UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
Department of Electrical and Computer Engineering

ECE 310 Digital Signal Processing

Homework 9 Solutions

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Problem 1

(20 points) In this problem we consider windowed DFT spectral analysis. A continuous-time signal $x_a(t)$ is sampled with period T to produce the sequence $x[n] = x_a(nT)$. An N-point window $w[n]$ is applied to $x[n]$ for $n = 0, 1, ..., N-1$ and $X[k]$, $k = 0, 1, ..., N-1$ is the N-point DFT of the resulting sequence.

(a) (10 points) You are given that $x_a(t) = \cos(\Omega_0 t)$. Assuming that $w[n]$ is a rectangular window, and Ω_0, N, and k_0 are fixed, how should T be chosen so that $X[k_0]$ and $X[N-k_0]$ are non-zero and $X[k] = 0$ for all other values of k? Is your answer unique? If not, give another value of T that satisfies the conditions given.

(b) (5 points) Suppose now that $X_a(\Omega) = 0$ for $|\Omega| \geq 10000\pi$, $w[n]$ is a rectangular window, and the window length N is constrained to be an integer power of 2. Recall that the $X[k]$ correspond to samples of the Fourier transform of a windowed version of $x_a(t)$. Determine T and the minimum value of $N = 2^\eta$ such that the following conditions are satisfied: (i) no aliasing occurs when you sample $x_a(t)$ to obtain $x[n]$; and (ii) the spacing between the $X[k]$ corresponds to analog frequency spacing of no more than 5 Hz.

(c) (5 points) Suppose instead that $w[n]$ is a length-N Hamming window, and $T = 50\mu$sec. A conservative rule of thumb for the frequency resolution of windowed DFT analysis is that the frequency resolution for the DTFT is equal to the width of the main lobe of $W_d(\omega)$. You wish to be able to resolve analog sinusoidal signals that are separated by as little as 15 Hz in frequency. In addition, your window length N is constrained to be an integer power of 2. What is the minimum length $N = 2^\mu$ that will meet your resolution requirement?

Solution

(a) This problem is similar to Problem 1c in Homework 8. Assuming no aliasing takes place, the sampled discrete-time signal will be $x[n] = \cos(\omega_0 n)$, where $\omega_0 = \Omega_0 T$. We get a finite number of samples, so we actually receive $x[n] = \cos(\omega_0 n)w[n]$, where $w[n]$ is the windowing function. Since we're multiplying in the time domain, we get a convolution in the frequency domain:

$$X_d(\omega) = \frac{1}{2\pi}(\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)) * W_d(\omega)$$

We can calculate the DTFT of the windowing function using the geometric series:

$$\begin{aligned}
W_d(\omega) &= \sum_{n=0}^{N-1} e^{-j\omega n} \\
&= \frac{1-e^{-j\omega N}}{1-e^{-j\omega}} \\
&= \frac{e^{-j\omega\frac{N}{2}}(e^{j\omega\frac{N}{2}}-e^{-j\omega\frac{N}{2}})}{e^{-j\omega\frac{1}{2}}(e^{j\omega\frac{1}{2}}-e^{-j\omega\frac{1}{2}})} \\
&= e^{-j\omega\frac{N-1}{2}}\frac{\sin(\frac{N\omega}{2})}{\sin(\frac{\omega}{2})}
\end{aligned}$$

Therefore, the DTFT of the windowed cosine is given by

$$X_d(\omega) = \frac{1}{2}e^{-j(\omega-\omega_0)\frac{N-1}{2}}\frac{\sin(\frac{N}{2}(\omega-\omega_0))}{\sin(\frac{1}{2}(\omega-\omega_0))} + \frac{1}{2}e^{-j(\omega+\omega_0)\frac{N-1}{2}}\frac{\sin(\frac{N}{2}(\omega+\omega_0))}{\sin(\frac{1}{2}(\omega+\omega_0))}$$

We see that we get two digital sincs, centered at ω_0 and $-\omega_0$. The peak locations will be at ω_0 and $-\omega_0$, and we find that zeros occur whenever

$$\begin{aligned}
\frac{N}{2}(\omega-\omega_0) = k\pi \to \omega-\omega_0 = \frac{2\pi k}{N} \to \omega = \omega_0 + \frac{2\pi k}{N} \\
\frac{N}{2}(\omega+\omega_0) = k\pi \to \omega+\omega_0 = \frac{2\pi k}{N} \to \omega = -\omega_0 + \frac{2\pi k}{N}
\end{aligned}$$

Therefore, we get zeros every $\frac{2\pi k}{N}$ - this corresponds *exactly* to the DFT samples, *if we sample the peaks of the sincs.* If we do so, then we get two nonzero samples - one at $\omega_0 = \frac{2\pi k_0}{N}$, and one at $2\pi - \omega_0 = \frac{2\pi(N-k_0)}{N}$, as the DTFT is 2π-periodic. Every other sample will be $\frac{2\pi k}{N}$ away from a peak, and therefore will be zero.

If we set $\omega_0 = \frac{2\pi k_0}{N}$ (sampling the peak at ω_0 when we take the DFT), then the DFT is given by

$$\begin{aligned}
X[k] &= \frac{1}{2}e^{-j(\frac{2\pi k}{N}-\frac{2\pi k_0}{N})\frac{N-1}{2}}\frac{\sin(\frac{N}{2}(\frac{2\pi k}{N}-\frac{2\pi k_0}{N}))}{\sin(\frac{1}{2}(\frac{2\pi k}{N}-\frac{2\pi k_0}{N}))} + \frac{1}{2}e^{-j(\frac{2\pi k}{N}+\frac{2\pi k_0}{N})\frac{N-1}{2}}\frac{\sin(\frac{N}{2}(\frac{2\pi k}{N}+\frac{2\pi k_0}{N}))}{\sin(\frac{1}{2}(\frac{2\pi k}{N}+\frac{2\pi k_0}{N}))} \\
&= \frac{1}{2}e^{-j\frac{(N-1)\pi}{N}(k-k_0)}\frac{\sin(\pi(k-k_0))}{\sin(\frac{\pi}{N}(k-k_0))} + \frac{1}{2}e^{-j\frac{(N-1)\pi}{N}(k+k_0)}\frac{\sin(\pi(k+k_0))}{\sin(\frac{\pi}{N}(k+k_0))}
\end{aligned}$$

Indeed, since $\sin(\pi k) = 0 \; \forall \; k$, the only potential nonzero samples are when the expression is indeterminate, which occurs at $k = k_0$ and $k = N - k_0$. But we know these correspond to sampling the peaks, so they're the two nonzero samples.

Therefore, in order to capture the peaks, we need to set $\omega_0 = \frac{2\pi k_0}{N}$. Since $\omega_0 = \Omega_0 T$, this allows us to solve for T:

$$\Omega_0 T = \frac{2\pi k_0}{N} \to \boxed{T = \frac{2\pi k_0}{N\Omega_0}}$$

However, this value of T is **not** unique because aliasing can occur when sampling. If, when we sample, we get $\omega_0 \pm 2\pi l$ instead of ω_0, nothing changes due to the 2π-periodicity of the DTFT. So, there are infinitely many possible values of T:

$$\Omega_0 T = \frac{2\pi k_0}{N} \pm 2\pi l \rightarrow \boxed{T = \frac{2\pi k_0}{N\Omega_0} \pm \frac{2\pi l}{\Omega_0}, l \in \mathbb{Z}}$$

(b) The first condition implies that we must sample at or above the Nyquist rate. Since the maximum analog frequency is 10000π, and $\omega = \Omega T$, this means that we need to choose T such that $10000\pi T \leq \pi$; this gives

$$\boxed{T \leq \frac{1}{10000} \text{ s}}$$

Since 5 Hz $= 10\pi$ rad/sec, we can find the corresponding spacing $\Delta\omega$ in the digital frequency domain:

$$\Delta\Omega = 10\pi \rightarrow \Delta\omega = \Delta\Omega T$$

Additionally, since we know that $\Delta\omega$ corresponds to the distance between two DFT samples, we can set $\Delta\omega = \frac{2\pi}{N}$ to get a relationship between N and T:

$$\frac{2\pi}{N} = 10\pi T$$

To make N as small as possible, we want to make T as large as possible; therefore, we choose to sample at the Nyquist rate. In that case,

$$\frac{2\pi}{N} = \frac{10\pi}{10000} \rightarrow \boxed{N = 2000 \rightarrow 2048 = 2^{11}}$$

since we're restricted to window lengths that are a power of two. Therefore, we choose the closest power of two that's larger than the window length.

(c) We're told that $T = 50 \times 10^{-6}$ s, so if the required analog frequency resolution is $\Delta\Omega = 15$ Hz $= 30\pi$ rad/sec, then the corresponding digital frequency resolution is $\Delta\omega = \Delta\Omega T = \frac{3\pi}{2000}$. Therefore, using the rule of thumb given in the problem, we need $W_d(\omega)$ to have a main lobe width of $\frac{3\pi}{2000}$ or less. Using the table on Page 563 of the textbook, the main lobe width for the Hamming window is approximately $\frac{8\pi}{L}$. So, we can solve for L:

$$\frac{8\pi}{L} = \frac{3\pi}{2000} \rightarrow \boxed{L = 5333.3 \rightarrow 5334 \rightarrow 8192 = 2^{13}}$$

where the first rounding comes from the fact that L must be an integer, and the second rounding comes from the fact that L must be a power of two; again, we choose the closest power of two that's larger than the window length.

Problem 2

(10 points) Let $x[n] = \cos(\frac{\pi n}{4})$ and $v[n] = \begin{cases} x[n], & n \text{ even}, 0 \le n \le 27 \\ 0, & \text{otherwise} \end{cases}$.

Sketch $|V_d(\omega)|$ for $-\pi \le \omega \le \pi$, labeling the frequencies of the peaks and the first nulls on either side of the peak. In addition, label the amplitudes of the peaks and the strongest side lobe of each peak.

Hint 1: Let $w[n]$ be a window function and $v[n] = x[n]w[n]$.

Hint 2: You can use the approximation $|X_1(\omega) + X_2(\omega)| \approx |X_1(\omega)| + |X_2(\omega)|$ if for $\omega < 0$, $|X_1(\omega)| \approx 0$ and for $\omega > 0$, $|X_2(\omega)| \approx 0$.

Solution

Using the hint, we write $v[n] = x[n]w[n]$, where $w[n]$ is a *windowing* function given by

$$w[n] = \begin{cases} 1, & n \text{ even}, 0 \le n \le 27 \\ 0, & \text{else} \end{cases}$$

Therefore, we can take its DTFT by applying a geometric series, changing ω to 2ω to account for the fact that the only nonzero values are when n is even:

$$\begin{aligned} W_d(\omega) &= \sum_{n=0}^{27} w[n]e^{-j\omega n} \\ &= \sum_{n=0}^{13} e^{-j2\omega n} \\ &= \frac{1 - e^{-j2\omega(14)}}{1 - e^{-j2\omega}} \\ &= \frac{e^{-j\omega 14}(e^{j\omega 14} - e^{-j\omega 14})}{e^{-j\omega}(e^{j\omega} - e^{-j\omega})} \\ &= e^{-j\omega 13}\frac{\sin(14\omega)}{\sin(\omega)} \end{aligned}$$

Therefore, as in Problem 1a, we can find $V_d(\omega)$ as a convolution with $W_d(\omega)$:

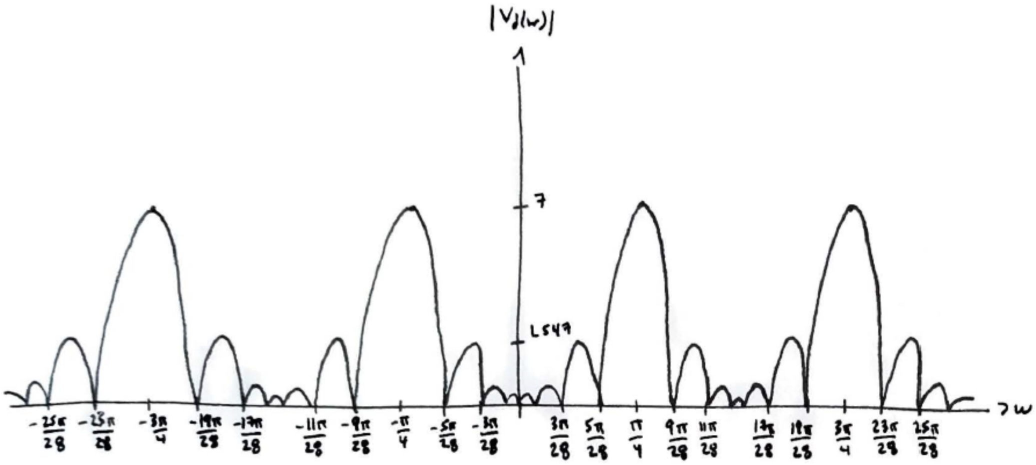
$$\begin{aligned} V_d(\omega) &= \frac{1}{2\pi}(X_d(\omega) * W_d(\omega)) \\ &= \frac{1}{2\pi}(\pi\delta(\omega - \frac{\pi}{4}) + \pi\delta(\omega + \frac{\pi}{4})) * W_d(\omega) \\ &= \frac{1}{2}W_d(\omega - \frac{\pi}{4}) + \frac{1}{2}W_d(\omega + \frac{\pi}{4}) \\ &= \boxed{\frac{1}{2}e^{-j(\omega - \frac{\pi}{4}13)}\frac{\sin(14(\omega - \frac{\pi}{4}))}{\sin(\omega - \frac{\pi}{4})} + \frac{1}{2}e^{-j(\omega + \frac{\pi}{4}13)}\frac{\sin(14(\omega + \frac{\pi}{4}))}{\sin(\omega + \frac{\pi}{4})}} \end{aligned}$$

Using the hint, the magnitude can be written approximately as

$$\boxed{|V_d(\omega)| \approx \frac{1}{2}\left|\frac{\sin(14(\omega - \frac{\pi}{4}))}{\sin(\omega - \frac{\pi}{4})}\right| + \frac{1}{2}\left|\frac{\sin(14(\omega + \frac{\pi}{4}))}{\sin(\omega + \frac{\pi}{4})}\right|}$$

Note that this approximation is quite inaccurate in this case, due to the additional peaks. In general, it's not true that $|a + b| = |a| + |b|$, but the approximation was made in this problem to allow you to simplify your plots.

The resulting plot of the magnitude can be seen below. The peaks are at $\frac{\pi}{4}$, $-\frac{\pi}{4}$, $\frac{3\pi}{4}$, and $-\frac{3\pi}{4}$, and have magnitude 7. There will be zeros every time $14(\omega - \frac{\pi}{4}) = \pi l \rightarrow \omega = \frac{\pi}{4} + \frac{\pi l}{14}$, and when $14(\omega + \frac{\pi}{4}) = \pi l \rightarrow \omega = -\frac{\pi}{4} + \frac{\pi l}{14}$. The maximum sidelobe height was calculated to be 1.547, which is 13 dB down from the height of the main lobe, as expected for a rectangular window. Since -13 dB represents a ratio of $10^{-\frac{13}{20}}$, the sidelobe height is $7 \times 10^{-\frac{13}{20}}$.



Problem 3

(10 points) Given two sequences $\{x[n]\}_{n=0}^{514}$ and $\{h[n]\}_{n=0}^{127}$ you are asked to compute their linear convolution $y[n] = x[n] * h[n]$. You decide to use the DFT to speed up the computation.

(a) (3 points) What is the length of the sequence $y[n]$?

(b) (3 points) Find the smallest number of zeros that should be padded to each sequence so that the linear convolution can be computed using the DFT.

(c) (4 points) To further speed computation, you decide to use a radix-2 FFT to compute the DFT. How should the sequences be padded so that their linear convolution can be computed using the smallest possible radix-2 FFT?

Solution

(a) If $x[n]$ is length L and $y[n]$ is length M, then the length of the resulting convolution will always be $L + M - 1$. In this case, $L = 515$ and $M = 128$, so $y[n]$ will be of length

$$515 + 128 - 1 = \boxed{642}$$

(b) If we want to use the DFT to compute the convolution, then both signals need to be zero-padded to length $L + M - 1$. That means that **127** zeros need to be added to $x[n]$, and **514** zeros need to be added to $h[n]$.

(c) If we want to use the radix-2 FFT to compute the convolution, then the lengths need to be a power of 2; we choose the closest power of two that's larger than $L + M - 1$. In this case, that's 1024. So we need to add **509** zeros to $x[n]$, and **896** zeros to $h[n]$.

Problem 4

(15 points) You are to compute by hand the DFT of the following sequence:

$$\{x[n]\}_{n=0}^{3} = \{1, 4, 3, 6\}$$

(a) (5 points) Compute the DFT of $x[n]$ using the definition of the DFT. Show your work and give exact answers.

(b) (5 points) Use a decimation-in-time radix-2 FFT to compute the DFT of $x[n]$. Draw the corresponding flow diagram.

(c) (5 points) Repeat part (b) using decimation-in-frequency.

Solution

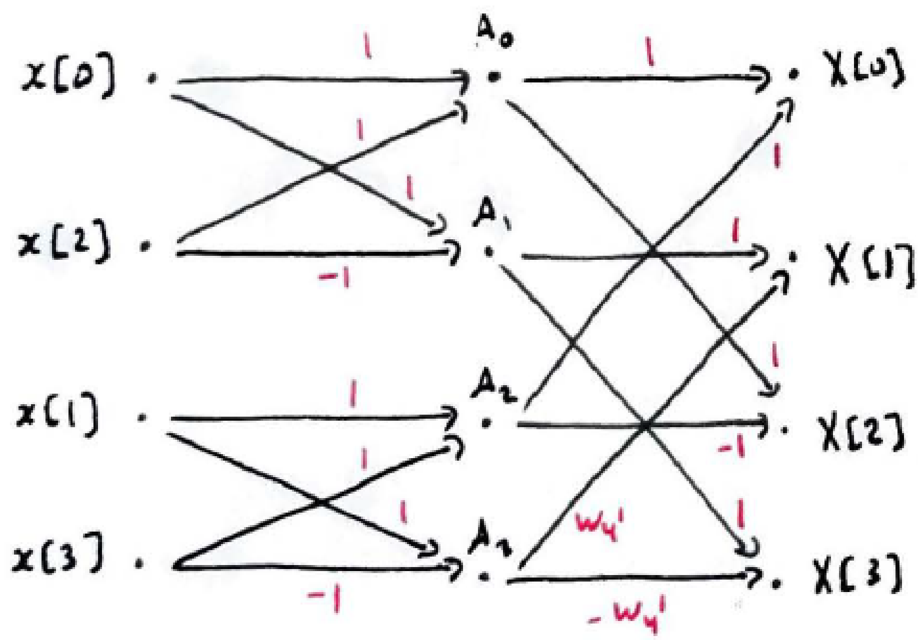
(a) Using the definition, we just plug in:

$$\begin{aligned}
X[k] &= \sum_{n=0}^{3} x[n] e^{-j\frac{2\pi kn}{4}} \\
X[0] &= \sum_{n=0}^{3} x[n] = 1 + 4 + 3 + 6 = 14 \\
X[1] &= \sum_{n=0}^{3} x[n] e^{-j\frac{\pi n}{2}} = 1 - 4j - 3 + 6j = -2 + 2j \\
X[2] &= \sum_{n=0}^{3} x[n] e^{-j\pi n} = 1 - 4 + 3 - 6 = -6 \\
X[3] &= \sum_{n=0}^{3} x[n] e^{-j\frac{3\pi n}{2}} = 1 + 4j - 3 - 6j = -2 - 2j
\end{aligned}$$

Therefore, the DFT is given by

$$\boxed{\{X[k]\}_{k=0}^{3} = \{14, -2 + 2j, -6, -2 - 2j\}}$$

(b) The corresponding flowgraph is seen below.



Calculating the values gives:

$$
\begin{aligned}
A_0 &= x[0] + x[2] = 4 \\
A_1 &= x[0] - x[2] = -2 \\
A_2 &= x[1] + x[3] = 10 \\
A_3 &= x[1] - x[3] = -2
\end{aligned}
$$

Therefore,

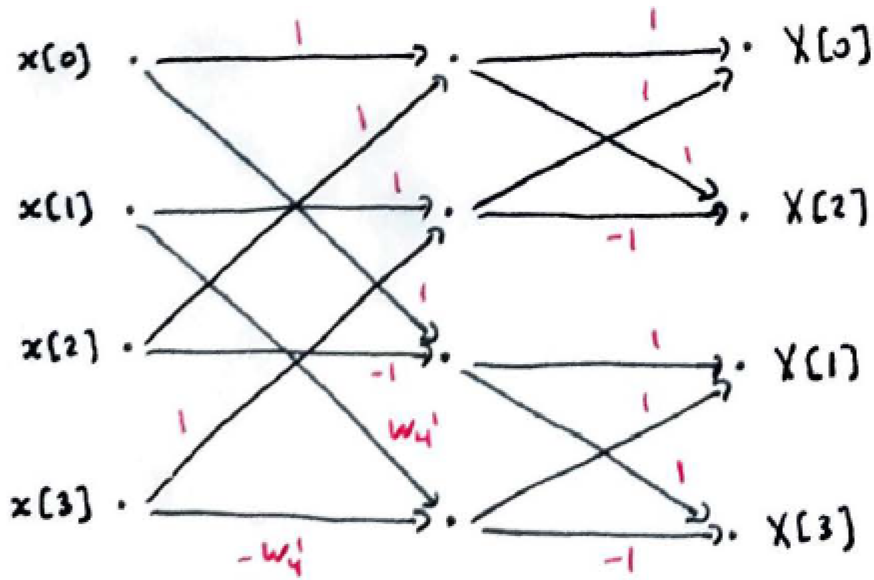
$$
\begin{aligned}
X[0] &= A_0 + A_2 = 14 \\
X[1] &= A_1 + e^{-j\frac{2\pi}{4}} A_3 = -2 + 2j \\
X[2] &= A_0 - A_2 = -6 \\
X[3] &= A_1 - e^{-j\frac{2\pi}{4}} A_3 = -2 - 2j
\end{aligned}
$$

Again, we see that the DFT is given by

$$\boxed{\{X[k]\}_{k=0}^{3} = \{14, -2+2j, -6, -2-2j\}}$$

(c) The corresponding flowgraph is seen below. Note the similarities between the DIF and DIT flowgraphs:

- The DIT flowgraph gives the input in bit-reversed order, while the DIF flowgraph gives the output in bit-reversed order.
- The DIT flowgraph increases the DFT size performed at each step from 2 to N, while the DIF flowgraph decreases the DFT size performed at each step from N to 2.
- The weights of the diagonal branches switch when going from DIT to DIF. In DIT, the diagonal branches going down from the top half take the value 1, while the diagonal branches going up from the bottom half take the value W_N^m. In DIF, the diagonal branches going down from the top half take the value W_N^m, and the diagonal branches going up from the bottom half take the value 1.



Calculating the values gives:

$$B_0 = x[0] + x[2] = 4$$
$$B_1 = x[1] + x[3] = 10$$
$$B_2 = x[0] - x[2] = -2$$
$$B_3 = e^{-j\frac{\pi}{2}}(x[1] - x[3]) = 2j$$

Therefore,

$$X[0] = B_0 + B_1 = 14$$
$$X[1] = B_2 + B_3 = -2 + 2j$$
$$X[2] = B_0 - B_1 = -6$$
$$X[3] = B_2 - B_3 = -2 - 2j$$

which again gives the DFT to be

$$\boxed{\{X[k]\}_{k=0}^{3} = \{14, -2 + 2j, -6, -2 - 2j\}}$$

Problem 5

(15 points) Suppose that you wish to design a circuit to perform a length-8 DFT. You are given a pair of chips that compute the DFT of a length-4 complex input sequence. The inputs and outputs of this chip are all complex numbers. You also have access to complex multiplication and addition circuits, each of which has two complex inputs and one complex output.

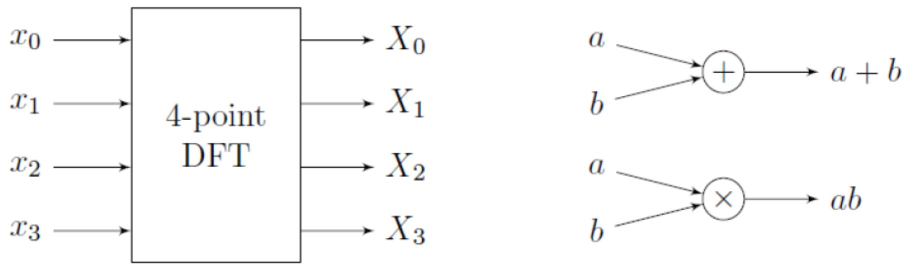


Figure 1: DFT chip and complex arithmetic circuits

(a) (3 points) Your goal is to use as few complex multiplication circuits as possible. Fortunately, you don't need to use a multiplier circuit to multiply by $+1, -1, +j$, or $-j$. Explain why these are trivial multiplications.

(b) (6 points) Show how you would connect two chips and the multiplication and addition circuits to compute a length-8 decimation-in-time FFT. Explain your solution. How many nontrivial multiplications are required?

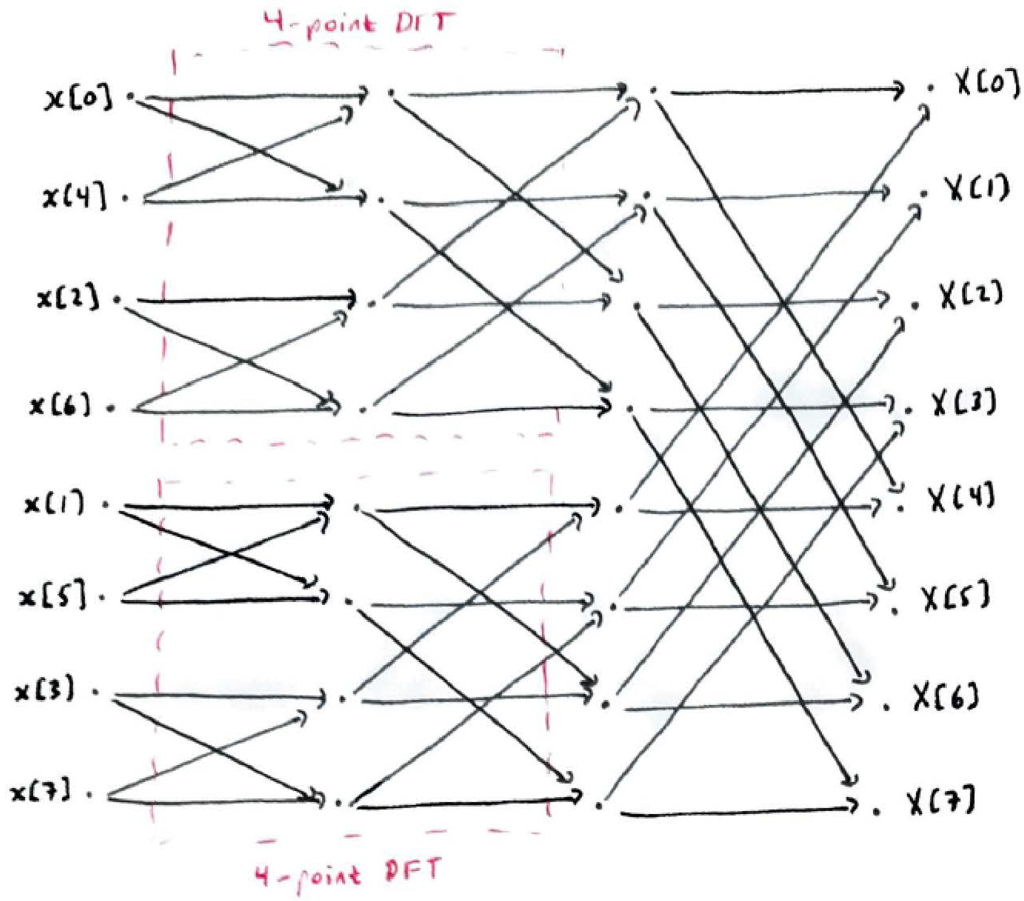
(c) (6 points) Repeat part (b) using decimation-in-frequency.

Solution

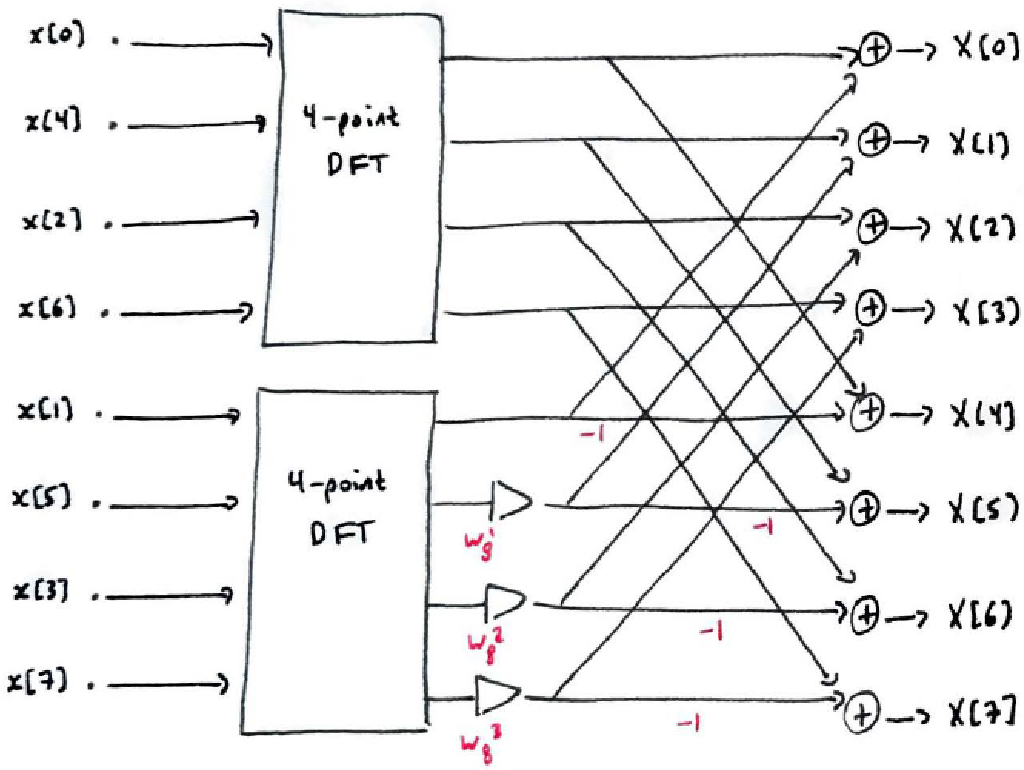
(a) Obviously, multiplying by 1 is a trivial multiplication; we just have to pass the signal through. To realize why multiplying by -1 is trivial, recall that we can represent numbers in two's complement notation. Then multiplying by -1 is equivalent to flipping the bits and adding 1; no multiplication is actually required.

To examine the effect of multiplying by j, it's simpler to represent $x = a + jb$. Then $jx = -b + ja$ - all we have to do is *switch* the real and imaginary parts, and multiply the new real part by -1, which was already shown to be a trivial multiplication. Similarly, since $-jx = b - ja$, we switch the real and imaginary parts, and multiply the new imaginary part by -1. No actual multiplications are needed for either.

(b) To figure out where the chips can be placed, it's helpful to sketch out the full 8-point FFT flowgraph (weights omitted), which can be seen below.

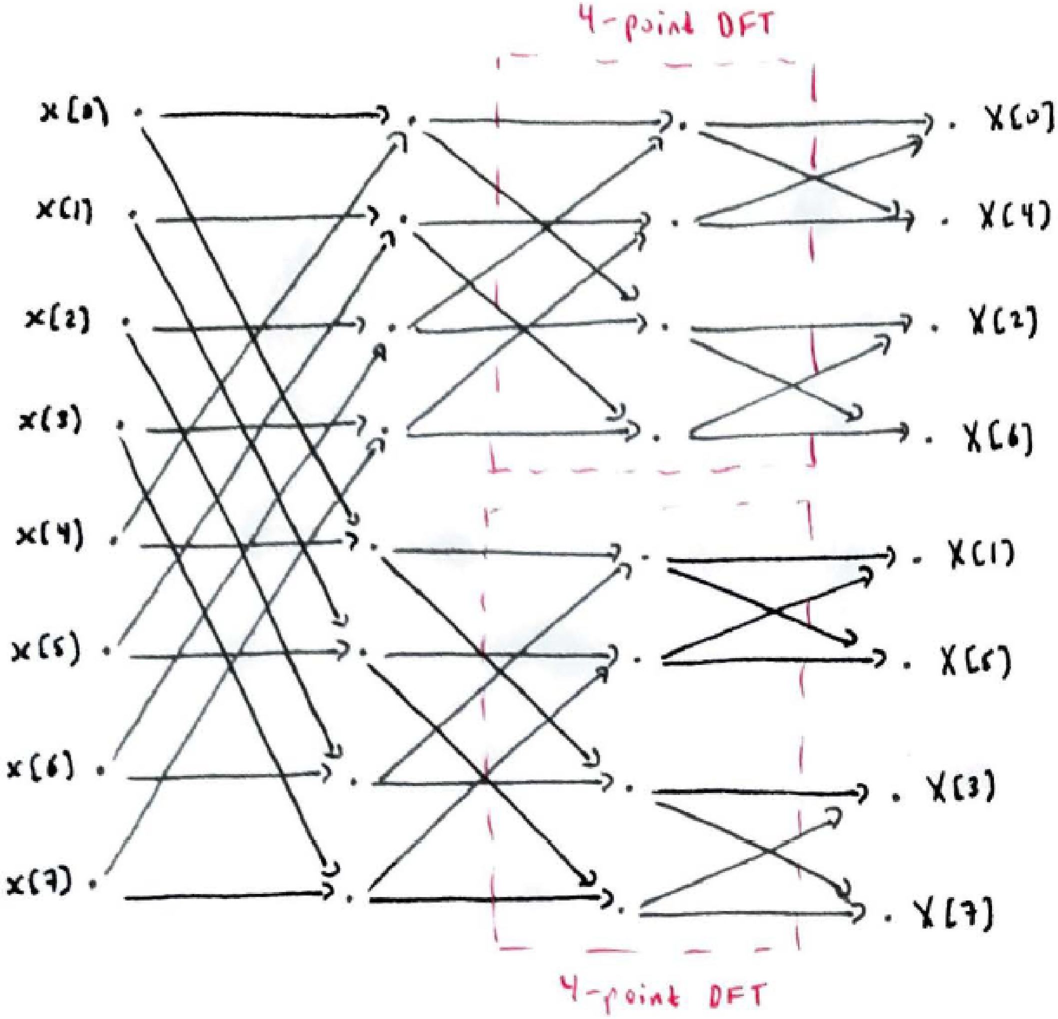


Now, comparing with the flowgraph in Problem 4b, we see that the 8-point FFT flowgraph contains two full 4-point DFTs; therefore, we can replace the 4-point DFT flowgraphs with the given chips. We can also reduce the number of multiplications by "pre-multiplying" the outputs; for example, in Problem 4b, we could have multiplied A_3 by W_4^1, and then passed it through branches with weights 1 and -1, both of which are trivial multiplications. So each pair of branches with complex weights actually only requires one multiplication, instead of two. The resulting circuit is shown below.

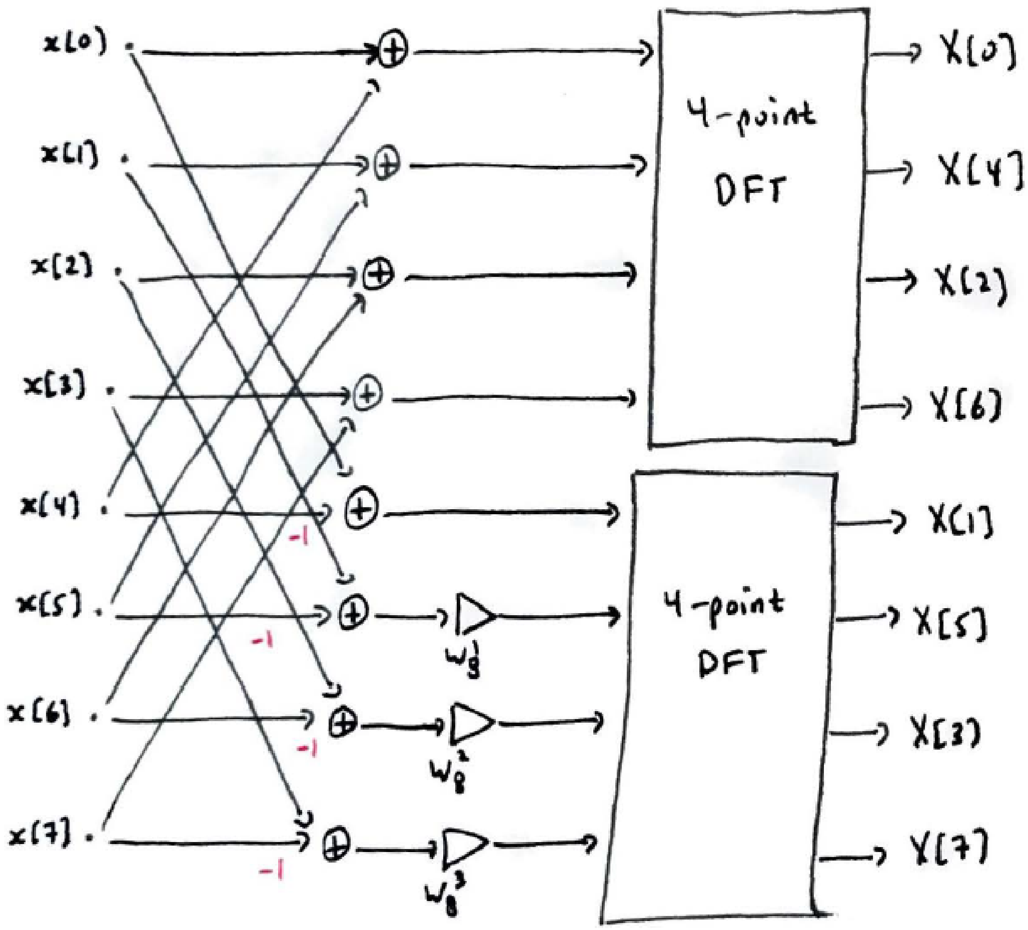


Furthermore, multiplying by $W_8^2 = -j$ is another trivial multiplication. Therefore, only **two** nontrivial multiplications are actually required.

(c) We perform a similar procedure that we did in Part (b). First, we sketch the full 8-point DIF FFT flowgraph (again, with the weights omitted), given below:



Comparing with the flowgraph in Problem 4c, we again see where the two full 4-point DFTs are located; this is where the chips are inserted. In this case, instead of pre-multiplying the branch, we can "post-multiply" by W_N^m instead. For example, in Problem 4c, we could have multiplied B_3 by W_4^1 before passing it onto the next step, multiplying $x[1]$ by 1 and $x[3]$ by -1 instead; both are trivial multiplications. The resulting circuit is shown below.



Since multiplying by W_8^2 is still a trivial multiplication, only **two** nontrivial multiplications will be required.

Problem 6

(10 points) Determine $y[n]$, the cyclic convolution of $x[n]$ and $h[n]$ for the following cases:

(a) (5 points) $\{x[n]\}_{n=0}^{4} = \{2, 4, 6, 8, 10\}$ and $\{h[n]\}_{n=0}^{4} = \{1, 0, 0, 0, 1\}$

(b) (5 points) $\{x[n]\}_{n=0}^{7} = \{1, 2, 3, 4, 5, 0, 0, 0\}$ and $\{h[n]\}_{n=0}^{7} = \{1, 0, 0, 1, 0, 0, 0, 1\}$

Solution

(a) There are two approaches to performing circular convolution. One is to use the table method, but with *periodic* signals; we keep one signal the same, flip the the other, and shift it to the right, but the signal we flip is assumed to be periodic. Therefore, the output will also be periodic, with the period equal to the length of the inputs. Doing this here gives:

$$\begin{array}{ccccccccccc}
 & & & & 2 & 4 & 6 & 8 & 10 & & \\
1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & & \to y[0] = 6 \\
 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & & \to y[1] = 10 \\
 & & 1 & 0 & 0 & 0 & 1 & 1 & 0 & & \to y[2] = 14 \\
 & & & 1 & 0 & 0 & 0 & 1 & 1 & & \to y[3] = 18 \\
 & & & & 1 & 0 & 0 & 0 & 1 & & \to y[4] = 12
\end{array}$$

Therefore, the output is given by

$$\boxed{\{y[n]\}_{n=0}^{4} = \{6, 10, 14, 18, 12\}}$$

Alternatively, the circular convolution could have been performed using a matrix multiplication. We can write $y = Hx$, where H is a *circulant* $N \times N$ matrix, formed as follows:

- Set the first column of H as $h[n]$.
- Get the other columns *recursively*; the n^{th} column is the $(n-1)^{th}$ column circularly shifted down by 1.

In this case, we could write

$$y = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \\ 10 \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ 14 \\ 18 \\ 12 \end{bmatrix}$$

(b) Using the matrix multiplication approach, we find that

$$y = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 7 \\ 10 \\ 7 \\ 3 \\ 4 \\ 6 \end{bmatrix} \to \boxed{\{y[n]\}_{n=0}^{7} = \{3, 5, 7, 10, 7, 3, 4, 6\}}$$

Problem 7

(20 points) Let $\{x[n]\}_{n=0}^{N-1}$ be a finite-length signal and $\{X[k]\}_{k=0}^{N-1}$ be the corresponding N-point DFT. Define the following two sequences:

$$\begin{aligned} s[n] &= \{x[0], x[1], ..., x[N-1], x[0], x[1], ..., x[N-1]\} \\ y[n] &= \{x[0], x[1], ..., x[N-1], \underbrace{0, 0, ..., 0}_{N \text{ zeros}}\} \end{aligned}$$

Let $\{S[k]\}_{k=0}^{2N-1}$ and $\{Y[k]\}_{k=0}^{2N-1}$ be the corresponding $2N$-point DFTs.

(a) (6 points) Show that $S[2k] = 2X[k]$ and $S[2k+1] = 0$ for $k = 0, 1, ..., N-1$.

(b) (7 points) Show that $Y[2k] = X[k]$ for $k = 0, 1, ..., N-1$.

(c) (7 points) Find an expression for $Y[2k+1], k = 0, 1, ..., N-1$ in terms of $X[k]$ using the results in part (a).

Hint: Use the fact that $y[n] = s[n]w[n]$, where $w[n] = \{\underbrace{1, 1, ..., 1}_{N \text{ ones}}, \underbrace{0, 0, ..., 0}_{N \text{ zeros}}\}$ is an N-point rectangular window zero-padded to length $2N$.

Solution

(a) We can write the formula for $S[k]$ using the definition:

$$S[k] = \sum_{n=0}^{2N-1} s[n] e^{-j\frac{2\pi kn}{2N}}$$

Therefore,

$$\begin{aligned} S[2k] &= \sum_{n=0}^{2N-1} s[n] e^{-j\frac{4\pi kn}{2N}} \\ &= \sum_{n=0}^{2N-1} s[n] e^{-j\frac{2\pi kn}{N}} \\ &= \sum_{n=0}^{N-1} s[n] e^{-j\frac{2\pi kn}{N}} + \sum_{n=N}^{2N-1} s[n] e^{-j\frac{2\pi kn}{N}} \\ &= \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi kn}{N}} + \sum_{l=0}^{N-1} s[l+N] e^{-j\frac{2\pi k(l+N)}{N}} \\ &= \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi kn}{N}} + \sum_{l=0}^{N-1} x[n] e^{-j\frac{2\pi kl}{N}} \\ &= \boxed{2X[k]} \end{aligned}$$

where we used a variable transformation $l = n - N$ to change the bounds of the second sum, the fact that $s[l+N] = s[l] = x[l]$ for $l = 0, 1, ..., N-1$, and the fact that $e^{-j\frac{2\pi k(l+N)}{N}} = e^{-j\frac{2\pi kl}{N}} e^{-j2\pi k} = e^{-j\frac{2\pi kl}{N}}$, since k must be an integer.

Similarly, we can show that $S[2k+1] = 0$ by plugging into the summation:

$$\begin{aligned}
S[2k+1] &= \sum_{n=0}^{2N-1} s[n]e^{-j\frac{2\pi kn}{N}}e^{-j\frac{\pi n}{N}} \\
&= \sum_{n=0}^{N-1} s[n]e^{-j\frac{2\pi kn}{N}}e^{-j\frac{\pi n}{N}} + \sum_{n=N}^{2N-1} s[n]e^{-j\frac{2\pi kn}{N}}e^{-j\frac{\pi n}{N}} \\
&= \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi kn}{N}}e^{-j\frac{\pi n}{N}} + \sum_{l=0}^{N-1} s[l+N]e^{-j\frac{2\pi k(l+N)}{N}}e^{-j\frac{\pi(l+N)}{N}} \\
&= \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi kn}{N}}e^{-j\frac{\pi n}{N}} + \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi kn}{N}}e^{-j\frac{\pi n}{N}}e^{-j\pi} \\
&= \boxed{0}
\end{aligned}$$

since $e^{-j\pi} = -1$.

(b) Similarly, we know that

$$Y[k] = \sum_{n=0}^{2N-1} y[n]e^{-j\frac{2\pi kn}{2N}}$$

Therefore,

$$\begin{aligned}
Y[2k] &= \sum_{n=0}^{2N-1} y[n]e^{-j\frac{4\pi kn}{2N}} \\
&= \sum_{n=0}^{2N-1} y[n]e^{-j\frac{2\pi kn}{N}} \\
&= \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi kn}{N}} \\
&= \boxed{X[k]}
\end{aligned}$$

since $y[n] = x[n]$ for $n = 0, 1, ..., N-1$, and 0 otherwise.

(c) Using the hint, and realizing that multiplication in the time domain corresponds to circular convolution in the frequency domain when dealing with DFTs, we can write

$$Y[k] = \frac{1}{2N}(S[k] \otimes W[k]) = \frac{1}{2N}\sum_{m=0}^{2N-1} W[m]S[\langle k-m\rangle_{2N}]$$

Here, $W[k]$ is the DFT of the windowing function; since

$$W_d(\omega) = e^{-j\omega\frac{N-1}{2}}\frac{\sin(\frac{N\omega}{2})}{\sin(\frac{\omega}{2})}$$

we can get its DFT by substituting $\omega = \frac{2\pi k}{2N}$; that is,

$$W[k] = e^{-j\pi k\frac{N-1}{2N}} \frac{\sin(\frac{\pi k}{2})}{\sin(\frac{\pi k}{2N})}, k = 0, 1, ..., 2N-1$$

Therefore, we have that

$$Y[2k+1] = \frac{1}{2N} \sum_{m=0}^{2N-1} W[m]S[\langle 2k+1-m \rangle_{2N}]$$

Now, we can use our result from (a), which tells us that $S[2k+1] = 0$; i.e., that $S[k] = 0$ whenever k is odd. This tells us that half of the terms in the circular convolution will be zero, which are the ones corresponding to $\langle 2k+1-m \rangle_{2N}$ being odd, which will happen whenever m is even. Therefore, we can change the summation:

$$Y[2k+1] = \frac{1}{2N} \sum_{m=0, m \text{ odd}}^{2N-1} W[m]S[\langle 2k+1-m \rangle_{2N}]$$

Now, we can make a change of variables; let $m = 2l+1$. Then the summation from $m = 0$ to $2N-1$ of all odd m becomes the summation from $l = 0$ to $N-1$:

$$Y[2k+1] = \frac{1}{2N} \sum_{l=0}^{N-1} W[2l+1]S[\langle 2(k-l) \rangle_{2N}]$$

Finally, since we know that $S[2k] = 2X[k]$, $S[\langle 2(k-l) \rangle_{2N}] = 2X[\langle k-l \rangle_N]$. Here, we use the fact that the modulo over an even number doesn't change whether the value is even or odd. This gives

$$\boxed{Y[2k+1] = \frac{1}{N} \sum_{l=0}^{N-1} W[2l+1]X[\langle k-m \rangle_N], \qquad W[k] = e^{-j\pi k\frac{N-1}{2N}} \frac{\sin(\frac{\pi k}{2})}{\sin(\frac{\pi k}{2N})}, k = 0, 1, ..., 2N-1}$$

Since we created $y[n]$ by zero-padding $x[n]$, the DFT samples the DTFT at twice as many points. Therefore, we can view the DFT as a "sinc-interpolated" version of $X[k]$.