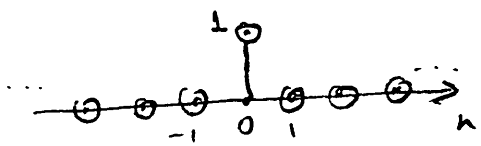
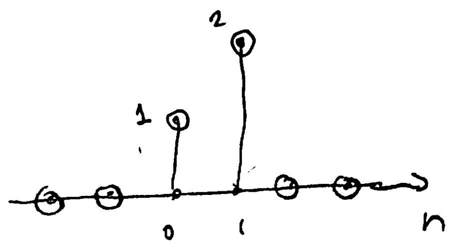
# Lecture 4

LTI systems (linear and time-invariant)

→ response to any signal determined by the impulse response
(LTI system is completely "described" by impulse resp.)



$\{\delta[n]\} \longrightarrow \boxed{S} \longrightarrow \{h[n]\}$ (impulse response)

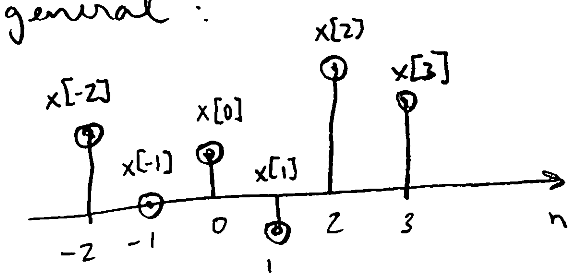


$\{x[n]\}$ $\longrightarrow \underset{\text{LTI}}{\boxed{S}} \longrightarrow$

$\{x[n]\} = \{1 \cdot \delta[n] + 2\delta[n-1]\}$

$$\begin{aligned} y[n] &= S(\{x[n]\}) \\ &= S(\{1 \cdot \delta[n] + 2\delta[n-1]\}) \\ \text{linearity} \rightarrow &= 1 \cdot S\{\delta[n]\} + 2 \cdot S\{\delta[n-1]\} \\ &= 1 \{h[n]\} + 2 \cdot \{h[n-1]\} \end{aligned}$$

In general:



$\{x[n]\}$ $\longrightarrow \underset{\text{LTI}}{\boxed{S}} \longrightarrow$

$$\{x[n]\} = \{\ldots + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + \ldots\}$$

$$\begin{aligned} y[n] &= \\ &= \ldots + x[-1]h[n+1] + x[0]h[n] + \ldots \\ &= \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k] \end{aligned}$$

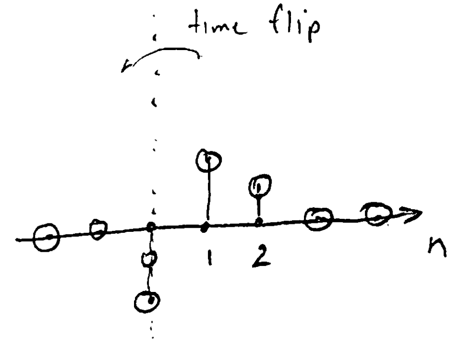
**Theorem**: If $S$ is an LTI system, then the output $y$ to an input $x$ can be written as

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\cdot h[n-k] = \underbrace{(x * h)[n]}_{\text{convolution operation}}$$

where $\{h[n]\} = S\{\delta[n]\}$.

**Convolution Operation**

Ex: $\{h[n]\} = \{\underset{\uparrow}{-1}, 1, \tfrac{1}{2}\}$ =




$$y[n] = (x * h)[n] = x[n] * h[n]$$

$$= \sum_{k=-\infty}^{\infty} \underbrace{x[k]\, h[n-k]}_{n} = x[n-2]\,h[2] + x[n-1]\,h[1] + x[n]\,h[0]$$

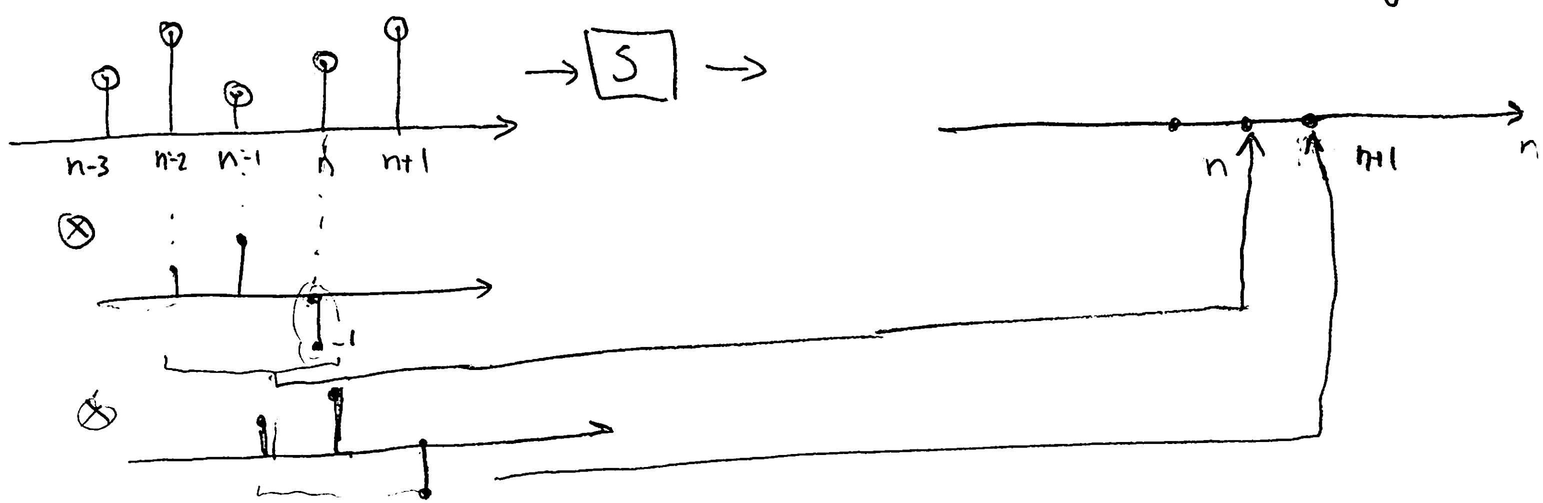
$$= \frac{1}{2} x[n-2] + x[n-1] - x[n]$$

$$\left(\tfrac{1}{2}, 1, -1\right)$$

Pictorially,

$\{x[n]\}$

flipped $h[n]$ and shift




$\{y[n]\}$

Interesting properties :

① $(x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] \overset{u=n-k}{=} \sum_{u=-\infty}^{\infty} x[n-u]\, h[u] = \sum_{u=-\infty}^{\infty} h[u]\, x[n-u]$

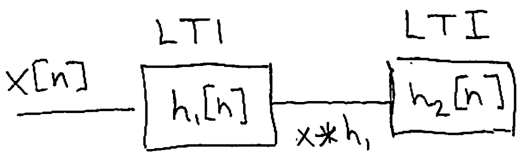
$= (h * x)[n]$

· Convolution IS commutative

② Convolution with delayed impulse

$x[n] * \delta[n] = x[n]$

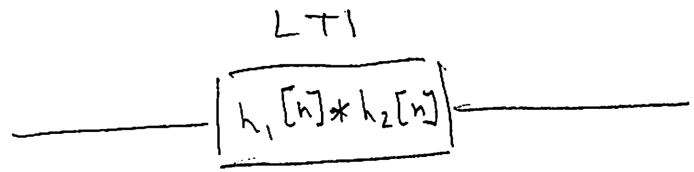
$x[n] * \delta[n-n_0] = x[n-n_0]$

③ 

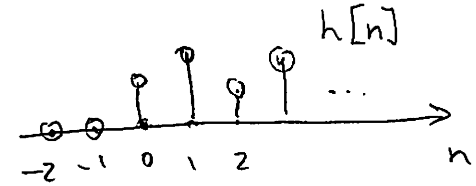

$(x * h_1) * h_2$

$= x * (h_1 * h_2)$

$\Longleftrightarrow$




Properties of LTI system can be deduced from $h[n]$



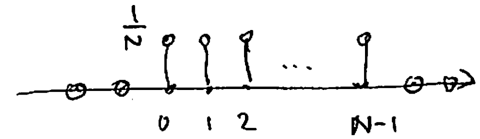

· causality : $h[n] = 0$ for $n < 0$

· stability : impulse response is absolutely summable

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

Recursive systems

Consider LTI system with $h[k] = \begin{cases} \frac{1}{N} & 0 \le k \le N-1 \\ 0 & \text{otherwise} \end{cases}$




moving average

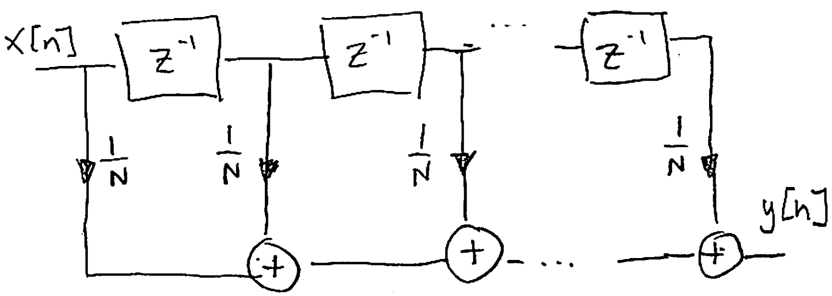
$$y[n] = (x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k] = \sum_{k=-\infty}^{\infty} x[n-k]\,h[k]$$

$$= \sum_{k=0}^{N-1} x[n-k] \cdot \frac{1}{N} = \frac{1}{N}\Big(x[n] + \underbrace{x[n-1] + \dots + x[n-(N-1)]}\Big)$$

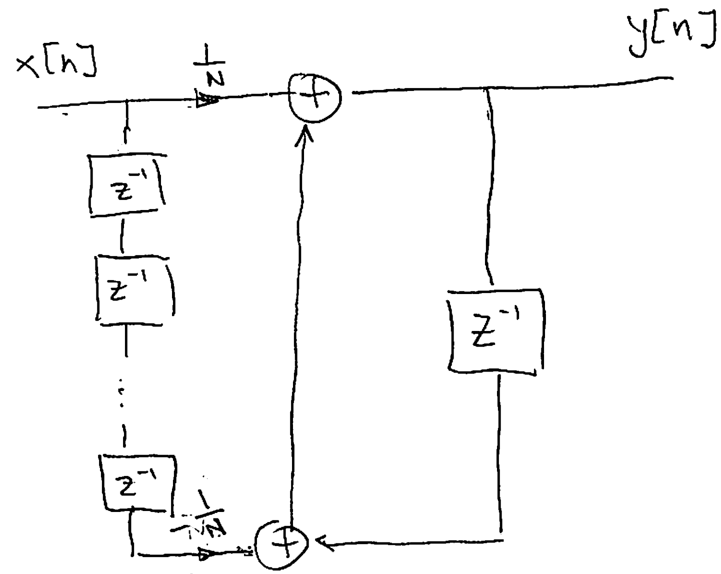
$$y[n-1] = \frac{1}{N}\Big(\underbrace{x[n-1] + x[n-2] + \dots +}\ x[n-N]\Big)$$

Hence $\quad y[n] = y[n-1] + \frac{1}{N}\big(x[n] - x[n-N]\big)$

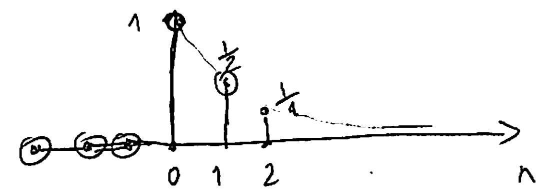
recursive implementation




$\Longleftrightarrow$



Ex: LTI system with $h[n] = \left(\frac{1}{2}\right)^n u[n]$




$$y[n] = \sum_{k=-\infty}^{\infty} x[n-k]\,h[k] = \sum_{k=0}^{\infty} x[n-k]\left(\frac{1}{2}\right)^n$$

$$= x[n] + \underbrace{\frac{1}{2}x[n-1] + \frac{1}{4}x[n-2] + \cdots}_{\frac{1}{2}\left(x[n-1] + \frac{1}{2}x[n-2] + \cdots\right) = \frac{1}{2}y[n-1]}$$

$$y[n-1] = x[n-1] + \frac{1}{2}x[n-2] + \frac{1}{4}x[n-3] + \cdots$$

Hence $y[n] = x[n] + \frac{1}{2}y[n-1]$

Can only be implemented recursively

Requires initial condition. Typically, $y[-1] = 0$

In general: Linear constant-coefficient difference equation (LCCDE)

$$y[n] = -\sum_{k=1}^{N} a_k\, y[n-k] + \sum_{k=0}^{M} b_k\, x[n-k]$$