Lecture 8

Review LCCDE:

$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]$$

We assume system is causal, zero initial conditions ($y[k]=0$ for $k<0$)

Transfer function:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + \dots + b_n z^{-M}}{1 + a_1 z^{-1} + \dots + a_N z^{-N}} = \frac{b_0 (1-z_1 z^{-1})(1-z_2 z^{-1}) \dots (1-z_n z^{-1})}{(1-p_1 z^{-1}) \dots (1-p_N z^{-1})}$$

We can get the impulse response $h[n]$

P.F.E.:

$$H(z) = \underbrace{\frac{A_1}{1-p_1 z^{-1}} + \frac{A_2}{1-p_2 z^{-1}} + \dots + \frac{A_N}{1-p_N z^{-1}}} + \overbrace{k_0 + k_1 z^{-1} + \dots + k_L z^{-L}}^{\text{if } \deg B \geq \deg A}$$

inverse z-transform

(causality)

$$h[n] = A_1 p_1^n u[n] + A_2 p_2^n u[n] + \dots + A_N p_N^n u[n] + k_0 \delta[n] + k_1 \delta[n-1] + \dots + k_L \delta[n-L]$$

here we assumed all poles are distinct

What if $H(z)$ has double poles?

$$\text{Ex: } H(z) = \frac{1}{(1-z^{-1})(1-3z^{-1})^2} \overset{\text{PFE}}{=} \frac{A_1}{1-z^{-1}} + \frac{A_2}{1-3z^{-1}} + \frac{A_3}{(1-3z^{-1})^2}$$

$$A_1(1-3z^{-1})^2 + A_2(1-z^{-1})(1-3z^{-1}) + A_3(1-z^{-1}) = 1$$

$z = 1$: $\quad A_1(1-3)^2 = 1 \Rightarrow A_1 = 1/4$

$z = 3$: $\quad A_3(1-1/3) = 1 \Rightarrow A_3 = 3/2$

$$\frac{1}{4}(1-3z^{-1})^2 + A_2(1-z^{-1})(1-3z^{-1}) + \frac{3}{2}(1-z^{-1}) = 1$$

pick arbitrary z, say $z = 2$.

$$\frac{1}{4}(1-3/2)^2 + A_2(1-1/2)(1-3/2) + \frac{3}{2}(1-1/2) = 1 \Rightarrow A_2 = -3/4$$

$$H(z) = \frac{1}{4}\cdot\frac{1}{1-z^{-1}} + \left(\frac{-3}{4}\right)\frac{1}{1-3z^{-1}} + \frac{3}{2}\,\frac{1}{(1-3z^{-1})^2}$$

$x[n]$	$X(z)$
$a^n u[n]$	$\frac{1}{1-az^{-1}}$
$n a^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$

$1 = \frac{z}{a}\cdot(az^{-1})$

$$\frac{(1-3z^{-1}) + 3z^{-1}}{(1-3z^{-1})^2} = \frac{1}{1-3z^{-1}} + \frac{3z^{-1}}{(1-3z^{-1})^2}$$

z^{-1}

$$3^n u[n] + n3^n u[n]$$

$$h[n] = \frac{1}{4}u[n] - \frac{3}{4}\cdot 3^n u[n] + \frac{3}{2}\left(3^n u[n] + n3^n u[n]\right)$$

Back to BIBO stability:

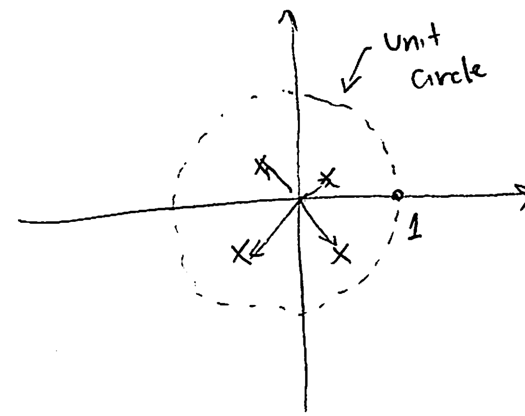
Recall: $\{x[n]\} \rightarrow \boxed{S} \rightarrow \{y[n]\}$

Defn 1: S is BIBO stable if and only if

$$|x[n]| < B_{in} \text{ for all } n \implies |y[n]| < B_{out} \text{ for all } n$$

Defn 2: Suppose S is LTI. S is BIBO stable if and only if

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty .$$



For LCCDE system:

$$\sum_{n=-\infty}^{\infty} \left| A_1 p_1^n u[n] + \dots + A_N p_N^n u[n] + k_0 \delta[n] + \dots + k_L \delta[n-L] \right|$$

$$\leq \sum_{n=0}^{\infty} \left(|A_1 p_1^n| + |A_2 p_2^n| + \dots + |A_N p_N^n| + |k_0 \delta[n]| + \dots + |k_L \delta[n-L]| \right)$$

$$= |A_1| \sum_{n=0}^{\infty} |p_1|^n + \dots + |A_N| \sum_{n=0}^{\infty} |p_N|^n + |k_0| \left(\cancelto{1}{\sum_{n=0}^{\infty} |\delta[n]|} \right) + \dots + |k_L| \left(\cancelto{1}{\sum_{n=0}^{\infty} |\delta[n-L]|} \right)$$

$$\frac{1}{1-|p_1|} \text{ if } |p_1| < 1$$

all poles need to be strictly inside the unit circle

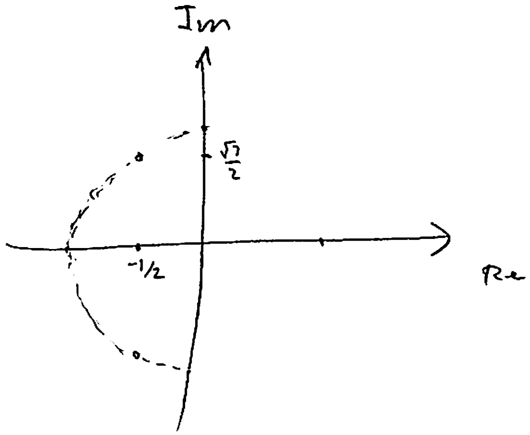
System is stable if $|p_1| < 1, |p_2| < 1, \dots, |p_N| < 1$

Defn 3: Suppose S is an LCCDE system (causal, zero initial conditions).

S is BIBO stable if and only if all poles are inside unit circle.

Ex: $H(z) = \dfrac{1}{1+z^{-1}+z^{-2}}$, Is this system stable?

$1+z^{-1}+z^{-2} = 0 \Rightarrow z^2 + z + 1 = 0 \Rightarrow z = \dfrac{-1 \pm \sqrt{1-4}}{2} = -\dfrac{1}{2} \pm j\dfrac{\sqrt{3}}{2}$ (poles)



$|p_1| = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$

$\Rightarrow$ S is not stable.