# FFT - FAST DFT

Thursday, October 25, 2018 8:59 PM

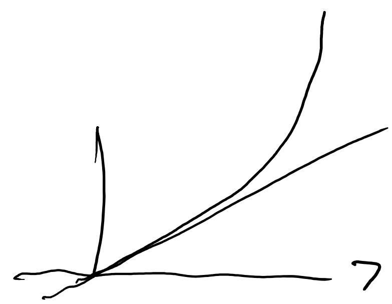
$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk}, \quad k = 0, 1, \cdots N-1$$

COST ? For $k=1$ $N \otimes$ $N-1 \oplus$

$N^2 \otimes$ $\sim N^2 \oplus$ $N^2$ MACs

<u>Example</u>

$x_c(t)$ = audio $\xrightarrow{F_s}$ $x[n]$ $\boxed{\text{DFT Spectral Analysis}}$ →




$F_s = 40$ kHz

1 sec - $N = 40 \cdot 10^3$ ⇒

10 sec - $N = 40 \cdot 10^4$ ⇒

Problem: QUADRATIC growth of cost with problem size!

Opportunity - Divide and conquer!

Break problem into 2 problems of half the size

Entire problem: $N^2$ MACs

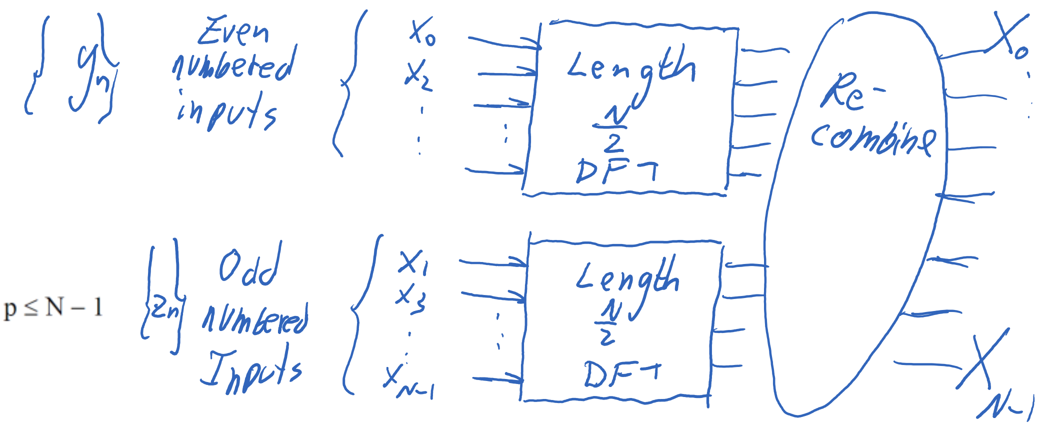
Half: $\left(\frac{N}{2}\right)^2$ " $\times 2 = \frac{N^2}{2}$ MACs

This works! - So keep doing it - break up the problem recursively.

# Radix 2 DIT (Decimation in time)

Friday, October 26, 2018 9:16 AM

$$\left.\begin{array}{l} y_n = x_{2n} \\ z_n = x_{2n+1} \end{array}\right\} \quad 0 \le n \le \frac{N}{2} - 1$$




DFT:

$$X_p = \sum_{n=0}^{N-1} x_n \bullet W_N^{np} \qquad 0 \le p \le N-1$$

$$W_N \triangleq e^{-j\frac{2\pi}{N}}$$

# Radix-2

$N = 2^\nu$ $\quad \nu$ - integer

$$X_p = \sum_{k=0}^{\frac{N}{2}-1} \left( x_{2k} W_N^{2kp} + x_{2k+1} W_N^{(2k+1)p} \right)$$

$$= \sum_{k=0}^{\frac{N}{2}-1} y_k W_{N/2}^{kp} + W_N^p \sum_{k=0}^{\frac{N}{2}-1} z_k W_{N/2}^{kp}$$

$$W_N^{2kp} = e^{-j\frac{2\pi}{N}2kp} = e^{-j\frac{2\pi}{N/2}kp} = W_{N/2}^{kp}$$

For $p = 0, 1, \ldots, \frac{N}{2} - 1$, $\Rightarrow$ $\left[ X_p = Y_p + W_N^p Z_p \quad 0 \le p \le \frac{N}{2} - 1 \right]$ (2)

# Divide and Conquer 2

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$$X_{p+\frac{N}{2}} = \sum_{k=0}^{\frac{N}{2}-1} y_k \, W_{N/2}^{k\left(p+\frac{N}{2}\right)} + W_N^{p+\frac{N}{2}} \sum_{k=0}^{\frac{N}{2}-1} z_k \, W_{N/2}^{k\left(p+\frac{N}{2}\right)}$$

$0 < p < \frac{N}{2} - 1$

$\frac{N}{2} < p + \frac{N}{2} < N - 1$

$$W_{N/2}^{k\left(p+\frac{N}{2}\right)} = W_{N/2}^{kp} \, W_{N/2}^{k\frac{N}{2}} = W_{N/2}^{kp} \bullet 1$$

$$W_N^{p+\frac{N}{2}} = W_N^p \, e^{-j\frac{2\pi}{N}\frac{N}{2}} = -W_N^p$$

$$\cancel{X}_{p+\frac{N}{2}} = \sum_{k=0}^{\frac{N}{2}-1} y_k \, W_{N/2}^{kp} - W_N^p \sum_{k=0}^{\frac{N}{2}-1} z_k \, W_{N/2}^{kp}$$

$$\Rightarrow \quad \left[ X_{p+\frac{N}{2}} = Y_p - W_N^p Z_p \qquad 0 \le p \le \frac{N}{2} - 1 \right] \qquad (3)$$

# The DIT Butterfly

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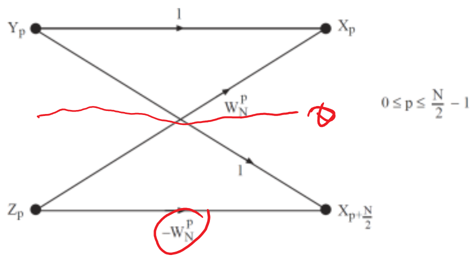
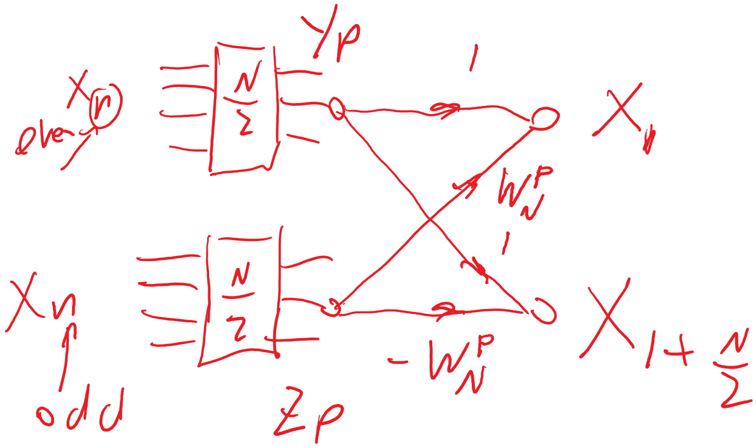



$$X_p = Y_p + W_N^p Z_p \qquad 0 \le p \le \frac{N}{2} - 1$$

$$X_{p+\frac{N}{2}} = Y_p - W_N^p Z_p$$




$2\otimes + 2\oplus$ per butterfly

# Divide and Conquer

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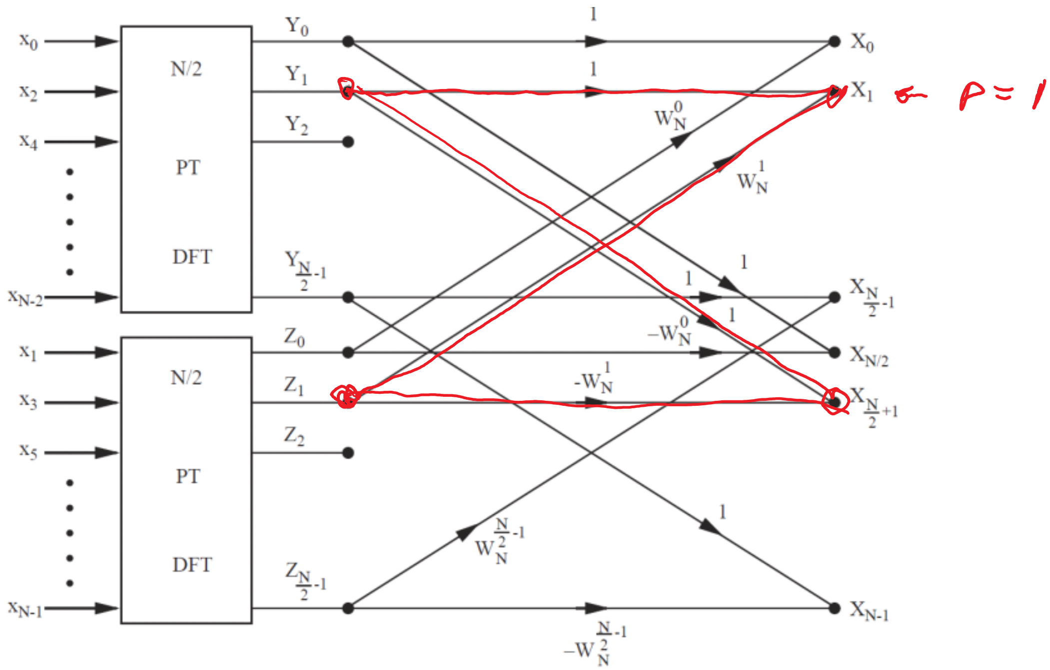


Work: $\frac{N}{2}$ butterf. × (2⊗ + 2⊕) =

N⊗ + N⊕

# 8-Point Radix2 FFT

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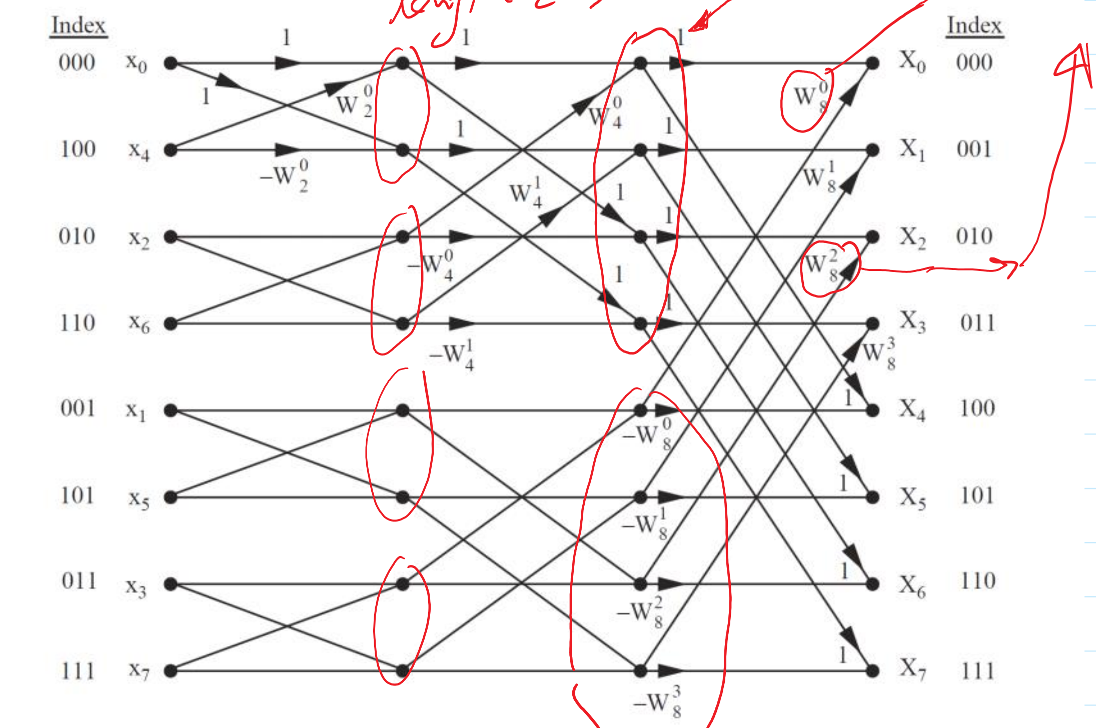
**Example** (N = 8, DIT FFT)

length 2 DFT

Length 4 DFT

Trivial ⊗




Length $\frac{N}{2} = 4$ DFT

$N \cdot \log_2 N$

$e^{-j\frac{2\pi}{8}\cdot 2}$

$= e^{-j\frac{\pi}{2}} = -j$

$-j(a+jb)$

$= -ja + b$

# Reordering the input in Radix2 DIT

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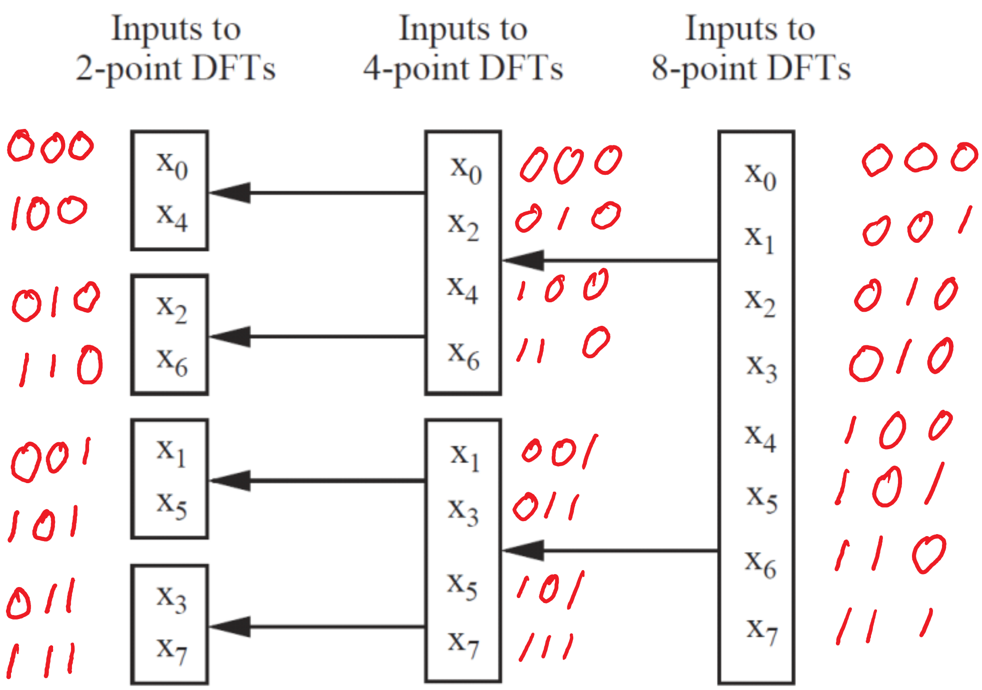
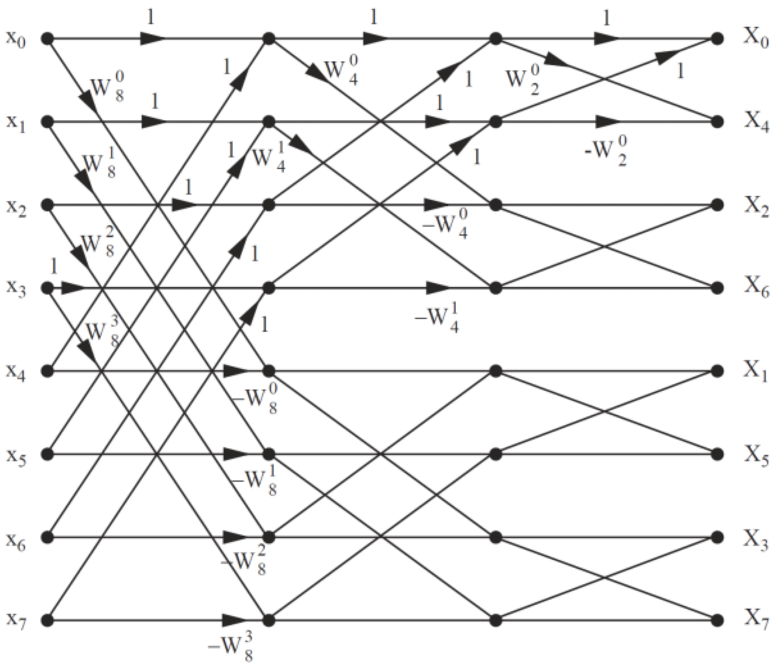



Bit Reversal

DIF - Decimation in Frequency

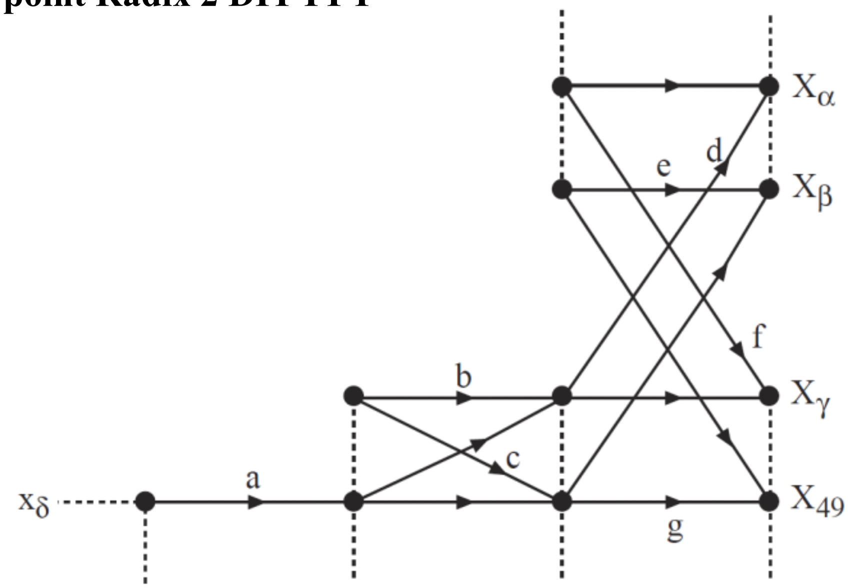
Friday, October 26, 2018 9:57 AM




Derived in text. But:
Can obtain DIF from DIT by $\boxed{\text{Flow Graph Transposition:}}$

1. Switch inputs & outputs
2. Reverse arrows
3. Replace summation nodes by distribution nodes and v.v.

## Example: 64 point Radix 2 DIT FFT




**Solution**: Use Eqs. (2) and (3) from p. 47.2 in course notes:

$$X_p = Y_p + W_N^p Z_p \qquad 0 \le p \le \frac{N}{2} - 1$$

$$X_{p+\frac{N}{2}} = Y_p - W_N^p Z_p \qquad 0 \le p \le \frac{N}{2} - 1$$

$N = 64,\ \beta + \dfrac{N}{2} = 49 \ \Rightarrow\ \beta = \underline{17}$

$\gamma = 49 - \dfrac{N}{4} = \underline{33}$

$\alpha = 33 - \dfrac{N}{2} = \underline{1}$

$\delta$ is bit reversal of $49 = (110001)_2 \Rightarrow \delta = (100011)_2 = \underline{35}$

$d = W_{64}^1 = e^{-j\frac{2\pi}{64}}$ $\qquad$ $g = -W_{64}^{17} = -e^{-j\frac{34\pi}{64}}$

$e = 1$ $\qquad$ $b = 1$

$f = 1$ $\qquad$ $c = 1$ $\qquad$ $a = 1$ $\qquad$ $\leftarrow$ since this is a top branch in butterfly of 16 pt DFT

# Computational Cost - Example

Friday, October 26, 2018 10:43 AM

$N = 2^{10}$

$\text{DFT} \sim N^2$ complex MACs $= 2^{20} \approx 10^6$

$\text{FFT} \sim N \log_2 N = 2^{10} \cdot 10 \sim 10^4$ - Factor of 100 improvement!


Acknowledgement: some material taken from course notes by D.C Munson, Jr.