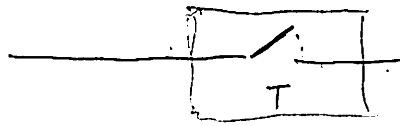
Lecture 13

Sampling of continuous - time (CT) signals

CT signal $x_c(t)$ $(t \in \mathbb{R})$

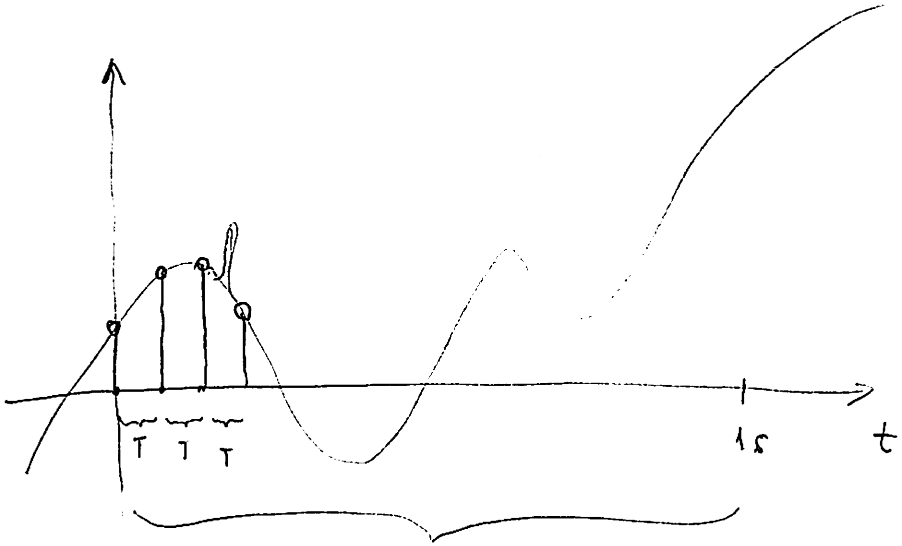


sampling period

discrete - time signal

$x[n] = x_c(nT)$

sampling frequency $F_s = \frac{1}{T}$ (samples / second)



F_s samples

Key question: Are the samples $\{x[0], x[1], x[2], \ldots\}$ sufficient to uniquely characterize $x_c(t)$? How large does F_s have to be?

Ex: $x_c(t) = \cos(\underbrace{2\pi f_0}_{\Omega_0} t)$ $\quad T,\ F_s = \frac{1}{T} \quad$ $x[n] = \cos(2\pi f_0 \cdot nT)$

$$= \cos\Big(\underbrace{2\pi \frac{f_0}{F_s}}_{\omega_0} n\Big)$$

Consider another signal:

$y_c(t) = \cos(2\pi(f_0 + kF_s))$ $\quad k \in \mathbb{Z} \quad$ F_s $\quad$ $y[n] = \cos(2\pi(f_0 + kF_s)nT)$

$$= \cos\Big(2\pi \frac{f_0}{F_s} n + 2\pi k n\Big)$$

$$= \cos\Big(2\pi \frac{f_0}{F_s} n\Big) = x[n]$$

signal @ freq. f_0

signal @ freq $f_0 + kF_s$

F_s $\quad$ same discrete-time signal!

CT: $\underbrace{f_0 + kF_s}_{-\frac{F_s}{2} \le \,\cdot\, \le \frac{F_s}{2}}$ $\xrightarrow[]{\text{sampled at } F_s}$ DT: $\overbrace{2\pi \frac{f_0}{F_s}}^{\omega_0}$ (angular frequency)

(fundamental frequency)

Ex: Video shot at 30 fps (frames per second) ($F_s = 30\,Hz$)

of an airplane propeller

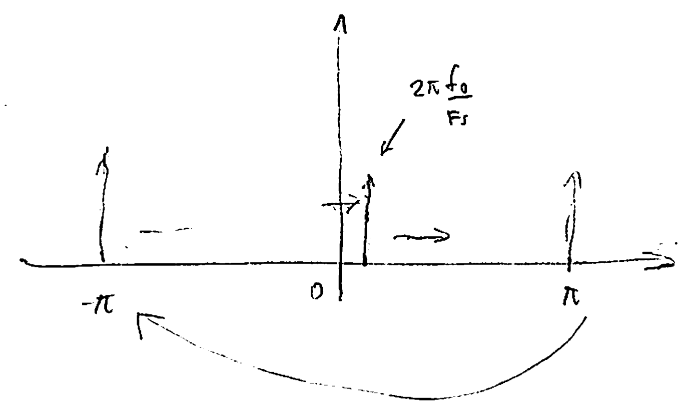
① $f_0 = 3\,Hz$: Fundamental freq : $3\,Hz$

② $f_0 = 27\,Hz$: Fund. freq : $-3\,Hz$

③ $f_0 = 43\,Hz$: Fund freq : $13\,Hz$

④ $f_0 = 327\,Hz$: Fund freq : $-3\,Hz$

propeller rotating backwards



In frequency domain:

Recall: $x_c(t) = \cos(\Omega_0 t) \xleftrightarrow{\text{CTFT}} \pi\left[\delta(\Omega-\Omega_0) + \delta(\Omega+\Omega_0)\right]$

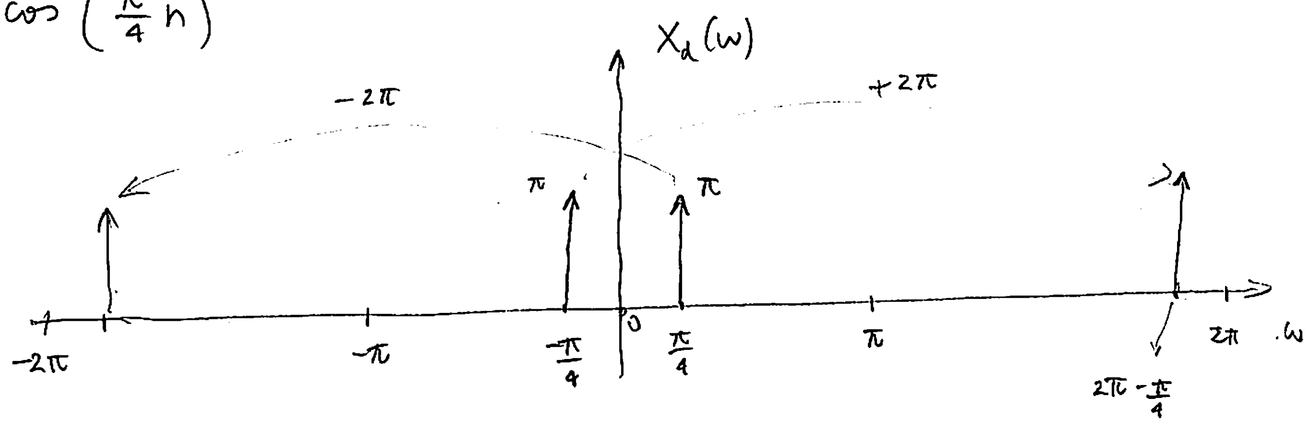
$$= \frac{e^{j\Omega_0 t} + e^{-j\Omega_0 t}}{2}$$

In DT: $x[n] = \cos(\omega_0 n)$

$$= \frac{e^{j\omega_0 n} + e^{-j\omega_0 n}}{2} \xleftrightarrow{\text{DTFT}} \underbrace{\pi\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]}_{\text{repeated at every interval of length } 2\pi}$$

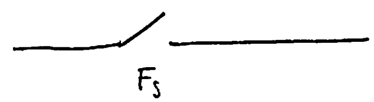
always 2π-periodic

Ex: $x[n] = \cos\left(\frac{\pi}{4} n\right)$



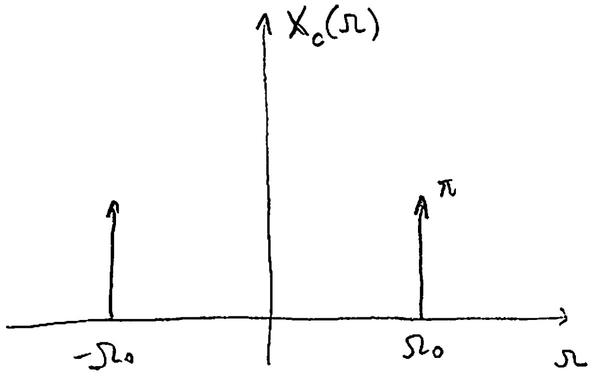
Sampling at $F_s = \frac{1}{T}$

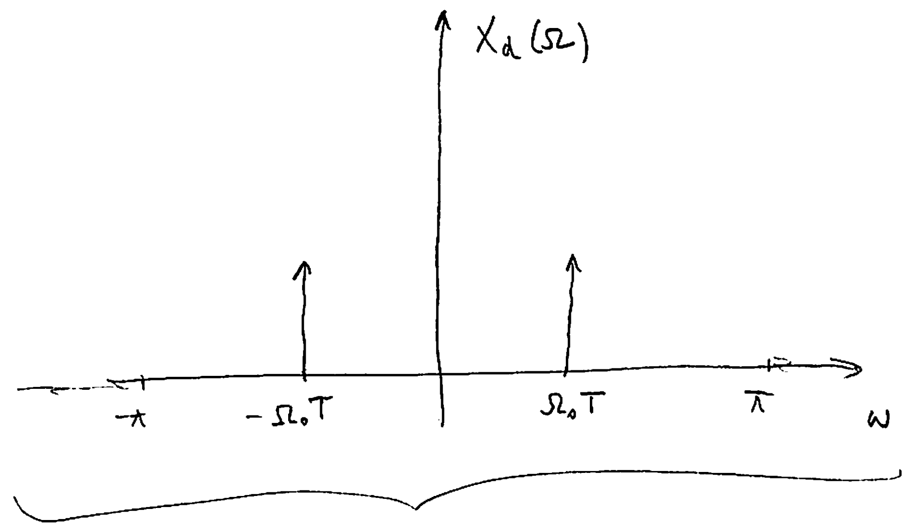
$x_c(t) = \cos(\Omega_0 t)$



$x[n] = \cos(\Omega_0 n T) = \cos(\underbrace{\Omega_0 T}^{w_0} \cdot n)$

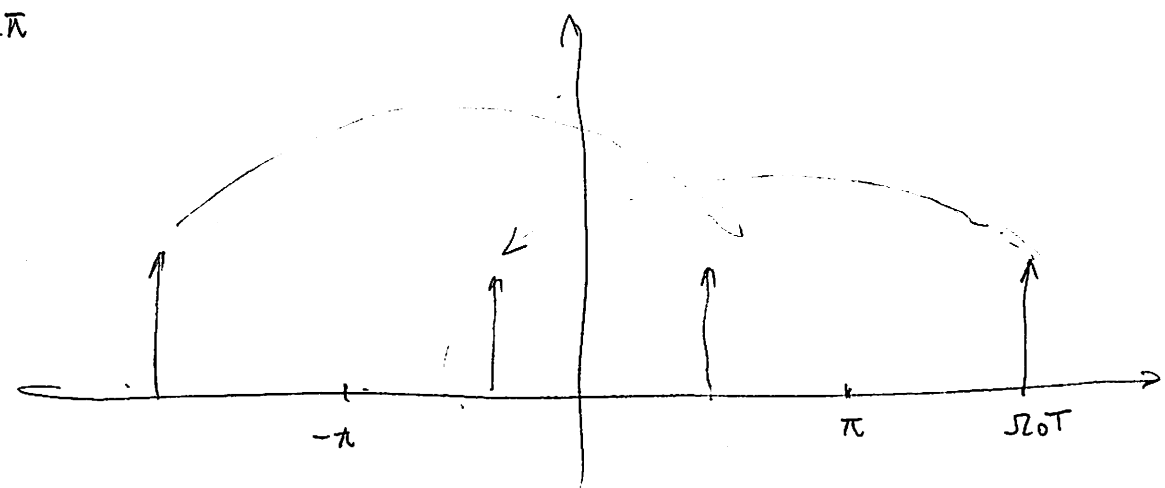
In freq. domain:





if $\Omega_0 T < \pi$

If $\pi < \Omega_0 t < 2\pi$

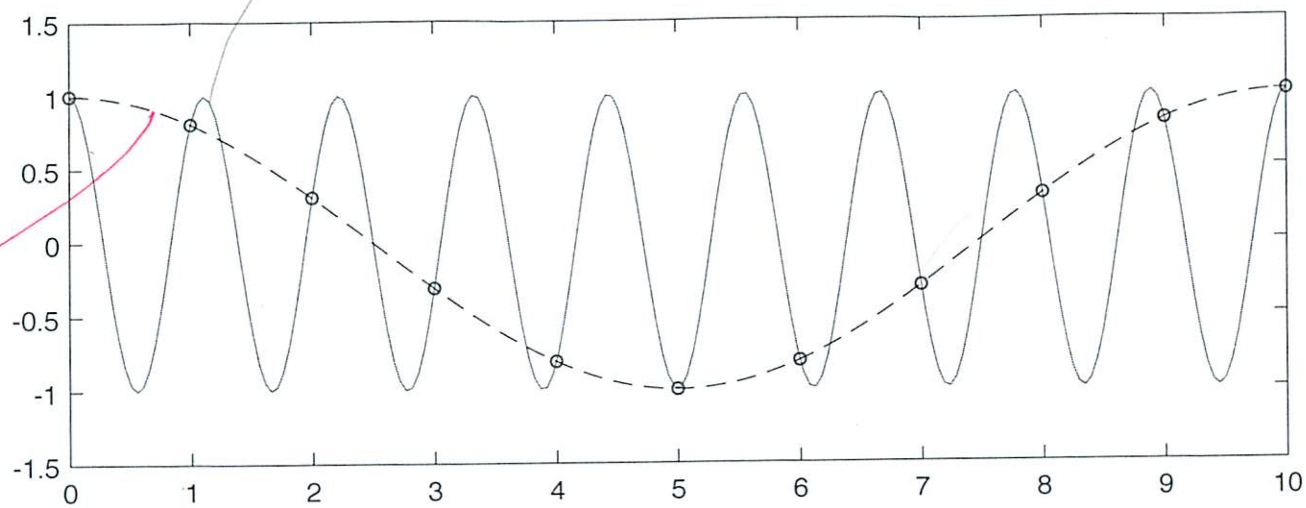


$f_0 = 0.9$

$x_c(t) = \cos(\underbrace{2\pi(0.9)}_{\Omega_0} t)$

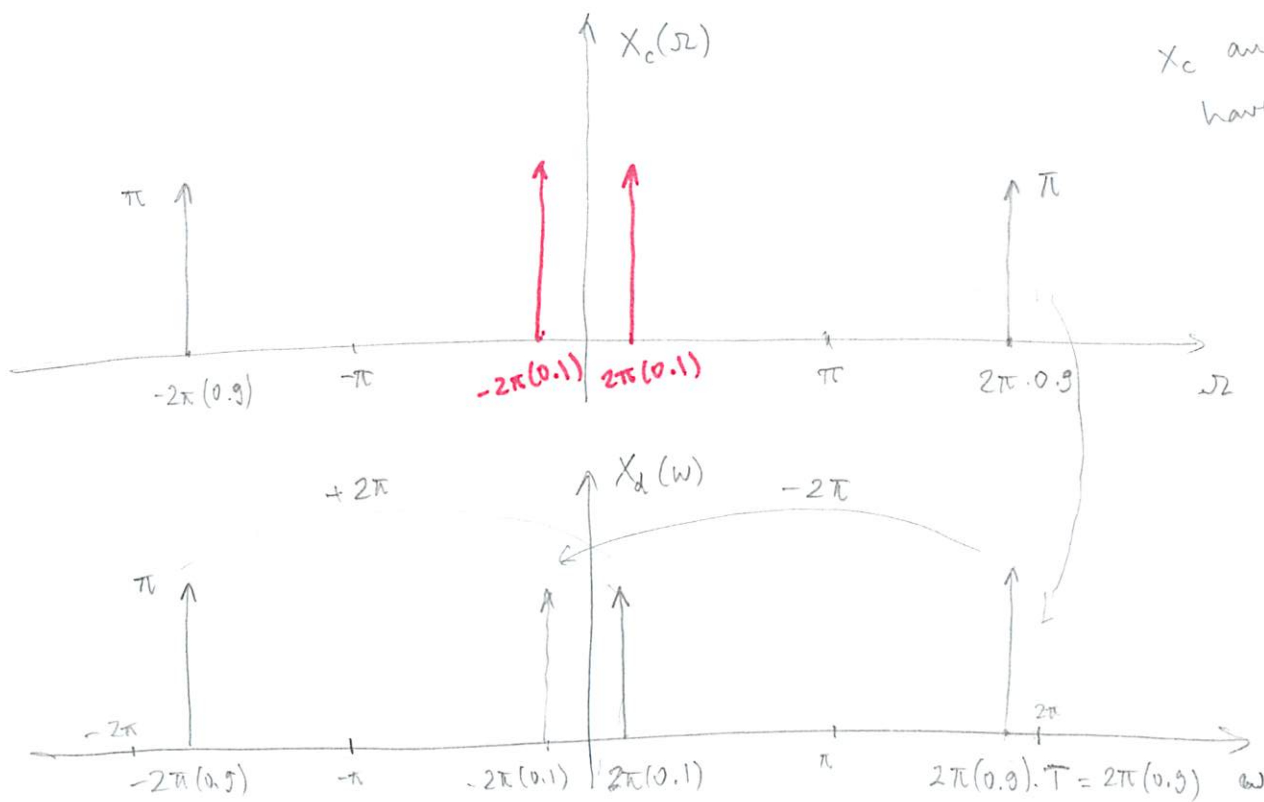
$F_s = 1$

$T = 1$



$y_c(t) = \cos(2\pi(0.1)t)$

$x[n] = \cos(2\pi(0.9)n)$



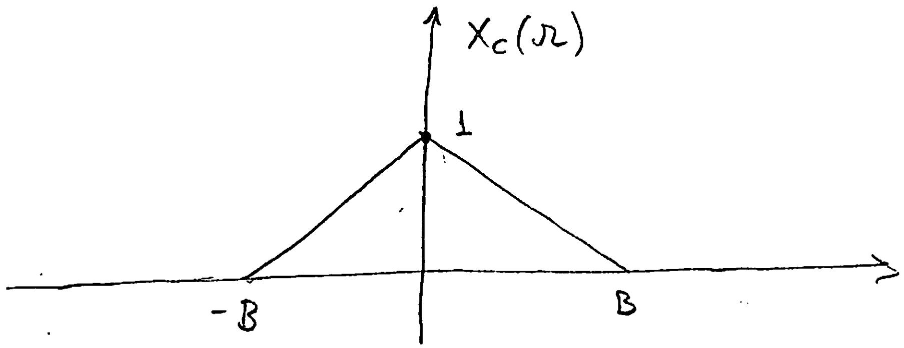
X_c and y_c have different CTFT

but the same DTFT

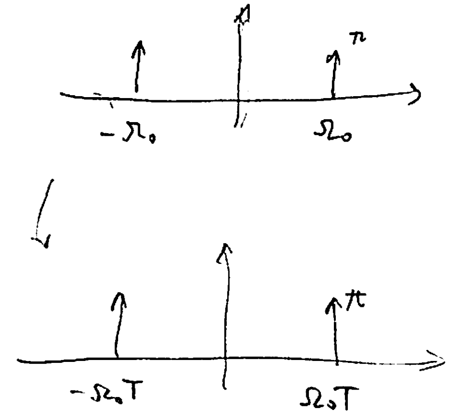
Connecting continuous and discrete

$$\begin{tikzcd}[column sep=huge, row sep=huge]
x_c(t) \arrow[r, bend left, "\text{sampling}"] \arrow[d, leftrightarrow, "\text{CTFT}"'] & x[n] = x_c(nT) \arrow[d, leftrightarrow, "\text{DTFT}"] \\
X_c(\Omega) \arrow[r, bend right, "?"'] & X_d(\omega)
\end{tikzcd}$$

Suppose $X_c(\Omega)$ is given below

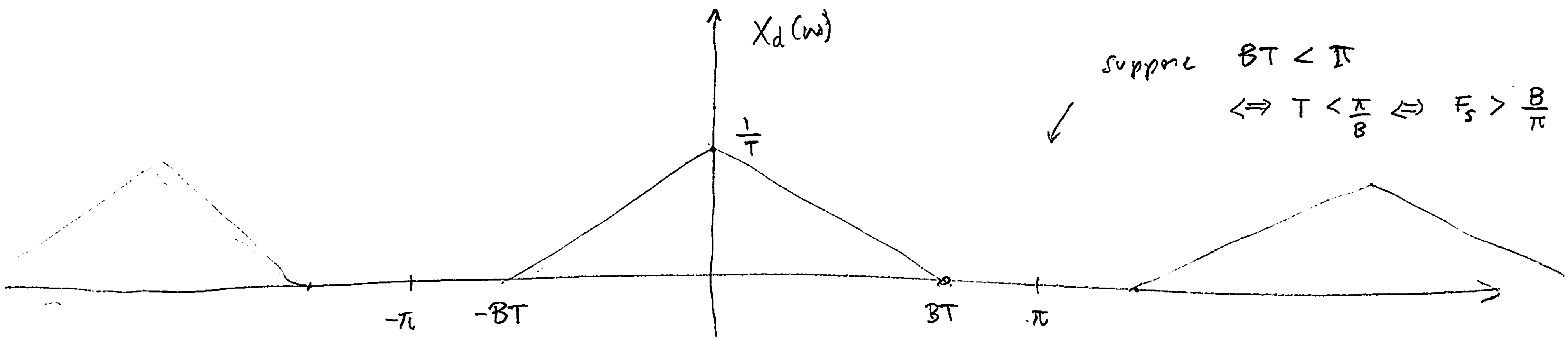


For $x(t) = \cos(\Omega_0 t)$



suppose $BT < \pi$

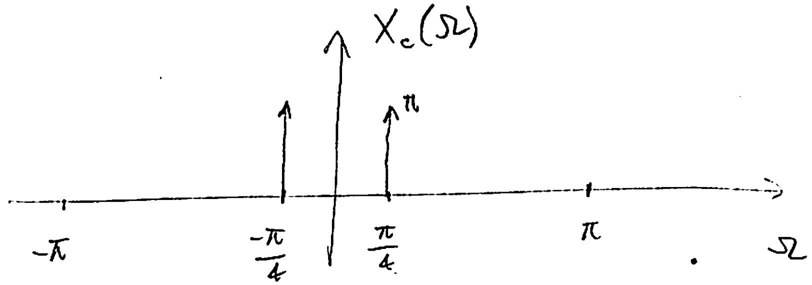
$\Leftrightarrow T < \frac{\pi}{B} \Leftrightarrow F_s > \frac{B}{\pi}$

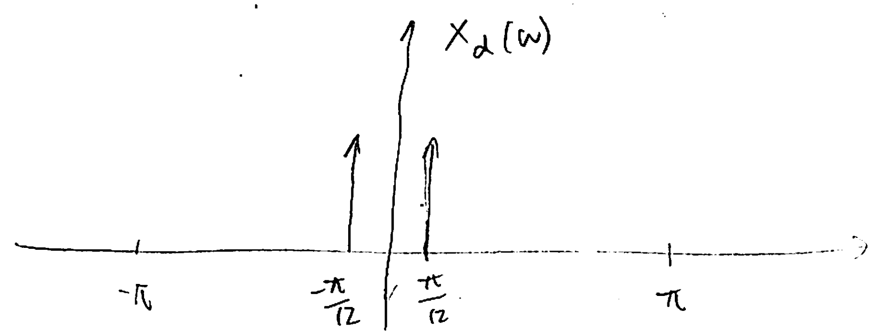


In general:

$$\boxed{X_d(\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(\frac{\omega+2k\pi}{T}\right)}$$

Ex: $x(t) = \cos\left(\frac{\pi}{4}t\right) \xrightarrow{\text{sampled at } F_s = 3,\ T = 1/3} \cos\left(\frac{\pi}{4}\cdot nT\right) = \cos\left(\frac{\pi}{12}n\right)$





$$X_c(\Omega) = \pi\left[\delta\left(\Omega - \frac{\pi}{4}\right) + \delta\left(\Omega + \frac{\pi}{4}\right)\right]$$

$$X_d(\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(\frac{\omega + 2k\pi}{T}\right) = \frac{1}{T}\sum_{k=-\infty}^{\infty} \pi\left[\delta\left(\frac{\omega+2k\pi}{T} - \frac{\pi}{4}\right) + \delta\left(\frac{\omega+2k\pi}{T} + \frac{\pi}{4}\right)\right]$$

$$= \frac{\pi}{T}\sum_{k=-\infty}^{\infty}\left[\delta\left(\frac{\omega + 2k\pi - T\frac{\pi}{4}}{T}\right) + \delta\left(\frac{\omega + 2k\pi + T\frac{\pi}{4}}{T}\right)\right]$$

Recall that $\delta(a\omega) = \frac{1}{a}\delta(\omega)$

$$= \pi\sum_{k=-\infty}^{\infty}\left[\delta\left(\omega + 2k\pi - \frac{\pi}{12}\right) + \delta\left(\omega + 2k\pi + \frac{\pi}{12}\right)\right]$$