ECE310: Quiz#4 (6pm Section CSS) Fall 2018 Solutions

1. (6 pts) A causal linear time invariant system has the transfer function $H(z) = \frac{1+z^{-1}}{(1+3z^{-1})(1-z^{-1})}$.

   (a) Determine a difference equation relating the input $x[n]$ to the output $y[n]$ for this system.

   (b) Find the impulse response of this system.

Solution

(a) We write

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1+z^{-1}}{(1+3z^{-1})(1-z^{-1})} = \frac{1+z^{-1}}{1+2z^{-1}-3z^{-2}}$$

Cross-multiplying gives

$$Y(z)(1+2z^{-1}-3z^{-2}) = X(z)(1+z^{-1})$$

and taking the inverse z-transform of both sides gives the difference equation,

$$\boxed{y[n] + 2y[n-1] - 3y[n-2] = x[n] + x[n-1]}$$

(b) Note that the system is causal; this means that the ROC of the z-transform is the outside of the circle formed by the largest pole. In this case, that is $|z| > 3$. Because the "negative order" of the numerator is one, while the negative order of the denominator is two, we can perform the partial fraction expansion:

$$H(z) = \frac{1+z^{-1}}{(1+3z^{-1})(1-z^{-1})} = \frac{A}{1+3z^{-1}} + \frac{B}{1-z^{-1}}, |z| > 3$$

Using the cover-up method gives:

$$A = \left.\frac{1+z^{-1}}{1-z^{-1}}\right|_{z=-3} = \frac{1-\frac{1}{3}}{1+\frac{1}{3}} = \frac{1}{2}, \quad B = \left.\frac{1+z^{-1}}{1+3z^{-1}}\right|_{z=1} = \frac{1}{2}$$

Therefore,

$$H(z) = \frac{1}{2}\left(\frac{1}{1-(-3)z^{-1}}\right) + \frac{1}{2}\left(\frac{1}{1-z^{-1}}\right), |z| > 3$$

Taking the inverse z-transform gives

$$\boxed{h[n] = \frac{1}{2}(-3)^n u[n] + \frac{1}{2}u[n]}$$

2. (4 pts) Determine in each case whether or not the system is BIBO stable. Justify your answer.

   (a) A causal system described by the difference equation $y[n] + jy[n-17] = 17x[n] + x[n-1]$.

   (b) A system described by the equation $y[n] = h[n] * x[n]$, with $h[n] = \cos(\pi n/3)u[n]$.

Solution

(a) Because the system is represented by an LCCDE, we can determine its stability by examining the z-transform of the impulse response. Taking the z-transform of both sides gives

$$Y(z) + jz^{-17}Y(z) = 17X(z) + z^{-1}X(z) \rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{17 + z^{-1}}{1 + jz^{-17}}$$

Because the system is causal, the ROC will be the outside of the circle formed by the largest pole, so we need to solve $1 + jz^{-17} = 0$:

$$\begin{aligned} z^{17} &= -j \\ z^{17} &= e^{-j(\frac{\pi}{2}+2\pi k)} \\ z &= e^{-j(\frac{\pi+4\pi k}{34})}, k = 0, 1, ..., 16 \end{aligned}$$

All of these poles will have magnitude one; therefore, they lie on the unit circle, and the ROC is $|z| > 1$. This means that the system is **not BIBO stable.** As an alternative to solving the above equation, one could note by inspection that $z = -j$ is a pole. Therefore, the ROC must be $|z| > 1$, meaning the system is not stable.

(b) Because the system is characterized by a convolution relationship, the system is LTI. Now, to examine stability, we examine the absolute summability of the impulse response, $h[n] = \cos(\pi n/3)u[n]$. Note that

$$\cos(\frac{n\pi}{3})u[n] = \{1, 0.5, -0.5, -1, -0.5, 0.5, 1, ...\}$$

so

$$|\cos(\frac{n\pi}{3})u[n]| = \{1, 0.5, 0.5, 1, 0.5, 0.5, 1, ...\}$$

Both of these sequences go on forever, since cosine is periodic. From here, it's obvious that

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=0}^{\infty} |\cos(\frac{n\pi}{3})| = \infty$$

meaning that the system is **not BIBO stable**.

Alternatively, we could have seen this by taking the z-transform of the impulse response:

$$H(z) = \frac{1 - \frac{1}{2}z^{-1}}{1 - z^{-1} + 2z^{-2}}, |z| > 1$$

where the ROC comes from the fact that the system is causal. The ROC does not contain the unit circle, so the system is **not BIBO stable**. *Note:* Even if you don't remember the z-transform, recall that the z-transform of $\cos(\omega_0 n)u[n]$ or $\sin(\omega_0 n)u[n]$ has poles at $e^{j\omega_0}$ and $e^{-j\omega_0}$.