ECE 310: Problem Set 5

Due: 5pm, Friday October 5, 2018

1. Evaluate the following expressions: (12pts, 3pts each)

   (a) $\int_{-\infty}^{\infty}(t^2+t-9)\delta(t+2)dt$

   $$\int_{-\infty}^{\infty}(t^2+t-9)\delta(t+2)dt = (-2)^2-2-9 = \boxed{-7}$$

   (b) $\int_{3}^{\infty}(t^2+t-9)\delta(t-2)dt$

   $$\int_{3}^{\infty}(t^2+t-9)\delta(t-2)dt = \boxed{0}$$

   Recall that $\delta(t-t_0)$ is non-zero only at $t=t_0$, and 2 is off the integration limits.

   (c) $\int_{-\infty}^{1}(t^2+t-9)\delta(3t-2)dt$

   $$\begin{aligned}\int_{-\infty}^{1}(t^2+t-9)\delta(3t-2)dt &= \int_{-\infty}^{1}(t^2+t-9)\frac{\delta(t-2/3)}{3}dt \\ &= \frac{1}{3}\left[\left(\frac{2}{3}\right)^2+\frac{2}{3}-9\right] \\ &= \boxed{-\frac{71}{27}}\end{aligned}$$

   (d) $[\cos(\omega t)u(t)]*\delta(3t-2)$, where $u(t)$ is a unit step function.

   $$\begin{aligned}x(t) &= [\cos(\omega t)u(t)]*\delta(3t-2) \\ &= [\cos(\omega t)u(t)]*\frac{1}{3}\delta(t-\frac{2}{3}) \\ &= \frac{1}{3}\int_{-\infty}^{\infty}[\cos(\omega\tau)u(\tau)]\cdot\delta((t-\tau)-\frac{2}{3})d\tau \\ &= \boxed{\frac{1}{3}\cos(\omega(t-\frac{2}{3}))u(t-\frac{2}{3})}\end{aligned}$$

   Note that this problem showcases the sifting property of the Dirac delta function. When evaluating the integral, for any given t, the value inside the integral is nonzero only when $\tau = t-\frac{2}{3}$.

2. Determine the Fourier transform of the following function: (16pts, 4pts each)

   (a) $\delta(3t-2)$

   $$X(\Omega) = \mathcal{F}(\delta(3t-2)) = \int_{-\infty}^{\infty}\delta(3t-2)e^{-j\Omega t}dt$$

$$= \frac{1}{3}\int_{-\infty}^{\infty} \delta(t - \frac{2}{3})e^{-j\Omega t}dt$$

$$= \boxed{\frac{1}{3}e^{-j\frac{2}{3}\Omega}}$$

(b) $\sin(2\Omega_0 t + \phi_0)$, where Ω_0 and ϕ_0 are known real numbers.

$$X(\Omega) = \mathcal{F}\left(\sin(2\Omega_0 t + \phi_0)\right) = \int_{-\infty}^{\infty} \sin(2\Omega_0 t + \phi_0)e^{-j\Omega t}dt$$

$$= \int_{-\infty}^{\infty} \frac{e^{j(2\Omega_0 t+\phi_0)} - e^{-j(2\Omega_0 t+\phi_0)}}{2j}e^{-j\Omega t}dt$$

$$= \int_{-\infty}^{\infty} \frac{1}{2j}e^{j\phi_0}e^{-j(\Omega-2\Omega_0)t}dt - \int_{-\infty}^{\infty} \frac{1}{2j}e^{-j\phi_0}e^{-j(2\Omega_0+\Omega)t}dt$$

$$= \frac{1}{2j}2\pi e^{j\phi_0}\delta(\Omega - 2\Omega_0) - \frac{1}{2j}2\pi e^{-j\phi_0}\delta(\Omega + 2\Omega_0)$$

$$= \boxed{\frac{\pi}{j}\left(e^{j\phi_0}\delta(\Omega - 2\Omega_0) - e^{-j\phi_0}\delta(\Omega + 2\Omega_0)\right)}$$

Note that we cannot actually compute the infinite integral, since the sin term oscillates but does not converge. Thus, after decomposing the sin term using Euler's formula, we guess that the Fourier transform of a complex exponential is a delta function, which we can confirm using the inverse Fourier transform as follows:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\Omega - \Omega_0)e^{j\Omega t}d\Omega$$

$$= \frac{1}{2\pi}e^{j\Omega_0 t}$$

Now that the Fourier transform of a complex exponential is known, we can solve the Fourier transform of any sinusoid by decomposing it into complex exponentials and then applying Fourier transform properties.

(c) $e^{-\frac{1}{2}|t|}$

$$X(\Omega) = \mathcal{F}\left(e^{-\frac{1}{2}|t|}\right) = \int_{-\infty}^{\infty} e^{-\frac{1}{2}|t|}e^{-j\Omega t}dt$$

$$= \int_{-\infty}^{0} e^{\frac{1}{2}t}e^{-j\Omega t}dt + \int_{0}^{\infty} e^{-\frac{1}{2}t}e^{-j\Omega t}dt$$

$$= \int_{-\infty}^{0} e^{(\frac{1}{2}-j\Omega)t}dt + \int_{0}^{\infty} e^{-(\frac{1}{2}+j\Omega)t}$$

$$= \left.\frac{e^{(\frac{1}{2}-j\Omega)t}}{\frac{1}{2}-j\Omega}\right|_{-\infty}^{0} + \left.\frac{e^{-(\frac{1}{2}+j\Omega)t}}{-(\frac{1}{2}+j\Omega)}\right|_{0}^{\infty}$$

$$= \frac{1}{\frac{1}{2}-j\Omega} + \frac{-1}{-(\frac{1}{2}+j\Omega)}$$

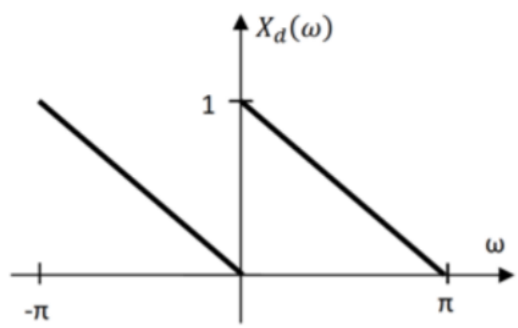
$$= \boxed{\frac{1}{\Omega^2 + \frac{1}{4}}}$$

(d) $(u(t-1) - u(t-3))e^{j2\pi t}$

$$\begin{aligned}
X(\Omega) &= \mathcal{F}\left((u(t-1) - u(t-3))e^{j2\pi t}\right) \\
&= \int_{-\infty}^{\infty} (u(t-1) - u(t-3))e^{j2\pi t} e^{-j\Omega t} dt \\
&= \int_{1}^{3} e^{j2\pi t} e^{-j\Omega t} dt \\
&= \frac{1}{j(2\pi - \Omega)} e^{j(2\pi - \Omega)t} \Big|_1^3 \\
&= \frac{1}{j(2\pi - \Omega)} e^{j2(2\pi - \Omega)} \left[e^{j(2\pi - \Omega)} - e^{-j(2\pi - \Omega)} \right] \\
&= \frac{1}{j(2\pi - \Omega)} e^{j2(2\pi - \Omega)} \cdot 2j \sin(2\pi - \Omega) \\
&= 2\frac{\sin(\Omega - 2\pi)}{\Omega - 2\pi} e^{j4\pi} e^{-j2\Omega} \\
&= \boxed{2\text{sinc}(\Omega - 2\pi)e^{-j2\Omega}}
\end{aligned}$$

Note that the input was a time-shifted, modulated rect function, and the Fourier transform is a frequency-shifted, modulated sinc function.

3. Let $x[n]$ be a signal with DTFT as shown in the following figure. Determine and sketch the DTFT of $y[n] = x[n]\cos(\pi n/3)$. (16pts)

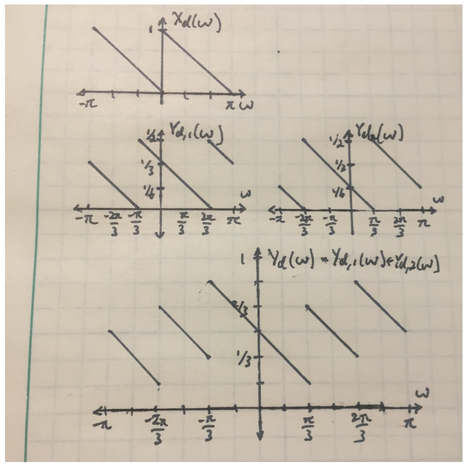


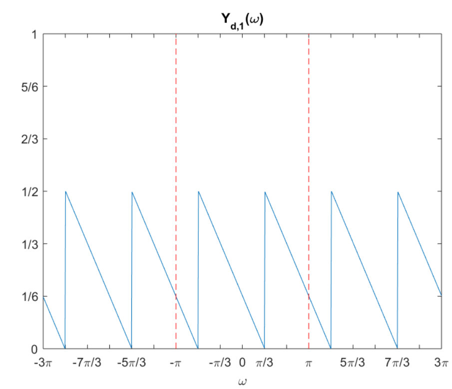
Use the <u>modulation property</u> of the DTFT which is described in (4.142) in the textbook:

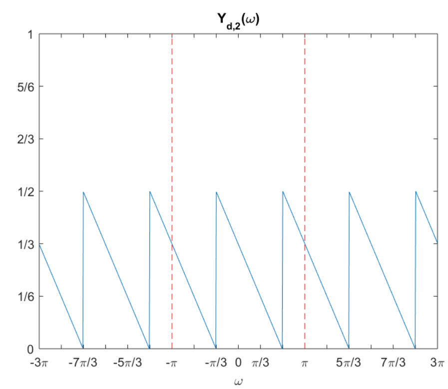
$$x[n]\cos(\omega_c n) \xleftrightarrow{\text{DTFT}} \frac{1}{2} X_d\left(e^{j(\omega + \omega_c)}\right) + \frac{1}{2} X_d\left(e^{j(\omega - \omega_c)}\right)$$

This tells us that the resulting DTFT will be the summation of two signals which are the original signal, scaled by a half, and shifted by $\pm\pi/3$. Remember that the DTFT is periodic with period 2π so it will 'wrap-around' at the boundaries, $-\pi$ and π.

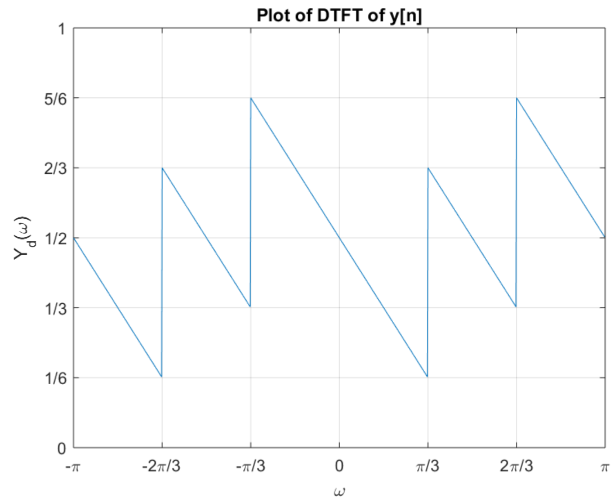
In the following images, we define $Y_{d,1}(\omega)$ as the left-shifted DTFT, $Y_{d,2}(\omega)$ as the right-shifted DTFT, and $Y_d(\omega) = Y_{d,1}(\omega) + Y_{d,2}(\omega)$. The MATLAB plots of $Y_{d,1}(\omega)$ and $Y_{d,2}(\omega)$ each







have three frequency periods plotted with dotted lines separating each period, showing the 'wrap-around' phenomenon.



4. Derive closed-form expressions for the DTFT of the following sequences. Sketch the magnitude and phase for parts (a) and (b). (20pts, 4pts each)

   (a) $x[n] = \delta[n+3] + \delta[n-3]$

$$\begin{aligned} X_d(\omega) &= \sum_{n=-\infty}^{\infty} (\delta[n+3] + \delta[n-3])e^{-j\omega n} \\ &= e^{-j3\omega} + e^{j3\omega} \\ &= \boxed{2\cos(3\omega)} \end{aligned}$$

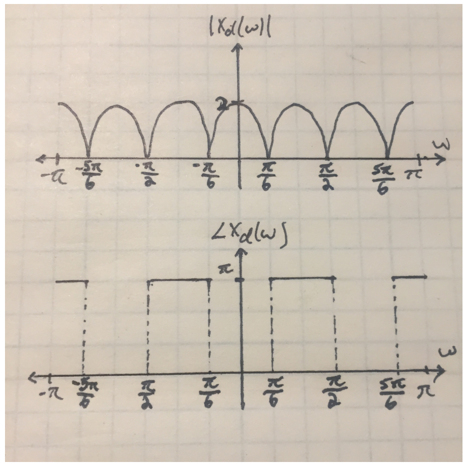
Magnitude:

$$\begin{aligned} |X_d(\omega)| &= |2\cos(3\omega)| \\ &= 2|\cos(3\omega)| \end{aligned}$$

Phase:

$$\begin{aligned} \angle X_d(\omega) &= \angle 2\cos(3\omega) \\ &= \arctan \frac{0}{2\cos(3\omega)} \\ &= \begin{cases} 0 & -\frac{\pi}{6} \leq \omega + \frac{2\pi}{3}N \leq \frac{\pi}{6} \ \forall \ N \in \mathbb{Z} \\ \pi & \text{else} \end{cases} \end{aligned}$$

In other words, $X_d(\omega)$ alternates between π and 0 at intervals of $\frac{\pi}{3}$ because of the sign change in the cos term.



(b) $x[n] = u[n] - u[n-7]$

$$
\begin{aligned}
X_d(\omega) &= \sum_{n=-\infty}^{\infty} (u[n] - u[n-7])e^{-j\omega n} \\
&= \sum_{n=0}^{6} e^{-j\omega n} \\
&= \frac{1 - e^{-j7\omega}}{1 - e^{-j\omega}} \\
&= \frac{e^{-j\frac{7}{2}\omega}(e^{j\frac{7}{2}\omega} - e^{-j\frac{7}{2}\omega})}{e^{-j\frac{1}{2}\omega}(e^{j\frac{1}{2}\omega} - e^{-j\frac{1}{2}\omega})} \\
&= \boxed{\begin{cases} e^{-j3\omega}\frac{\sin(\frac{7}{2}\omega)}{\sin(\frac{1}{2}\omega)} & , \ \omega \neq 0 \\ 7 & , \ \omega = 0 \end{cases}}
\end{aligned}
$$

The first function would be undefined at $\omega = 0$ so we must solve for it separately, which can be done in one of two ways. The first is by solving the summation separately at $\omega = 0$.

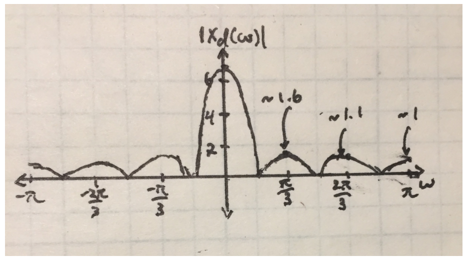
$$
\begin{aligned}
X_d(0) &= \sum_{n=-\infty}^{\infty} (u[n] - u[n-7]e^{-jn\cdot 0} \\
&= \sum_{n=0}^{6} 1 \\
&= 7
\end{aligned}
$$

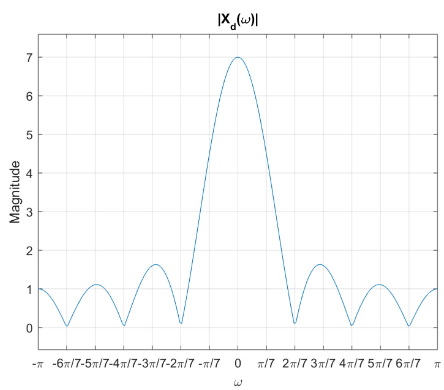
We can also verify this by using L'Hopital's rule at $\omega = 0$.

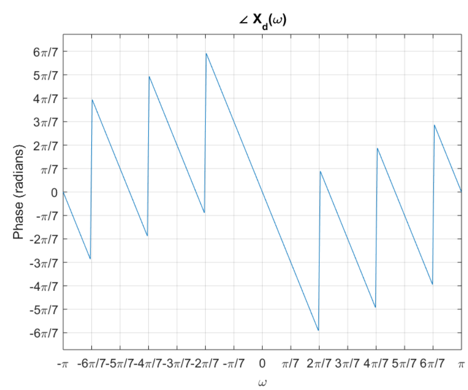
$$
\lim_{\omega \to 0} e^{-j3\omega}\frac{\sin(\frac{7}{2}\omega)}{\sin(\frac{1}{2}\omega)} = \lim_{\omega \to 0} e^{-j3\omega}\frac{-\frac{7}{2}\cos(\frac{7}{2}\omega)}{-\frac{1}{2}\cos(\frac{1}{2}\omega)}
$$

$$= 7$$

Note that this DTFT is a phase-shifted Dirichlet sinc function, which is the Fourier transform of a time-shifted rect function input. The Dirichlet sinc differs from a normal sinc in that it is periodic and takes the form $D_n(x) = \sin((n + \frac{1}{2})x)/\sin(\frac{1}{2}x)$.







(c) $x[n] = \left(\frac{1}{4}\right)^n u[n-3]$

$$\begin{aligned} X_d(\omega) &= \sum_{n=-\infty}^{\infty} \left(\frac{1}{4}\right)^n u[n-3] e^{-j\omega n} \\ &= \sum_{n=3}^{\infty} \left(\frac{1}{4}\right)^n e^{-j\omega n} \\ &= \left(\frac{1}{4}\right)^3 e^{-j3\omega} \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n e^{-j\omega n} \\ &= \boxed{\frac{e^{-j3\omega}}{64} \frac{1}{1 - \frac{1}{4}e^{-j\omega}}} \end{aligned}$$

Note that the input is a time-shifted power series.

(d) $x[n] = \left(\frac{1}{4}\right)^n e^{j\pi n/3} u[n-5]$

$$\begin{aligned}
X_d(\omega) &= \sum_{n=-\infty}^{\infty} \left(\frac{1}{4}\right)^n e^{j\pi n/3} u[n-5] e^{-j\omega n} \\
&= \sum_{n=5}^{\infty} \left(\frac{1}{4}\right)^n e^{j(\frac{\pi}{3}-\omega)n} \\
&= \boxed{\left(\frac{1}{4}\right)^5 \frac{e^{-j5(\omega-\pi/3)}}{1-\frac{1}{4}e^{-j(\omega-\pi/3)}}}
\end{aligned}$$

Note that this DTFT takes the same form as the z-transform $a^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} 1/(1-az^{-1})$ with the substitute $z = e^{jw}$, but that here we have a frequency shift of $\pi/3$ from the modulated input.

(e) $x[n] = n\left(\frac{1}{4}\right)^n u[n-3]$

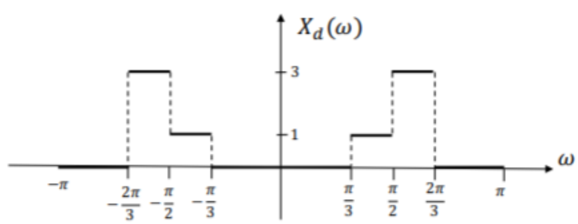
$$\begin{aligned}
X_d(\omega) &= \sum_{n=-\infty}^{\infty} n \left(\frac{1}{4}\right)^n u[n-3] e^{-j\omega n} \\
&= \sum_{n=3}^{\infty} n \left(\frac{1}{4}e^{-j\omega}\right)^n \\
&= \sum_{n=3}^{\infty} nx^n \text{ , letting } x = \frac{1}{4}e^{-j\omega} \text{ and } \frac{dx}{d\omega} = \frac{-j}{4}e^{-j\omega} \\
&= x^3 \sum_{n=0}^{\infty} (n+3)x^n \\
&= x^3 \sum_{n=0}^{\infty} nx^n + 3x^3 \sum_{n=0}^{\infty} x^n \\
&= x^3 \sum_{n=0}^{\infty} x\frac{d}{dx}x^n + 3x^3 \sum_{n=0}^{\infty} x^n \\
&= x^4 \frac{d}{dx} \sum_{n=0}^{\infty} x^n + 3x^3 \sum_{n=0}^{\infty} x^n \\
&= x^4 \frac{d}{dx}\frac{1}{1-x} + 3x^3 \frac{1}{1-x} \\
&= x^4 \frac{1}{(1-x)^2} + 3x^3 \frac{1}{1-x} \\
&= x^3 \frac{-2x+3}{(1-x)^2} \\
&= \boxed{\left(\frac{1}{4}e^{-j\omega}\right)^3 \frac{-\frac{1}{2}e^{-j\omega}+3}{\left(1-\frac{1}{4}e^{-j\omega}\right)^2}}
\end{aligned}$$

Alternative: Use the given z-transform pairs, $na^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} az^{-1}/(1-az^{-1})^2$ and $a^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} 1/(1-az^{-1})$, and then substitute $z = e^{jw}$.

$$\begin{aligned}
x[n] &= n\left(\frac{1}{4}\right)^n u[n-3] \\
&= (n-3)\left(\frac{1}{4}\right)^3\left(\frac{1}{4}\right)^{n-3} u[n-3] + 3\left(\frac{1}{4}\right)^3\left(\frac{1}{4}\right)^{n-3} u[n-3] \\
&\overset{\mathcal{Z}}{\longleftrightarrow} z^{-3}\left(\frac{1}{4}\right)^3 \frac{\frac{1}{4}z^{-1}}{\left(1-\frac{1}{4}z^{-1}\right)^2} + 3z^{-3}\left(\frac{1}{4}\right)^3 \frac{1}{1-\frac{1}{4}z^{-1}}, \quad \text{ROC: } |z| \geq \frac{1}{4} \\
&= \left(\frac{1}{4}e^{-j\omega}\right)^3 \left(\frac{\frac{1}{4}e^{-j\omega}}{\left(1-\frac{1}{4}e^{-j\omega}\right)^2} + \frac{3}{1-\frac{1}{4}e^{-j\omega}}\right) \text{ applying the substitution} \\
&= \boxed{\left(\frac{1}{4}e^{-j\omega}\right)^3 \frac{-\frac{1}{2}e^{-j\omega}+3}{\left(1-\frac{1}{4}e^{-j\omega}\right)^2}}
\end{aligned}$$

Note that the ROC for the z-transform contains the unit circle; thus, the DTFT exists.

5. The DTFT of $x[n]$ is as shown below. Determine $x[n]$. (20pts)



To solve, recognize that this DTFT is the sum of four distinct rect functions. In other words, we can write it as follows:

$$X_d(\omega) = \text{rect}(\frac{\omega - \pi/2}{\pi/3}) + \text{rect}(\frac{\omega + \pi/2}{\pi/3}) + 2\text{rect}(\frac{\omega - 7\pi/12}{\pi/6}) + 2\text{rect}(\frac{\omega - 7\pi/12}{\pi/6})$$

Where $\text{rect}(\frac{\omega-\omega_c}{\omega_0})$ is a rectangular pulse centered at ω_c and with a width of ω_0. We define it in terms of the unit step function as follows:

$$\text{rect}(\frac{\omega - \omega_c}{\omega_0}) = u\left(\omega - \omega_c + \frac{\omega_0}{2}\right) - u\left(\omega - \omega_c - \frac{\omega_0}{2}\right)$$

Make use of the Fourier transform pair

$$\frac{\omega_0}{2\pi}\text{sinc}(\frac{\omega_0}{2}n) \overset{\text{DTFT}}{\longleftrightarrow} \text{rect}(\frac{\omega}{\omega_0})$$

and recall the modulation property, which we saw in Problem 3,

$$x[n]\cos(\omega_c n) \overset{\text{DTFT}}{\longleftrightarrow} \frac{1}{2}X_d\left(e^{j(\omega+\omega_c)}\right) + \frac{1}{2}X_d\left(e^{j(\omega-\omega_c)}\right)$$

This helps us recognize that the first two rect functions result from a time-domain modulation with $\cos(\frac{\pi}{2}n)$ and the latter two rect functions result from a modulation with $\cos(\frac{7\pi}{12}n)$.

Finally we arrive at

$$\boxed{x[n] = \frac{1}{3}\text{sinc}(\frac{\pi}{6}n)\cos(\frac{\pi}{2}n) + \frac{1}{3}\text{sinc}(\frac{\pi}{12}n)\cos(\frac{7\pi}{12}n)}$$

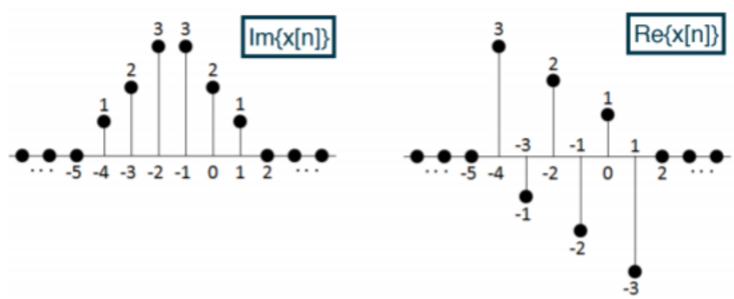
<u>Alternative</u>: Recognize that this DTFT is the sum of three rect functions, one positive and two negative. Then we can write it as follows, using the same definition of the rect function:

$$X_d(\omega) = 3\text{rect}(\frac{\omega}{4\pi/3}) - 2\text{rect}(\frac{\omega}{\pi}) - \text{rect}(\frac{\omega}{2\pi/3})$$

Now make the same rect to sinc transform as above, which gives

$$\boxed{x[n] = 2\text{sinc}(\frac{2\pi}{3}n) - \text{sinc}(\frac{\pi}{2}n) - \frac{1}{3}\text{sinc}(\frac{\pi}{3}n)}$$

6. Let $X_d(\omega)$ denote the DTFT of the complex valued signal $x[n]$, where the real and imaginary parts of $x[n]$ are given below. Perform the following calculations **without** explicitly evaluating $X_d(\omega)$. (16pts, 4pts each)



(a) Evaluate $X_d(0)$

$$\begin{aligned} X_d(0) &= \sum_{n=-\infty}^{\infty} x[n]e^{jn\cdot 0} \\ &= \sum_{n=-\infty}^{\infty} x[n] \\ &= j(1+2+3+3+2+1) + 3 - 1 + 2 - 2 + 1 - 3 \\ &= \boxed{j12} \end{aligned}$$

(b) Evaluate $X_d(\pi)$

$$\begin{aligned}
X_d(\pi) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\pi n} \\
&= \sum_{n=-\infty}^{\infty} x[n](-1)^n \\
&= j(1-2+3-3+2-1)+3+1+2+2+1+3 \\
&= \boxed{12}
\end{aligned}$$

(c) Evaluate $\int_{-\pi}^{\pi} X_d(\omega)d\omega$

$$\begin{aligned}
\int_{-\pi}^{\pi} X_d(\omega)d\omega &= 2\pi \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi} X_d(\omega)e^{j\omega 0}d\omega \\
&= 2\pi \cdot x[0] \\
&= \boxed{2\pi + j4\pi}
\end{aligned}$$

(d) Determine and sketch the signal whose DTFT is $X_d^*(-\omega)$

$$\begin{aligned}
X_d^*(-\omega) \leftrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} X_d^*(-\omega)e^{j\omega n}d\omega \\
&= \frac{1}{2\pi}\left(\int_{-\infty}^{\infty} X_d(-\omega)e^{-j\omega n}d\omega\right)^* \\
&= \frac{1}{2\pi}\left(\int_{-\infty}^{\infty} X_d(\omega')e^{j\omega' n}d\omega'\right)^* \\
&= x^*[n]
\end{aligned}$$