[n]$ ↗ impulse response

From convolution property of z - transforms,

$$Y(z) = X(z)\boxed{H(z)}$$ → transfer function (z-transf. of $h[n]$)

Working in z-domain is convenient:

e.g. cascade of systems:

$$\xrightarrow[X(z)]{x[n]} \overset{\text{LTI}}{\boxed{H_1(z)}} \xrightarrow[X(z)H_1(z)]{} \overset{\text{LTI}}{\boxed{H_2(z)}} \xrightarrow[X(z)(H_1(z)H_2(z))]{}$$

equivalent to

$$\longrightarrow \underbrace{\boxed{H_1(z)\, H_2(z)}}_{\text{also LTI}} \longrightarrow$$

LTI system

$Y(z) = H(z) \cdot X(z)$

$$\begin{cases} y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k] \\ (\text{Initial conditions: } y[n] = 0 \text{ for } n<0) \end{cases}$$

(LCCDE)

$$y[n] + a_1 y[n-1] + a_2 y[n-2] + \dots + a_N y[n-N] = b_0 x[n] + \dots + b_M x[n-M]$$

$\mathcal{Z}$

$$Y(z) + a_1 z^{-1} Y(z) + a_2 z^{-2} Y(z) + \dots + a_N z^{-N} Y(z) = b_0 X(z) + \dots + b_M z^{-M} X(z)$$

$$Y(z) \big(\underbrace{1 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_N z^{-N}}_{A(z)} \big) = X(z) \big(\underbrace{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}_{B(z)} \big)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{B(z)}{A(z)} \rightsquigarrow \text{rational function}$$

Ex. from HW 2 :

$$y[n] = 5y[n-1] + x[n], \quad y[-1] = 0$$

What is $h[n]$ for this system ? $\quad (h[n] = 5^n u[n])$

One approach: set $x[n] = \delta[n]$ and compute $y[n]$.

Another approach:

$$Y(z)(1 - 5z^{-1}) = X(z) \Rightarrow Y(z) = \underbrace{\left(\frac{1}{1-5z^{-1}}\right)}_{H(z) \text{ transfer function}} X(z)$$

$$h[n] = \mathcal{Z}^{-1}\left\{\frac{1}{1-5z^{-1}}\right\} = 5^n u[n] \quad \text{(system is causal)}$$

In this course, we consider causal systems.

$h[n] = 0$ for $n < 0$

$$y[n] - 5y[n-1] = x[n]$$

$$\mathcal{Z} \downarrow$$

$$Y(z) - 5z^{-1}Y(z) = X(z)$$

Ex 2: $4y[n] - y[n-2] = x[n] + x[n-2]$ (causal, zero initial conditions)

$\mathcal{Z} \downarrow$

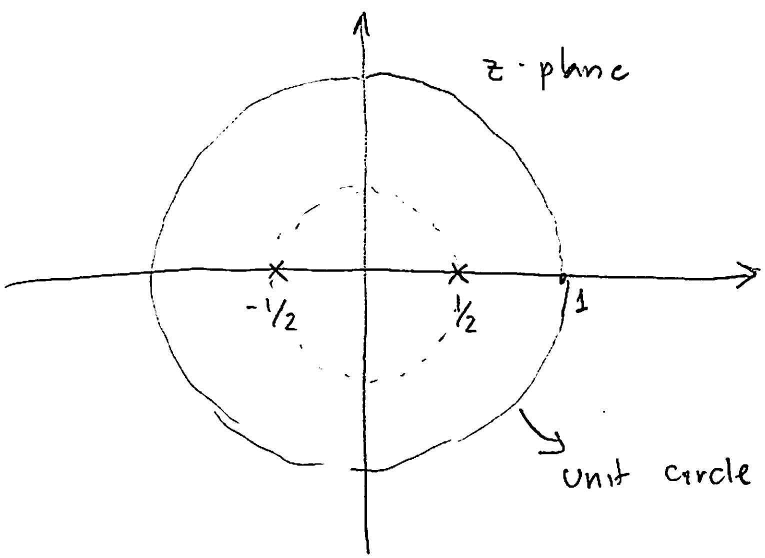
$$4Y(z) - z^{-2}Y(z) = X(z) + z^{-2}X(z)$$

$$Y(z)(4 - z^{-2}) = X(z)(1 + z^{-2})$$

$$\Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{1+z^{-2}}{4-z^{-2}}$$ (poles $\pm\frac{1}{2}$)

$\mathcal{Z}^{-1} \downarrow$

$$h[n] = \frac{5}{8}\left(\frac{1}{2}\right)^n u[n] + \frac{5}{8}\left(-\frac{1}{2}\right)^n u[n] - \delta[n]$$



ROC: $|z| > \frac{1}{2}$

Fact 1: For a causal LTI system, the ROC is of the form $|z| > |a|$

Fact 2: stability?

For a stable system, all poles are inside unit circle