ECE 310 Fall 2018 Homework 6 Solution

Due: Friday October 12, 2018

1.

(a) First, take the z-transform of the difference equation:

$$y[n] - \frac{1}{2}y[n-1] = 3x[n] \overset{Z}{\longleftrightarrow} Y(z) - \frac{1}{2}z^{-1}Y(z) = 3X(z)$$

Rearrange into the form $H(z) = \frac{Y(z)}{X(z)}$:

$$\begin{aligned} Y(z)\left(1 - \frac{1}{2}z^{-1}\right) &= 3X(z) \\ \frac{Y(z)}{X(z)} &= \frac{3}{1-\frac{1}{2}z^{-1}} \\ H(z) &= \frac{3}{1-\frac{1}{2}z^{-1}} \end{aligned}$$

Since the system is causal, the ROC will be outside any poles. The one pole here is at $z = \frac{1}{2}$ so the ROC is $|z| > \frac{1}{2}$.

(b) To find $h[n]$ take the inverse z-transform of $H(z)$. since the system is causal, use the relation: $a^n u[n] \leftrightarrow \frac{1}{1-az^{-1}}$, ROC: $|z| > |a|$:

$$H(z) = \frac{3}{1-\frac{1}{2}z^{-1}} \leftrightarrow h[n] = 3\left(\frac{1}{2}\right)^n u[n]$$

(c) Because the unit circle is in the ROC of $H(z)$, to find $H_d(\omega)$, set the magnitude of z in $H(z)$ to 1, replacing z with $e^{j\omega}$:

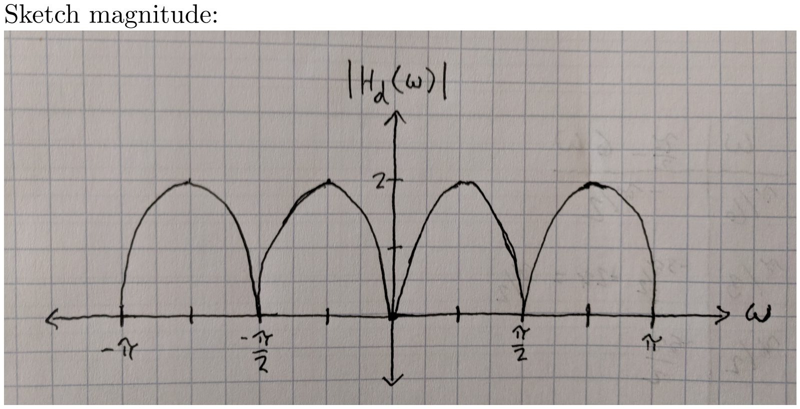
$$H_d(\omega) = \frac{3}{1-\frac{1}{2}e^{-j\omega}}$$

2.

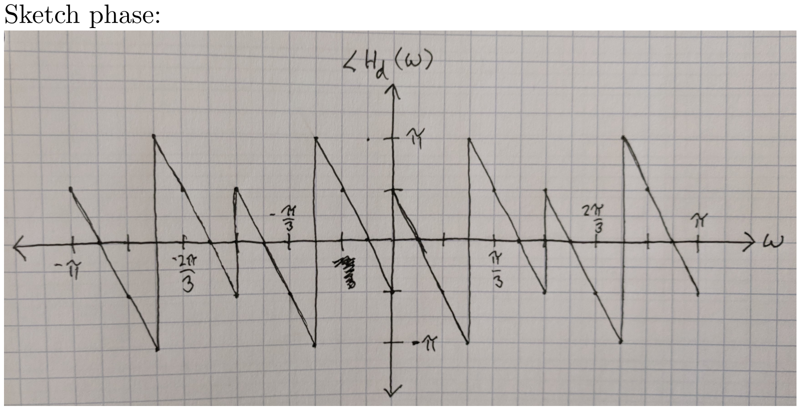
(a) Take the DTFT of both sides and apply the shifting property:

$$
\begin{aligned}
Y_d(\omega) &= X_d(\omega)\left(e^{-4j\omega} - e^{-8j\omega}\right) \\
H_d(\omega) &= \frac{Y_d(\omega)}{X_d(\omega)} = e^{-4j\omega} - e^{-8j\omega} \\
&= e^{-j6\omega}\left(e^{j2\omega} - e^{-j2\omega}\right) \\
&= e^{-j6\omega}\left(2j\sin(2\omega)\right) \\
&= 2\sin(2\omega)\, e^{j\left(\frac{\pi}{2}-6\omega\right)} \\
|H_d(\omega)| &= 2\,|\sin(2\omega)| \\
\angle H_d(\omega) &= \begin{cases} \frac{\pi}{2} - 6\omega & , 0 \le \omega \pm \pi k \le \frac{\pi}{2} \\ \frac{3\pi}{2} - 6\omega & , \text{else} \end{cases}
\end{aligned}
$$

Sketch magnitude:



Sketch phase:



(b)

i. Write $y[n]$ based on input $x[n]$ and frequency response $H_d(\omega)$, using the relationship for real systems:

$$
y[n] = 2\left|H_d\left(\frac{\pi}{5}\right)\right| \cos\left(\frac{\pi n}{5} + \angle H_d\left(\frac{\pi}{5}\right)\right) + \left|H_d\left(\frac{\pi}{4}\right)\right| \sin\left(\frac{\pi n}{4} + \frac{\pi}{10} + \angle H_d\left(\frac{\pi}{4}\right)\right)
$$

Calculate individual values of the DTFT:

$$\begin{aligned}
\left|H_d\left(\frac{\pi}{5}\right)\right| &= 2\sin\left(\frac{2\pi}{5}\right)\\
\angle H_d\left(\frac{\pi}{5}\right) &= \frac{\pi}{2} - 6\left(\frac{\pi}{5}\right) = -\frac{7\pi}{10}\\
\left|H_d\left(\frac{\pi}{4}\right)\right| &= 2\sin\left(2\left(\frac{\pi}{4}\right)\right) = 2\\
\angle H_d\left(\frac{\pi}{4}\right) &= \frac{\pi}{2} - 6\left(\frac{\pi}{4}\right) = -\pi
\end{aligned}$$

Plug into the equation for $y[n]$:

$$\begin{aligned}
y[n] &= 2\left(2\sin\left(\frac{2\pi}{5}\right)\right)\cos\left(\frac{\pi n}{5} + \left(-\frac{7\pi}{10}\right)\right) + (2)\sin\left(\frac{\pi n}{4} + \frac{\pi}{10} + (-\pi)\right)\\
&= 4\sin\left(\frac{2\pi}{5}\right)\cos\left(\frac{\pi n}{5} - \frac{7\pi}{10}\right) + 2\sin\left(\frac{\pi n}{4} - \frac{9\pi}{10}\right)
\end{aligned}$$

ii. Write $y[n]$ based on input $x[n]$ and frequency response $H_d(\omega)$:

$$\begin{aligned}
y[n] =& 2H_d(0) + 4\left|H_d\left(\frac{2\pi}{3}\right)\right|\cos\left(\frac{2\pi n}{3} + \frac{\pi}{4} + \angle H_d\left(\frac{2\pi}{3}\right)\right)\\
&+ 3\left|H_d\left(\frac{\pi}{3}\right)\right|\cos\left(\frac{\pi n}{3} + \frac{\pi}{5} + \angle H_d\left(\frac{\pi}{3}\right)\right)
\end{aligned}$$

Calculate individual values of DTFT:

$$\begin{aligned}
H_d(0) &= 2\sin(2(0))\,e^{j\left(\frac{\pi}{2} - 6(0)\right)} = 0\\
\left|H_d\left(\frac{\pi}{3}\right)\right| &= 2\left|\sin\left(2\left(\frac{\pi}{3}\right)\right)\right| = \sqrt{3}\\
\angle H_d\left(\frac{\pi}{3}\right) &= \frac{\pi}{2} - 6\left(\frac{\pi}{3}\right) = \frac{\pi}{2} - 2\pi = \frac{\pi}{2}\\
\left|H_d\left(\frac{2\pi}{3}\right)\right| &= 2\left|\sin\left(2\left(\frac{2\pi}{3}\right)\right)\right| = \sqrt{3}\\
\angle H_d\left(\frac{2\pi}{3}\right) &= \frac{3\pi}{2} - 6\left(\frac{2\pi}{3}\right) = \frac{3\pi}{2} - 4\pi = \frac{3\pi}{2}
\end{aligned}$$

Plug into the equation for $y[n]$:

$$\begin{aligned}
y[n] &= 2(0) + 4\left(\sqrt{3}\right)\cos\left(\frac{2\pi n}{3} + \frac{\pi}{4} + \left(\frac{3\pi}{2}\right)\right) + 3\left(\sqrt{3}\right)\cos\left(\frac{\pi n}{3} + \frac{\pi}{5} + \left(\frac{\pi}{2}\right)\right)\\
&= 4\sqrt{3}\cos\left(\frac{2\pi n}{3} + \frac{7\pi}{4}\right) + 3\sqrt{3}\cos\left(\frac{\pi n}{3} + \frac{7\pi}{10}\right)
\end{aligned}$$

3.

(a) Rewriting $x[n]$ as complex exponentials:

$$x[n] = 2 + 10e^{j\left(\frac{\pi}{8}n + 45^\circ\right)} - j^n = 2 + 10e^{j\frac{\pi}{8}n}e^{j\frac{\pi}{4}} - e^{j\frac{\pi}{2}n}$$

Using the eigensequence property:

$$y[n] = 2H_d(0) + 10e^{j\frac{\pi}{8}n}e^{j\frac{\pi}{4}}H_d\left(\frac{\pi}{8}\right) - e^{j\frac{\pi}{2}n}H_d\left(\frac{\pi}{2}\right)$$

Calculating the DTFT values:

$$\begin{aligned} H_d(0) &= (0)^2 e^{j\cos(0)} = 0 \\ H_d\left(\frac{\pi}{8}\right) &= \left(\frac{\pi}{8}\right)^2 e^{j\cos\left(\frac{\pi}{8}\right)} \\ H_d\left(\frac{\pi}{2}\right) &= \left(\frac{\pi}{2}\right)^2 e^{j\cos\left(\frac{\pi}{2}\right)} = \left(\frac{\pi}{2}\right)^2 e^0 = \left(\frac{\pi}{2}\right)^2 \end{aligned}$$

Calculating the output:

$$\begin{aligned} y[n] &= 2(0) + 10e^{j\frac{\pi}{8}n}e^{j\frac{\pi}{4}}\left(\left(\frac{\pi}{8}\right)^2 e^{j\cos\left(\frac{\pi}{8}\right)}\right) - e^{j\frac{\pi}{2}n}\left(\frac{\pi}{2}\right)^2 \\ &= \frac{10\pi^2}{64}e^{j\left(\frac{\pi}{8}n+\frac{\pi}{4}+\cos\left(\frac{\pi}{8}\right)\right)} - \frac{\pi^2}{4}e^{j\frac{\pi}{2}n} \end{aligned}$$

(b) Rewriting as complex exponentials:

$$\begin{aligned} x[n] &= 4 + 10\left(\frac{1}{2}\left(e^{j\frac{3\pi}{4}n+j\frac{\pi}{4}} + e^{-j\frac{3\pi}{4}n-j\frac{\pi}{4}}\right)\right) + 2e^{j\frac{\pi}{2}n} \\ &= 4 + 5\left(e^{j\frac{3\pi}{4}n}e^{j\frac{\pi}{4}} + e^{-j\frac{3\pi}{4}n}e^{-j\frac{\pi}{4}}\right) + 2e^{j\frac{\pi}{2}n} \end{aligned}$$

Using the eigensequence property:

$$y[n] = 4H_d(0) + 5\left(e^{j\frac{3\pi}{4}n}e^{j\frac{\pi}{4}}H_d\left(\frac{3\pi}{4}\right) + e^{-j\frac{3\pi}{4}n}e^{-j\frac{\pi}{4}}H_d\left(-\frac{3\pi}{4}\right)\right) + 2e^{j\frac{\pi}{2}n}H_d\left(\frac{\pi}{2}\right)$$

Caclulating the DTFT values:

$$\begin{aligned} H_d(0) &= (0)^2 e^{j\cos(0)} = 0 \\ H_d\left(\frac{\pi}{2}\right) &= \left(\frac{\pi}{2}\right)^2 e^{j\cos\left(\frac{\pi}{2}\right)} = \left(\frac{\pi}{2}\right)^2 e^0 = \left(\frac{\pi}{2}\right)^2 \\ H_d\left(\frac{3\pi}{4}\right) &= 0 \\ H_d\left(-\frac{3\pi}{4}\right) &= 0 \end{aligned}$$

Calculating the output:

$$\begin{aligned} y[n] &= 4(0) + 5\left(e^{j\frac{3\pi}{4}n}e^{j\frac{\pi}{4}}(0) + e^{-j\frac{3\pi}{4}n}e^{-j\frac{\pi}{4}}(0)\right) + 2e^{j\frac{\pi}{2}n}\left(\frac{\pi}{2}\right)^2 \\ &= \frac{\pi^2}{2}j^n \end{aligned}$$

4. Take the z-transform of both sides:

$$\begin{aligned}
y[n] - \frac{1}{\sqrt{2}} y[n-2] &= 2x[n] \\
Y(z) - \frac{1}{\sqrt{2}} z^{-2} Y(z) &= 2X(z) \\
Y(z)\left(1 - \frac{1}{\sqrt{2}} z^{-2}\right) &= 2X(z) \\
\frac{Y(z)}{X(z)} &= \frac{2}{1 - \frac{1}{\sqrt{2}} z^{-2}} \\
H(z) &= \frac{2}{1 - \frac{1}{\sqrt{2}} z^{-2}}
\end{aligned}$$

Since the system is causal, the ROC is outside the poles. The poles are at:

$$\begin{aligned}
z^2 &= \frac{1}{\sqrt{2}} \\
z &= \pm \frac{1}{\sqrt[4]{2}}
\end{aligned}$$

So the ROC is $|z| > \frac{1}{\sqrt[4]{2}}$. Because the ROC includes the unit circle, we can take the DTFT by substituting $z = e^{j\omega}$:

$$H_d(\omega) = \frac{2}{1 - \frac{1}{\sqrt{2}} e^{-j2\omega}}$$

Write $y[n]$ based on input $x[n] = 3 + \cos\left(\frac{\pi}{8}\right) \sin\left(\frac{\pi n}{2}\right) + (-1)^n$ and the frequency response, using the fact that the system is real to make the sin transformation easier:

$$y[n] = 3H_d(0) + \cos\left(\frac{\pi}{8}\right) \left|H_d\left(\frac{\pi}{2}\right)\right| \sin\left(\frac{\pi n}{2} + \angle H_d\left(\frac{\pi}{2}\right)\right) + e^{j\pi n} H_d(\pi)$$

Calculate the DTFT values:

$$\begin{aligned}
H_d(0) &= \frac{2}{1 - \frac{1}{\sqrt{2}} e^{-j2(0)}} = \frac{2}{1 - \frac{1}{\sqrt{2}}} = \frac{2}{\frac{\sqrt{2}-1}{\sqrt{2}}} = \frac{2\sqrt{2}}{\sqrt{2}-1} \frac{\sqrt{2}+1}{\sqrt{2}+1} = 4 + 2\sqrt{2} \\
H_d\left(\frac{\pi}{2}\right) &= \frac{2}{1 - \frac{1}{\sqrt{2}} e^{-j2\frac{\pi}{2}}} = \frac{2}{1 - \frac{1}{\sqrt{2}}(-1)} = \frac{2}{1 + \frac{1}{\sqrt{2}}} = \frac{2}{\frac{\sqrt{2}+1}{\sqrt{2}}} = \frac{2\sqrt{2}}{\sqrt{2}+1} \frac{\sqrt{2}-1}{\sqrt{2}-1} = 4 - 2\sqrt{2} \\
H_d(\pi) &= \frac{2}{1 - \frac{1}{\sqrt{2}} e^{-j2\pi}} = \frac{2}{1 - \frac{1}{\sqrt{2}}(1)} = 4 + 2\sqrt{2}
\end{aligned}$$

Plug into $y[n]$:

$$\begin{aligned}
y[n] &= 3\left(4 + 2\sqrt{2}\right) + \cos\left(\frac{\pi}{8}\right)\left(4 - 2\sqrt{2}\right) \sin\left(\frac{\pi n}{2} + 0\right) + e^{j\pi n}\left(4 + 2\sqrt{2}\right) \\
&= 12 + 6\sqrt{2} + \left(4 - 2\sqrt{2}\right) \cos\left(\frac{\pi}{8}\right) \sin\left(\frac{\pi n}{2}\right) + \left(4 + 2\sqrt{2}\right)(-1)^n
\end{aligned}$$

5. Based on the input and output pair given, we calculate values for $H_d(\omega)$ at given points:

$$
\begin{aligned}
H_d(0) &= \frac{4}{1} = 4 \\
\left|H_d\left(\frac{\pi n}{8}\right)\right| &= \frac{2}{1} = 2 \\
\angle H_d\left(\frac{\pi n}{8}\right) &= -90^\circ \\
H_d\left(\frac{\pi n}{4}\right) &= 0
\end{aligned}
$$

For the new input sequence, the output is:

$$
\begin{aligned}
\tilde{y}[n] =& 3H_d(0) + 2\left|H_d\left(\frac{\pi n}{8}\right)\right| \sin\left(\frac{\pi n}{8} + 15^\circ + \angle H_d\left(\frac{\pi n}{8}\right)\right) \\
&+ 10\left|H_d\left(\frac{\pi n}{4}\right)\right| \cos\left(\frac{\pi n}{4} + 25^\circ + \angle H_d\left(\frac{\pi n}{4}\right)\right) \\
\tilde{y}[n] =& 3(4) + 2(2)\sin\left(\frac{\pi n}{8} + 15^\circ + (-90^\circ)\right) + 10(0)\cos\left(\frac{\pi n}{4} + 25^\circ + (0)\right) \\
=& 12 - 4\cos\left(\frac{\pi n}{8} + 15^\circ\right)
\end{aligned}
$$