ECE 310: Recitation 4

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Notes

BIBO Stability:

$$|x[n]| \leq M_x < \infty \Rightarrow |y[n]| \leq M_y < \infty$$

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

$$|H(z)| = \sum_{n=-\infty}^{\infty} \left|h[n] z^{-n}\right| < \infty, |z| = 1$$

Causality:

$$y[n] = \sum_{k=0}^{M} a_k x[n - b_k] \Rightarrow b_k \geq 0 \ \forall k$$

$$h[n] = 0 \ \forall n < 0$$

$$H[z] = \sum_{n=0}^{\infty} h[n] z^{-n} \Rightarrow \text{only negative powers of z}$$

Region of convergence:

Stability: ROC must include the unit circle (all poles must be inside)

Causality: ROC must extend outside some circle of radius r, or $|z| > r$

Table 3.1 Some common z-transform pairs

	Sequence $x[n]$	z-Transform $X(z)$	ROC
1.	$\delta[n]$	1	All z
2.	$u[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3.	$a^n u[n]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
4.	$-a^n u[-n-1]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
5.	$na^n u[n]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
6.	$-na^n u[-n-1]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
7.	$(\cos\omega_0 n)u[n]$	$\dfrac{1-(\cos\omega_0)z^{-1}}{1-2(\cos\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
8.	$(\sin\omega_0 n)u[n]$	$\dfrac{(\sin\omega_0)z^{-1}}{1-2(\cos\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
9.	$(r^n\cos\omega_0 n)u[n]$	$\dfrac{1-(r\cos\omega_0)z^{-1}}{1-2(r\cos\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
10.	$(r^n\sin\omega_0 n)u[n]$	$\dfrac{(\sin\omega_0)z^{-1}}{1-2(r\cos\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$

Table 3.2 Some z-transform properties.

	Property	Sequence	Transform	ROC
		$x[n]$	$X(z)$	R_x
		$x_1[n]$	$X_1(z)$	R_{x_1}
		$x_2[n]$	$X_2(z)$	R_{x_2}
1.	Linearity	$a_1x_1[n]+a_2x_2[n]$	$a_1X_1(z)+a_2X_2(z)$	At least $R_{x_1}\bigcap R_{x_2}$
2.	Time shifting	$x[n-k]$	$z^{-k}X(z)$	R_x except $z=0$ or ∞
3.	Scaling	$a^n x[n]$	$X(a^{-1}z)$	$\lvert a\rvert R_x$
4.	Differentation	$nx[n]$	$-z\dfrac{dX(z)}{dz}$	R_x
5.	Conjugation	$x^*[n]$	$X^*(z^*)$	R_x
6.	Real-part	$\mathrm{Re}\{x[n]\}$	$\frac{1}{2}[X(z)+X^*(z^*)]$	At least R_x
7.	Imaginary part	$\mathrm{Im}\{x[n]\}$	$\frac{1}{2}[X(z)-X^*(z^*)]$	At least R_x
8.	Folding	$x[-n]$	$X(1/z)$	$1/R_x$
9.	Convolution	$x_1[n]*x_2[n]$	$X_1(z)X_2(z)$	At least $R_{x_1}\bigcap R_{x_2}$
10.	Initial-value theorem	$x[n]=0$ for $n<0$	$x[0]=\lim\limits_{z\to\infty}X(z)$	

1

Use the Z-transform to find the output $y[n]$ for a system with impulse response $h[n] = 2^{-n}u[n] + 3^n u[-n-1]$ given input $x[n] = \left(\frac{3}{4}\right)^n u[n]$

$$\begin{aligned}
H(x) &= \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{1}{1-3z^{-1}} \\
X(z) &= \frac{1}{1-\frac{3}{4}z^{-1}} \\
Y(z) &= H(x)\,X(z) \\
&= \left(\frac{1}{1-\frac{1}{2}z^{-1}} - \frac{1}{1-3z^{-1}}\right)\frac{1}{1-\frac{3}{4}z^{-1}} \\
&= \frac{1}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{3}{4}z^{-1}\right)} - \frac{1}{\left(1-3z^{-1}\right)\left(1-\frac{3}{4}z^{-1}\right)} \\
&= \left(\frac{A}{1-\frac{1}{2}z^{-1}} + \frac{B}{1-\frac{3}{4}z^{-1}}\right) - \left(\frac{C}{1-3z^{-1}} + \frac{D}{1-\frac{3}{4}z^{-1}}\right)
\end{aligned}$$

$$\begin{aligned}
A\left(1-\frac{3}{4}z^{-1}\right) + B\left(1-\frac{1}{2}z^{-1}\right) &= 1 \\
z^{-1} = \frac{4}{3} &\Rightarrow B = 3 \\
z^{-1} = 2 &\Rightarrow A = -2 \\
C\left(1-\frac{3}{4}z^{-1}\right) + D\left(1-3z^{-1}\right) &= 1 \\
z^{-1} = \frac{4}{3} &\Rightarrow D = -\frac{1}{3} \\
z^{-1} = frac13 &\Rightarrow C = \frac{4}{3}
\end{aligned}$$

$$\begin{aligned}
H(z) &= \left(\frac{-2}{1-\frac{1}{2}z^{-1}} + \frac{3}{1-\frac{3}{4}z^{-1}}\right) - \left(\frac{\frac{4}{3}}{1-3z^{-1}} + \frac{-\frac{1}{3}}{1-\frac{3}{4}z^{-1}}\right) \\
h[n] &= -2\left(\frac{1}{2}\right)^n u[n] + 3\left(\frac{3}{4}\right)^n u[n] - \frac{4}{3}(3)^n u[n] + \frac{1}{3}\left(\frac{3}{4}\right)^n u[n]
\end{aligned}$$

2

Find the impulse response $h[n]$ for a system with output $y[n] = 4(2)^2 y[n] - \left(\frac{1}{2}\right)^n u[n]$ when the input is $x[n] = (-3)^n u[n]$

$$\begin{aligned} Y(z) &= \frac{4}{1-2z^{-1}} - \frac{1}{1-\frac{1}{2}z^{-1}} \\ X(z) &= \frac{1}{1+3z^{-1}} \\ H(z) &= \frac{Y(z)}{X(z)} \\ &= \frac{\left(1+3z^{-1}\right)\left(4\left(1-\frac{1}{2}z^{-1}\right)-\left(1-2z^{-1}\right)\right)}{\left(1-2z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)} \\ &= \frac{\left(1+3z^{-1}\right)(3)}{\left(1-2z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)} \\ &= \frac{\left(3+9z^{-1}\right)}{\left(1-2z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)} = \frac{A}{1-2z^{-1}} + \frac{B}{1-\frac{1}{2}z^{-1}} \end{aligned}$$

$$3+9z^{-1} = A\left(1-\frac{1}{2}z^{-1}\right) + B\left(1-2z^{-1}\right)$$

$$\begin{aligned} z^{-1} &= 2 \Rightarrow B = -7 \\ z^{-1} &= \frac{1}{2} \Rightarrow A = 10 \end{aligned}$$

$$\begin{aligned} H(z) &= \frac{10}{1-2z^{-1}} + \frac{-7}{1-\frac{1}{2}z^{-1}} \\ h[n] &= 10\left(2^n u[n]\right) - 7\left(\left(\frac{1}{2}\right)^n u[n]\right) \end{aligned}$$

3

Determine if the systems with input $x[n]$ and output $y[n]$ are BIBO stable:

a) $y[n] = (n-3)\,x[2n+1]$
For $|x[n]| < M < \infty$, $|x[2n+1]| < M < \infty$. But as $n \to \infty$, $n-3 \to \infty$ and the system is unstable.

b) $y[n] = \frac{1}{n-4}x[n]$
For $n = 4$ the denominator is 0 and $yn = \infty$. The system is unstable.

c) $y[n] = x^2[n]\sin(x[n])$
For $|x[n]| < M < \infty$, $\left|x^2[n]\right| < M^2 < \infty$ and $|\sin(x[n])| \leq 1$ so the system is stable.

4

Determine if the system function $H(z)$ of a causal system represents a BIBO stable system:

a) $H(z) = \frac{z^2+3z-4}{z^2-\frac{1}{4}}$
The poles are at $z = \frac{1}{2}, -\frac{1}{2}$ and the system is stable.

b) $H(z) = \frac{3z+1}{z^2+1/16}$
The poles are at $z = \frac{1}{4}e^{j\frac{\pi}{2}}, -\frac{1}{4}e^{j\frac{\pi}{2}}$ and the system is stable.

c) $H(z) = \frac{1}{\left(z^2-\frac{4}{9}\right)(z^2-8)}$
The poles are at $z = \frac{2}{3}, -\frac{2}{3}, 2\sqrt{2}, -2\sqrt{2}$ and the system is unstable.