ECE310: Quiz#3 (10am Section G) Fall 2018 Solutions

1. (5 pts) Determine the z-transform and sketch the pole-zero plot with the ROC for the signal

$$x[n] = \frac{1}{2}\left(\frac{1}{2}\right)^n u[n] - \delta[n-1]$$

Solution: Use the z-transform pair:

$$\alpha a^n u[n] \leftrightarrow \frac{\alpha}{a - z^{-1}}, |z| > a$$

alongside the transform pair $\delta[n-n_0] \leftrightarrow z^{-n_0}, |z| > 0$, assuming $n_0 > 0$. This gives

$$\boxed{X(z) = \frac{1}{2}\left(\frac{1}{1-\frac{1}{2}z^{-1}}\right) - z^{-1}, |z| > \frac{1}{2}}$$

This gives

The ROC is $|z| > \frac{1}{2}$ because the individual z-transforms will have ROCs of $|z| > \frac{1}{2}$ and $|z| > 0$; we must take the stricter of the two. To find the poles and zeros, we simplify the expression into a ratio of two polynomials in z:

$$\begin{aligned}
X(z) &= \frac{\frac{1}{2}z}{z-\frac{1}{2}} - \frac{z^{-1}(z-\frac{1}{2})}{z-\frac{1}{2}} \\
&= \frac{\frac{1}{2}z - 1 + \frac{1}{2}z^{-1}}{z-\frac{1}{2}} \\
&= \frac{\frac{1}{2}z^2 - z + \frac{1}{2}}{z(z-\frac{1}{2})} \\
&= \frac{z^2 - 2z + 1}{2z(z-\frac{1}{2})}, |z| > \frac{1}{2}
\end{aligned}$$

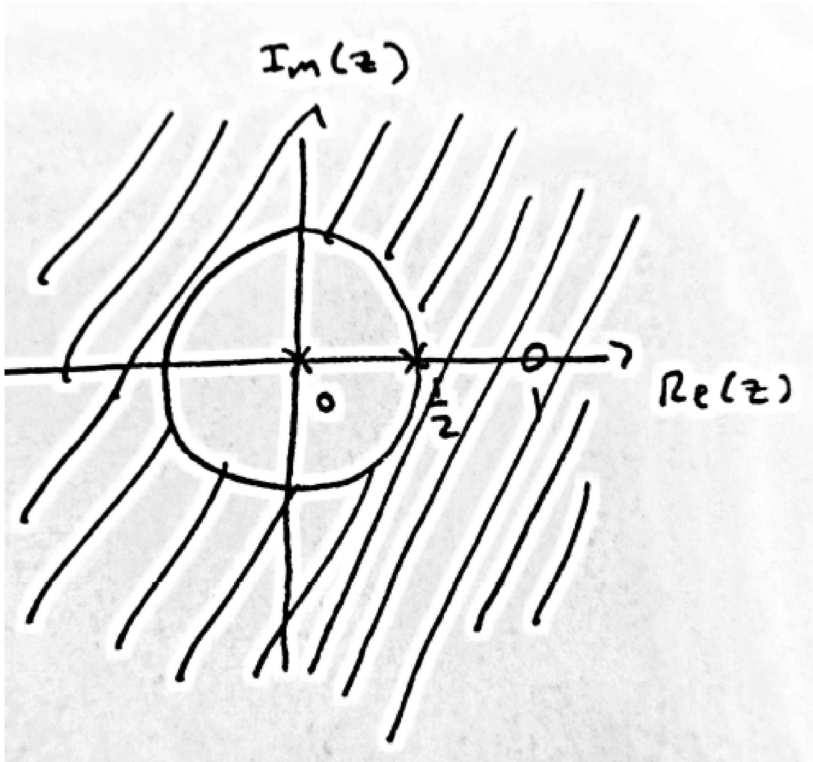
We can now solve for the zeros by setting the numerator equal to zero:

$$z^2 - 2z + 1 = 0 \to z = 1, z = 1$$

Similarly, we can solve for the poles by setting the denominator equal to zero:

$$2z(z - \frac{1}{2}) = 0 \to z = 0, z = \frac{1}{2}$$

This leads to the pole-zero plot given below.


Im(z)
Re(z)
0
1/2
1

2. (5 pts) Given the z-transform pair $x[n] \leftrightarrow X(z) = \frac{1}{1-\frac{1}{4}z^{-1}}$ with ROC: $|z| > 1/4$, use the z-transform properties to determine the z-transform of the signal $y[n] = 2^{n-1}x[n-1]$.

Solution, Approach 1: Apply the shifting and scaling properties directly to the z-transform. If we let $w[n] = 2^n x[n]$, then

$$W(z) = X(\frac{1}{2}z) = \frac{1}{1-\frac{1}{4}(\frac{1}{2}z)^{-1}} = \frac{1}{1-\frac{1}{2}z^{-1}}, |z| > \frac{1}{2}$$

Then, noting that $y[n] = w[n-1]$, we get

$$Y(z) = z^{-1}W(z) = \boxed{\frac{z^{-1}}{1-\frac{1}{2}z^{-1}}, |z| > \frac{1}{2}}$$

Solution, Approach 2: Taking the inverse z-transform gives

$$x[n] = \left(\frac{1}{4}\right)^n u[n]$$

So,

$$y[n] = 2^{n-1}\left(\frac{1}{4}\right)^{n-1} u[n-1]$$

Using the transform pair

$$a^n u[n] \leftrightarrow \frac{1}{1-az^{-1}}, |z| > a$$

and the shifting property, we get

$$Y(z) = z^{-1}\left(\frac{1}{1-\frac{1}{2}z^{-1}}\right) = \boxed{\frac{z^{-1}}{1-\frac{1}{2}z^{-1}}, |z| > \frac{1}{2}}$$