Lecture 2:

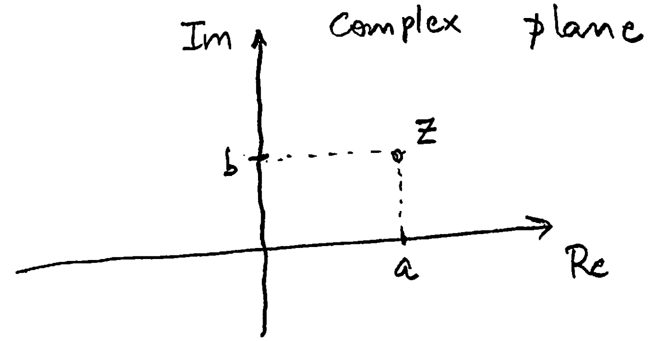
- Complex numbers (review)

  - $z = a + jb$, $a, b \in \mathbb{R}$, $j = \sqrt{-1}$

    $a = \mathrm{Re}\{z\}$, $b = \mathrm{Im}\{z\}$

    rectangular / cartesian form

    mosted suited for $+, -$



$z_1 = a_1 + jb_1$

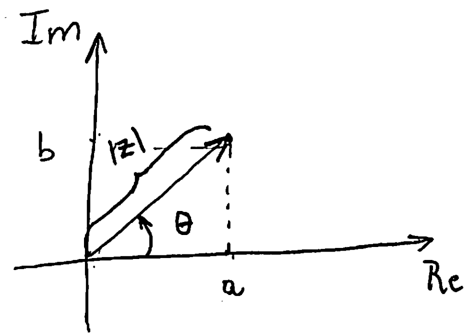
$z_2 = a_2 + jb_2$

$z_1 + z_2 = (a_1 + a_2) + j(b_1 + b_2)$

$\theta = \angle z$

$r = |z|$

Polar form



$z = |z| e^{j\theta}$

polar form is most suited for $*$, /

$z_1 = |z_1| e^{j\theta_1}$, $z_2 = |z_2| e^{j\theta_2}$

$z_1 z_2 = |z_1||z_2| e^{j\theta_1} e^{j\theta_2} = |z_1||z_2| e^{j(\theta_1 + \theta_2)}$

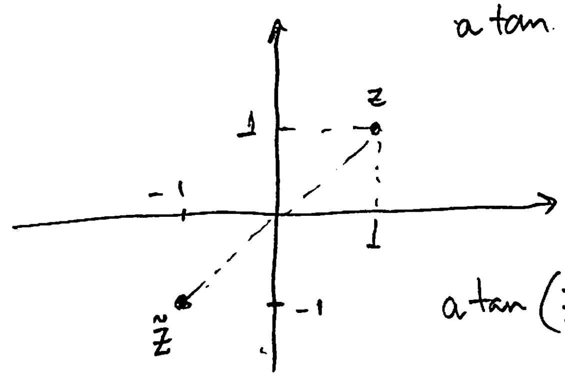
Euler's identity: $e^{j\theta} = \cos\theta + j\sin\theta$

$$a + jb = re^{j\theta} = r\cos\theta + j\,r\sin\theta$$

$a = r\cos\theta$, $b = r\sin\theta$ (to convert polar to rect)

$\operatorname{atan}(1/1) = \frac{\pi}{4}$

From rectangular to polar:



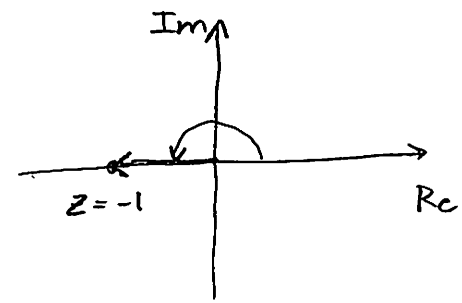
$$|z| = r = \sqrt{a^2 + b^2}$$

$$\theta = \operatorname{atan2}(b, a) \qquad e^{j\frac{5\pi}{4}} = e^{j\left(\frac{-3\pi}{4}\right)}$$

$\operatorname{atan}\left(\frac{-1}{-1}\right) = \frac{\pi}{4}$

Ex 1: Convert $z = -1 + 0j$ to polar form.

$|z| = 1$, $z = e^{j\pi}$



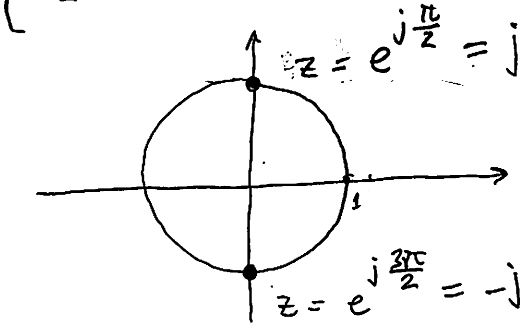
Ex 2: Solve $z^2 = -1$ (using polar form)

Let $z = re^{j\theta}$. $(re^{j\theta})^2 = e^{j(\pi + 2k\pi)}$

$$\Leftrightarrow r^2 e^{j(2\theta)} = e^{j(\pi + 2k\pi)} \Leftrightarrow \begin{cases} r^2 = 1 \\ 2\theta = \pi + 2k\pi \end{cases}$$

$$\Leftrightarrow \begin{cases} r = 1 \quad (r > 0) \\ \theta = \frac{\pi}{2} + k\pi, \quad k = 0, 1, 2, \ldots \end{cases}$$

$$\Leftrightarrow z = j \quad \text{or} \quad z = -j$$



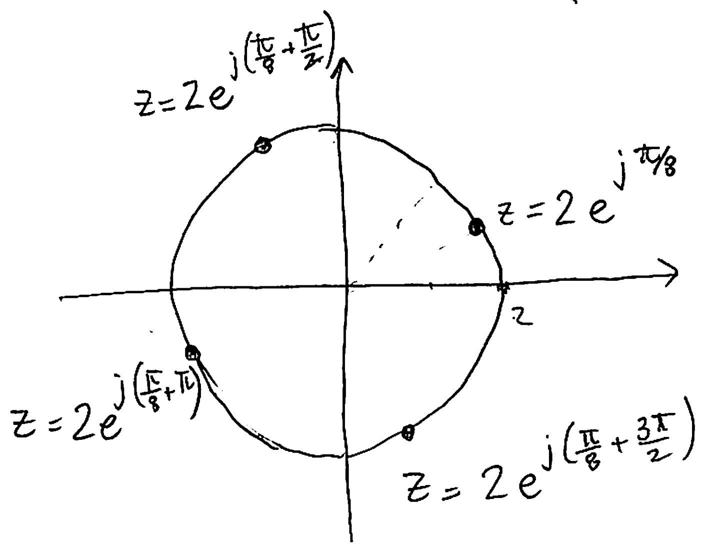
Ex 3: Solve $z^4 = 16j$

Let $z = re^{j\theta}$

$$(re^{j\theta})^4 = 16 \cdot e^{j\pi/2}$$

$$\iff r^4 \cdot e^{j4\theta} = 16 \cdot e^{j(\pi/2 + 2k\pi)}$$

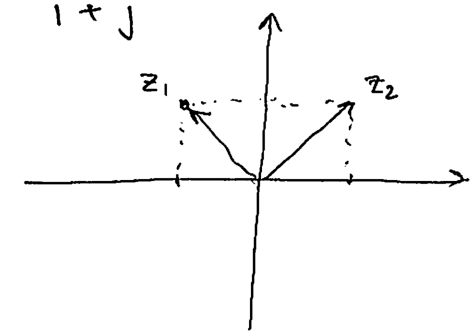
$$\iff \begin{cases} r^4 = 16 \\ 4\theta = \frac{\pi}{2} + 2k\pi \end{cases} \iff \begin{cases} r = 2 \\ \theta = \frac{\pi}{8} + k\frac{\pi}{2} \end{cases}$$



Ex 4. Simplify $z = \dfrac{(-1+j)^5}{1+j}$

$z_1 = -1+j = \sqrt{2} \cdot e^{j\frac{3\pi}{4}}$

$z_2 = 1+j = \sqrt{2} \cdot e^{j\frac{\pi}{4}}$

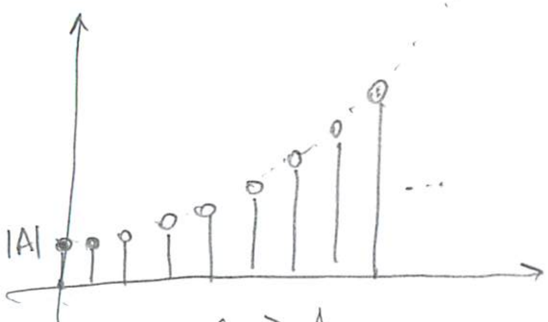


$$z = \frac{\left(\sqrt{2} \cdot e^{j\left(\frac{3\pi}{4}\right)}\right)^5}{\sqrt{2} \cdot e^{j\pi/4}} = 4 \cdot e^{j\left(5 \cdot \frac{3\pi}{4} - \frac{\pi}{4}\right)} = 4e^{j\frac{7\pi}{2}}$$

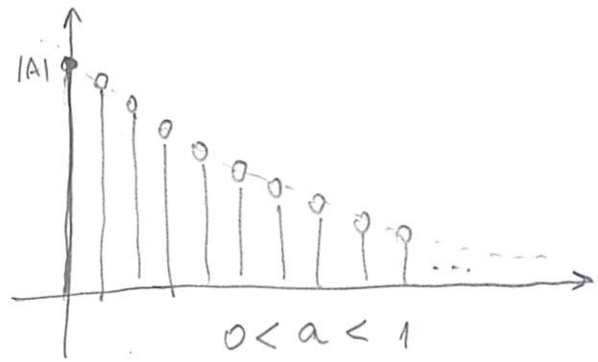
Back to signals and systems:

- Exponential sequence: $x[n] = |A|\, a^n \quad , \quad -\infty < n < \infty$

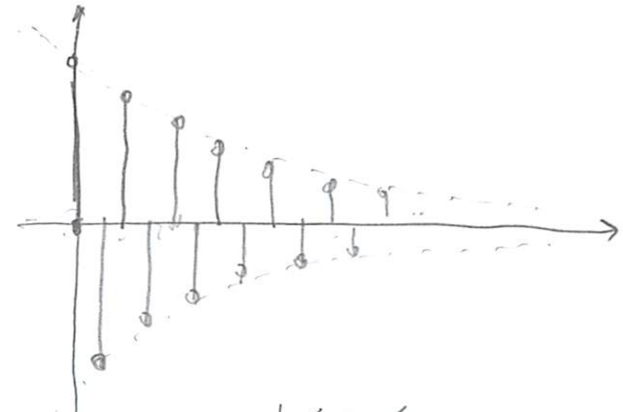
  - If a is real



$a > 1$



$0 < a < 1$



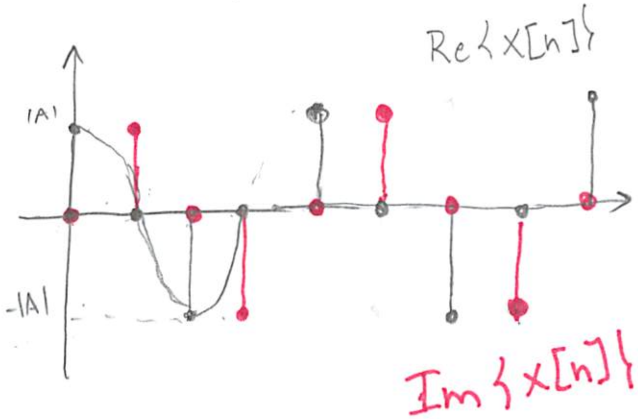
$-1 < a < 0$

  - $a = e^{j\omega_0}$ (complex sinusoid)

$$x[n] = |A|\, e^{j\omega_0 n} \underset{\text{Euler's Id}}{=} |A| \cos(\omega_0 n) + j\,|A| \sin(\omega_0 n)$$

$\omega_0 = \frac{\pi}{2}$

$Re\{x[n]\}$
$= |A| \cos(\omega_0 n)$



$Im\{x[n]\} = |A| \sin(\omega_0 n)$

Definition : A signal $x[n]$ is periodic with period $N \in \mathbb{Z}$ if $x[n+N] = x[n]$ for all $n \in \mathbb{Z}$

When is $x[n] = Ae^{j\omega_0 n}$ periodic ?

$$x[n+N] = x[n] \iff \cancel{A} \cdot e^{j\omega_0(n+N)} = \cancel{A}\, e^{j(\omega_0 n + 2k\pi)}$$

$$\iff \cancel{\omega_0 n} + \omega_0 N = \cancel{\omega_0 n} + 2k\pi$$

$$\iff N = \frac{2k\pi}{\omega_0}$$

$\frac{\pi}{\omega_0}$ is rational $\iff$ ω_0 is a rational multiple of π

Example : Complex sinusoid through a system

$$\{x[n]\} \rightarrow \boxed{S} \rightarrow \{y[n]\}$$

$$y[n] = x[n] + \frac{1}{2}x[n-1] + \frac{1}{4}x[n-2]$$

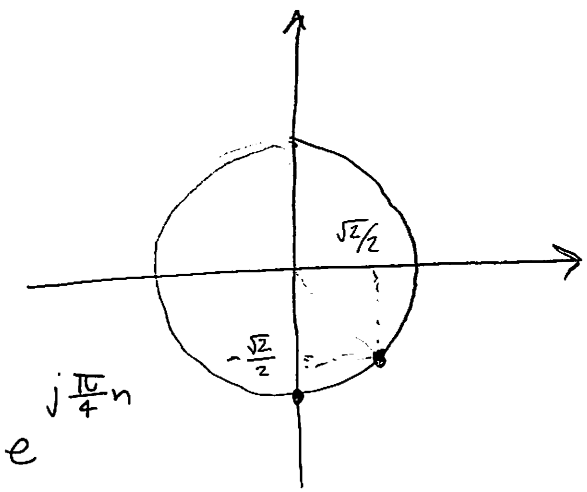
$$x[n] = e^{j\frac{\pi}{4}n}$$

$$y[n] = e^{j\frac{\pi}{4}n} + \frac{1}{2}e^{j\frac{\pi}{4}(n-1)} + \frac{1}{4}e^{j\frac{\pi}{4}(n-2)}$$

$$= e^{j\frac{\pi}{4}n} + \frac{1}{2}e^{j\frac{\pi}{4}n}e^{-j\frac{\pi}{4}} + \frac{1}{4}e^{j\frac{\pi}{4}n}e^{-j\frac{\pi}{2}}$$

$$= e^{j\frac{\pi}{4}n}\left(1 + \frac{1}{2}e^{-j\frac{\pi}{4}} + \frac{1}{4}e^{-j\frac{\pi}{2}}\right)$$

$$= e^{j\frac{\pi}{4}n}\underbrace{\left(1 + \frac{\sqrt{2} - j\sqrt{2}}{4} + \frac{1}{4}(-j)\right)}_{z_0} = z_0 \cdot e^{j\frac{\pi}{4}n}$$



$$\left|a\,e^{j\omega_0 n}\right| = a$$

Ex 2: $y[n] = \sum_{k=0}^{\infty} 2^{-k} x[n-k]$, $x[n] = e^{j\frac{\pi}{2}n}$

$$|2^{-1} e^{-j\frac{\pi}{2}}| = \frac{1}{2} < 1$$

$$y[n] = \sum_{k=0}^{\infty} 2^{-k} e^{j\frac{\pi}{2}(n-k)}$$

$$= \sum_{k=0}^{\infty} 2^{-k} e^{j\frac{\pi}{2}n} e^{-j\frac{\pi}{2}k} = e^{j\frac{\pi}{2}n} \sum_{k=0}^{\infty} \left(2^{-1} \cdot e^{-j\frac{\pi}{2}}\right)^k$$

$$= e^{j\frac{\pi}{2}n} \frac{1}{1-\frac{1}{2}e^{-j\frac{\pi}{2}}} = \left(\frac{2}{3} + \frac{j}{3}\right) e^{j\frac{\pi}{2}n}$$

Check

$$\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}$$

if $|a| < 1$

(a can be complex)