University of Illinois Fall 2018

ECE 310: Recitation 5

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Eisensignals and LTI systems

An input in the form $e^{j\omega_0}$, which is a complex exponential, is called an 'eisensignal' for the LTI system. This is because an output will just be a scaling for the input.

$$x[n] = e^{j\omega_0 n} \leftrightarrow y[n] = H(\omega_0)e^{j\omega_0 n}$$

where $H(\omega)$ is the transfer function of our LTI system. This concept motivates the usage of the Fourier Transforms. *(Recall that an eigenvector of a linear transform is a vector that is only changed by a scalar factor when the transform is applied.)*

Fourier Synthesis and Fourier Analysis

Because of the above property of an eigensignal, and the fact that any signal sequence can be viewed as a composition of eigensignals, <u>Fourier synthesis</u> is derived as

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega$$

otherwise known as the inverse Fourier transform. Here, we can view $X(\omega)$ as the coefficients for eigensignals of different frequencies, i.e. $e^{j\omega_1 n}$, $e^{j\omega_2 n}\dots$

The counterpart to Fourier synthesis is <u>Fourier analysis</u>, also known as the Fourier tarnsform.

$$X(\omega) = \sum_{n=\infty}^{\infty} x[n]e^{-j\omega n}$$

We can see that the above equation 'analyzes' our sequence $x[n]$ to find $X(\omega)$, the coefficients of the eigensignals. Similarly, for the continuous time Fourier Transform (CTFT), we have

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega)e^{j\Omega t}d\Omega$$
$$X(\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}d\omega$$

Note that the unit for Ω the continuous time case is Hertz (Hz), while the unit for ω in the discrete time case is radians/sample.

CTFT : Laplace transform :: DTFT : z-transform

The CTFT is to the Laplace transform what the DTFT is to the z-transform. Why? Recall that the Laplace transform is defined as

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

Letting $s = j\Omega$ gives us

$$X(\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt$$

which is nothing but a scaled CTFT.

For a z-transform, we have

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Letting $z = e^{j\omega}$ gives us

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

which is the definition of the DTFT. Thus we observe that the CTFT and DTFT are special cases of the Laplace Transform and the z-transform, respectively. We must take a look at the s-plane and z-plane to know the significance of these special cases.

First, when we say that the CTFT is the Laplace transform with $s = j\Omega$, we are looking at only the imaginary axis of the s-plane. Thus, we care whether or not the region of convergence (ROC) of a transfer function includes the imaginary axis.

Similarly for the z-transform, when we say the DTFT is the z-transform when $z = e^{jw}$, we are taking the z-transform only at the unit circle of the z-plane. This is why we care whether or not the ROC of a transfer function includes the unit circle.

Note: The Fourier Transform does not always exist - there are a few conditions which ensure its existence (absolute integrability and square integrability of the function).

Practice Problems

1. True or False: the DTFT is defined only for real-valued signals.

2. True or False: if $x[n]$ is a band-limited sequence, there must be a finite ω_{max} such that

$$|X_d(\omega)| = 0, \ \forall \ |\omega| > |\omega_{max}|$$

3. True or False: The input and output relationship of **any** system is completely determined by the system's unit pulse response.

4. True or False: the output $y[n]$ of a system for an arbitrary signal $x[n]$ is given by $y[n] = x[n] * h[n]$, where $h[n]$ is the unit pulse response of the system. The system must by linear and shift invariant.

5. Determine $x[n]$ whose DTFT is given by:

$$X(\omega) = \begin{cases} 1, & -\pi/3 < \omega < -\pi/9 \\ 2, & -\pi/9 < \omega < \pi/9 \\ 1, & \pi/9 < \omega < \pi/3 \\ 0, & \pi/3 < |\omega| \leq \pi \end{cases}$$

6. Compute the CTFT of $\cos(\Omega_0 t)$

Solutions

1. **False**. We know that the DTFT is linear, so multiplying a function by a complex constant results in a DTFT which is multiplied by the same complex constant. Thus, we can multiply any real function that has a DTFT by j and create an imaginary-valued sequence with an existing DTFT.

2. **False**. The DTFT is 2π-periodic. Therefore, it will never be bandlimited in frequency - we always get copies of what's between $-\pi$ and π over the entire spectrum.

3. **False**. This holds only for LTI systems. Because they are linear and time-invariant, we can represent any discrete-time signal as a summation of shifted and scaled delta functions. If we know the output of any LTI system to the delta function, we can use ethe time-invariance and linearity to obtain the output for any discrete-time signal. Therefore, the impulse response completely characterizes LTI systems.

4. **True**. We know that the impulse response characterizes an LTI system. So if the output is given by the convolution with the impulse response, it must be LTI.

5. There are two approaches to the problem:
   Method 1 Using the inverse DTFT formula, we have

$$\begin{aligned} x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\left(\int_{-\pi/3}^{-\pi/9} 1\cdot e^{j\omega n}d\omega + \int_{-\pi/9}^{\pi/9} 2\cdot e^{j\omega n} + \int_{\pi/9}^{\pi/3} 1\cdot e^{j\omega n}\right) \end{aligned}$$

   Using Euler's method, we can get our answer in two more steps. This should be the standard way to solve a sequence $x[n]$ from the DTFT.
   Method 2 Observe that we can solve this using the linearity property of DTFT. In particular, we can view $X(\omega)$ as a sum of two rectangular functions:

$$X_1(\omega) = \begin{cases} 1, & -\pi/3 < \omega < \omega/3 \\ 0, & \text{else} \end{cases} \quad X_2(\omega) = \begin{cases} 1, & -\pi/9 < \omega < \omega/9 \\ 0, & \text{else} \end{cases}$$

   Note that $\mathcal{F}(a+b) = \mathcal{F}(a) + \mathcal{F}(b)$, and reading from the DTFT table:

$$z[n] = \frac{\omega_0}{\pi}\text{sinc}(\omega_0 n) \leftrightarrow Z(\omega) = \begin{cases} 1, & |\omega| < \omega_0 \\ 0, & \text{else} \end{cases}$$

   We see that $x[n]$ is a sum of two sinc functions.

$$x[n] = \frac{1}{3}\text{sinc}(\frac{\pi}{3}n) + \frac{1}{9}\text{sinc}(\frac{\pi}{9}n)$$

6. The Fourier transform of $\cos(\Omega_0 t)$ can be obtained as

$$\begin{aligned} X(\Omega) &= \int_{-\infty}^{\infty} cos(\Omega_0 t)e^{-j\Omega t}dt \\ &= \int_{-\infty}^{\infty} \frac{e^{j\Omega_0 t} + e^{-j\Omega_0 t}}{2}e^{-j\Omega t}dt \\ &= \int_{-\infty}^{\infty} \frac{1}{2}e^{-j(\Omega-\Omega_0)t}dt + \int_{-\infty}^{\infty} \frac{1}{2}e^{-j(\Omega+\Omega_0)t}dt \end{aligned}$$

Now we can use the fact that $\int_{-\infty}^{\infty} e^{j\Omega t} dt = d\pi\delta(\Omega)$ and that $\mathcal{F}(e^{j\Omega_0 t}x(t)) = X(\Omega - \Omega_0)$ to get

$$\begin{aligned} X(\Omega) &= \int_{-\infty}^{\infty} \frac{1}{2} e^{-j(\Omega - \Omega_0)t} dt + \int_{-\infty}^{\infty} \frac{1}{2} e^{-j(\Omega + \Omega_0)t} dt \\ &= \pi\delta(\Omega - \Omega_0) + \pi\delta(\Omega + \Omega_0) \end{aligned}$$