

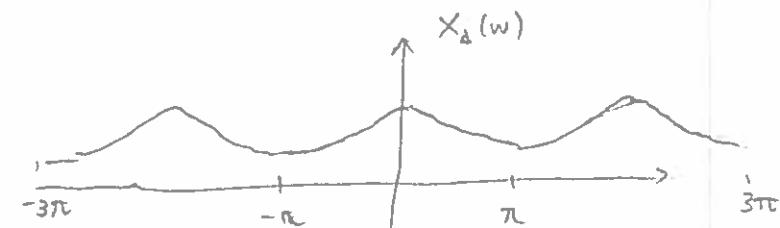
Lecture 11

DTFT

$$\{x[n]\} \xleftrightarrow{\text{DTFT}} X_d(w)$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_d(w) e^{jwn} dw$$

$$X_d(w) = \sum_{n=-\infty}^{\infty} x[n] e^{-jwn}$$



* $X_d(w)$ is 2π -periodic. So we

can write $x[n] = \frac{1}{2\pi} \int_{-\pi}^{2\pi} X_d(w) e^{jwn} dw$ for
any interval of length 2π

Ex: DTFT of $x[n] = 1$ for all n ?

Given: $1 \xleftrightarrow{\text{DTFT}} 2\pi, \delta(w) ?$
 Let $Y_d(w) = \delta(w)$. Then $y[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} Y_d(w) e^{jwn} dw = \frac{1}{2\pi} e^{jwn} = \frac{1}{2\pi} \cdot \frac{1}{2\pi} \xleftrightarrow{\text{DTFT}} \delta(w)$

\hookrightarrow not 2π -periodic! Not a valid DTFT.

Looks right!

Correct DTFT pair:

$$1 \xleftrightarrow{\text{DTFT}} 2\pi \sum_{k=-\infty}^{\infty} \delta(w - 2k\pi)$$

Connection between DTFT and the z-transform

$$X_d(w) = X(z) \Big|_{z=e^{jw}} \quad \text{if ROC of z-transform contains unit circle}$$

What if it doesn't? DTFT may still exist for some values of w if you have poles on the unit circle

$$\text{Ex: } x[n] = \cos\left(\frac{\pi}{2}n\right)u[n] = \frac{e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi}{2}n}}{2} u[n] = \frac{1}{2} e^{j\frac{\pi}{2}n} u[n] + \frac{1}{2} e^{-j\frac{\pi}{2}n} u[n]$$

$$\text{z-transform: } X(z) = \frac{1}{2} \frac{1}{1-jz^{-1}} + \frac{1}{2} \frac{1}{1+jz^{-1}}, \quad \text{ROC: } |z| > 1$$

To compute DTFT, we will use properties. $x[n] = \{1, 0, -1, 0, 1, \dots\}$

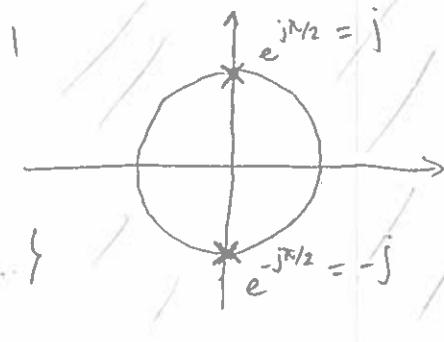
$$x[n] + x[n-2] = \delta[n]$$

DTFT ↓

$$X_d(w) + e^{-j2w} X_d(w) = 1$$

$$\Rightarrow X_d(w)(1 + e^{-j2w}) = 1$$

$$\Rightarrow X_d(w) = \frac{1}{1 + e^{-j2w}} \quad \text{if } e^{-j2w} \neq -1 \iff w \neq \frac{\pi}{2} + k\pi, \quad k=0, 1, 2, \dots$$



Back to LTI systems



time: impulse response $h[n]$

$$y[n] = x[n] * h[n]$$

z -domain: transfer function $H(z)$

$$Y(z) = X(z) H(z)$$

DTFT (frequency): frequency domain $H_d(\omega)$

$$Y_d(\omega) = X_d(\omega) H_d(\omega)$$

Response to a sinusoidal input:

$$x[n] = e^{j\omega_0 n} \rightarrow \boxed{S \begin{matrix} h, H \end{matrix}} \rightarrow y[n] = (x * h)[n]$$

$$= \sum_{k=-\infty}^{\infty} h[k] x[n-k] = \sum_{k=-\infty}^{\infty} h[k] e^{j\omega_0(n-k)}$$

$$= \underbrace{\sum_{k=-\infty}^{\infty} h[k]}_{x[n]} e^{j\omega_0 n} / e^{-j\omega_0 k}$$

$$= \underbrace{e^{j\omega_0 n}}_{\text{freq. response}} \underbrace{\sum_{k=-\infty}^{\infty} h[k]}_{H_d(\omega_0)} e^{-j\omega_0 k}$$

$$H_d(\omega_0)$$

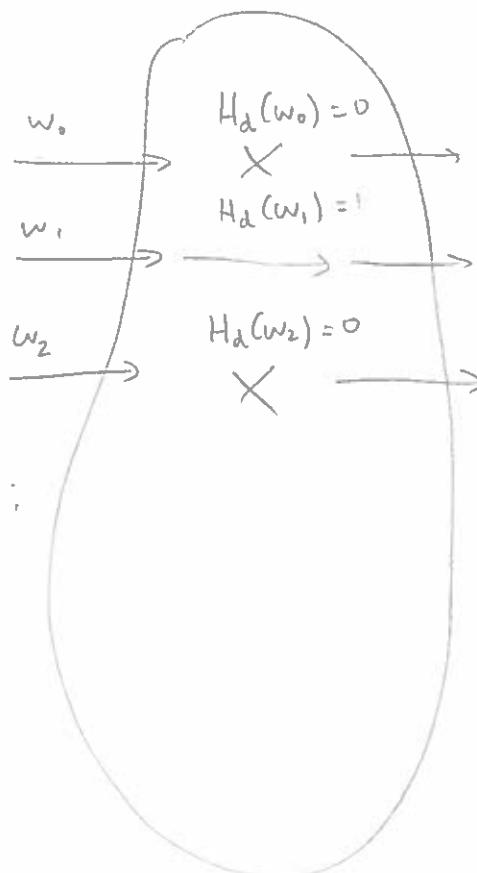


$x[n]$ is essentially a
"sum" of complex sinusoids

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_d(w) e^{jwn} dw$$

∴

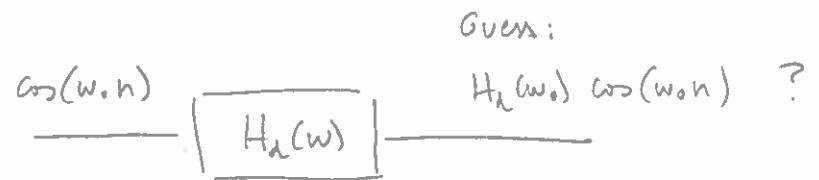
$$\sum_k \left(\frac{X_d(w_k)}{2\pi} \right) \cdot e^{jw_k n}$$



$$y[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_d(w) H_d(w) e^{jwn} dw$$

system is acting
as a filter!

What if input is a real sinusoid?



First, let's consider a signal $x[n] = a[n] + j b[n]$ for all n .

$$x[n] = a[n] + j b[n] \xrightarrow{\boxed{h}} y[n]$$

Suppose $h[n]$ is real-valued.

$$y[n] = (a[n] + j b[n]) * h[n] = \underbrace{(a[n] * h[n])}_{\text{real-valued}} + j \underbrace{(b[n] * h[n])}_{\text{real-valued}}$$

System processes real and imaginary parts independently

$$x[n] = (A e^{j\phi}) e^{j\omega_0 n} \xrightarrow{\boxed{h}} y[n] = (A e^{j\phi}) H_d(\omega_0) \cdot e^{j\omega_0 n}$$

$$= A e^{j(\omega_0 n + \phi)}$$

$$= A \underbrace{\cos(\omega_0 n + \phi)}_{\text{real part}} + j \underbrace{A \sin(\omega_0 n + \phi)}_{\text{imaginary part}}$$

$$= A e^{j\phi} |H_d(\omega_0)| e^{j\angle H_d(\omega_0)} e^{j\omega_0 n}$$

$$= A |H_d(\omega_0)| e^{j(\omega_0 n + \phi + \angle H_d(\omega_0))} \underbrace{\cos(\omega_0 n + \phi + \angle H_d(\omega_0))}_{\text{real part}}$$

$$= A |H_d(\omega_0)| \cos(\omega_0 n + \phi + \angle H_d(\omega_0))$$

$$+ j A |H_d(\omega_0)| \underbrace{\sin(\omega_0 n + \phi + \angle H_d(\omega_0))}_{\text{imaginary part}}$$

In conclusion:

$$A \cos(\omega_0 n + \phi) \xrightarrow{\boxed{h}} A \underbrace{|H_d(\omega_0)|}_{\text{magnitude response}} \cos(\omega_0 n + \phi + \underbrace{\angle H_d(\omega_0)}_{\text{phase response}})$$

$$\text{Ex. } x[n] = 1 + 2 \cos(\pi n) \rightarrow \boxed{h} \rightarrow y[n] = ?$$

$$h[n] = \left(\frac{1}{2}\right)^n u[n]$$

\Leftrightarrow

$$H(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}$$

\Leftrightarrow

$$H_a(\omega) = \frac{1}{1 - \frac{1}{2}e^{-j\omega}}$$

$$H_d(0) = \frac{1}{1 - \frac{1}{2}e^{-j0}} = \frac{1}{1 - \frac{1}{2}} = 2$$



$$1 = 1 \cdot \cos(0 \cdot n), \quad \text{so we need } H_d(0) \text{ and } H_d(\pi)$$

$$H_a(0) = 2, \quad H_a(\pi) = 2/3 \quad \left(|H_d(0)| = 2, \angle H_d(0) = 0, |H_d(\pi)| = \frac{2}{3}, \angle H_d(\pi) = 0 \right)$$

$$\Rightarrow y[n] = 1 \cdot |H_d(0)| + 2 \cdot \underbrace{|H_d(\pi)|}_{2/3} \cos(\pi n + 0) = 2 + \frac{4}{3} \cos(\pi n)$$