ECE 310: Quiz #10 (10am Section G) Fall 2018 Solutions

1. (5 pts) A music signal $x_c(t)$ is assumed to be bandlimited to 20 kHz. It is desired to filter this signal with a **highpass** filter that will pass the frequencies above 5 kHz by using a digital filter with frequency response $H_d(\omega)$ sandwiched between an ideal A/D and an ideal D/A.

   (a) Determine the Nyquist sampling rate for the input signal, and specify the frequency response $H_d(\omega)$ for the necessary discrete-time filter, when sampling at the Nyquist rate.

   Solution:
   The Nyquist rate, the minimum sampling rate needed to avoid aliasing, is twice the highest frequency in Hz, F_H, in a bandlimited signal.

   $$F_s = 2F_H = 40 \text{ kHz}$$

   Now that we know the sampling rate and the sampling period $T = 1/F_s$, we can use the relation $\omega = \Omega T$ to find the cutoff frequency, ω_c, in the digital domain.

   $$\omega_c = \Omega_c T = 2\pi F_c T = \frac{2\pi \cdot 5000}{40000} = \frac{\pi}{4} \text{ rads}$$

   For an ideal HPF we have the following frequency response:

   $$H_d(\omega) = \begin{cases} 1 & |\omega_c| > \frac{\pi}{4} \\ 0 & \text{otherwise} \end{cases}$$

   (b) Smart Alec claims that the system can perform the desired filtering function even when the sampling rate is lower than the Nyquist rate. Is this true? Justify your answer.

   Solution:
   False. The digital filter passes all of the highest frequencies, so using a lower sampling rate results in aliasing at the A/D, which contributes aliased frequency components to the digital filter, and therefore to the output of the system.

2. (5 pts) A system for processing analog signals $x_c(t)$ is composed of the following parts, connected in cascade: (i) an ideal analog LPF with cutoff frequency F_c; followed by (ii) a causal digital system whose input $x[n]$ and output $y[n]$ are related as $y[n] = 0.3y[n-1] + x[n]$, which is sandwiched between an ideal A/D and an ideal D/A operating at a sampling rate of 10 kHz. The output of the entire system is denoted by $y_c(t)$

   (a) What is the largest value of F_c for which the entire system will act as an analog LTI system, from input $x_c(t)$ to output $y_c(t)$? Justify your answer.

   Solution:
   Because the digital filter passes all frequency components in the sampled signals, we need to ensure that there is no aliasing that occurs from the sampling operation. Thus, we choose a cutoff frequency F_c such that the given sampling rate will satisfy the Nyquist criterion. When there is no aliasing, the system will be a valid analog LTI system.

   $$F_c = \frac{F_s}{2} = 5 \text{ kHz}$$

(b) For the F_c determined in (a) determine the analog frequency response $H_c(\Omega)$ of the entire system.

Solution:
The analog frequency response of the entire system, which we call $H_c(\Omega)$, will be composed of the analog LPF, $G_c(\Omega)$; and the analog frequency response of the system composed of the A/D, digital filter, and D/A, which will call $C_d(\Omega)$:

$$H_c(\Omega) = G_c(\Omega)C_c(\Omega),$$

where $C_c(\Omega) = C_d(\Omega T)$ for $|\Omega| \leq \pi/T$ and $C_c(\Omega) = 0$ otherwise. First, we find $C_d(\omega)$ using the difference equation.

$$\begin{aligned} y[n] - 0.3y[n-1] &= x[n] \\ (1 - 0.3e^{-j\omega})Y_d(\omega) &= X_d(\omega) \\ C_d(\omega) = \frac{Y_d(\omega)}{X_d(\omega)} &= \frac{1}{1 - 0.3e^{-j\omega}} \end{aligned}$$

Using the above relation gives us the following analog frequency response:

$$C_d(\Omega) = \begin{cases} \frac{1}{1-0.3e^{-j\Omega/10000}} & |\Omega| \leq 10000\pi \\ 0 & \text{otherwise} \end{cases}$$

Lastly, we apply the analog LPF, $G_c(\Omega)$, noting that $\Omega_c = 2\pi F_c = 10000\pi$ rad/sec.

$$H_d(\Omega) = G_c(\Omega)C_c(\Omega) = \begin{cases} \frac{1}{1-0.3e^{-j\Omega/10000}} & |\Omega| \leq 10000\pi \\ 0 & \text{otherwise} \end{cases}$$