University of Illinois
ECE 310

Fall 2018
Profs. Bresler, Katselis, and Shomorony

Midterm Exam

7:00-8:30PM, Monday, October 15, 2018

Problem	Pts.	Score
1	5	
2	5	
3	3	
4	8	
5	6	
6	5	
7	6	
8	5	
9	5	
10	5	
11	5	
12	5	
13	5	
14	3	
15	7	
16	5	
17	5	
18	7	
19	5	
Total		

Name Key

UIN: ______________________

Section: 10:00 AM 3:00 PM 6:00PM Chicago

Score ____________

Please do not turn this page over until told to do so.

You may not use any books, electronic devices, or notes other than **one** handwritten two-sided sheet of 8.5" x 11" paper.

GOOD LUCK!

1. **(5 Pts.)** The output $y[n]$ and input $x[n]$ of a causal system are related by the equation below. Find the values of the impulse response $h[n]$ for the indicated values of n.

$$y[n] = -y[n-6] + 2x[n] + x[n-5]$$

$\downarrow$

$h[n] = -h[n-6] + 2\delta[n] + \delta[n-5]$

$h[0] = -h[-6] + 2\delta[0] + \delta[-5] = 2$

$h[1] = -h[-5] + 2\delta[1] + \delta[-4] = 0$

$h[5] = -h[-1] + 2\delta[5] + \delta[0] = 1$

$h[6] = -h[0] + 2\delta[6] + \delta[1] = -2$

$h[11] = -h[5] + 2\delta[11] + \delta[6] = -1$

$h[12] = -h[6] + 2\delta[12] + \delta[7] = 2$

n	0	1	5	6	11	12
$h[n]$	2	0	1	-2	-1	2
	+0.5	+0.5	+1	+1	+1	+1

2. **(5 Pts.)** Let $h[n] = (n-1)^2\,(u[n] - u[n-3])$ for all n. Let $x_1[n] = (-3)^n u[n-1]$. Determine the signal $y[n]$ given by the convolution $y[n] = x_1[n] * h[n]$.

(a) $y[n] = (-3)^n u[n-1] + (-3)^{n-1}u[n-2] + (-3)^{n-2}u[n-3]$

(b) $y[n] = (-3)^{n-1}u[n-1] - (-3)^{n-2}u[n-3]$

(c) $y[n] = (-3)^n u[n] + (-3)^{n-2}u[n-2]$

(d) $y[n] = (-3)^{n-3}\left((-3)^3 u[n-1] - 3u[n-3] + 4u[n-4]\right)$

5 (e) $y[n] = (-3)^n u[n-1] + (-3)^{n-2}u[n-3]$

$u[n] - u[n-3]$

$= \delta[n] + \delta[n-1] + \delta[n-2]$

$(n-1)^2 (u[n] - u[n-3])$

$= (n-1)^2\delta[n] + (n-1)^2\delta[n-1] + (n-1)^2\delta[n-2]$

$= (0-1)^2\delta[n] + (1-1)^2\delta[n-1] + (2-1)^2\delta[n-2]$

$= \delta[n] + \delta[n-2]$

So $y[n] = x_1[n] * (\delta[n] + \delta[n-2])$

$= x_1[n] + x_1[n-2]$

$= (-3)^n u[n-1] + (-3)^{n-2} u[n-3]$

3. **(3 Pts.)** An LTI system has impulse response $h[n] = \cos(\pi\sqrt{n})u[n]$.
Is this system BIBO stable? **Yes** **No** +1

Why? The impulse response is not absolutely summable: $\sum_{n=0}^{\infty} |\cos(\pi\sqrt{n})| = \infty$ +2

$\cos(\pi\sqrt{n})$ is always nonzero (since n must be an integer) and oscillates; it never converges to 0

4. **(8 Pts.)** For each of the systems with input $x[n]$ and output $y[n]$ in the table, indicate with YES or NO whether the properties indicated apply to the system.

	Linear	Time-Invariant	Causal	Stable
$y[n] = 3y[n-1] + x[n+1]$	Yes $2/3$	Yes $2/3$	No $2/3$	No $2/3$
$y[n] = x[2]\cos(x[n])$	No $2/3$	No $2/3$	No $2/3$	Yes $2/3$
$y[n] = \frac{2}{\lvert n\rvert+1} y[n-1] + x[n] + \cos(n+3)$	No $2/3$	No $2/3$	Yes $2/3$	Yes $2/3$

5. **(6 Pts.)** Given the z-transform pair $x[n] \leftrightarrow X(z) = 1/(1-0.3z^{-1})^3$ with ROC: $|z| > 0.3$, determine the z-transform with ROC of $y[n] = x[n-2] * \left(5^n x[n-1]\right)$

$x[n-2] \leftrightarrow z^{-2}X(z)$ (+1)

$5^n x[n-1] = 5 \cdot 5^{n-1} x[n-1] \leftrightarrow 5z^{-1} X(\tfrac{1}{5}z)$ (+1)

So $Y(z) = (z^{-2}X(z))(5z^{-1}X(\tfrac{1}{5}z)) = \dfrac{5z^{-3}}{(1-0.3z^{-1})^3(1-1.5z^{-1})^3}$, $|z| > 1.5$ (+2)

(+2)

$Y(z) = \dfrac{5z^{-3}}{(1-0.3z^{-1})^3(1-1.5z^{-1})^3}$ 4

ROC_Y : $|z| > 1.5$ 2

6. **(5 Pts.)** The one-sided z-transform of a right-sided sequence $x[n]$ is

$$X(z) = \frac{1}{z^7(z+3)^2}, \qquad |z| > 3$$

Find $x[n]$ for all n.

(a) $n3^{n-9}u[n]$

(b) $(n-8)3^{n-8}u[n-8]$

5 (c) $(n-8)(-3)^{n-9}u[n-8]$

(d) $(n-6)3^{n-6}u[n-6]$

(e) $(n-6)(-3)^{n-6}u[n-6]$

$X(z) = \dfrac{z^{-7}}{(z+3)^2} = -\dfrac{1}{3}z^{-8}\left(\dfrac{-3z}{(z+3)^2}\right)$, $|z|>3$

$\rightarrow x[n] = -\frac{1}{3}(n-8)(-3)^{n-8}u[n-8]$

$= (n-8)(-3)^{n-9}u[n-8]$

7. **(6 Pts.)** The one-sided z-transform of a right-sided *real-valued* sequence $x[n]$ is

$$\frac{(1-j)}{1+(1/4+\sqrt{3}/4j)z^{-1}} + \frac{B}{1+(1/4-\sqrt{3}/4j)z^{-1}}, \qquad |z|>1/2$$

(a) Find B. $\boxed{B = 1+j}$ 1

Need conjugate symmetry since $x[n]$ is real

(b) $x[n]$ has the form $x[n] = a\cos(bn+c)d^n u[n]$. Find the unknown constants.

$$X(z) = \frac{\sqrt{2}e^{-j\pi/4}}{1+\frac{1}{2}e^{j\pi/3}z^{-1}} + \frac{\sqrt{2}e^{j\pi/4}}{1+\frac{1}{2}e^{-j\pi/3}z^{-1}}, \quad |z|>1/2 \rightarrow x[n] = (-\tfrac{1}{2})^n\sqrt{2}e^{-j\pi/4}e^{j\pi/3 n}u[n]$$

$$+(-\tfrac{1}{2})^n\sqrt{2}e^{j\pi/4}e^{-j\pi/3 n}u[n]$$

$$=\sqrt{2}(-\tfrac{1}{2})^n\left(e^{j(\frac{\pi}{3}n-\frac{\pi}{4})}+e^{-j(\frac{\pi}{3}n-\frac{\pi}{4})}\right)u[n]$$

$$=2\sqrt{2}\cos(\tfrac{\pi}{3}n-\tfrac{\pi}{4})(-\tfrac{1}{2})^n u[n]$$

$\boxed{a = 2\sqrt{2}}$ 2 $\boxed{b = \frac{\pi}{3}}$ 1 $\boxed{c = -\frac{\pi}{4}}$ 1 $\boxed{d = -\frac{1}{2}}$ 1

8. **(5 Pts.)** Find the partial fraction expansion for $X(z) = \dfrac{8z^{-1}}{(1-4z^{-1})(0.5+z^{-1})}$.

$$X(z) = \frac{8z^{-1}}{(1-4z^{-1})(0.5+z^{-1})} = \frac{A}{1-4z^{-1}} + \frac{B}{0.5+z^{-1}} \quad (+3)$$

$$A = \left.\frac{8z^{-1}}{0.5+z^{-1}}\right|_{z=4} = \frac{8(1/4)}{3/4} = \frac{8}{3} \quad (+1)$$

$$B = \left.\frac{8z^{-1}}{1-4z^{-1}}\right|_{z=2} = \frac{8(-1/2)}{1-4(-1/2)} = -\frac{4}{3} \quad (+1)$$

$$\boxed{X(z) = \frac{8}{3}\left(\frac{1}{1-4z^{-1}}\right) - \frac{4}{3}\left(\frac{1}{0.5+z^{-1}}\right)}$$

OR: $X(z) = \dfrac{8z}{(z-4)(0.5z+1)}$

$$= \frac{A}{z-4} + \frac{B}{0.5z+1} \quad (+3)$$

$$A = \left.\frac{8z}{0.5z+1}\right|_{z=4} = \frac{32}{3} \quad (+1)$$

$$B = \left.\frac{8z}{z-4}\right|_{z=-2} = \frac{8}{3} \quad (+1)$$

9. **(5 Pts.)** An LTI system has the impulse response $h[n] = (j)^n u[n-1]$. Determine a difference equation relating the input $x[n]$ and output $y[n]$ of this system.

$$h[n] = j(j)^{n-1}u[n-1] \rightarrow H(z) = \frac{jz^{-1}}{1-jz^{-1}} = \frac{Y(z)}{X(z)}$$

(+1) (+2)

$$Y(z)(1-jz^{-1}) = X(z)(jz^{-1})$$

$$y[n] = jy[n-1] + jx[n-1] \quad (+2)$$

$\boxed{y[n] = jy[n-1]+jx[n-1]}$

10. **(5 Pts.)** A causal system produces the output $y[n] = \{1, 3, 0, 0, \ldots\}_{n=0}^{\infty}$ when excited by the input signal $x[n] = \{1, 1/2, 0, 0, \ldots\}_{n=0}^{\infty}$. Determine the impulse response of the system.

(a) $\delta[n] - n(-3)^n u[n]$

(b) $(-0.5)^n u[n] + \delta[n]$

(c) $(-0.5)^n u[n] + (-3)^n u[n-1]$

(d) $5n(-3)^n u[n-1]$

5 (e) $(-0.5)^n u[n] + 3(-0.5)^{n-1} u[n-1]$

$y[n] = \delta[n] + 3\delta[n-1]$

$\hookrightarrow Y(z) = 1 + 3z^{-1}, \ |z| > 0$

$x[n] = \delta[n] + \frac{1}{2}\delta[n-1]$

$\hookrightarrow X(z) = 1 + \frac{1}{2}z^{-1}, \ |z| > 0$

$\hookrightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{1+3z^{-1}}{1+\frac{1}{2}z^{-1}} = \frac{1}{1+\frac{1}{2}z^{-1}} + \frac{3z^{-1}}{1+\frac{1}{2}z^{-1}}, \ |z| > 1/2$

$\hookrightarrow h[n] = (-\frac{1}{2})^n u[n] + 3(-\frac{1}{2})^{n-1} u[n-1]$

11. **(5 Pts.)** A system with input $x[n]$ and output $s[n]$ is described by the difference equation

$$s[n] = x[n] + cx[n-1].$$

The output $s[n]$ of the first system is the input to another causal system described by the difference equation:

$$y[n] - 2y[n-1] = s[n].$$

Find the value of c that guarantees that $y[n] = x[n]$.

$S(z) = (1 + cz^{-1})X(z)$

$Y(z)(1 - 2z^{-1}) = S(z) = (1 + cz^{-1})X(z)$

$\frac{Y(z)}{X(z)} = \frac{1+cz^{-1}}{1-2z^{-1}} = 1$ +3 $\rightarrow$ $c = -2$ +2

$\boxed{c = -2}$

12. **(5 Pts.)** The input and output of a causal system are related by the equation

$$y[n] + 1.5y[n-1] - y[n-2] = x[n]$$

Is this system BIBO stable? **Yes** **No** +1

Why? The system is causal, and one of the poles of $H(z)$ has magnitude larger than 1, so the ROC of $H(z)$ does not contain the unit circle. +4

13. **(5 Pts.)** The causal LTI system with transfer function given below is not BIBO stable. Find a *real valued* bounded input $x[n]$ that produces an unbounded output $y[n]$.

$$H(z) = \frac{1 - 3z^{-3}}{1 - e^{j\pi/3}z^{-1}}$$

(a) $e^{j\pi n/3}u[n] - e^{-j\pi n/3}u[n]$

(b) $n\cos(\pi n/3)u[n]$

5 (c) $e^{j\pi n/3}u[n] + e^{-j\pi n/3}u[n]$

(d) $ne^{j\pi n/3}u[n]$

(e) $\sin(\pi n/6 + \pi/3)$

$H(z)$ has a pole at $e^{j\pi/3}$, so $X(z)$ needs a pole at $e^{j\pi/3}$

(a) $e^{j\pi n/3}u[n] - e^{-j\pi n/3}u[n] = 2j\sin(\frac{n\pi}{3})u[n]$; not real

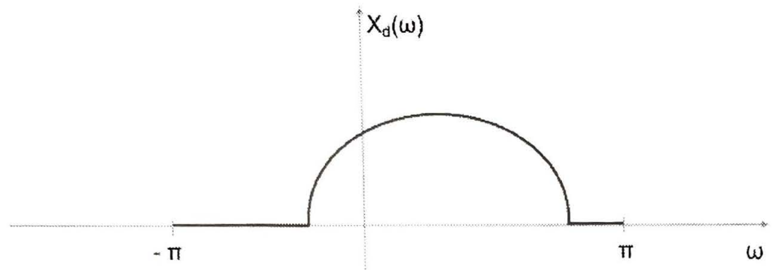
(b) Not bounded

(c) $e^{j\pi n/3}u[n] + e^{-j\pi n/3}u[n] = 2\cos(\frac{n\pi}{3})u[n]$; this works

(d) Not bounded

(e) Wrong pole locations

14. **(3 Pts.)** Let $x[n]$ have the real-valued DTFT given in the plot below. Is $x[n]$ a real-valued sequence?



Yes **No** +1

Why? The magnitude does not exhibit even symmetry +2

15. **(7 Pts.)** Let $x[n] = \delta[n] + \delta[n-2] + \delta[n-4]$.

(a) What is the DTFT $X_d(\omega)$ of $x[n]$?

(a) $X_d(\omega) = 1 + e^{-j2w} + e^{j2w}$

(b) $X_d(\omega) = \sin^2(4\omega)$

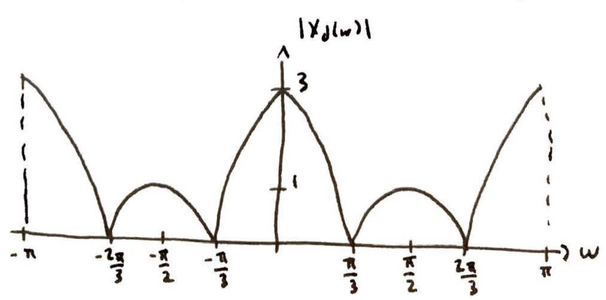
4 (c) $X_d(\omega) = e^{-j2w}(1 + 2\cos(2\omega))$

(d) $X_d(\omega) = \frac{\sin(3\omega)}{\omega}$

(e) $X_d(\omega) = e^{-j2w}\sin(4\omega)$

$$X_d(w) = 1 + e^{-jw2} + e^{-jw4}$$
$$= e^{-jw2}(e^{jw2} + 1 + e^{-jw2})$$
$$= e^{-jw2}(1 + 2\cos(2w))$$

(b) **Sketch** the magnitude of $X_d(\omega)$, labeling the axes and "important points" on your sketch.



Height of 3: +1

Sinusoidal shape: +1

Correct graph: +1

16. **(5 Pts.)** Determine the signal $x[n]$ whose DTFT is $X_d(\omega) = -1 + 3e^{-j(\omega+\pi/2)} + j\sin(4\omega)$. **Note:** The arrow indicates $n = 0$.

(a) $x[n] = \{0.5\pi, 0, 0, 0, \underset{\uparrow}{-1}, 3, 0, 0, -0.5\pi\}$

5 (b) $x[n] = \{0.5, 0, 0, 0, \underset{\uparrow}{-1}, -3j, 0, 0, -0.5\}$

(c) $x[n] = \{0.5j, 0, 0, 0, \underset{\uparrow}{1}, -3j, 0, 0, 0.5j\}$

(d) $x[n] = \{-2\pi, 0, 0, 0, \underset{\uparrow}{-1}, 3e^{-j\pi/2}, 0, 0, 2\pi\}$

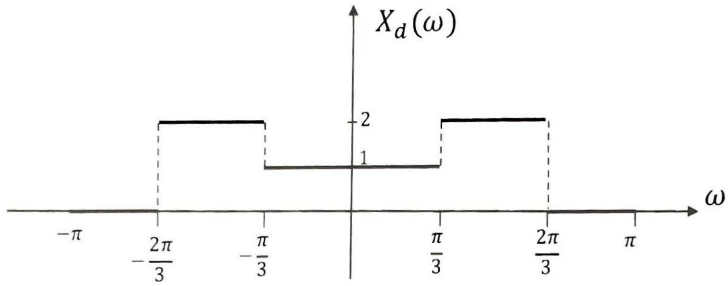
(e) $x[n] = \{0.5\pi j, 0, 0, 0, \underset{\uparrow}{-1}, 3j, 0, 0, 0.5\pi j\}$

$X_d(w) = -1 + 3e^{-j\pi/2}e^{-jw} + j\left(\frac{e^{j4w} - e^{-j4w}}{2j}\right)$

$= -1 - 3je^{-jw} + 0.5e^{j4w} - 0.5e^{-j4w}$

$\rightarrow x[n] = \{0.5, 0, 0, 0, \underset{\uparrow}{-1}, -3j, 0, 0, -0.5\}$

17. **(5 Pts.)** The DTFT of $x[n]$ is as shown below. Determine $x[n]$.



- Note: None of the given options were correct due to a typo. So everyone received 5/5, and those closest to the correct solution (i.e., those that chose (b)) received 8/5.

- Note: The problem must be attempted to receive the points.

5 (a) $x[n] = \text{sinc}(\pi n)\cos(\pi n/2) + \text{sinc}(2\pi n/3)$

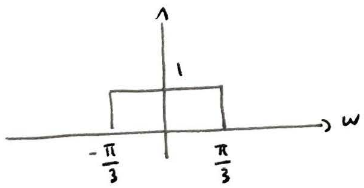
8 (b) $x[n] = \frac{1}{3}\text{sinc}(\pi n/3) + \frac{2}{3}\text{sinc}(\pi n/6)\cos(\pi n/2)$

5 (c) $x[n] = 2\text{sinc}(\pi n/3) + \text{sinc}(2\pi n/3)$

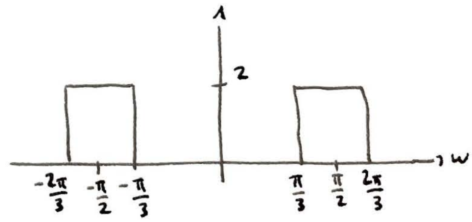
5 (d) $x[n] = \frac{1}{3}\text{sinc}(\pi n/3)\cos(\pi n/3) + 3\text{sinc}(2\pi n/3)$

5 (e) $x[n] = \frac{1}{3}\text{sinc}(\pi n/3) + \frac{1}{3}\text{sinc}(\pi n/6)\cos(\pi n/5)$

$X_d(w) =$



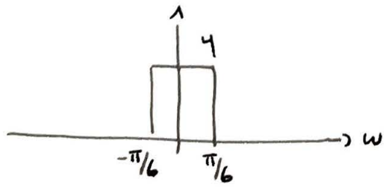
$+$



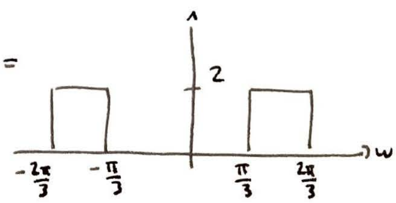
$\updownarrow$

$\frac{1}{3}\text{sinc}\left(\frac{\pi}{3}n\right)$

↳ **Modulation:**



$* \left(\delta(w - \frac{\pi}{2}) + \delta(w + \frac{\pi}{2})\right)\frac{1}{2} =$



$\updownarrow$

$\frac{2}{3}\text{sinc}\left(\frac{\pi}{6}n\right) \times \cos\left(\frac{\pi}{2}n\right)$

$\rightarrow x[n] = \frac{1}{3}\text{sinc}\left(\frac{\pi}{3}n\right) + \frac{2}{3}\text{sinc}\left(\frac{\pi}{6}n\right)\cos\left(\frac{\pi}{2}n\right)$

18. **(7 Pts.)** Consider the connection of LTI systems given by Fig. ??, where the impulse response of the first system is $h_1[n] = 2\delta[n] - \delta[n-2]$, and the frequency response of the second system for $|\omega| \leq \pi$ is given by

$$H_2(e^{j\omega}) = \begin{cases} 1 & \text{for } |\omega| \leq \frac{3\pi}{4} \\ 0 & \text{for } \frac{3\pi}{4} < |\omega| \leq \pi \end{cases}$$

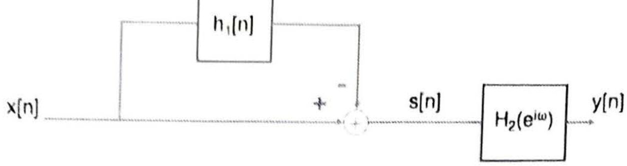


Figure 1: Interconnection of LTI Systems

(a) Find the impulse response of the interconnected system with input $x[n]$ and output $y[n]$

$x[n] * h_1[n] = 2x[n] - x[n-2]$, so $s[n] = x[n] - (2x[n] - x[n-2])$

$= -x[n] + x[n-2]$ +1

$\rightarrow y[n] = x[n] * (-\delta[n] + \delta[n-2]) * h_2[n]$

$\rightarrow h[n] = (-\delta[n] + \delta[n-2]) * h_2[n]$

$= -h_2[n] + h_2[n-2]$ +1

$\rightarrow h_2[n] = \frac{3}{4}\,\text{sinc}\left(\frac{3\pi}{4}n\right)$ +2

$$\boxed{h[n] = -\frac{3}{4}\,\text{sinc}\left(\frac{3\pi}{4}n\right) + \frac{3}{4}\,\text{sinc}\left(\frac{3\pi}{4}(n-2)\right)}$$ 4

(b) Find the frequency response of the interconnected system

$h[n] = -h_2[n] + h_2[n-2]$

$H(e^{j\omega}) = -H_2(e^{j\omega}) + e^{-j\omega 2} H_2(e^{j\omega}) = H_2(e^{j\omega})(-1 + e^{-j\omega 2})$

$$\boxed{H(e^{j\omega}) = \begin{cases} -1 + e^{-j\omega 2}, & |\omega| \leq 3\pi/4 \\ 0, & 3\pi/4 < |\omega| \leq \pi \end{cases}}$$ 3

19. **(5 Pts.)** A causal system is described by the difference equation

$$y[n] + \frac{\sqrt{3}}{3} y[n-1] = x[n].$$

Determine the response of the system to the input $x[n] = 2\cos(\frac{\pi}{2}n)$, $-\infty < n < \infty$.

$Y(z)\left(1 + \frac{\sqrt{3}}{3}z^{-1}\right) = X(z) \rightarrow H(z) = \dfrac{1}{1 + \frac{\sqrt{3}}{3}z^{-1}}, \; |z| > \frac{\sqrt{3}}{3}$ +1

$\rightarrow H_d(\omega) = \dfrac{1}{1 + \frac{\sqrt{3}}{3}e^{-j\omega}}$

$$\boxed{y[n] = \sqrt{3}\cos\left(\frac{\pi}{2}n + \frac{\pi}{6}\right)}$$ 5

System is real, so $y[n] = 2|H_d(\frac{\pi}{2})|\cos\left(\frac{\pi}{2}n + \measuredangle H_d(\frac{\pi}{2})\right)$ +1

$\hookrightarrow H_d\left(\frac{\pi}{2}\right) = \dfrac{1}{1 - j\frac{\sqrt{3}}{3}}\left(\dfrac{1 + j\sqrt{3}/3}{1 + j\sqrt{3}/3}\right) = \frac{3}{4} + j\frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{2}e^{j\pi/6}$ +1

$= \dfrac{1}{\frac{2}{\sqrt{3}}e^{-j\pi/6}}$ +1

+3 both