

UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN  
 Department of Electrical and Computer Engineering  
 ECE 310 DIGITAL SIGNAL PROCESSING  
**Homework 12 Solutions**

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**Due: 5 pm, Friday, November 30, 2018**

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## Problem 1

A speech signal  $x_a(t)$  is assumed to be bandlimited to 10 kHz. It is desired to filter this signal with a bandpass filter that will pass the frequencies between 200 Hz and 4 kHz by using a digital filter  $H_d(\omega)$  sandwiched between an ideal A/D and an ideal D/A.

- Determine the Nyquist sampling rate for the input signal.
- Sketch the frequency response  $H_{d,1}(\omega)$  for the necessary discrete-time filter, when sampling at the Nyquist rate.
- Find the largest sampling period  $T$  for which the A/D, digital filter response ( $H_{d,2}(\omega)$ ), and D/A can perform the desired filtering function. (Hint: some amount of aliasing may be permissible during A/D conversion for this part.)
- For the system using  $T$  from part (c), sketch the necessary  $H_d(\omega)$ .

## Solution

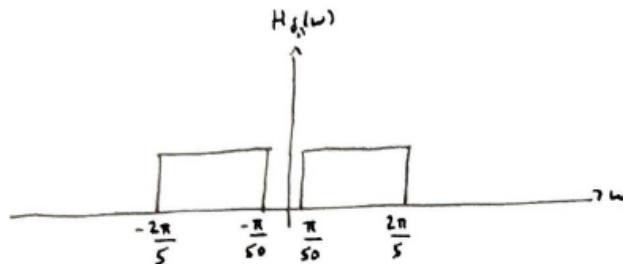
- (a) The Nyquist sampling rate is simply twice the maximum frequency of the signal. Since the signal is bandlimited to 10 kHz, we find that

$$f_s = 20 \text{ kHz or } T_s = \frac{1}{20000} \text{ s}$$

- (b) If we're sampling at the Nyquist rate, using  $\omega = \Omega T$ , we can map the analog frequencies to digital frequencies:

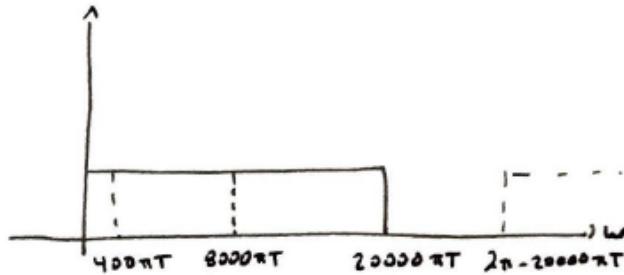
$$\begin{aligned} 200 \text{ Hz} &= 400\pi \text{ rad/s} \rightarrow \omega_1 = \frac{400\pi}{20000} = \frac{\pi}{50} \\ 4000 \text{ Hz} &= 8000\pi \text{ rad/s} \rightarrow \omega_2 = \frac{8000\pi}{20000} = \frac{2\pi}{5} \end{aligned}$$

Assuming an ideal filter, the response is plotted below.



(c) As per the hint, we want to examine whether or not some level of aliasing will be permissible. Intuitively, it makes sense that it should be, since we only need to pass through frequencies up to a certain range in the input, instead of the entire input. If only frequencies beyond 4 kHz are distorted, the filter still performs its desired operation.

Suppose we sample at some arbitrary rate  $T$ . Then the maximum frequency we need to preserve is  $8000\pi T$ , and, since the signal extends out to  $20000\pi T$ , the first aliased frequency will occur at  $2\pi - 20000\pi T$ . Therefore, we can sample below the Nyquist rate as long as the first aliased frequency is beyond the maximum frequency that needs to be preserved; the situation is as in the picture below.



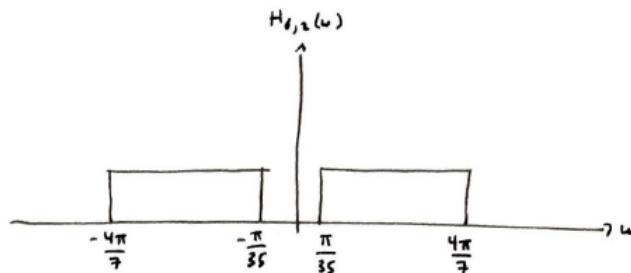
Equating the two gives:

$$2\pi - 20000\pi T = 8000\pi T \rightarrow 2\pi = 28000\pi T \rightarrow T = \frac{1}{14000} \text{ s}$$

(d) We again convert the desired analog frequencies to digital ones:

$$\begin{aligned} 200 \text{ Hz} &= 400\pi \text{ rad/s} \rightarrow \omega_1 = \frac{400\pi}{14000} = \frac{\pi}{35} \\ 4000 \text{ Hz} &= 8000\pi \text{ rad/s} \rightarrow \omega_2 = \frac{8000\pi}{14000} = \frac{4\pi}{7} \end{aligned}$$

Again assuming an ideal filter, the response is shown below.



## Problem 2

You are given the task of implementing an analog double-echo generator using a system composed of a digital filter sandwiched between an ideal A/D and D/A, both operating at a common sampling period  $T$ . The echo generator is specified by the following system equation:

$$y_a(t) = x_a(t) + \alpha x_a(t - \tau) + \beta x_a(t - 2\tau)$$

where  $x_a(t)$  and  $y_a(t)$  are the input and output of the analog system, respectively, and  $\tau$  is the time delay constant. Assume  $x_a(t)$  is bandlimited to 20 kHz.

- (a) Find the desired analog frequency response  $H_a(\Omega)$ . (Your answer will be in terms of  $\tau, \alpha, \beta$ , which are assumed to be known.)
- (b) Find the appropriate maximum sampling period  $T = T_0$  making this design possible.
- (c) Find the digital filter response  $H_d(\omega)$  that is needed to implement the analog system.
- (d) Assuming that  $\tau = 100T_0$ , find the transfer function  $H(z)$  corresponding to  $H_d(\omega)$ , and draw a block diagram for its implementation.
- (e) Suppose the input to your system is  $x_a(t) = \text{sinc}(2000\pi t)$ . Determine the output  $y_a(t)$ .
- (f) Suppose the input to your system is  $x_a(t) = \cos(50 \cdot 10^3 \pi t)$ . Determine the output  $y_a(t)$ .

### Solution

- (a) Taking the CTFT of both sides gives

$$Y_a(\Omega) = X_a(\Omega) + \alpha X_a(\Omega)e^{-j\Omega\tau} + \beta X_a(\Omega)e^{-j\Omega2\tau}$$

Therefore, the desired analog frequency response is

$$H_a(\Omega) = \frac{Y_a(\Omega)}{X_a(\Omega)} = \boxed{1 + \alpha e^{-j\Omega\tau} + \beta e^{-j\Omega2\tau}}$$

- (b) Since  $x_a(t)$  is bandlimited to 20 kHz, then the appropriate maximum sampling period is the Nyquist rate:

$$\boxed{T_0 = \frac{1}{40000} \text{ s}}$$

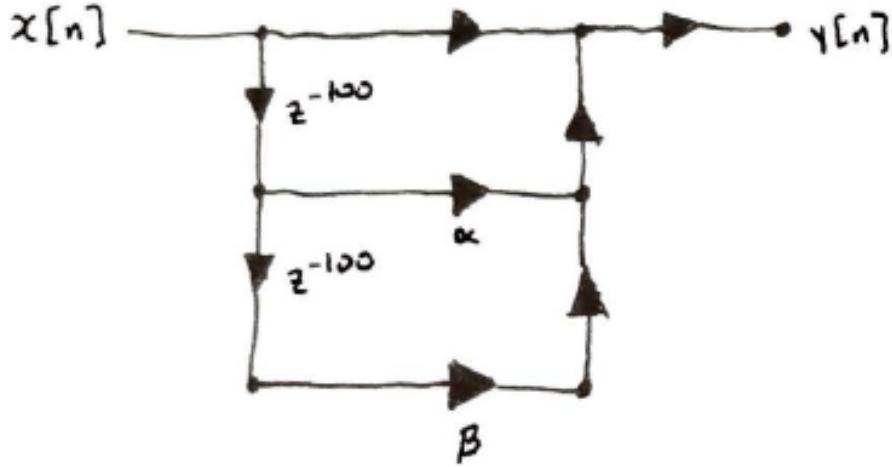
- (c) Since we're sampling at the Nyquist rate, no aliasing will occur. Furthermore, since we're using an ideal D/A and A/D, we can simply say that  $H_d(\omega) = H_a(\frac{\omega}{T})$ . Therefore, required digital filter response is

$$\boxed{H_d(\omega) = 1 + \alpha e^{-j40000\omega\tau} + \beta e^{-j80000\omega\tau}}$$

(d) If we assume  $\tau = 100T_0 = \frac{1}{400}$ , then  $H_d(\omega) = 1 + \alpha e^{-j100\omega} + \beta e^{-j200\omega}$ . We get the  $z$ -transform by setting  $z = e^{j\omega}$ . This gives

$$H(z) = 1 + \alpha z^{-100} + \beta z^{-200}$$

The block diagram is shown below.



(e) Since  $x[n] = x_a(nT)$ , and  $x_a(t)$  is a bandlimited signal with a maximum frequency below 20 kHz, no aliasing will occur; therefore, we can immediately write  $x[n] = \text{sinc}(\frac{\pi n}{20})$ . Then, since

$$y[n] = x[n] + \alpha x[n - 100] + \beta x[n - 200]$$

we note that the system is FIR; therefore, we don't need to take any  $z$ -transforms, and can immediately write that

$$\begin{aligned} y[n] &= \text{sinc}\left(\frac{\pi n}{20}\right) + \alpha \text{sinc}\left(\frac{\pi}{20}(n - 100)\right) + \beta \text{sinc}\left(\frac{\pi}{20}(n - 200)\right) \\ &= \text{sinc}\left(\frac{\pi n}{20}\right) + \alpha \text{sinc}\left(\frac{\pi n}{20} - 5\pi\right) + \beta \text{sinc}\left(\frac{\pi n}{20} - 10\pi\right) \end{aligned}$$

Using  $\omega = \Omega T$ , the output of the ideal D/A converter is

$$y_a(t) = \text{sinc}(2000\pi t) + \alpha \text{sinc}(2000\pi t - 5\pi) + \beta \text{sinc}(2000\pi t - 10\pi)$$

For this input, the system performed the desired operation, which we can verify by plugging into the analog system's difference equation:

$$\begin{aligned} y_a(t) &= x_a(t) + \alpha x_a(t - \tau) + \beta x_a(t - 2\tau) \\ &= \text{sinc}(2000\pi t) + \alpha \text{sinc}\left(2000\pi\left(t - \frac{1}{400}\right)\right) + \beta \text{sinc}\left(2000\pi\left(t - \frac{2}{400}\right)\right) \\ &= \text{sinc}(2000\pi t) + \alpha \text{sinc}(2000\pi t - 5\pi) + \beta \text{sinc}(2000\pi t - 10\pi) \end{aligned}$$

(f) Writing  $x[n] = x_a(nT)$  gives

$$x[n] = \cos\left(\frac{5\pi n}{4}\right) = \cos\left(-\frac{3\pi n}{4}\right) = \cos\left(\frac{3\pi n}{4}\right)$$

We observe that aliasing occurs, since  $\omega = \frac{5\pi}{4}$  is outside of  $[-\pi, \pi]$ . Using the method in (e), we can write

$$y[n] = \cos\left(\frac{3\pi n}{4}\right) + \alpha \cos\left(\frac{3\pi}{4}(n - 100)\right) + \beta \cos\left(\frac{3\pi}{4}(n - 200)\right)$$

Using  $\omega = \Omega T$ , the output of the ideal D/A converter is

$$y_a(t) = \cos(30000\pi t) + \alpha \cos\left(\left(30000\pi t - \frac{300\pi}{4}\right)\right) + \beta \cos(30000\pi t - 150\pi)$$

We can compare this to the desired result:

$$\begin{aligned} y_a(t) &= x_a(t) + \alpha x_a(t - \tau) + \beta x_a(t - 2\tau) \\ &= \cos(50000\pi t) + \alpha \cos\left(50000\pi\left(t - \frac{1}{400}\right)\right) + \beta \cos\left(50000\pi\left(t - \frac{2}{400}\right)\right) \\ &= \cos(50000\pi t) + \alpha \cos(50000\pi t - 125\pi) + \beta \cos(50000\pi t - 250\pi) \end{aligned}$$

In this case, because aliasing occurred, the output of the digital implementation does not match the desired analog output.

### Problem 3

Consider the system in Figure 1. Suppose that  $x_a(t)$  ( $X_a(f)$ ) is given in Figure 2) is bandlimited to  $F_c = 20$  kHz and  $H(\omega)$  is an ideal low-pass filter with cutoff frequency  $\omega_c = \frac{\pi}{2}$ . Sketch and label the Fourier Transform of  $y_a(t)$  for each of the following cases:

- (a)  $T_1 = T_2 = 25 \cdot 10^{-6}$
- (b)  $T_1 = T_2 = 12.5 \cdot 10^{-6}$
- (c)  $T_1 = 12.5 \cdot 10^{-6}, T_2 = 25 \cdot 10^{-6}$
- (d)  $T_1 = 25 \cdot 10^{-6}, T_2 = 12.5 \cdot 10^{-6}$

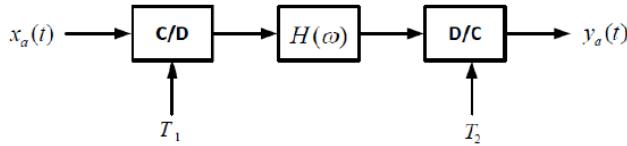


Figure 1: Discrete-Time Implementation of Analog Filtering. The C/D and D/C blocks correspond to ideal A/D and D/A converters, respectively.

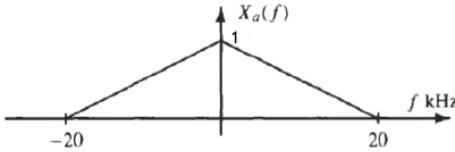
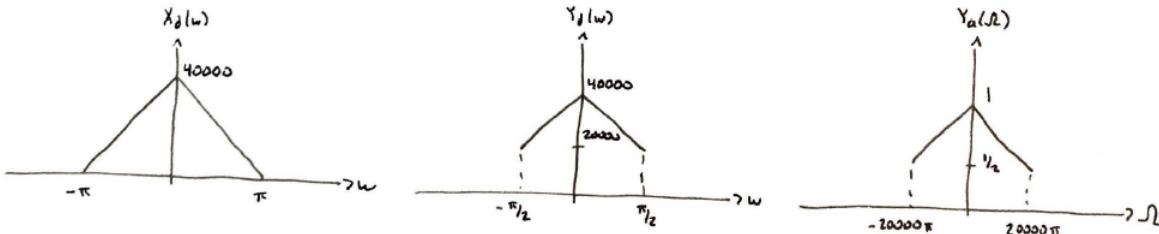


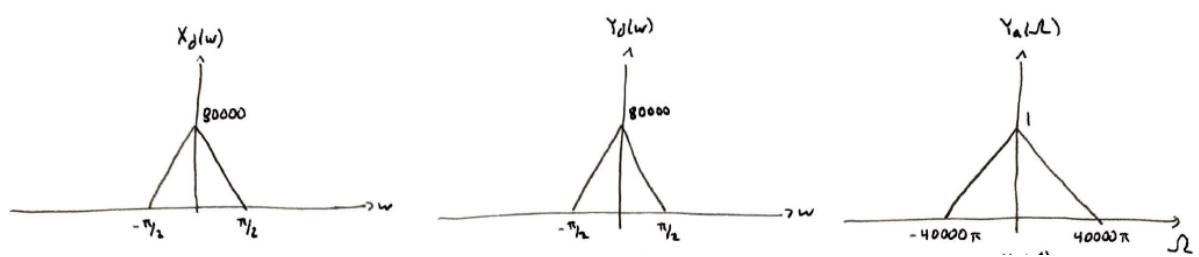
Figure 2:  $X_a(f)$  for the input signal  $x_a(t)$ .

### Solution

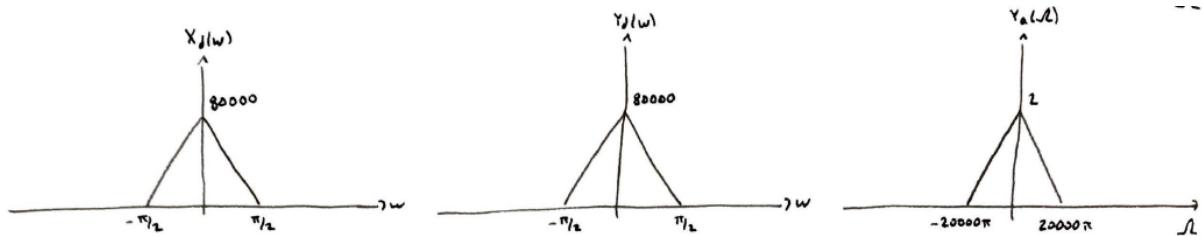
- (a)** The sketches of  $X_d(\omega)$ ,  $Y_d(\omega)$ , and  $Y_a(\Omega)$  can be seen below. Recall that the ideal A/D converter scales the amplitude by  $\frac{1}{T_1}$ , compresses the spectrum by a factor of  $\frac{1}{T_1}$ , and adds  $2\pi$ -periodicity, while the ideal D/A converter scales the amplitude by  $T_2$ , expands the spectrum by a factor of  $\frac{1}{T_2}$ , and removes the  $2\pi$ -periodicity, only keeping the copy of the DTFT that lies between  $-\pi$  and  $\pi$ .



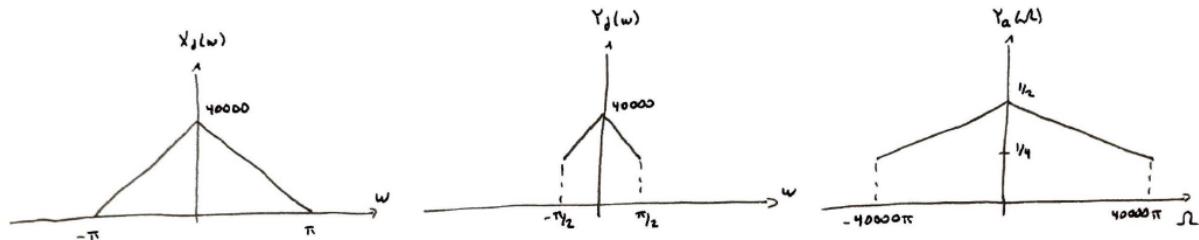
(b) The sketches can be seen below. In this case, the original input is perfectly recovered, as this value of  $T_1$  compresses the spectrum to lie in  $\omega \in [-\frac{\pi}{2}, \frac{\pi}{2}]$  - the passband of the filter.



(c) The sketches can be seen below. Note that the "mismatch" between  $T_1$  and  $T_2$  causes the analog output to differ, despite having  $Y_d(\omega) = X_d(\omega)$ .



(d) The sketches can be seen below. Again, because there's a mismatch between  $T_1$  and  $T_2$ , despite  $Y_a(\Omega)$  having the same support as  $X_a(\Omega)$ , the two are quite different.



## Problem 4

Suppose that an analog signal  $x_a(t)$  is to be processed using an analog high-pass filter with a passband starting at 2.5 kHz, stopband being the interval [0, 2 kHz], and a stopband attenuation of -40 dB. This filter is to be implemented digitally, as illustrated in Figure 1, with  $T_1 = T_2 = T_s$ .

- (a) The block  $H(\omega)$  represents the frequency response of a digital FIR filter. Design this filter to have the shortest length possible using the windowing method with one of the standard windows (rectangular, triangular, Hanning, or Hamming) to meet the analog filter specifications. Assume a sampling frequency of  $F_s = 10$  kHz. Your answer should be a closed-form expression for the designed impulse response  $h[n]$ .
- (b) Suppose that in your design procedure, you forgot to apply the window you have chosen for the design, and used instead a rectangular window of the same length. Sketch (by hand) the approximate analog magnitude frequency response  $|H_a(\Omega)|$  for your entire system, over the range of frequencies for which the system functions as an LTI system. Discuss what happens for an input with frequencies outside this range. Your sketch should show the approximate shapes of the passband and stopband magnitude responses, and should be labeled to show the height or ripples if any, and band edge frequencies. Do the sketch twice: once using a linear scale, and once using a dB scale for the vertical axis.

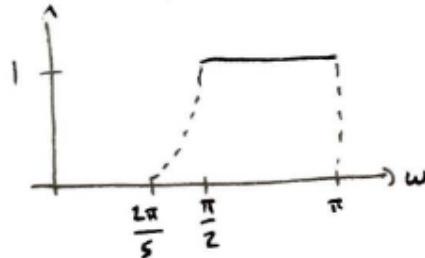
## Solution

- (a) Converting the desired analog frequencies to digital gives

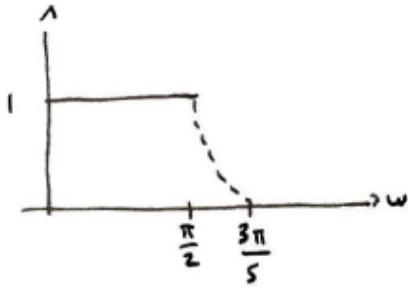
$$2 \text{ kHz} = 4000\pi \text{ rad/s} \rightarrow 0.4\pi$$

$$2.5 \text{ kHz} = 5000\pi \text{ rad/s} \rightarrow 0.5\pi$$

Therefore, the desired digital prototype filter is as shown below.



One way to design the high-pass filter is to use the method described in Homework 11 Problem 2b. Instead of trying to design a high-pass filter, we design a low-pass filter, and modulate it by multiplying by  $e^{j\pi n} = (-1)^n$ . The resulting low-pass prototype is shown below.



We choose the cutoff frequency to be the midpoint between the end of the passband ( $0.5\pi$ ) and the beginning of the stopband ( $0.6\pi$ ), so  $\omega_c = 0.55\pi$ . Because we require at least 40 dB of attenuation in the stopband, we can choose between the Hann, Hamming, and Blackman windows; we choose the **Hann** window, since it will provide the required attenuation using the shortest length,  $L = \frac{6.2\pi}{0.1\pi} = 62$ .

The length-L GLP approximation of the LPF (before windowing) is given by

$$g_{LPF}[n] = \frac{\omega_c}{\pi} \text{sinc}\left(\omega_c \left(n - \frac{L-1}{2}\right)\right) = 0.55 \text{sinc}(0.55\pi(n - 30.5)).$$

So, we obtain our high-pass filter by setting  $g_{HPF}[n] = (-1)^n g_{LPF}[n]$ , and obtain  $h[n]$  by multiplying  $g_{HPF}[n]$  by the Hann window. This gives

$$h[n] = \begin{cases} (-1)^n 0.55 \text{sinc}(0.55\pi(n - 30.5)) (0.5 - 0.5 \cos(\frac{2\pi n}{61})) & 0 \leq n \leq 61 \\ 0 & \text{else} \end{cases}$$

**(b)** If we were to use a rectangular window instead of the Hann window, we would expect a sharper transition region, but higher sidelobes. Indeed, we see that the new transition region width is

$$\Delta\omega = \frac{1.8\pi}{62} \approx 0.03\pi$$

Therefore, the end of the stopband is now  $0.45\pi - 0.015\pi = 0.435\pi$ , and the beginning of the passband is now  $0.45\pi + 0.015\pi = 0.465\pi$ .

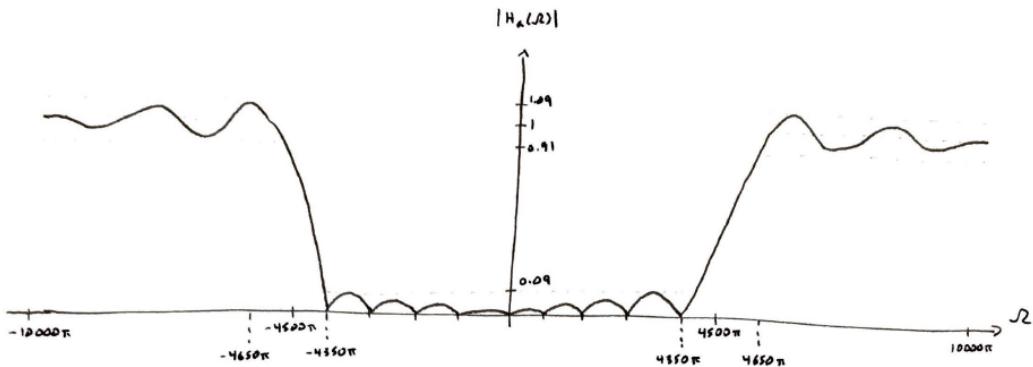
However, the sharper transition region comes with a cost - the maximum attenuation in the stopband is now only 21 dB. Furthermore, we would expect much more passband ripple - 0.75 dB for the rectangular window, compared to 0.055 dB for the Hann window. These parameters are given by  $\delta_p = \delta_s$  on page 563 of the textbook. We get 0.75 dB of passband ripple since  $20 \log_{10}(1.09) \approx 0.75$ , and 21 dB of stopband attenuation since  $20 \log_{10}(0.09) \approx -21$ .

We can use  $\omega = \Omega T$  to plot  $|H_a(\Omega)|$ , but **only** for  $|\Omega| \leq 10000\pi$ . If the input has any frequencies beyond  $10000\pi$ , aliasing will occur, and the system will no longer act as an LTI system. This is because LTI systems cannot create new frequencies - if  $X_d(\omega_0) = 0$ , and LTI system must give  $Y_d(\omega_0) = 0$ , since  $Y_d(\omega_0) = X_d(\omega_0)H_d(\omega_0)$ . However, if aliasing were to occur, new frequencies will be created, for example in Problem 2, Part (f).

To plot the approximate filter response on a linear scale, we:

- Determine the locations of the passband and stopband. Using the rectangular window, the stopband ends at  $0.435\pi$ , which corresponds to  $\Omega = 4350\pi$ . Similarly, the passband begins at  $0.465\pi$ , which corresponds to  $\Omega = 4650\pi$ . Since we used the exact definition of the transition region width to determine the filter length,  $H_a(4350\pi) = 0$ , and  $H_a(4650\pi) = 1$ .
- Determine the size of the ripple in passband and stopband. Since  $\delta_p = \delta_s \approx 0.09$ , this means that the passband ripples fluctuate between 0.91 and 1.09, and the stopband ripples fluctuate between 0 and 0.09 in magnitude. Additionally, we know the size of the ripples will decrease as we move further away from the transition region, due to the sidelobes of the "sinc" we're convolving with when we apply the window decreasing in size. So the ripples in the passband converge to 1 as  $\Omega \rightarrow 10000\pi$ , and the ripples in the stopband converge to 0 as  $\Omega \rightarrow 0$ .

The resulting plot is shown below.



If we want to plot on a dB scale, then the largest stopband ripple takes a value of -21 dB, and the stopband ripples fluctuate between -21 dB and  $-\infty$  dB (which corresponds to a zero), decreasing to  $-\infty$  dB as  $\Omega \rightarrow 0$ . Similarly, since 1 = 0 dB, the values in the stopband fluctuate between -0.75 dB and 0.75 dB, converging to 0 dB as  $\Omega \rightarrow 10000\pi$ . The resulting plot is shown below.

