ECE 310: Problem Set 11

Due: 5pm, Friday November 16, 2018

1. Design a length-18 FIR low pass filter with cutoff frequency $\omega_c = \frac{\pi}{3}$ radians, using the window design method.

   (a) Find an expression for the filter coefficients $\{h_n\}_{n=0}^{17}$ if the rectangular window is used for the design. (10 pts)

   The impulse response of an ideal LPF is given by

$$h_{\text{lp}}[n] = \frac{\sin(\omega_c n)}{\pi n}$$

   .

   Before applying the window, we need to shift this filter by a factor of $\frac{M}{2}$ so that our filter will be causal and exhibit generalized linear phase. This lines up with the definition of a LPF given by (10.22) in the textbook, where we set $\omega_c = \frac{\pi}{3}$ and $\alpha = 17/2$. Our shifted ideal LPF is then

$$\begin{aligned} h_{\text{lp}}[n] &= \frac{\sin(\frac{\pi}{3}(n - \frac{17}{2}))}{\pi(n - \frac{17}{2})} \\ &= \frac{1}{3}\text{sinc}(\frac{\pi}{3}(n - \frac{17}{2})) \end{aligned}$$

   .

   Here, and in following problems, we define the sinc function as $\text{sinc}(x) = \sin(x)/x$. Applying the rectangular window gives the solution

$$\boxed{\{h[n]\}_{n=0}^{17} = \begin{cases} \frac{1}{3}\text{sinc}(\frac{\pi}{3}(n - \frac{17}{2})) & 0 \leq n \leq 17 \\ 0 & \text{else} \end{cases}}$$

   (b) Find an expression for the filter coefficients $\{h_n\}_{n=0}^{17}$ if the Hamming window is used for the design. (5 pts)

   (10.75) in the textbook gives the definition of the Hamming window. We simply apply this to $h_{\text{lp}}[n]$ in place of the rectangular window.

$$\boxed{\{h[n]\}_{n=0}^{17} = \begin{cases} \left(0.54 - 0.46\cos(2\pi n/17)\right)\frac{1}{3}\text{sinc}(\frac{\pi}{3}(n - \frac{17}{2})) & 0 \leq n \leq 17 \\ 0 & \text{else} \end{cases}}$$

2. Find an expression for the filter coefficients $\{h_n\}_{n=0}^{29}$ of a FIR high pass filter with cutoff frequency $\omega_c = \frac{2\pi}{3}$ radians, using the following methods:

   (a) Directly using window design with a Hann window. (10 pts)

   Because $M = 29$ is odd order, we are restricted to a choice of Type II and Type IV filters. But Type II filters are not ideal for HPF because even symmetry, with odd order of M, means that we will always have a 0 at $\omega = \pi$, so we must choose a Type IV filter with odd symmetry.

$$H_{\text{hp}}(\omega) = \begin{cases} e^{j(\frac{\pi}{2} - \frac{29}{2}\omega)} & |\omega| \geq \frac{2\pi}{3} \\ 0 & \text{else} \end{cases}$$

We then take the IDTFT to find the impulse response.

$$\begin{aligned}
h[n] &= \frac{1}{2\pi}\left[\int_{-\pi}^{-2\pi/3} e^{-j(\frac{\pi}{2}+\frac{29}{2}\omega)} e^{j\omega n} d\omega + \int_{2\pi/3}^{\pi} e^{j(\frac{\pi}{2}-\frac{29}{2}\omega)} e^{j\omega n} d\omega\right] \\
&= \frac{1}{2\pi} e^{j\frac{\pi}{2}} \int_{2\pi/3}^{4\pi/3} e^{-j\frac{29}{2}\omega} e^{j\omega n} d\omega \\
&= \frac{1}{2\pi} e^{j\frac{\pi}{2}} \frac{e^{j\omega(n-29/2)}}{j(n-29/2)}\Bigg|_{2\pi/3}^{4\pi/3} \\
&= \frac{1}{2\pi} e^{j\frac{\pi}{2}} \frac{e^{j4\pi/3(n-29/2)} - e^{j2\pi/3(n-29/2)}}{j(n-29/2)} \\
&= \frac{e^{j\pi n} e^{-j14\pi} 2j\sin(\frac{\pi}{3}(n-29/2))}{2\pi j(n-29/2)} \\
&= (-1)^n \frac{1}{3}\text{sinc}(\frac{\pi}{3}(n - \frac{29}{2}))
\end{aligned}$$

Note that the second step, where we change the bounds of integration and combine the integrals, can be performed because the DTFT is 2π-periodic. Thus, integrating in the interval $[-\pi, \pi]$ is equivalent to integrating in the interval $[0, 2\pi]$.

Applying the Hann window, which is given by (10.74) in the textbook, we arrive at:

$$\boxed{\{h[n]\}_{n=0}^{29} = \begin{cases} \left(0.5 - 0.5\cos(\frac{2\pi n}{29})\right)(-1)^n \frac{1}{3}\text{sinc}(\frac{\pi}{3}(n - \frac{29}{2})) & 0 \leq n \leq 29 \\ 0 & \text{else} \end{cases}}$$

(b) First designing a low-pass filter and then converting it to a high-pass filter (and again using a Hann window). (5 pts)

Our ideal HPF has a total passband region of $\frac{2\pi}{3}$, which is analagous to a LPF with cutoff frequency $w_c = \frac{\pi}{3}$.As in Question 1.a, a length 30 windowed LPF with $w_c = \frac{\pi}{3}$ is given by

$$\{h_{\text{lp}}[n]\}_{n=0}^{29} = \begin{cases} \left(0.5 - 0.5\cos(\frac{2\pi n}{29})\right) \frac{1}{3}\text{sinc}(\frac{\pi}{3}(n - \frac{29}{2})) & 0 \leq n \leq 29 \\ 0 & \text{else} \end{cases}$$

We now modulate this by π to turn this from a low pass to a high pass filter. Note that this modulation also takes advantage of the 2π-periodicity of the DTFT.

$$\begin{aligned}
h_{\text{hp}}[n] &= e^{j\pi n} h_{\text{LP}}[n] \\
&= \boxed{\begin{cases} \left(0.5 - 0.5\cos(\frac{2\pi n}{29})\right)(-1)^n \frac{1}{3}\text{sinc}(\frac{\pi}{3}(n - \frac{29}{2})) & 0 \leq n \leq 29 \\ 0 & \text{else} \end{cases}}
\end{aligned}$$

We arrive at the same solution as in Question 2.a.

3. Design a length-8, anti-symmetric (i.e., having odd symmetry) differentiating FIR filter ($D_d(\omega) = j\omega$ before shifting) using the window design method with a simple truncation (i.e., rectangular/boxcar) window. Give the filter coefficients as your answer. (20 pts)

   Since $M = 7$ is odd and the filter has odd symmetry, this is a Type IV filter. (Note that if M were even, we can also implement a differentiator using a Type I filter.) The frequency response is given by:

$$\begin{aligned} D_d(\omega) &= (j\omega)e^{-j\frac{7}{2}\omega} \\ &= \omega e^{j(\frac{\pi}{2}-\frac{7}{2}\omega)} \end{aligned}$$

   Take the IDTFT to find the impulse response of this filter.

$$\begin{aligned} d[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \omega e^{j(\frac{\pi}{2}-\frac{7}{2}\omega)} e^{-j\omega n} d\omega \\ &= \frac{j}{2\pi}\int_{-\pi}^{\pi} \omega e^{j\omega(n-\frac{7}{2})} d\omega \end{aligned}$$

   Here, use integration by parts, letting $u = \omega$ and $dv = e^{j\omega(n-\frac{7}{2})}d\omega$.

$$\begin{aligned} d[n] &= \frac{j}{2\pi}\left[\frac{\omega e^{j\omega(n-3.5)}}{j(n-3.5)}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{e^{j\omega(n-3.5)}}{j(n-3.5)}d\omega\right] \\ &= \frac{j}{2\pi}\left[\frac{\pi e^{j\pi(n-3.5)} + \pi e^{-j\pi(n-3.5)}}{j(n-3.5)} - \frac{e^{j\omega(n-3.5)}}{-(n-\frac{7}{2})^2}\Big|_{-\pi}^{\pi}\right] \\ &= \frac{j}{2\pi}\left[\frac{\pi(2\cos(\pi(n-\frac{7}{2})))}{j(n-\frac{7}{2})} - \frac{2j\sin(\pi(n-\frac{7}{2}))}{-(n-\frac{7}{2})^2}\right] \\ &= \frac{\cos[\pi(n-\frac{7}{2})]}{(n-\frac{7}{2})} - \frac{\sin[\pi(n-\frac{7}{2})]}{\pi(n-\frac{7}{2})^2} \end{aligned}$$

   Then, applying the rectangular window, we have the following windowed impulse response.

$$\{d[n]\}_{n=0}^{7} = \begin{cases} \frac{\cos[\pi(n-\frac{7}{2})]}{(n-\frac{7}{2})} - \frac{\sin[\pi(n-\frac{7}{2})]}{\pi(n-\frac{7}{2})^2} & 0 \le n \le 7 \\ 0 & \text{else} \end{cases}$$

   Filter coefficients are given by:

$$\boxed{\{d[n]\}_{n=0}^{7} = \{-.0260, .0509, -.1415, 1.2732, -1.2732, .1415, -.0509, .0260\}}$$

4. The Hann window function can be written as

$$w[n] = [0.5 - 0.5\cos(2\pi n/N)]w_R[n]$$

   where $w_R[n]$ is the rectangular window of length $N + 1$.

(a) Express the DTFT of $w[n]$ in terms if the DTFT of $w_R[n]$ (10 pts)

We express the frequency response of $w_R[n]$ as

$$W_R(\omega) = \sum_{n=-\infty}^{\infty} w_R[n]e^{-j\omega n}$$

. Now find the DTFT of $w[n]$.

$$\begin{aligned} W_d(\omega) &= 0.5 \sum_{n=-\infty}^{\infty} w_R[n]e^{-j\omega n} - 0.5 \sum_{n=-\infty}^{\infty} \cos(2\pi n/N) w_R[n]e^{-j\omega n} \\ &= \boxed{0.5W_R(\omega) - 0.25W_R(\omega + \frac{2\pi}{N}) - 0.25W_R(\omega - \frac{2\pi}{N})} \end{aligned}$$

The final step follows from the cosine modulation property.

(b) Express the DTFT of $w[n]$ as $W_d(\omega) = A(\omega)e^{-j\omega N/2}$, where $A(\omega)$ is a real-valued function. By analyzing the shape of $A(\omega)$, explain why the Hann window has a wider mainlobe but lower sidelobes than the rectangular of the same length. (10 pts)

$$W_R(\omega) = \frac{\sin(\frac{N+1}{2}\omega)}{\sin(\frac{\omega}{2})}e^{-j\omega N/2}$$

$$\begin{aligned} W_d(\omega) = 0.5\frac{\sin(\frac{N+1}{2}\omega)}{\sin(\frac{\omega}{2})}e^{-j\omega N/2} &- 0.25\frac{\sin(\frac{N+1}{2}(\omega + \frac{2\pi}{N}))}{\sin(\frac{\omega+2\pi/N}{2})}e^{-j(\omega+\frac{2\pi}{N})N/2} \\ &- 0.25\frac{\sin(\frac{N+1}{2}(\omega - \frac{2\pi}{N}))}{\sin(\frac{\omega-2\pi/N}{2})}e^{-j(\omega-\frac{2\pi}{N})N/2} \end{aligned}$$

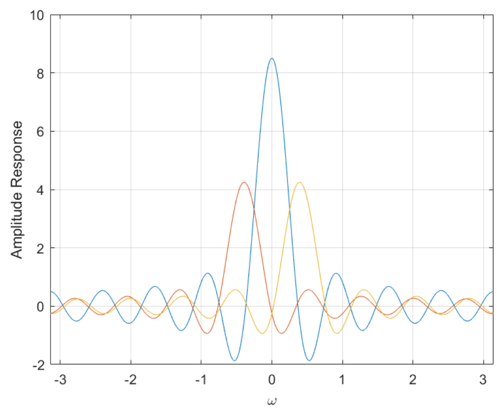
$$\begin{aligned} = 0.5\frac{\sin(\frac{N+1}{2}\omega)}{\sin(\frac{\omega}{2})}e^{-j\omega N/2} &- 0.25\frac{\sin(\frac{N+1}{2}(\omega + \frac{2\pi}{N}))}{\sin(\frac{\omega+2\pi/N}{2})}e^{-j\omega N/2}e^{-j\pi} \\ &- 0.25\frac{\sin(\frac{N+1}{2}(\omega - \frac{2\pi}{N}))}{\sin(\frac{\omega-2\pi/N}{2})}e^{-j\omega N/2}e^{j\pi} \end{aligned}$$

$$= \boxed{\left[0.5\frac{\sin(\frac{N+1}{2}\omega)}{\sin(\frac{\omega}{2})} + 0.25\frac{\sin(\frac{N+1}{2}(\omega + \frac{2\pi}{N}))}{\sin(\frac{\omega+2\pi/N}{2})} + 0.25\frac{\sin(\frac{N+1}{2}(\omega - \frac{2\pi}{N}))}{\sin(\frac{\omega-2\pi/N}{2})}\right]e^{-j\omega N/2}}$$

Looking at $W_R(\omega)$, we find that the first zeros occur at

$$\begin{aligned} \frac{N+1}{2}\omega &= \pm\pi \\ \omega &= \pm\frac{2\pi}{N+1} \end{aligned}$$

Meanwhile, the two modulated sinc functions of $W_d(\omega)$ are located at $\pm\frac{2\pi}{N}$. For N large, $\frac{2\pi}{N+1} \approx \frac{2\pi}{N}$. Thus, if we were to superimpose the three sinc functions, we would find that the peaks of the modulated sinc functions nearly line up with the zero crossings of the original sinc function. The three individual mainlobes are merged together, which widens the mainlobe of $W_d(\omega)$. Meanwhile, since the three sinc functions have the same frequency, the peaks and valleys will somewhat cancel each out, which reduces the sidelobes.



This figure shows the three individual sinc functions superimposed for $N = 16$. We can see how the two modulated sinc functions will destructively interfere with the original sinc function, which will widen the mainlobe but dampen the sidelobes.

5. Using the windowing method, you are supposed to design a GLP FIR bandpass filter to approximate a filter with the following specifications:

   - Stop band 1: $[0, 0.2\pi]$, attenuation of at least 30dB;
   - Stop band 2: $[\pi/2, \pi]$, attenuation of at least 45dB;
   - Pass band: $[0.3\pi, 0.45\pi]$, Passband ripple: at most 1dB.

   Suppose you are only given four choices of windows: rectangular, Hann, Hamming, and Blackman. Which window choice would allow you to meet the design specifications with the shortest filter length? Determine the filter length and filter cutoffs to be used in your design. Explain your answers. (20 pts)

   Table 10.3 on p.563 in the textbook gives us the necessary information for this problem. We must satisfy the strictest requirements, meaning the stopband must have a minimum attenuation of 45dB. This limits us to using the Hamming and Blackman windows.

   Next, we want to have the shortest filter length, which will determine the width of the transition region. A Hamming window requires fewer taps (or a shorter length) for a narrow transition region, so we will choose that window. Once again we must satisfy the strictest requirement of a transition region of 0.05π. So we must have $L = \lceil \frac{6.6\pi}{0.05\pi} \rceil = 132$.

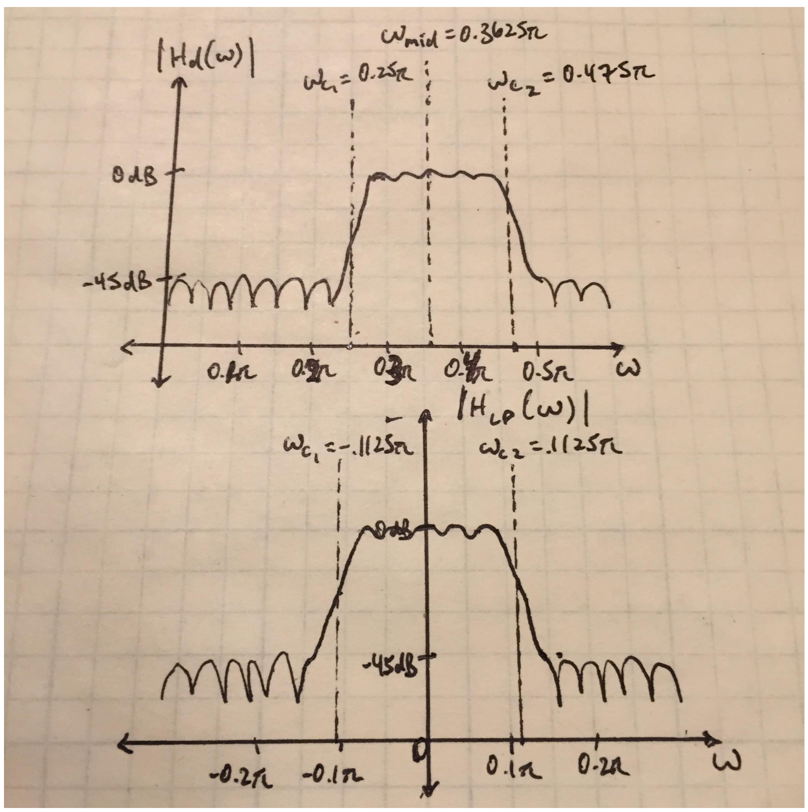
Lastly, we need to determine where the cutoff frequencies will be.

$$\omega_{c,1} = \frac{0.3\pi + 0.2\pi}{2} = 0.25\pi$$
$$\omega_{c,2} = \frac{0.45\pi + 0.5\pi}{2} = 0.475\pi$$

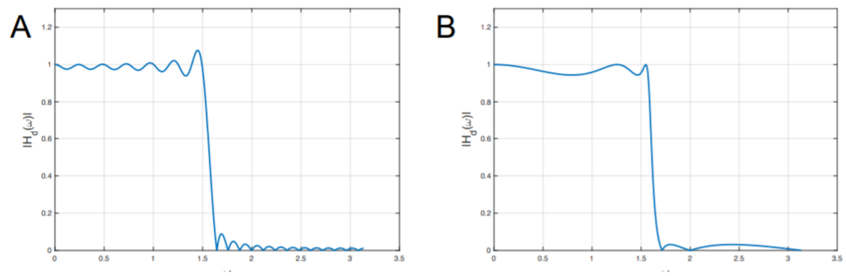
One method of generating this bandpass filter would be to modulate a lowpass filter. In that case, the single cutoff frequency is found as follows.

$$\omega_{c,\text{lp}} = \frac{\omega_{c,1} - \omega_{c,2}}{2} = .1125\pi$$

This figure shows an approximation of our filter with labelled cutoff frequencies and midpoint frequency of the filter. The second graph shows what we would use for the LPF before modulation.



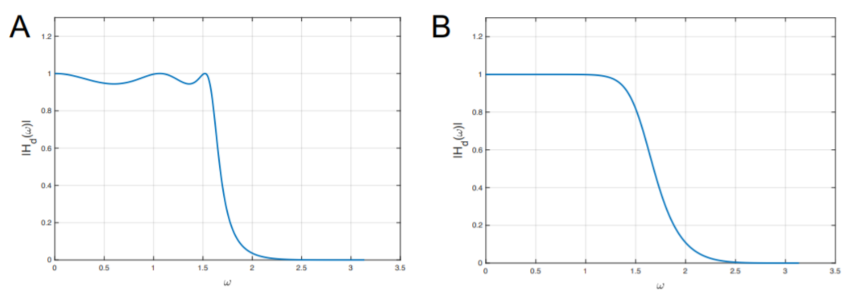
6. (a) Determine which of the following magnitude frequency responses corresponds to an FIR filter and which one corresponds to an IIR Elliptic filter. Explain your answer. Which considerations should you use if you had to decide between these two types of filter? (5 pts)



A: FIR filter
B: IIR Elliptic filter

We know that **A** must be the FIR filter because it exhibits the Gibbs phenomenon. Choose the FIR filter if generalized linear phase or stability are important concerns, but choose the elliptic filter for a lower filter order and potentially less ripple.

(b) Both magnitude frequency responses below correspond to IIR filters. Determine the IIR filter types, explaining your resasoning. Which considerations should you use if you had to decide between these two types of filter? (5 pts)



A: Chebychev filter (Type I filter, odd order of N)
B: Butterworth

We know that **A** is a Chebychev filter because it has passband ripple but no stopband ripple. Meanwhile **B**, the Butterworth filter, has no ripple but a wider transition band. The tradeoff in choice here is between less ripple in the passband for the Butterworth vs a narrower transition band for the Chebychev.