Lecture 10

More examples

Ex (CTFT) : Recall that $\mathcal{F}\{\delta(t-t_0)\} = \int_{-\infty}^{\infty} \delta(t-t_0)\, e^{-j\Omega t}\, dt = e^{-j\Omega t_0}$

So $\quad \delta(t-t_0) \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\Omega t_0}$

What about $\quad ? \overset{\mathcal{F}}{\longleftrightarrow} \delta(\Omega-\Omega_0)$

Use inverse formula. Let $X(\Omega) = \delta(\Omega-\Omega_0)$. Then

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega)\, e^{j\Omega t}\, d\Omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\Omega-\Omega_0)\, e^{j\Omega t}\, d\Omega = \frac{1}{2\pi} e^{j\Omega_0 t}$$

Hence $\quad e^{j\Omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\, \delta(\Omega-\Omega_0)$

What about $\quad \cos(\Omega_0 t) \overset{\mathcal{F}}{\longleftrightarrow} \pi\delta(\Omega-\Omega_0) + \pi\delta(\Omega+\Omega_0)$

$$\cos(\Omega_0 t) = \frac{e^{j\Omega_0 t} + e^{-j\Omega_0 t}}{2}$$

Ex (DTFT) Let $x[n] = \{ \underset{\uparrow}{1}, 1, 1, 1 \}$. Compute DTFT of $x[n]$, and sketch magnitude & phase.

$$X_d(w) = \sum_{n=-\infty}^{\infty} x[n] e^{-jwn} = e^{0} + e^{-jw} + e^{-j(2w)} + e^{-j(3w)}$$

$\swarrow$ (equiv. to $X(e^{jw})$)

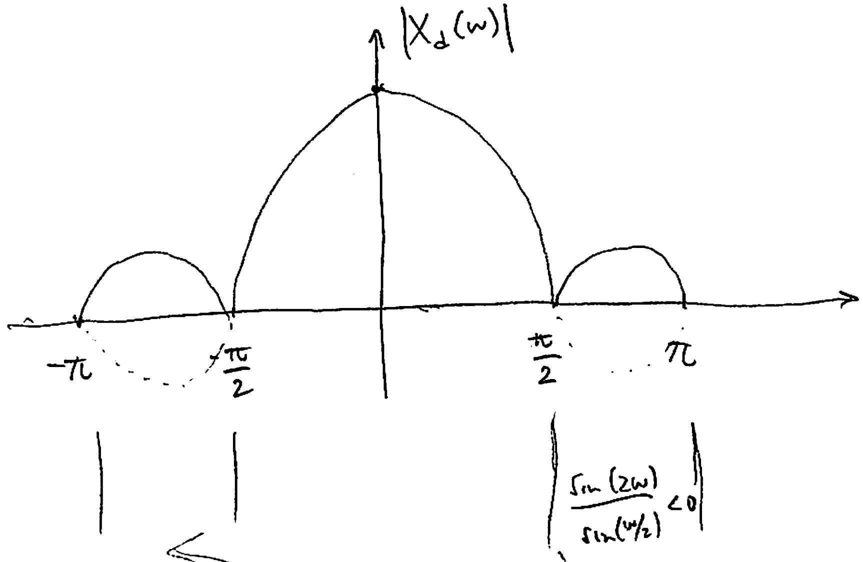
$$\boxed{\sum_{n=0}^{N} b^n = \frac{1-b^{N+1}}{1-b}}$$

$$= \frac{1-e^{-j4w}}{1-e^{-jw}} = \frac{e^{-j2w}\left(e^{j2w} - e^{-j2w}\right)/2j}{e^{-j\frac{w}{2}}\left(e^{j\frac{w}{2}} - e^{-j\frac{w}{2}}\right)/2j}$$

$$= \frac{e^{-j2w}}{e^{-jw/2}} \frac{\sin(2w)}{\sin(w/2)} = e^{-j\frac{3w}{2}} \frac{\sin(2w)}{\sin(w/2)}$$

$$|X_d(w)| = \left| \frac{\sin(2w)}{\sin(w/2)} \right| ,\qquad \angle X_d(w) = \begin{cases} -\frac{3w}{2} & \text{if } \frac{\sin(2w)}{\sin(w/2)} > 0 \\ \pi - \frac{3w}{2} & \text{if } \frac{\sin(2w)}{\sin(w/2)} < 0 \end{cases}$$

$X_d(0) = 4$

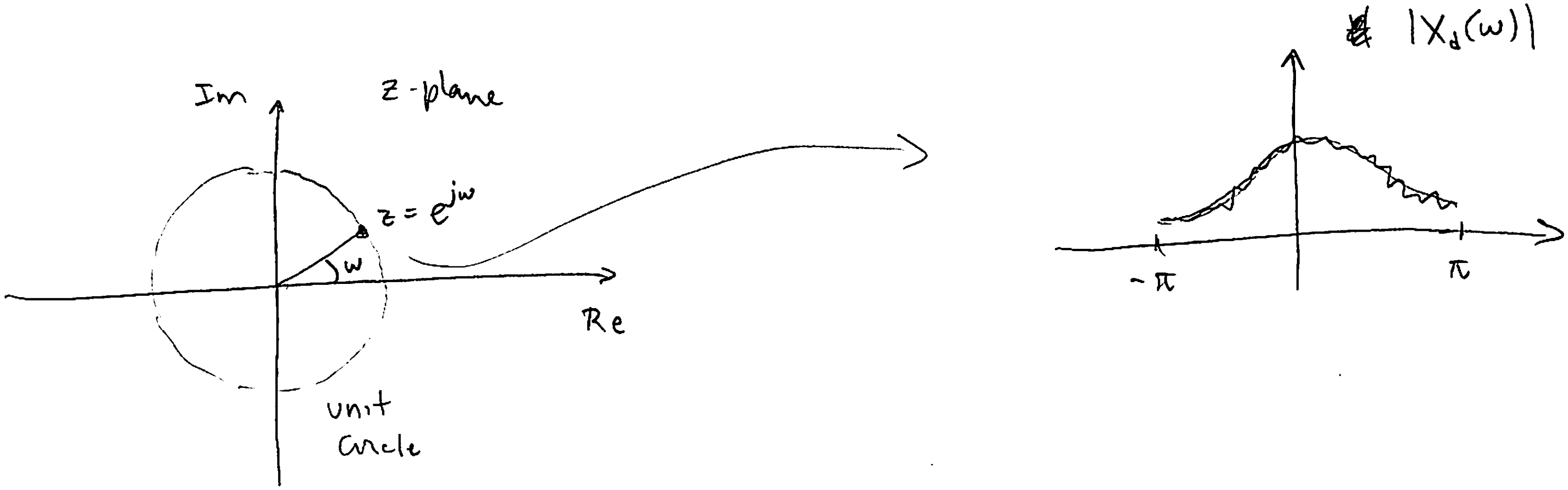


phase?

Relationship between DTFT and z-transform:

Recall z-transform: $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$

DTFT: $X_d(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = X(z)\Big|_{z=e^{j\omega}} = \sum x[n] (e^{j\omega})^{-n}$



Properties of DTFT

⊛ Linearity

⊛ time-shifting

$x[n] \overset{Z}{\longleftrightarrow} X(z) \xrightarrow{z=e^{j\omega}} X_d(\omega) = X(e^{j\omega})$

$x[n-n_0] \overset{Z}{\longleftrightarrow} z^{-n_0} X(z) \xrightarrow{z=e^{j\omega}} e^{-j\omega n_0} X_d(\omega)$

(*) Frequency shifting

$$x[n] \xleftrightarrow{\text{DTFT}} X_d(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$\underbrace{e^{j\omega_0 n} x[n]}_{y[n]} \longleftrightarrow X_d(\omega-\omega_0)$$

Proof: $Y_d(\omega) = \sum_{n=-\infty}^{\infty} y[n] e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x[n] e^{-j(\omega-\omega_0)n} = X_d(\omega-\omega_0)$

(*) Convolution

$$x[n] * h[n] \xleftrightarrow{\text{DTFT}} X_d(\omega) H_d(\omega) \qquad \text{(application: filtering)}$$

(*) Symmetry properties:

Recall: complex conjugate: $z = a + j\cdot b \Rightarrow z^* = a - jb$

$z = re^{j\theta} \Rightarrow z^* = re^{-j\theta}$

Suppose $\{x[n]\}$ is <u>real-valued</u>:

DTFT: $X_d(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$. Take complex conjugate on both sides:

$$(X_d(\omega))^* = \left(\sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}\right)^* = \sum_{n=-\infty}^{\infty} \left(x[n] e^{-j\omega n}\right)^* = \sum_{n=-\infty}^{\infty} (x[n])^* \left(e^{-j\omega n}\right)^*$$

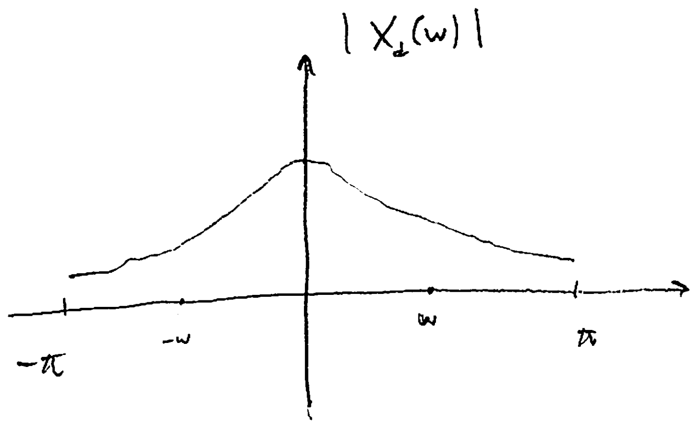
$$= \sum_{n=-\infty}^{\infty} x[n] e^{j\omega n} = \sum_{n=-\infty}^{\infty} x[n] e^{-j(-\omega)n} = X_d(-\omega)$$

$$\left(X_d(\omega)\right)^* = X_d(-\omega) \qquad \text{(Hermitian symmetry)}$$

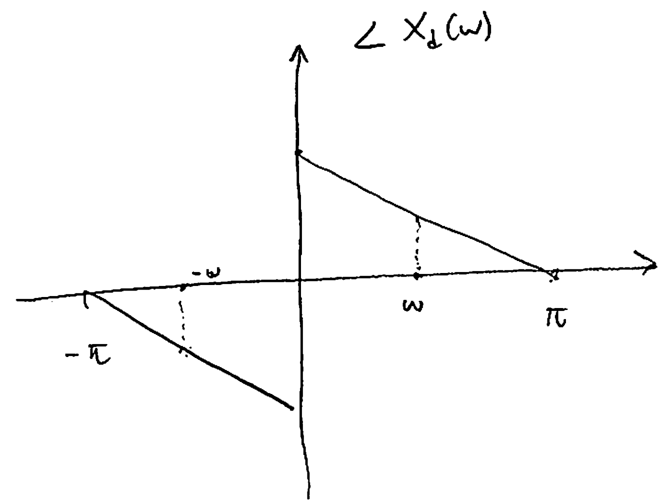
If $\{x[n]\}$ is real-valued:

$$\left|\left(X_d(\omega)\right)^*\right| = \boxed{|X_d(\omega)| = |X_d(-\omega)|} \qquad \text{magnitude is symmetric}$$

$$\angle\left(\left(X_d(\omega)\right)^*\right) = \boxed{-\angle X_d(\omega) = \angle X_d(-\omega)} \qquad \text{phase is anti-symmetric}$$



symmetric



anti-symmetric

⊛ Modulation

$$x[n] \xleftrightarrow{DTFT} X_d(\omega)$$

$$x[n]\cos(\omega_c n) \xleftrightarrow{DTFT}$$

$$\cos(\omega_c n) = \frac{e^{j\omega_c n} + e^{-j\omega_c n}}{2}$$

$$\frac{1}{2}x[n]e^{j\omega_c n} + \frac{1}{2}x[n]e^{-j\omega_c n} \xleftrightarrow{DTFT} \frac{1}{2}X_d(\omega - \omega_c) + \frac{1}{2}X_d(\omega + \omega_c)$$