ECE 310: Problem Set 2

Due: 5pm, Friday September 14, 2018

Problem 1

(10pts, 5pts each)

$y[n] = 5y[n-1] + x[n], y[-1] = 0$

(a) Recall that the impulse response is defined as

$$h[n] = S\{\delta[n]\},$$

where S is the system. Therefore, we can derive the impulse response for this system by inputting $x[n] = \delta[n]$; this means that $x[0] = 1$, and $x[n] = 0$ for all other n. From here, the impulse response can be derived through simple iteration:

$$\begin{aligned} h[0] &= 5h[-1] + \delta[0] = 1 \\ h[1] &= 5h[0] + \delta[1] = 5 \\ h[2] &= 5h[1] + \delta[2] = 25 \end{aligned}$$

For $n > 2$, we see that $h[n] = 5^n$. Therefore, we can write the impulse response as

$$\boxed{h[n] = 5^n u[n]}$$

where the unit step function is necessary to account for the fact that $h[n] = 0$ for all $n < 0$.

(b) This system is **not stable**. The easiest way to see this is to analyze the impulse response; note that the bounded input $x[n] = \delta[n]$ produces an unbounded output $h[n] = 5^n u[n]$. Therefore, we've found a bounded input that produces an unbounded output. Alternatively, we know that an LTI system is BIBO stable if and only if its impulse response is absolutely summable. This is not the case here, as

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{\infty} |5^n u[n]| = \sum_{n=0}^{\infty} 5^n = \infty$$

Problem 2

(20pts, 3pts for linearity and 2pts for time-invariance each part)

(a) $y[n] = y[n-3] + x[n] - 2x[n-2]$

(i) Consider two arbitrary input signals $x_1[n]$ and $x_2[n]$ with outputs $y_1[n]$ and $y_2[n]$ respectively. Then we have:

$$y_1[n] - y_1[n-3] = x_1[n] - 2x_1[n-2] \tag{1}$$
$$y_2[n] - y_2[n-3] = x_2[n] - 2x_2[n-2] \tag{2}$$

Now if we multiply equation 1 by a_1 and equation 2 by a_2, and sum the two resulting equations, we obtain

$$(a_1y_1[n] + a_2y_2[n]) - (a_1y_1[n-3] + a_2y_2[n-3]) = (a_1x_1[n] + a_2x_2[n]) - 2\left(a_1x_1[n-2] + a_2x_2[n-2]\right).$$

If we let $x_3[n] = a_1x_1[n] + a_2x_2[n]$ and $z[n] = a_1y_1[n] + a_2y_2[n]$ and substitute in the equation above, we obtain

$$z[n] - z[n-3] = x_3[n] - 2x_3[n-2].$$

Notice that this is the original recursion expression, but with $x_3[n]$ as input and $z[n]$ as output. Therefore, the output to $x_3[n] = a_1x_1[n] + a_2x_2[n]$ is $z[n] = a_1y_1[n] + a_2y_2[n]$.

(ii) For an input $x[n]$, the expression

$$y[n] - y[n-3] = x[n] - 2x[n-2]$$

holds for every n. So we can make a change of variables, and write $m = n + n_0$, or $n = m - n_0$:

$$y[m-n_0] - y[m-n_0-3] = x[m-n_0] - 2x[m-n_0-2].$$

Now if we let $x_2[m] = x[m-n_0]$ and $z[m] = y[m-n_0]$ and substitute in the equation above, we obtain

$$z[m] - z[m-3] = x_2[m] - 2x_2[m-2],$$

which holds for every m. This is the original recursion expression, but with $x_2[m]$ as input and $z[m]$ as output. Therefore, the output to $x_2[n] = x[n-n_0]$ is $z[n] = y[n-n_0]$.

(b) $y[n-2] + 2y[n] = e^{j\pi n}x[n]$

(i) Since we have another recurrence relation, we proceed as we did in the previous problem, receiving two recurrence relations for the two inputs:

$$x_1[n] \rightarrow y_1[n-2] + 2y_1[n] = e^{j\pi n}x_1[n]$$
$$x_2[n] \rightarrow y_2[n-2] + 2y_2[n] = e^{j\pi n}x_2[n]$$

If we multiply the first equation by a_1 and the second equation by a_2, and sum the two resulting equations, we obtain

$$(a_1y_1[n-2] + 2a_1y_1[n]) + (a_2y_2[n-2] + 2a_2y_2[n]) = (a_1e^{j\pi n}x_1[n]) + (a_2e^{j\pi n}x_2[n])$$

If we let $x_3[n] = a_1x_1[n] + a_2x_2[n]$ and $z[n] = a_1y_1[n] + a_2y_2[n]$, substituting in the above equation gives

$$z[n-2] + 2z[n] = e^{j\pi n}x_3[n]$$

This is again the original recurrence relation with $x_3[n]$ as the input and $z[n]$ as the output. Therefore, the output to $x_3[n] = a_1x_1[n] + a_2x_2[n]$ is $z[n] = a_1y_1[n] + a_2y_2[n]$ - the system is **linear**.

(ii) We can show the system is time-varying with a counterexample. If we assume zero initial conditions ($y[n] = 0$ for all $n < 0$), then we can run through a few terms of the recursion:

$$\begin{aligned}
y[0] &= \frac{1}{2}e^{j\pi 0}\delta[0] - \frac{1}{2}y[-2] = \frac{1}{2} \\
y[1] &= \frac{1}{2}e^{j\pi}\delta[1] - \frac{1}{2}y[-1] = 0 \\
y[2] &= \frac{1}{2}e^{j\pi 2}\delta[2] - \frac{1}{2}y[0] = -\frac{1}{4} \\
y[3] &= \frac{1}{2}e^{j\pi 3}\delta[3] - \frac{1}{2}y[1] = 0 \\
y[4] &= \frac{1}{2}e^{j\pi 4}\delta[4] - \frac{1}{2}y[2] = \frac{1}{8}
\end{aligned}$$

So, starting at $n = 0$, $y[n] = \{\frac{1}{2}, 0, -\frac{1}{4}, 0, \frac{1}{8}, ...\}$. Now, if we were to input $x_1[n] = x[n-1] = \delta[n-1]$, if the system were to be time-invariant, then the output should be $y_1[n] = y[n-1]$. We can check this by again running through the recursion with the new input:

$$\begin{aligned}
y_1[0] &= \frac{1}{2}e^{j\pi 0}\delta[-1] - \frac{1}{2}y[-2] = 0 \\
y_1[1] &= \frac{1}{2}e^{j\pi}\delta[0] - \frac{1}{2}y[-1] = -\frac{1}{2} \\
y_1[2] &= \frac{1}{2}e^{j\pi 2}\delta[1] - \frac{1}{2}y[0] = 0 \\
y_1[3] &= \frac{1}{2}e^{j\pi 3}\delta[2] - \frac{1}{2}y[1] = \frac{1}{4} \\
y_1[4] &= \frac{1}{2}e^{j\pi 4}\delta[3] - \frac{1}{2}y[2] = 0
\end{aligned}$$

We get $y_1[n] = \{0, -\frac{1}{2}, 0, \frac{1}{4}, 0, -\frac{1}{8}, ...\}$, which is not equal to $y[n-1]$ - the signs are flipped. Therefore, the system is **time-varying**.

Two more formal proof techniques are shown below. We can introduce an auxiliary signal $z[n] = y[n-2] + 2y[n]$, which creates two subsystems:

$$\begin{aligned}
z[n] &= y[n-2] + 2y[n] \\
z[n] &= e^{j\pi n}x[n]
\end{aligned}$$

In this case, it's easy to show that the second system is not time-invariant:

$x_1[n] = x[n - n_0] \to z_1[n] = e^{j\pi n} x_1[n] = e^{j\pi n} x[n - n_0] \neq z[n - n_0] = e^{j\pi(n - n_0)} x[n - n_0]$

Therefore, the system is **time-varying**.

A final approach would be to use the methodology developed in Problem 2a(i). We take the original recursion and perform a variable transformation, setting $m = n + n_0$, or $n = m - n_0$. This gives the recursion:

$$y[m - n_0 + 2] + 2y[m - n_0] = e^{j\pi(m - n_0)} x[m - n_0]$$

If we again defined shifted inputs/outputs as $x_2[m] = x[m - n_0]$ and $z[m] = x[m - n_0]$, we obtain

$$z[m + 2] + 2z[m] = e^{j\pi(m - n_0)} x_2[m]$$

This is **not** the same as the original recursive relationship - the complex exponential has a factor of $m - n_0$ instead of m. Therefore, the system is **time-varying**.

(c) $y[n] = 5x[n] + 1$

(i) Let S be the system s.t.

$$y[n] = S\{x[n]\} = 5x[n] + 1$$

Suppose

$$\begin{aligned} y_1[n] &= S\{x_1[n]\} = 5x_1[n] + 1 \\ y_2[n] &= S\{x_2[n]\} = 5x_2[n] + 1 \end{aligned}$$

Then for any arbitrary a_1, a_2

$$\begin{aligned} S\{a_1x_1[n] + a_2x_2[n]\} &= 5(a_1x_1[n] + a_2x_2[n]) + 1 = 5a_1x_1[n] + 5a_2x_2[n] + 1 \\ a_1S\{x_1[n]\} + a_2S\{x_2[n]\} &= a_1(5x_1[n] + 1) + a_2(5x_2[n]) + 1) = 5a_1x_1[n] + 5a_2x_2[n] + a_1 + a_2 \\ S\{a_1x_1[n] + a_2x_2[n]\} &\neq a_1S\{x_1[n]\} + a_2S\{x_2[n]\} \end{aligned}$$

Therefore, the system is $\boxed{\text{non-linear}}$.

(ii) For any arbitrary n_o

$$S\{x[n - n_o]\} = 5x[n - n_o] + 1 = y[n - n_o]$$

Therefore, the system is $\boxed{\text{time-invariant}}$.

(d) $y[n] = x[0] \cdot x[n]$
Note that $x[0]$ is a constant w.r.t. input signal $x[n]$.

(i) Let S be the system s.t.

$$y[n] = S\{x[n]\} = x[0] \cdot x[n]$$

Consider

$$x_1[n] = \{-2, \underset{\uparrow}{-1}, 0\}$$
$$x_2[n] = \{0, \underset{\uparrow}{1}, 2\}$$

where $\uparrow$ indicates index of 0. We see that

$$x_1[n] + x_2[n] = \{-2, \underset{\uparrow}{0}, 2\}$$
$$\Rightarrow S\{x_1[n] + x_2[n]\} = \{0, \underset{\uparrow}{0}, 0\}$$

However

$$y_1[n] = S\{x_1[n]\} = \{2, \underset{\uparrow}{1}, 0\}$$
$$y_2[n] = S\{x_2[n]\} = \{0, \underset{\uparrow}{1}, 2\}$$
$$\Rightarrow S\{x_1[n]\} + S\{x_2[n]\} = \{2, \underset{\uparrow}{2}, 2\}$$

Therefore, the system is $\boxed{\text{non-linear}}$.

(ii) Consider the input signal $x[n] = \{-1, \underset{\uparrow}{0}, 1\}$. Then

$$x[n-1] = \{\underset{\uparrow}{-1}, 0, 1\}$$
$$S\{x[n-1]\} = \{\underset{\uparrow}{1}, 0, 1\}$$

However,

$$y[n] = x[0]x[n] = \{0, \underset{\uparrow}{0}, 0\}$$
$$\Rightarrow y[n-1] = \{\underset{\uparrow}{0}, 0, 0\} \neq S\{x[n-1]\}$$

Therefore, the system is $\boxed{\text{time-varying}}$.

We can also use the techniques shown in the previous parts to show that the system is nonlinear and time-varying. To show non-linearity, suppose we input $x_1[n]$ and $x_2[n]$ into the system. This gives two relations:

$$y_1[n] = x_1[0]x_1[n]$$
$$y_2[n] = x_2[0]x_2[n]$$

Then, if we input $x_3[n] = \alpha x_1[n] + \beta x_2[n]$, we get a third relation:

$$\begin{aligned} y_3[n] &= x_3[0]x_3[n] \\ &= (\alpha x_1[0] + \beta x_2[0])(\alpha x_1[n] + \beta x_2[n]) \\ &= \alpha^2 x_1[0]x_1[n] + \alpha\beta(x_1[n]x_2[0] + x_1[0]x_2[n]) + \beta^2 x_2[0]x_2[n] \\ &\neq \alpha y_1[n] + \beta y_2[n] = \alpha x_1[0]x_1[n] + \beta x_2[0]x_2[n] \end{aligned}$$

Therefore, the system is nonlinear. Similarly, to prove that the system is time-varying, we input $x_1[n] = x[n - n_0]$:

$$x_1[n] = x[n - n_0] \to y_1[n] = x_1[0]x_1[n] = x[-n_0]x[n - n_0] \neq y[n - n_0] = x[0]x[n - n_0]$$

Therefore, the system is time-varying.

Problem 3

(15pts)

Recall that the unit-step signal is

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & \text{else} \end{cases} .$$

Its delayed version $u[n-1]$ is written as:

$$u[n-1] = \begin{cases} 1, & n \geq 1 \\ 0, & \text{else} \end{cases} .$$

Since impulse signal is $\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & \text{else} \end{cases}$, it can be derived as $\delta[n] = u[n] - u[n-1]$.

By linearity and shift-invariance, the impulse response is

$$h[n] = g[n] - g[n-1].$$

Hence,

$$y[n] = x[n] * h[n] = \boxed{x[n] * (g[n] - g[n-1])}.$$

Problem 4

(20pts)

Suppose S is the LTI system, s.t. $S\{x[n]\} = y[n]$.

$$\begin{aligned} &x[n] = 3^{-n}u[n-3] \\ \Rightarrow &x[n-1] = 3^{-(n-1)}u[(n-1)-3] = 3^{-n+1}u[n-4] \\ \Rightarrow &3^{-1}x[n-1] = 3^{-n}u[n-4] \\ \Rightarrow &x[n] - 3^{-1}x[n-1] = 3^{-n}(u[n-3] - u[n-4]) = 3^{-3}\delta[n-3] \\ \Rightarrow &\delta[n-3] = 3^3 x[n] - 3^2 x[n-1] \end{aligned}$$

Since the system is time-invariant,

$$\delta[n] = 3^3 x[n+3] - 3^2 x[n+2]$$

Recall that $S\{\delta[n]\} = h[n]$.

$$\begin{aligned}
h[n] = S\{\delta[n]\} &= S\left\{3^3 x[n+3] - 3^2 x[n+2]\right\} \\
&= 3^3 S\{x[n+3]\} - 3^2 S\{x[n+2]\} && \because \text{(linear)} \\
&= 3^3 y[n+3] - 3^2 y[n+2] && \because \text{(time-invariant)} \\
&= 3^3 2^{-(n+3)} u[(n+3)-5] - 3^2 2^{-(n+2)} u[(n+2)-5] && \because (y[n] = 2^{-n} u[n-5]) \\
&= \boxed{3^3 2^{-n-3} u[n-2] - 3^2 2^{-n-2} u[n-3]}
\end{aligned}$$

Problem 5

(20pts, 5pts each)

(a) $x[n] = \{1, \underset{\uparrow}{2}, 3\}, h[n] = \{\underset{\uparrow}{1}, 0, 3, -1\}$ We can compute the convolution by rewriting the terms as summations of scaled and shifted version of delta functions, then apply the property $x[n] * \delta[n-n_0] = x[n-n_0]$:

$$\begin{aligned}
(x * h)[n] =& (\delta[n+1] + 2\delta[n] + 3\delta[n-1]) * (\delta[n] + 0\delta[n-1] + 3\delta[n-2] - \delta[n-3]) \\
=& \delta[n+1] * (\delta[n] + 3\delta[n-2] - \delta[n-3]) + \\
& 2\delta[n] * (\delta[n] + 3\delta[n-2] - \delta[n-3]) + \\
& 3\delta[n-1] * (\delta[n] + 3\delta[n-2] - \delta[n-3]) \\
=& \delta[n+1] + 3\delta[n-1] - \delta[n-2] + \\
& 2\delta[n] + 6\delta[n-2] - 2\delta[n-3] + \\
& 3\delta[n-1] + 9\delta[n-3] - 3\delta[n-4] \\
=& \delta[n+1] + 2\delta[n] + 6\delta[n-1] + 5\delta[n-2] + 7\delta[n-3] - 3\delta[n-4] \\
=& \boxed{\{1, \underset{\uparrow}{2}, 6, 5, 7, -3\}}.
\end{aligned}$$

An alternate approach (if you don't like writing out so many delta functions) is to perform the convolution "graphically," such as you did in ECE 210. We simply "flip" one of the signals and shift it to the right, multiplying the overlap terms. Since the overlap starts at $n = -1$ and ends at $n = 4$, $y[n]$ will take nonzero values between $-1 \leq n \leq 4$.

$$
\begin{array}{cccccccccc}
 & & 1 & 0 & 3 & -1 & & & & \\
3 & 2 & 1 & & & & & & & y[-1]=1 \\
 & 3 & 2 & 1 & & & & & & y[0]=2 \\
 & & 3 & 2 & 1 & & & & & y[1]=6 \\
 & & & 3 & 2 & 1 & & & & y[2]=5 \\
 & & & & 3 & 2 & 1 & & & y[3]=7 \\
 & & & & & 3 & 2 & 1 & & y[4]=-3
\end{array}
$$

One final approach is to view the convolution as a matrix multiplication. We observe that

$$
\begin{bmatrix} -1 \\ 2 \\ 6 \\ 5 \\ 7 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 3 \\ -1 \end{bmatrix} \rightarrow \mathbf{y} = C\mathbf{x}
$$

We see that if we let $\mathbf{x}$ be a vector consisting of the elements of $h[n]$, and C be a matrix containing column-shifted versions of $x[n]$, then the convolution output $\mathbf{y}$ can be calculated as $\mathbf{y} = C\mathbf{x}$. If $x[n]$ has length L_1 and $h[n]$ has length L_2, then C has $L_1 + L_2 - 1$ rows and L_2 columns. This is also known as a Toeplitz matrix.

(b) $x[n] = 3^{(-n)}u[n], h[n] = \{\underset{\uparrow}{-1}, 2, 3\}$

Similar to (a), we rewrite $h[n]$ and apply the property:

$$
\begin{aligned}
(x * h)[n] &= x[n] * (-\delta[n] + 2\delta[n-1] + 3\delta[n-2]) \\
&= -x[n] + 2x[n-1] + 3x[n-2] \\
&= \boxed{-3^{-n}u[n] + 2 \cdot 3^{-(n-1)}u[n-1] + 3 \cdot 3^{-(n-2)}u[n-2]}
\end{aligned}
$$

(c) $x[n] = (0.1)^n u[n], h[n] = n(u[n] - u[n-3])$

Note that $u[n] - u[n-3] = \begin{cases} 1, 0 \leq n \leq 2 \\ 0, \text{else} \end{cases}$ is the summation of delta functions, i.e.

$$
u[n] - u[n-3] = \delta[n] + \delta[n-1] + \delta[n-2]
$$

Therefore

$$
\begin{aligned}
(x * h)[n] &= 0.1^n u[n] * n(\delta[n] + \delta[n-1] + \delta[n-2]) \\
&= 0.1^n u[n] * (\delta[n-1] + 2\delta[n-2]) \\
&= \boxed{0.1^{n-1}u[n-1] + 2 \cdot 0.1^{n-2}u[n-2]}.
\end{aligned}
$$

(d) $x[n] = (-1)^{(-n)}u[n], h[n] = e^{(-n)}u[n-2]$ Since $x[n]$ and $h[n]$ cannot be written as finite sum of weighted delta functions, we use the definition of convolution, i.e.

$$(x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k].$$

Plugging in $x[n] = (-1)^{(-n)}u[n]$ and $h[n] = e^{(-n)}u[n-2]$ gives:

$$(x * h)[n] = \sum_{k=-\infty}^{\infty} (-1)^{(-k)}u[k] \cdot e^{-(n-k)}u[n-k-2].$$

Notice that $u[k] = 0, \forall k < 0$ and $u[n-k-2] = 0, \forall k > n-2$. Also notice that the summation limits are only valid if $n \geq 2$; for $n < 2$, the convolution result is zero. This is because the first overlap between $x[n]$ and $h[n]$ will occur when $n = 2$. We also use a clever trick: $(-1)^{-n} = (\frac{1}{-1})^n = (-1)^n$. Therefore, the sum can be written as:

$$\begin{aligned}(x * h)[n] &= \sum_{k=0}^{n-2}(-1)^{(-k)}e^{-(n-k)} \\ &= e^{-n}\sum_{k=0}^{n-2}(-1)^{-k}e^{k} \\ &= e^{-n}\sum_{k=0}^{n-2}(-e)^{k}.\end{aligned}$$

Using the fact that:

$$\sum_{n=0}^{m} a^n = \frac{1-a^{m+1}}{1-a},$$

we get:

$$\boxed{(x * h)[n] = e^{-n}\left(\frac{1-(-e)^{n-1}}{1+e}\right)u[n-2]},$$

where the $u[n-2]$ term comes from the fact that the summation limits are only valid for $n \geq 2$.

Problem 6

(15pts)

Let $x[n] = x_a[n] + x_b[n]$, where $x_a[n] = n(u[n-1] - u[n-10])$ and $x_b[n] = 0.5^n u[n-30]$. Then we can write $(x * h)[n]$ and $(x_a * h)[n] + (x_b * h)[n]$ because of linearity of convolution. Notice that, for $n \in \mathbb{Z}$,

$$u[n-1] - u[n-10] = \begin{cases} 1 & n \in [1, 9] \\ 0 & \text{otherwise} \end{cases}$$

and

$$u[n-30] = \begin{cases} 1 & n \in [30, \infty) \\ 0 & \text{otherwise} \end{cases}$$

Now consider $y_a[n] = (x_a * h)[n]$, $y_a[n]$ only has non-zero values when there is overlap between $x_a[n]$ and the shifting $h[n]$. Since $h[n] = 0.3^n(u[n] - u[n-7])$ has 7 non-zero values, $y_a[n]$ will have non-zero values if $n \in [1, 9+7-1] \iff n \in [1, 15]$ (where 9 is from $x_a[n]$, 7 is from the "length" of $h[n]$, and 1 is from overlap condition).

Similarly, $y_b[n] = (x_b * h)[n]$ has non-zero values if $n \in [30, \infty)$.

Hence, $(x * h)[n] = y_a[n] + y_b[n]$ has non-zero values if $n \in [1, 15] \cup [30, \infty)$