

From Circular Back to Linear Convolution

$$(f \circledast g)[n] = \sum_{m=0}^{N-1} f[m]g[\langle n-m \rangle_N]$$

Where $f[n]$ and $g[n]$ are N long. Linear convolution is different from circular convolution. However, if we zero-pad f and g to at least length $N+N-1$, we can use circular convolution to calculate linear convolution.

Example

Let $\{x_n\}_{n=0}^3 = \{1, 2, 3\}$ and $\{h_n\}_{n=0}^3 = \{1, 1, 0\}$. Compute $z[n]$, the linear convolution of $x[n]$ and $h[n]$ using circular convolution.

Solution

To compute the cyclic convolution, we flip on signal and shift it to the right, but we're now assuming that the flipped signal is **periodic**. So we can use a modified version of the table method, shifting the signal over and determining where the terms overlap:

$$\begin{array}{ll} & |1\ 2\ 3| \\ \text{No shift: } & \mathbf{0\ 1\ 1\ 0\ 1} \mid 1 \qquad \rightarrow z[0] = 4 \\ \text{Shift by 1: } & 1\ \mathbf{0\ 1\ 1\ 0} \mid 1 \qquad \rightarrow z[1] = 3 \\ \text{Shift by 2: } & 1\ 1\ \mathbf{0\ 1\ 1} \mid 0 \qquad \rightarrow z[2] = 5 \end{array}$$

(The bold part is the flipped h_n). Because we've reached the length of the original signal, we can stop. Therefore:

$$x_n \circledast h_n = \{4, 3, 5\}$$

Now if we zero-pad both signals to length $3 + 2 - 1 = 4$ (note that h_n is has a trailing zero) :

$$\{x_n\}_{n=0}^3 = \{1, 2, 3, 0\} \quad \text{and} \quad \{h_n\}_{n=0}^3 = \{1, 1, 0, 0\}$$

and evaluate this new circular convolution, we will see it matches the result of the linear convolution of $\{1, 2, 3\}$ and $\{1, 1, 0\}$.

The Fast Fourier Transform (FFT)

$$\text{DFT: } X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi kn}{N}}$$

- N complex multiplications and additions for each k value
- N k values in total.
- The DFT has a complexity of $O(N^2)$

FFT (radix-2): reduces the computation complexity to $O(N \log_2 N)$.

$$\begin{aligned} X[k] &= \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi kn}{N}} \\ &= \sum_{n_{\text{even}}} x[n]e^{-j\frac{2\pi kn}{N}} + \sum_{n_{\text{odd}}} x[n]e^{-j\frac{2\pi kn}{N}} \end{aligned}$$

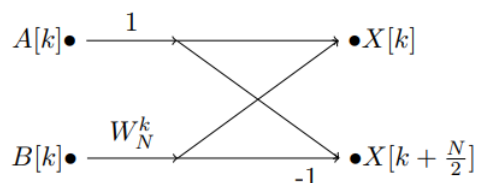
The even and odd parts can be rewritten and treated as $N/2$ point DFTs respectively and can be further broken down into its even and odd parts. This is why the length of $x[n]$ has to be 2^k (powers of 2).

The end result of the FFT is:

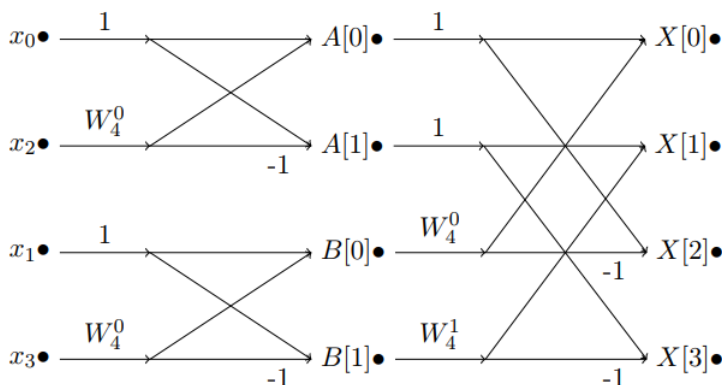
$$X[k] = A[k] + W_N^k B[k]$$

$$X[k + \frac{N}{2}] = A[k] - W_N^k B[k]$$

If interested, the derivation is on page 440 of your textbook.



Example of a 4-point DFT



Convolution via FFT

Convolution has a computation complexity of $O(N^2)$, but if we use FFT, we can improve it to $O(N \log N)$.

HINT: We know from the DFT properties, if x_n and h_n are of length N , then:

$$\text{DFT}\{x_n\} \cdot \text{DFT}\{h_n\} \leftrightarrow x_n \circledast h_n$$

steps:

If we are given x_n and h_n of different lengths N and M :

- We can zero-pad x_n and h_n to length $N + M - 1$. Now, the circular convolution will have the same result as a linear convolution.
- We can zero-pad x_n and h_n further to length $2^{\lceil \log_2(N+M-1) \rceil}$ to enable FFT on x_n and h_n
- We calculate linear convolution by: $\text{FFT}^{-1}\{\text{FFT}\{x_n\} \cdot \text{FFT}\{h_n\}\}$