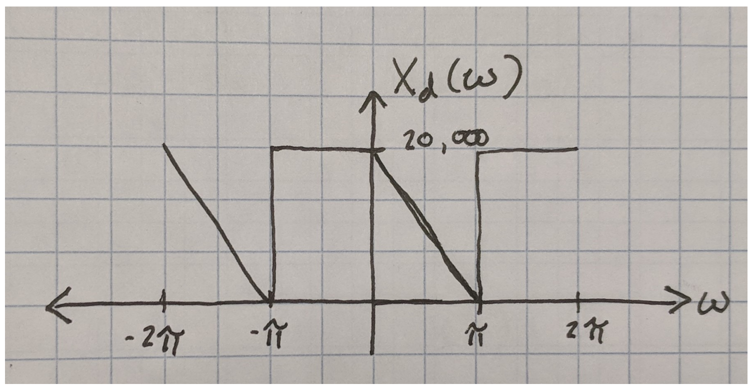
ECE 310: Homework #7 Solution

Due: October 19th, 2018

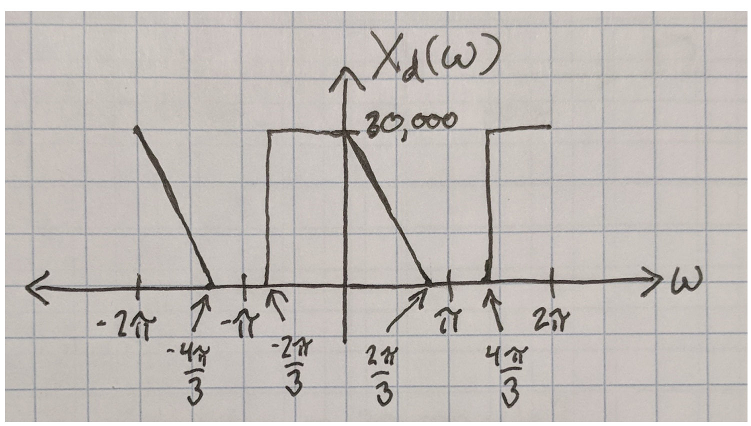
1.

(a) Apply equation 6.13 from the textbook: $X_d(\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(\frac{\omega}{T} - \frac{2\pi}{T}k\right)$

i.



ii.



(b)

$$F_H = 20000\pi = 10\text{kHz}$$
$$F_s = 2F_H = 20\text{kHz}$$

(c) The signal $x[n]$ is not real because its DTFT is not symmetric about the vertical axis.

2. The signal sampled with sampling rate T can be represented:

$$x[n] = x_a(nT)$$

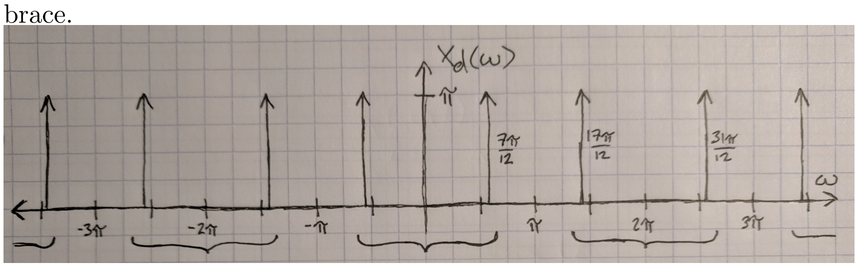
Plugging in $\frac{1}{80}$ for T:

$$\begin{aligned}
\cos\left(\frac{7\pi}{12}n\right) &= \cos\left(\Omega_0 n \frac{1}{80}\right) \\
\frac{7\pi}{12}n &= \Omega_0 n \frac{1}{80} \\
\Omega_0 &= \frac{140\pi}{3} \\
x_a(t) &= \cos\left(\frac{140\pi}{3}t\right)
\end{aligned}$$

Now find the first two signals that will alias to the same frequency. Because cosine is even, these signals will be at $\pm\Omega_0 + 2\pi F_s$:

$$\begin{aligned}
x_1(t) &= \cos\left(\left(-\frac{140\pi}{3} + 160\pi\right)t\right) = \cos\left(\frac{340\pi}{3}t\right) \\
x_2(t) &= \cos\left(\left(\frac{140\pi}{3} + 160\pi\right)t\right) = \cos\left(\frac{620\pi}{3}t\right)
\end{aligned}$$

To help illustrate this, below is a graph of the DTFT of $x[n]$. The DTFT is 2π periodic, and each period in this DTFT is illustrated with a curly brace.



3.

(a) Use integration to find the inverse fourier transform of $X_c(\Omega)$:

$$\begin{aligned}
x_c(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_c(\Omega)\, e^{j\Omega t} d\Omega \\
&= \frac{1}{2\pi}\int_{-500\pi}^{500\pi} (20)\, e^{j\Omega t} d\Omega - \frac{1}{2\pi}\int_{-250\pi}^{250\pi} (10)\, e^{j\Omega t} d\Omega \\
&= \frac{1}{2\pi}(20)\int_{-500\pi}^{500\pi} e^{j\Omega t} d\Omega - \frac{1}{2\pi}(10)\int_{-250\pi}^{250\pi} e^{j\Omega t} d\Omega \\
&= \frac{1}{2\pi}(20)\left[\frac{e^{j\Omega t}}{jt}\right]_{-500\pi}^{500\pi} - \frac{1}{2\pi}(10)\left[\frac{e^{j\Omega t}}{jt}\right]_{-250\pi}^{250\pi} \\
&= \frac{1}{2\pi}(20)\left(\frac{e^{j500\pi t} - e^{-j500\pi t}}{jt}\right) - \frac{1}{2\pi}(10)\left(\frac{e^{j250\pi t} - e^{-j250\pi t}}{jt}\right) \\
&= 20\frac{\sin(500\pi t)}{\pi t} - 10\frac{\sin(250\pi t)}{\pi t}
\end{aligned}$$

alternatively:
Use the CTFT transform pair:

$$\frac{W}{\pi}\text{sinc}\left(\frac{Wt}{\pi}\right) = \begin{cases} \frac{\sin(Wt)}{\pi t}, & t \neq 0 \\ \frac{W}{\pi}, & t = 0 \end{cases} \leftrightarrow \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}$$

Splitting $X_c(\Omega)$:

$$\begin{aligned}
X_c(\Omega) &= X_1(\Omega) - X_2(\Omega) \\
X_1(\Omega) &= \begin{cases} 20, & |\omega| < 500\pi \\ 0, & |\omega| > 500\pi \end{cases} \\
X_2(\Omega) &= \begin{cases} 10, & |\omega| < 250\pi \\ 0, & |\omega| > 250\pi \end{cases}
\end{aligned}$$

Applying the transform pair:

$$\begin{aligned}
x_1(t) &= 20\frac{500\pi}{\pi}\text{sinc}\left(\frac{500\pi t}{\pi}\right) \\
x_2(t) &= 10\frac{250\pi}{\pi}\text{sinc}\left(\frac{250\pi t}{\pi}\right) \\
x_c(t) &= x_1(t) - x_2(t) \\
x_c(t) &= 10000\text{sinc}(500t) - 2500\text{sinc}(250t)
\end{aligned}$$

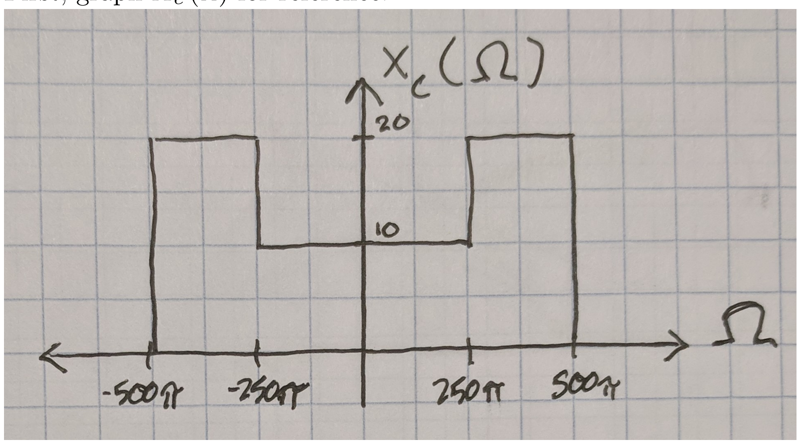
(b)

$$\begin{aligned}
x[n] &= x_c(nT) = x_c\left(\frac{n}{1000}\right) \\
&= 20000\frac{\sin\left(\frac{\pi}{2}n\right)}{\pi n} - 10000\frac{\sin\left(\frac{\pi}{4}n\right)}{\pi n}
\end{aligned}$$

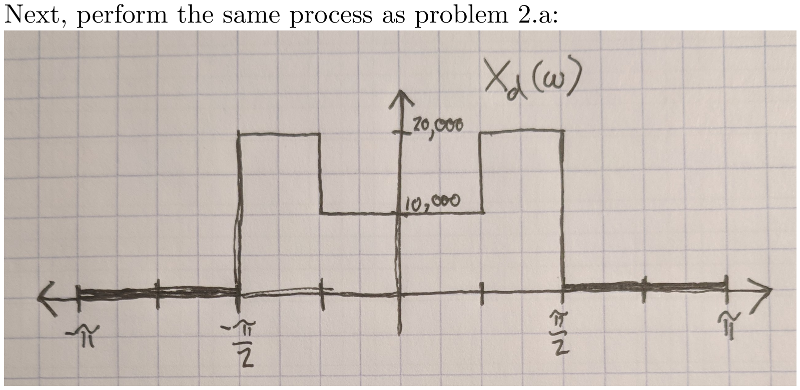
alternatively:

$$
\begin{aligned}
x[n] &= x_c(nT) = x_c\left(\frac{n}{1000}\right) \\
&= 10000\text{sinc}\left(\frac{n}{2}\right) - 2500\text{sinc}\left(\frac{n}{4}\right)
\end{aligned}
$$

(c) First, graph $X_c(\Omega)$ for reference:



Next, perform the same process as problem 2.a:



4.

(a) The output of an ideal D/A is represented by the equation:

$$Y_c(\Omega) = G(\Omega)\, X_d(\Omega T)$$

Where $G(\Omega)$ is defined:

$$G(\Omega) = \begin{cases} T, & |\Omega| \leq \frac{\Omega_s}{2} \\ 0, & |\Omega| > \frac{\Omega_s}{2} \end{cases}$$

And $X_d(\Omega T)$ is defined by the sampling relationship:

$$X_d(\Omega T) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(\Omega - \frac{2\pi}{T}k\right)$$

We use the relationship: $\cos(\Omega_0 t) \overset{CTFT}{\longleftrightarrow} \pi\left(\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)\right)$ to find $X_c(\Omega)$:

$$X_c(\Omega) = \pi\left(\delta(\Omega - 10\pi) + \delta(\Omega + 10\pi)\right)$$

Returning to the sampling relationship:

$$X_d(\Omega T) = \frac{1}{T}\sum_{k=-\infty}^{\infty} \pi\left(\delta\left(\Omega - \frac{2\pi}{T}k - 10\pi\right) + \delta\left(\Omega - \frac{2\pi}{T}k + 10\pi\right)\right)$$

Substituting $\frac{1}{3}$ for T and simplifying:

$$\begin{aligned} X_d(\Omega T) &= \frac{1}{T}\pi\left(\sum_{k=-\infty}^{\infty}\delta(\Omega - 6\pi k - 10\pi) + \sum_{k=-\infty}^{\infty}\delta(\Omega - 6\pi k + 10\pi)\right) \\ &= \frac{1}{T}\pi\left(\sum_{k=-\infty}^{\infty}\delta(\Omega - \pi(6k - 10)) + \sum_{k=-\infty}^{\infty}\delta(\Omega - \pi(6k + 10))\right) \end{aligned}$$

Recalling that multiplication by $G(\Omega)$ will cause any values where $|\Omega| > \frac{\Omega_s}{2}$ to disappear, and recognizing that $\Omega_s = 2\pi F_s = 6\pi$:

$$\begin{aligned} Y_c(\Omega) &= T\frac{1}{T}\pi\left(\delta(\Omega - \pi(6(2) - 10)) + \delta(\Omega - \pi(6(-2) + 10))\right) \\ &= \pi\left(\delta(\Omega - 2\pi) + \delta(\Omega + 2\pi)\right) \end{aligned}$$

Using the CTFT pair from earlier to return to the time domain:

$$y_c(t) = \cos(2\pi t)$$

Alternatively:
Compute $x[n]$:

$$\begin{aligned} x[n] &= x_c(nT) \\ &= \cos\left(\frac{10\pi}{3}n\right) \end{aligned}$$

Using the 2π periodicity of the cosine, $x[n] = \cos(-2\pi n/3) = \cos(2\pi n/3)$, therefore:

$$X_d(\omega) = \pi\delta\left(\omega - \frac{2\pi}{3}\right) + \pi\delta\left(\omega + \frac{2\pi}{3}\right), \text{for } |\omega| < \pi$$

From this we can see that the DTFT of $x[n]$ in the range $[-\pi, \pi]$ is the sum of two delta functions at $\omega = \pm\frac{2\pi}{3}$. Thus:

$$X_d(\Omega T) = \pi\delta\left(\Omega T - \frac{2\pi}{3}\right) + \pi\delta\left(\Omega T + \frac{2\pi}{3}\right)$$
$$= \frac{\pi}{T}\delta\left(\Omega - \frac{2\pi}{3T}\right) + \frac{\pi}{T}\delta\left(\Omega + \frac{2\pi}{3T}\right)$$

For $T = \frac{1}{3}$:

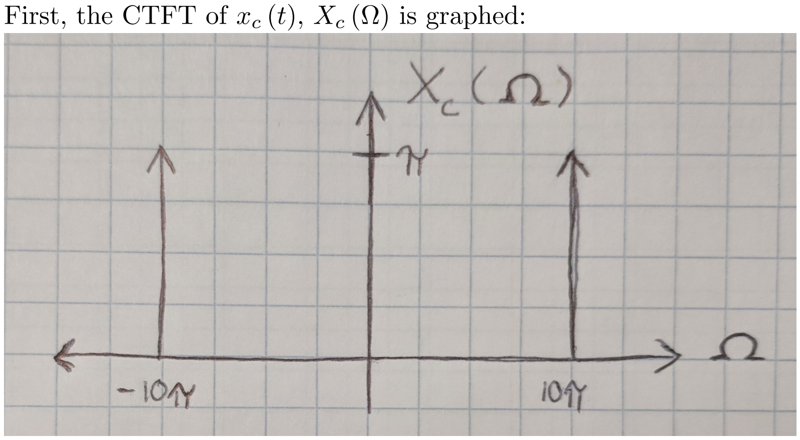
$$Y_c(\Omega) = G(\Omega)\, X_d(\Omega T)$$
$$= \pi\delta(\Omega - 2\pi) + \pi\delta(\Omega + 2\pi)$$

Taking this back to the time domain:

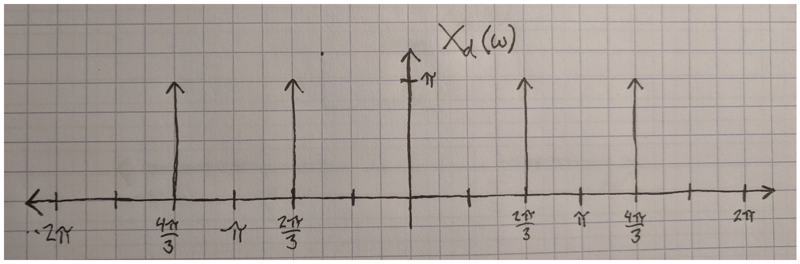
$$y_c(t) = \cos(2\pi t)$$

To confirm the results of either approach are correct, consider the following:
When fed with samples of a signal $x_c(t)$ that is bandlimited to under $\frac{\pi}{T}$ and sampled at interval T, the **ideal** D/A recovers $x_c(t)$ perfectly. So $y_c(t) = x_c(t)$. This leads to the expression $y_c(nT) = x_c(nT) = x[n]$. For this problem, we have $y_c(nT) = \cos\left(\frac{2\pi}{3}n\right)$ which is equal to the $x[n]$ determined before.

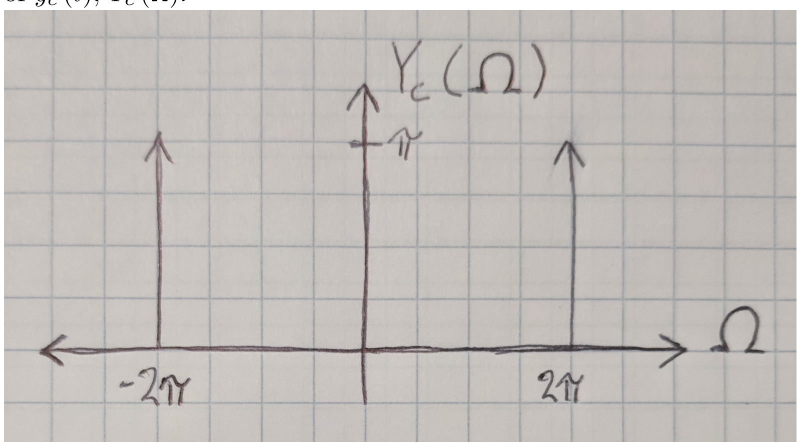
(b) First, the CTFT of $x_c(t)$, $X_c(\Omega)$ is graphed:



Next, the same process from question 1.a is used to graph the spectrum of $x[n]$, $X_d(\omega)$:



Finally, the function of an ideal D/A is used to graph the spectrum of $y_c\left(t\right)$, $Y_c\left(\Omega\right)$:



From the graphs, we can see that the effect of the aliasing on the cosine is to shift its frequency by steps of $/Omega_s = 6\pi$ until it is within the range $\Omega_0 \leq \left|\frac{\Omega_s}{2}\right| = 3\pi$.