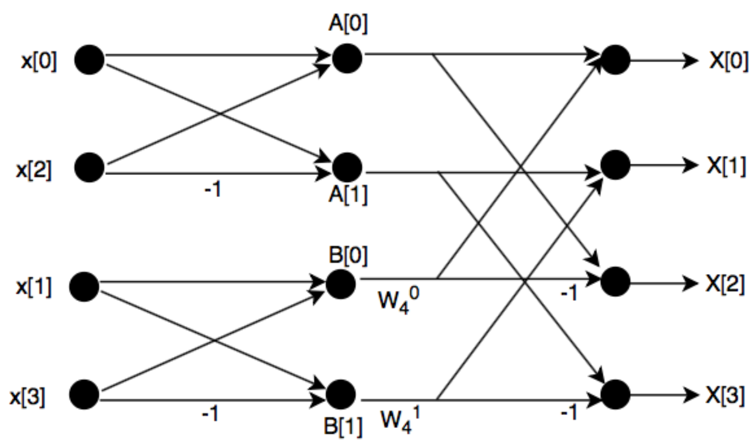
ECE310: Quiz#8 (6pm Section CSS) Fall 2018 Solutions

1. (4 pts) Consider the signal $\{x[n]\}_{n=0}^{3} = \{1, 2, -2, -1\}$. Use a decimation-in-time radix-2 FFT to compute the DFT of $x[n]$, i.e., calculate explicitly all the intermediate quantities that are computed in this FFT, and show how they are combined to produce the final output, which you should also give explicitly.

Solution

The corresponding butterfly diagram can be seen below.



We calculate the intermediate quantities (the outputs of the length-2 butterflies) as follows, since the 2-point DFT just outputs the sum and the difference:

$$
\begin{aligned}
A[0] &= x[0] + x[2] = -1 \\
A[1] &= x[0] - x[2] = 3 \\
B[0] &= x[1] + x[3] = 1 \\
B[1] &= x[1] - x[3] = 3
\end{aligned}
$$

Then we combine everything together using the butterfly structure. This gives

$$
\begin{aligned}
X[0] &= A[0] + B[0] = 0 \\
X[1] &= A[1] + W_4^1 B[1] = A[1] - jB[1] = 3 - 3j \\
X[2] &= A[0] - B[0] = -2 \\
X[3] &= A[1] - W_4^1 B[1] = A[1] + jB[1] = 3 + 3j
\end{aligned}
$$

Therefore,

$$\boxed{\{X[k]\}_{k=0}^{3} = \{0, 3-3j, -2, 3+3j\}}$$

Grading:

- 2 points for drawing the correct flowgraph.
- 1 point for calculating the intermediate values.
- 1 point for the correct output.

2. (3 pts) Given $\{x[n]\}_{n=0}^{N-1}$, with $x[n] = x_c(nT)$ and $T = 300\mu$sec, you compute a length-N FFT of $x[n]$ and plot the magnitude. Using this method, you wish to resolve analog sinusoidal signals that are separated by as little as 40 Hz in frequency. Assume that the frequency resolution for windowed DTFT-based spectral analysis is equal to the width of the main lobe of the DTFT of the window. Determine the minimum length $N = 2^\nu$ that will meet your resolution requirement. (**Hint:** The DTFT of the sequence $v[n] = u[n] - u[n-N]$ is $V_d(\omega) = e^{-j\omega(N-1)/2}\frac{\sin(N\omega/2)}{\sin(\omega/2)}$).

Solution

In order to find the main lobe of the DTFT, and therefore determine the resolution, we need to determine where the zeros of the DTFT are located. These will occur when

$$\frac{N\omega}{2} = k\pi \rightarrow \omega = \frac{2\pi k}{N}, k \in \mathbb{Z}$$

Since the peak occurs at $\omega = 0$, the first zero on the right will be at $\omega = \frac{2\pi}{N}$, and the first zero on the left will be at $\omega = -\frac{2\pi}{N}$. Therefore, the main lobe width is twice the distance between zeros; this is $\frac{4\pi}{N}$.

Since $\omega = \Omega T$, a frequency resolution of 40 Hz in the analog domain corresponds to a frequency resolution of $80\pi \times (300 \times 10^{-6}) = 2.4\pi \times 10^{-2}$ rad/sec in the digital domain. Therefore, we need

$$\frac{4\pi}{N} = 2.4\pi \times 10^{-2} \rightarrow N = 166.67 \rightarrow \boxed{256}$$

Because we're told N must be a power of two, to meet the resolution criterion while minimizing N, we need to choose the closest power of two that's larger than the actual result.

Grading:

- 1 point for finding the main lobe width.
- 1 point for using $\omega = \Omega T$ correctly.
- 1 point for the final result. -0.5 points if not giving a power of two.

3. (3 pts) Let $\{x[n]\}_{n=0}^{2} = \{3, 5, 7\}$, and $\{v[n]\}_{n=0}^{1} = \{-1, 1\}$. A new sequence $\{g[n]\}_{n=0}^{3}$ is generated as follows: $\{g[n]\}_{n=0}^{3} = \text{IFFT}(\{G[k]\}_{k=0}^{3})$ where the IFFT is a 4-point inverse FFT, $G[k] = X[k]V[k], k = 0, 1, 2, 3$, and the sequences $X[k]$ and $V[k]$ are generated each by a 4-point FFT of the sequences $\{x[n]\}$ and $\{v[n]\}$, respectively, after zero-padding them to length 4. Determine $g[1]$ and $g[3]$.

Solution

Note that the length of $x[n]$ is 3, and the length of $h[n]$ is 2; therefore, if we zero-pad both signals to length 3+2-1 = 4 and take the circular convolution, the result is the same as if we had taken the linear convolution. Furthermore, since the zero-padded length is a power of two, we can compute the circular convolution using the FFT. Therefore, $g[n]$ will just be the linear convolution between $x[n]$ and $v[n]$, since it's the fast convolution between the zero-padded versions. We can compute this convolution through a matrix multiplication:

$$g[n] = \begin{bmatrix} 3 & 0 \\ 5 & 3 \\ 7 & 5 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ -2 \\ 7 \end{bmatrix}$$

Therefore,

$$\boxed{g[1] = -2, g[3] = 7}$$

Note that the circular convolution between the zero-padded versions would give the same result:

$$g[n] = \begin{bmatrix} 3 & 0 & 7 & 5 \\ 5 & 3 & 0 & 7 \\ 7 & 5 & 3 & 0 \\ 0 & 7 & 5 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ -2 \\ 7 \end{bmatrix}$$

Grading:

- 2 points for realizing $g[n]$ is some form of convolution.
- 0.5 points each for $g[1]$ and $g[3]$.