University of Illinois Fall 2018

ECE 310: Recitation 12

Monday, October 3, 2018

1 Rate Conversion

A signal $x[n]$ is bandlimited to $2\pi/3$ radians. We want to change its bandwidth to 2π using a fractional sampling rate converter. This can be achieved by upsampling, followed by lowpass filtering, and finally downsampling the signal.

1. Determine the smallest values of L, the upsampling factor, and M, the downsampling factor.
2. For the obtained value of L above, sketch the spectrum of the upsampled signal.
3. Sketch a plot of the magnitude response of the required lowpass filter.
4. Finally, using the obtained minimum value of M, sketch the spectrum of the downsampled signal and verify its bandwidth.

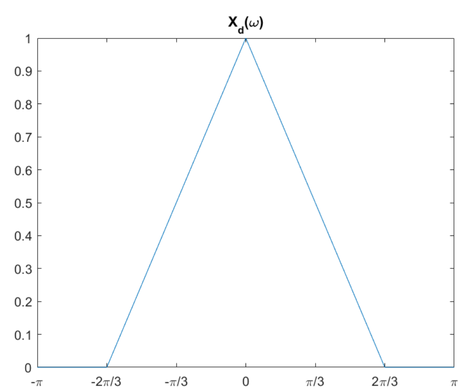
Solution

1. We have the following multirate relation,

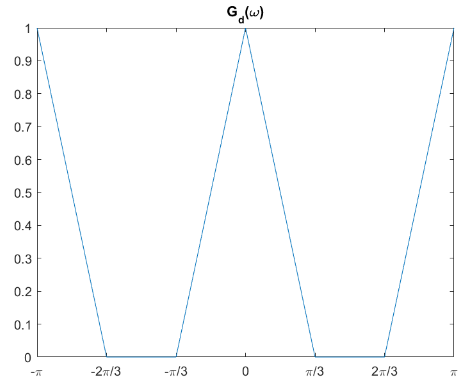
$$\frac{2\pi}{3} = \frac{L}{M}\pi$$

From which we can easily tell that the smallest values will be $L = 2$ and $M = 3$.

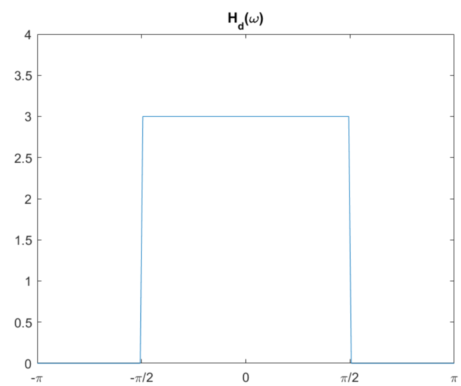
2. First, we sketch the spectrum of the bandlimited signal, $X_d(\omega)$.



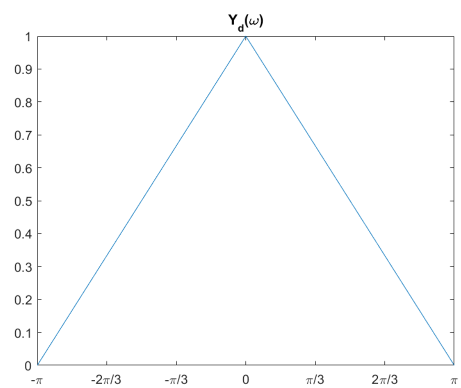
From this we can sketch the spectrum of the upsampled signal, $G_d(\omega)$, noting that the frequency axis will be compressed and we will introduce spectral copies in the region $\omega \in [-\pi, \pi]$.



3. Since we will be downsampling by $M = 3$, we need our filter's frequency response to have a gain of 3 in the passband. The LPF filter with cutoff frequency $\omega_c = \pi/2$ removes the additional spectral copies and appropriately scales the original copy.



4. Lastly, downsampling stretches the frequency axis and compresses the magnitude. Since we introduced a gain in the LPF, our resulting magnitude will have a maximum of 1. We can see that the bandwidth is indeed now 2π.



2 Rate Conversion with Cosines

The general expressions for upsampling and downsampling in the time and frequency domains are expressed in the following table. In general, upsampling by L compresses the spectrum by a factor of L but does not

	Time Domain	Frequency Domain
Upsampling by L	$y[n] = \begin{cases} x[\frac{n}{L}] & \frac{n}{L} \in \mathcal{Z} \\ 0 & \text{else} \end{cases}$	$Y_d(\omega) = X_d(L\omega)$
Downsampling by M	$y[n] = x[Mn]$	$Y_d(\omega) = \frac{1}{M}\sum_{k=0}^{M-1} X_d(\frac{\omega - 2\pi k}{M})$

change the amplitude. Similarly, downsampling by M expands the spectrum by a factor of M, but scales the amplitude by a factor of $\frac{1}{M}$. (In addition, downsampling adds copies to maintain the 2π-periodicity of the DTFT).

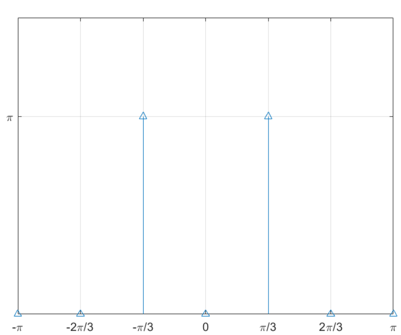
However, these relations are somewhat modified for a pure sine or cosine. Recall the notion of conservation of energy: if we upsample or downsample, the total area under the curve in the frequency domain between $-\pi$ and π shouldn't change. Because the DTFT of a cosine/sine is a delta function, and $\frac{1}{M}\delta(\frac{\omega}{M}) = \delta(\omega)$, we should expect the amplitudes to remain *unchanged* when downsampling. This makes sense when considering the time domain expression. For example, if $x[n] = \cos(\pi n/3)$, and we wish to downsample by 2, we can immediately write $y[n] = \cos(2\pi n/3) = x[2n]$.

A similar phenomenon happens when we upsample. While the area decreases when we compress the spectrum, copies previously centered around 2π will enter the $\omega \in [-\pi, \pi]$ band, offsetting the decrease in area in the original copy. However, this doesn't happen when we're dealing with delta functions, since compressing the spectrum will not decrease the area. Therefore, if we upsample by L, we need to *decrease* the amplitude by a factor of L.

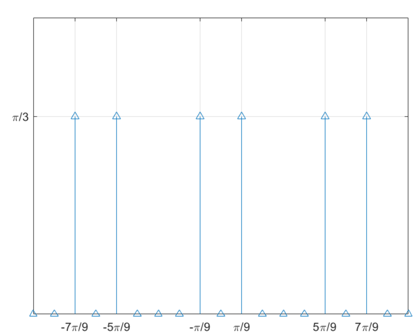
Take the example input

$$x[n] = \cos(\frac{\pi}{3}n)$$

and the upsampling factor $L = 3$. The original spectrum, $X_d(\omega)$, and the upsampled spectrum, $Y_d(\omega)$, can be seen below.



(a) $X_d(\omega)$



(b) $Y_d(\omega)$

If we didn't scale the amplitudes, we would have

$$y[n] = \cos(\frac{\pi}{9}n) + \cos(\frac{5\pi}{9}n) + \cos(\frac{7\pi}{9}n)$$

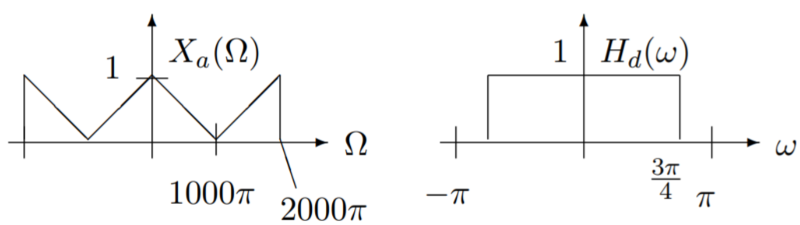
However, by looking at time domain sequence, we can tell that this won't be correct. Upsampling by $L = 3$, which simply adds two zeros between samples, gives us the following sequences:

$$x[n] = \{1, 0.5, -0.5, -1, -0.5, 0.5, 1, \ldots\}$$
$$y[n] = \{1, 0, 0, 0.5, 0, 0, -0.5, 0, 0, -1, \ldots\}$$

But we can see right away that, without if we didn't scale the amplitude in the frequency domain, we would have $y[0] = 3$, which does not match our known output sequence. This tells us that the amplitudes should be scaled by a factor of $\frac{1}{3}$.

3 Zero-Order Hold vs Ideal D/A Converter

Consider the digital system represented in the figure below with a sampling period $T = 0.5$ ms.



1. Sketch $X_d(\omega)$, $Y_d(\omega)$, and $Y_a(\Omega)$.

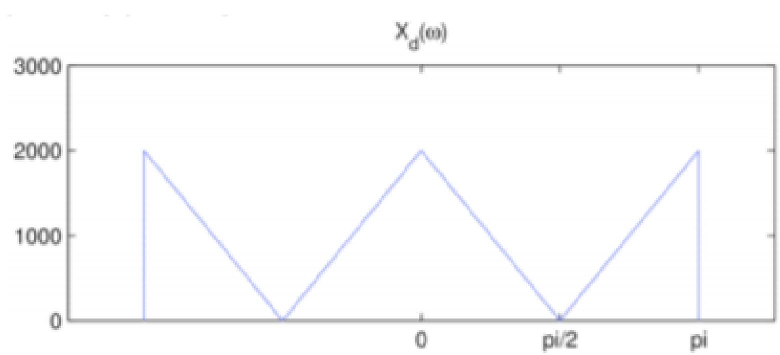
2. Suppose the ideal D/A is now replaced by a ZOH, using the pulse

$$g_a(t) = \begin{cases} 1, & 0 \leq t \leq T \\ 0, & \text{else} \end{cases}$$

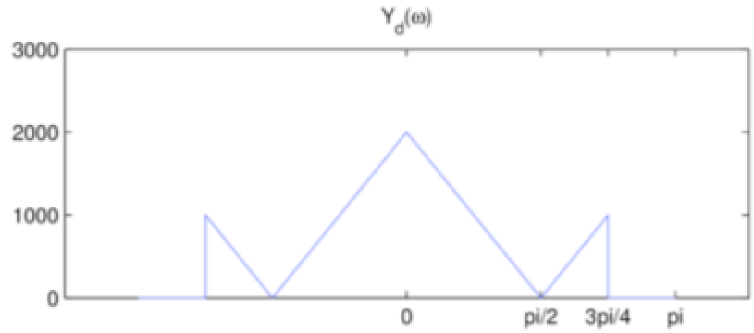
   Sketch $Y_a(\Omega)$ for $|\Omega| \leq 8000\pi$. Find the amplitude of the largest unwanted (out of the band $|\Omega| \leq \frac{\pi}{T}$) componet of $Y_a(\Omega)$, due to the nonideal D/A.

Solution

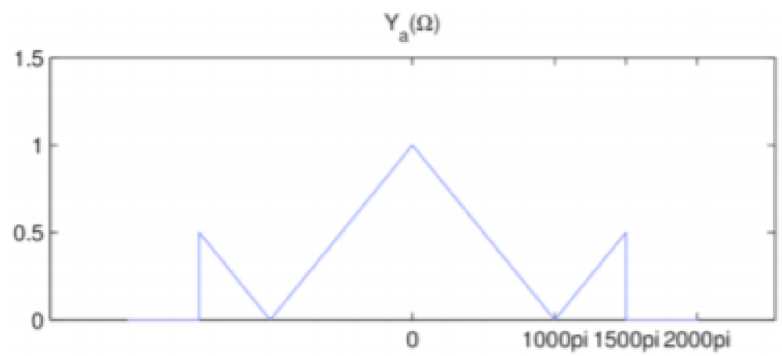
1. Because $T = \frac{1}{2000}$, and because the original signal is bandlimited to 2000π rad/sec, no aliasing will occur when the signal passes through the A/D converter. This means that all we have to do is compress the spectrum by a factor of $\frac{1}{T}$, and scale the amplitude by a factor of $\frac{1}{T}$. The resulting $X_d(\omega)$ can be seen below.



When this passes through the low-pass filter, everything beyond $|\omega| > \frac{3\pi}{4}$ is completely attenuated. The resulting output, $Y_d(\omega)$ is sketched below.



Finally, with an assumed ideal D/A converter, to convert back to analog, we just scale the amplitude by T and apply $\Omega = \frac{\omega}{T}$. Note that the copies that would be present from the DTFT are completely removed in the ideal case because of the LPF in the ideal D/A. The resulting $Y_a(\Omega)$ can be seen below.



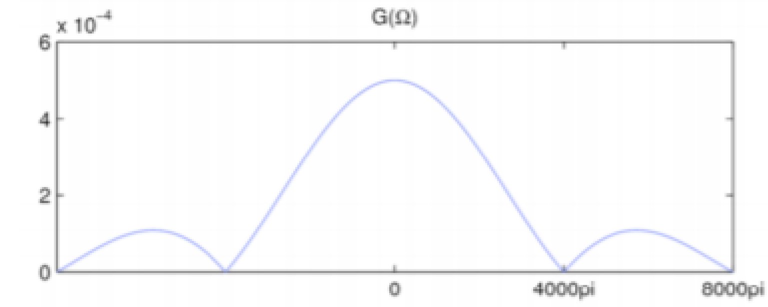
2. To see what happens with the ZOH, we first need to determine the frequency response of the ZOH by taking its CTFT:

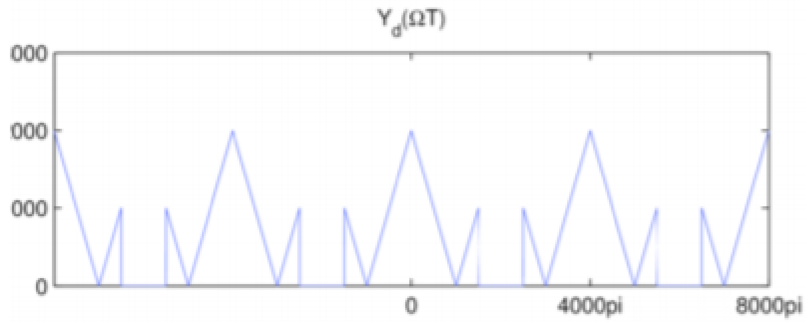
$$
\begin{aligned}
G_a(\Omega) &= \int_0^T e^{-j\Omega t} dt = \frac{1}{-j\Omega} e^{-j\Omega t}|_0^T \\
&= \frac{1 - e^{-j\Omega T}}{j\Omega} = \frac{e^{-j\Omega \frac{T}{2}}(e^{j\Omega \frac{T}{2}} - e^{-j\Omega \frac{T}{2}})}{j\Omega} \\
&= Te^{-j\Omega \frac{T}{2}} \text{sinc}\left(\frac{\Omega T}{2}\right)
\end{aligned}
$$

The shifted sinc lines up with our understanding of the ZOH as a shifted rectangle in the time domain. We can onw get the resulting analog output by applying the ZOH as a filter:

$$Y_a(\Omega) = Y_d(\Omega T) G_a(\Omega)$$

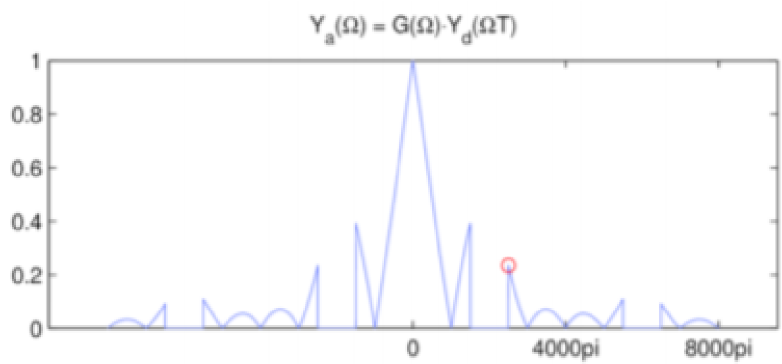
This multiplies the two functions together below.





Note that, while the ZOH zeros out the peaks in the copies, it does not completely attenuate the entire copies. This means that we leave "artifacts" at higher frequencies, which is nonideal behavior. The resulting $Y_a(\Omega)$ is plotted below, and the largest spurious component (the amplitude of the largest out of band component) is circled. Its value is given as:

$$1000T\text{sinc}\left(\frac{2500\pi T}{2}\right) = 0.2532$$



Also, note that these artifacts occur at very high frequencies. We can think of this resulting because the discontinuities introduced by the ZOH are very high frequency.