The DFT

Up to this point in the course, we've learned how to analyze systems using the DTFT, defined as

$$X_d(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

However, the DTFT is *impossible* to use in any actual system, since ω is a continuous variable. Therefore, infinite memory is required to store the DTFT, which is obviously infeasible. So we do the next best thing; we *sample* the DTFT in the hopes that we can maintain most of its characteristics and properties. This sampling is known as the DFT, or Discrete Fourier Transform, and represents a mapping from discrete time to *discrete* frequency.

The formal definition of the DFT is given below.

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi kn}{N}}$$

There are a few important things to note in this definition.

- The DFT is only defined for *finite-length* sequences (those with length N). This makes sense, as infinite-length sequences cannot be stored in a computer.
- The DFT *samples* the DTFT every $\omega = \frac{2\pi k}{N}$, from $\omega = 0$ to $\omega = 2\pi$. That is,

  $$\boxed{X[k] = X_d(\frac{2\pi k}{N})}$$

  Therefore, the signal length will change the number of DTFT samples that are taken. Note the different range of the DFT samples; we still get a full period, but it's shifted by π.
- The DFT treats both the input and the output as *periodic* with period N.

DFT properties

Most of the properties closely mirror those of the DTFT. No DFT table should be necessary, since you can always derive a DFT by taking the DTFT and substituting $\omega = \frac{2\pi k}{N}$.

DFT Property	DT Signal	DFT
Linearity	$ax[n] + by[n]$	aX[k] + bY[k]
Time Shift	$x[\langle\langle n-d \rangle\rangle_N]$	$X[k]e^{-j\frac{2\pi kd}{N}}$
Modulation	$x[n]e^{j\frac{2\pi ln}{N}}$	$X[\langle\langle k-l \rangle\rangle_N]$
Time-Reversal	$x[\langle\langle -n \rangle\rangle_N]$	$X[\langle\langle N-k \rangle\rangle_N]$
Conjugation	$x^*[n]$	$X^*[\langle\langle N-k \rangle\rangle_N]$

In addition, there are some properties that are satisfied when $x[n]$ is real valued:

- $X[k] = X^*[\langle\langle N-k \rangle\rangle_N]$
- $\text{Re}\{X[k]\} = \text{Re}\{X[\langle\langle N-k \rangle\rangle_N]$
- $\text{Im}\{X[k]\} = -\text{Im}\{X[\langle\langle N-k \rangle\rangle_N]$
- $|X[k]| = |X[\langle\langle N-k \rangle\rangle_N]|$
- $\angle X[k] = -\angle X[\langle\langle N-k \rangle\rangle_n]$

Modulo Arithmetic

Notice that most of the properties have a rather strange form; what does $x[\langle\langle n-d \rangle\rangle_N]$ mean? Because the DFT treats everything as periodic, we never really "shift" a signal; when we shift to the left, or to the right, the *other copies* of the signal come in. Therefore, all arithmetic performed with respect to the DFT is *modulo* arithmetic with respect to N.

The modulo operator is simple; it just returns the remainder with respect to the base. For example, if we're performing arithmetic with respect to 8, then the maximum representable number is 7, since 7 mod 8 = 7 and 8 mod 8 = 0. We effectively "loop" back around to the beginning. This means that shifting a signal modulo N corresponds to *circularly* shifting the signal, instead of adding zeros. For example, suppose $x[n] = \{1, 2, 3, 4, 5\}$, and we wanted to compute $x[\langle\langle n-2 \rangle\rangle_5]$. This means that we need to *circularly shift* everything in $x[n]$ to the right by 2; what would "pop off" of the output gets wrapped back around to the input. Therefore:

$$x[\langle\langle n-1 \rangle\rangle_5] = \{5, 1, 2, 3, 4\}$$
$$x[\langle\langle n-2 \rangle\rangle_5] = \{4, 5, 1, 2, 3\}$$

Using the DFT - Circular Convolution and Zero-Padding

Convolutions

When using the DFT, it is no longer the case that multiplication in one domain corresponds to convolution in the other domain. Instead, multiplication corresponds to *circular* convolution, since every signal is periodic. Performing circular convolution is very similar to performing linear convolution using the table method; we leave one signal untouched, we flip the other signal, and shift it to the right, but there's one major difference; the signal we flip is treated as *periodic*, so the output will also be periodic. This requires both input signals to be the *same length.* For example, suppose we have $x[n] = \{1, 2, 3\}$ and $\{h[n]\} = \{2, 3, 4\}$. If we wanted to perform circular convolution, we would leave $x[n]$ untouched, flip $h[n]$, but treat it as periodic:

$$\begin{array}{l}
\quad\;\; 1\ 2\ 3 \\
4\ 3\ 2\ 4\ 3 \rightarrow y[0] = 2 + 8 + 9 = 19 \\
2\ 4\ 3\ 2\ 4\ \ \rightarrow y[1] = 3 + 4 + 12 = 19 \\
3\ 2\ 4\ 3\ 2\ \ \rightarrow y[2] = 4 + 6 + 6 = 16
\end{array}$$

We stop here, since we've covered one full period of $h[n]$. Therefore, $y[n] = \{19, 19, 16\}$.

For the linear algebra-inclined, circular convolution can also be represented in matrix form. If $x[n]$ and $h[n]$ both have length N, then the circular convolution matrix H will be $N \times N$. The first column will contain $h[n]$, and each concurrent column is obtained by circularly shifting the previous column down by 1. In this example, we would have

$$H = \begin{bmatrix} 2 & 4 & 3 \\ 3 & 2 & 4 \\ 4 & 3 & 2 \end{bmatrix}$$

Then, the circular convolution can be performed by setting $y = Hx$, where x is a vector containing the input:

$$y = \begin{bmatrix} 2 & 4 & 3 \\ 3 & 2 & 4 \\ 4 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 19 \\ 19 \\ 16 \end{bmatrix}$$

Zero-Padding

We saw that, since the DFT samples the DTFT every $\omega = \frac{2\pi k}{N}$, increasing the length of the signal will increase the number of samples we take, therefore increasing the accuracy of the approximation. The simplest way to do so is to *zero-pad* the signal, which means adding zeros onto the *end* of the signal. This will not change the DTFT, since we're just adding zeros to the summation. However, this will improve the quality of the DFT. For example, if $x[n] = \{1, 2, 3\}$, taking the DFT directly only gives us 3 samples of the DTFT. However, if we zero pad, setting $y[n] = \{1, 2, 3, 0, 0, 0, 0, 0\}$, taking the DFT will give eight samples of the DTFT, a large improvement in quality. It's important to know how the sampled frequencies change upon zero-padding. Since $X[k]$ and $Y[k]$ both sample the same DTFT, we have

$$Y[k] = X_d(\frac{2\pi k}{8}) \quad \text{and} \quad X[k] = X_d(\frac{2\pi k}{3})$$

Practice Problems and Solutions

Problem 1

Given that the DFT of $x[n] = \{2, 0, 6, 4\}$ is $\{X_0, X_1, X_2, X_3\}$, determine the DFT of $y[n] = \{2, 1, 0, 3\}$ and express the result in terms of X_0, X_1, X_2, X_3.

Solution

Any time you're given a similar problem (where you want to express one DFT in terms of another DFT), you should immediate start thinking about applying some properties. First, we notice that $\frac{1}{2}x[n] = \{1, 0, 3, 2\}$, which is almost the desired sequence; the only problem is that it's in the wrong order. But we know how to fix this; we can *circularly shift* the signal to the right by 1 and end up with $y[n]$. Therefore,

$$y[n] = \frac{1}{2}x[\langle\langle n-1\rangle\rangle_4]$$

Computing the DFT requires the linearity and time-shift properties. Since $ax[n] \leftrightarrow aX[k]$, and $x[\langle\langle n-d\rangle\rangle_N] \leftrightarrow X[k]e^{-j\frac{2\pi kd}{N}}$, we get

$$Y[k] = \frac{1}{2}e^{-j\frac{2\pi k}{4}}X[k]$$

Now, we can examine the effect on each element:

$$\begin{aligned}
Y[0] &= \frac{1}{2}X[0] = \frac{1}{2}X_0 \\
Y[1] &= \frac{1}{2}e^{-j\frac{\pi}{2}}X[1] = -\frac{j}{2}X_1 \\
Y[2] &= \frac{1}{2}e^{-j\pi}X[2] = -\frac{1}{2}X_2 \\
Y[3] &= \frac{1}{2}e^{-j\frac{3\pi}{2}}X[3] = \frac{j}{2}X_3
\end{aligned}$$

which tells us that

$$\boxed{Y[k] = \left\{\frac{1}{2}X_0, -\frac{j}{2}X_1, -\frac{1}{2}X_2, \frac{j}{2}X_3\right\}}$$

Problem 2

A 3 second segment of $x_a(t) = \cos(0.2\pi t)$ is sampled at a rate of $T = \frac{1}{30}s$. The resulting 90 samples are zero-padded to 128, and the DFT $\{X[k]\}_{k=0}^{127}$ is computed. Determine the k_0 that maximizes $X[k]$ over $0 \leq k \leq 63$.

When we sample, we use the relation $\omega = \Omega T$, so the corresponding discrete-time signal is $x[n] = \cos(\frac{\pi}{150}n)$, from $n = 0$ to $n = 89$. If $x[n]$ were to be infinite-length, the DTFT would just be two delta functions. However, because we have to *truncate* the signal somewhere, we can think of the received signal as

$$x[n] = \cos(\frac{\pi}{150}n)w[n]$$

where $w[n]$ is a *windowing* function defined as $u[n] - u[n - 90]$. Since we're multiplying two signals together in the time domain, it corresponds to a convolution in the frequency domain. We know the DTFT of $w[n]$ - it corresponds to a discrete-time/Dirichlet sinc function. So, $X_d(\omega)$ will be two sinc functions located at $\frac{\pi}{150}$ and $-\frac{\pi}{150}$; most importantly, the peaks will remain at $\frac{\pi}{150}$ and $-\frac{\pi}{150}$.

Note the restriction on k - this is because the DTFT is 2π-periodic, so we would also get a delta function at $\frac{299\pi}{150}$ - there are two possible maxima. The restriction tells us that we only need to worry about the first peak.

Therefore, the problem reduces to finding the k that gets us closest to the peak, since this will maximize the value of the DFT. Since we're told the signal is zero-padded to length 128, the DFT will sample the DTFT at $\omega = \frac{\pi k}{64}$. Solving gives:

$$\frac{\pi k}{64} = \frac{\pi}{150} \to k = \frac{64}{150}$$

However, this is impossible, since k is a discrete variable! Therefore, we need to round to the *nearest integer*, which results in

$$\boxed{k_0 = 0}$$

Problem 3

Determine the DFT of

$$x[n] = \cos(\frac{\pi n}{8}), 0 \leq n \leq 15$$

This problem exhibits the windowing effect; because we only take 16 samples of the cosine, we won't just get two delta functions in the DTFT. As usual, if you're asked to compute a DFT, first compute the DTFT, and then just set $\omega = \frac{2\pi k}{N}$. Doing so using the definition gives:

$$\begin{aligned}
X_d(\omega) &= \frac{1}{2}\sum_{n=0}^{15} e^{-j(\omega-\frac{\pi}{8})n} + \frac{1}{2}\sum_{n=0}^{15} e^{-j(\omega+\frac{\pi}{8})n} \\
&= \frac{1}{2}\sum_{n=0}^{15} (e^{-j(\omega-\frac{\pi}{8})})^n + \frac{1}{2}\sum_{n=0}^{15} (e^{-j(\omega-\frac{\pi}{8})})^n \\
&= \frac{1}{2}\left[\frac{1-e^{-j16(\omega-\frac{\pi}{8})}}{1-e^{-j(\omega-\frac{\pi}{8})}}\right] + \frac{1}{2}\left[\frac{1-e^{-j16(\omega+\frac{\pi}{8})}}{1-e^{-j(\omega+\frac{\pi}{8})}}\right]
\end{aligned}$$

Splitting the phase in both numerator and denominator gives

$$\begin{aligned}
X_d(\omega) &= \frac{1}{2}\frac{e^{-j8(\omega-\frac{\pi}{8})}}{e^{-j0.5(\omega-\frac{\pi}{8})}}\frac{\sin(8(\omega-\frac{\pi}{8}))}{\sin(0.5(\omega-\frac{\pi}{8}))} + \frac{1}{2}\frac{e^{-j8(\omega+\frac{\pi}{8})}}{e^{-j0.5(\omega+\frac{\pi}{8})}}\frac{\sin(8(\omega+\frac{\pi}{8}))}{\sin(0.5(\omega+\frac{\pi}{8}))} \\
&= \frac{1}{2}e^{-j7.5(\omega-\frac{\pi}{8})}\frac{\sin(8(\omega-\frac{\pi}{8}))}{\sin(0.5(\omega-\frac{\pi}{8}))} + \frac{1}{2}e^{-j7.5(\omega+\frac{\pi}{8})}\frac{\sin(8(\omega+\frac{\pi}{8}))}{\sin(0.5(\omega+\frac{\pi}{8}))}
\end{aligned}$$

This is exactly what we would expect to get. Since the DTFT of $u[n] - u[n-15]$ is a Dirichlet sinc, and the DTFT of $\cos(\frac{\pi}{8}n)$ is just two delta functions at $\omega = \frac{\pi}{8}$ and $\omega = -\frac{\pi}{8}$, the result is two Dirichlet sincs centered at $\omega = \frac{\pi}{8}$ and $\omega = -\frac{\pi}{8}$. Finally, to take the DFT, we just replace $\omega = \frac{2\pi k}{16} = \frac{\pi k}{8}$ to end up with the result.

$$\boxed{X[k] = \frac{1}{2}e^{-j7.5(\frac{\pi k}{8}-\frac{\pi}{8})}\frac{\sin(8(\frac{\pi k}{8}-\frac{\pi}{8}))}{\sin(0.5(\frac{\pi k}{8}-\frac{\pi}{8}))} + \frac{1}{2}e^{-j7.5(\frac{\pi k}{8}+\frac{\pi}{8})}\frac{\sin(8(\frac{\pi k}{8}+\frac{\pi}{8}))}{\sin(0.5(\frac{\pi k}{8}+\frac{\pi}{8}))}, k = 0, 1, ..., 15}$$

Problem 4

Let $X[k]$ be the 6-point DFT of $x[n] = \{1, 2, 3, 4, 5, 6\}$. Determine the sequence $y[n]$ whose DFT is given by $Y[k] = X[\langle\langle -k\rangle\rangle_6]$.

This problem is really just a test of your understanding of modular arithmetic. We're given the time-reversal property of the DFT, which states that

$$x[\langle\langle -n\rangle\rangle_N] \leftrightarrow X[\langle\langle N-k\rangle\rangle_N]$$

However, what is $N-k$ mod N? Since the arithmetic is being performed with respect to N, we can add or subtract as many factors of N as we want without changing the result. Therefore, we can write $N-k$ mod N as $-k$ mod N, and $-n$ mod N as $N-n$ mod N. This gets everything into the form we need to solve the problem; we rewrite the property as

$$X[\langle\langle N-n\rangle\rangle_N] \leftrightarrow X[\langle\langle -k\rangle\rangle_N]$$

making it obvious that

$$y[n] = x[\langle\langle 6-n\rangle\rangle_6]$$

To calculate this, it's simplest just to plug in values of n:

$$\begin{aligned}
y[0] &= x[\langle\langle 6\rangle\rangle_6] = x[0] \\
y[1] &= x[\langle\langle 5\rangle\rangle_6] = x[5] \\
y[2] &= x[\langle\langle 4\rangle\rangle_6] = x[4] \\
&\vdots \\
y[5] &= x[\langle\langle 1\rangle\rangle_6] = x[1]
\end{aligned}$$

We see that the first element remains unchanged, and all of the other elements are flipped. Therefore,

$$\boxed{y[n] = \{1, 6, 5, 4, 3, 2\}}$$