The z-transform

Similar to how we use the Laplace transform to solve differential equations, we can use the z-transform to solve LCCDEs (linear constant-coefficient difference equations). This is important because every LTI system has an LCCDE representation, so the ability to take the z-transform will greatly simplify our job of finding the impulse response. The formal definition of the z-transform is as follows:

$$\boxed{X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}}$$

One can think of z as a "complex frequency", analogous to the complex variable s from the Laplace transform.

However, the above expression doesn't tell the entire story; when giving a z-transform, one must *always* include its corresponding region of convergence; i.e., the values of z for which the summation gives a finite result. Without the region of convergence, it is impossible to ascertain the discrete-time sequence that produced the transform. For example, consider the z-transform of $x[n] = \left(\frac{1}{2}\right)^n u[n]$:

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u[n]z^{-n} \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2}z^{-1}\right)^n \end{aligned}$$

This is the standard geometric series, however, it will only converge when the expression being exponentiated has magnitude less than 1. Therefore, we require $|\frac{1}{2}z^{-1}| < 1 \rightarrow |z| > \frac{1}{2}$ to arrive at the standard result,

$$X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}, |z| > \frac{1}{2}$$

However, what if we were to examine $x[n] = -\left(\frac{1}{2}\right)^n u[-n-1]$? Going through the same procedure gives

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} -\left(\frac{1}{2}\right)^n u[-n-1]z^{-n} \\ &= \sum_{n=-\infty}^{-1} -\left(\frac{1}{2}\right)^n z^{-n} \\ &= \sum_{n=-\infty}^{-1} -\left(\frac{1}{2}z^{-1}\right)^n \end{aligned}$$

This is the "anticausal", or left-sided, version of the previous geometric series, and will only converge if the expression being exponentiated has magnitude greater than 1. Therefore, we now require $|\frac{1}{2}z^{-1}| > 1 \rightarrow |z| < \frac{1}{2}$ to arrive at

$$X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}, |z| < \frac{1}{2}$$

Note that if the ROC was not provided, we would have no way of knowing which signal produced the transform! This is why the ROC is so important.

In general, the ROC of a z-transform will depend on the magnitude of its poles, or values of z where the denominator of $H(z)$ is zero. If the signal is causal, or right-sided, the ROC will be the outside of the circle formed by the largest pole. If the signal is anticausal, or left-sided, the ROC will be the inside of the circle formed by the smallest pole. If the signal is a mix of both causal and anticausal portions, the ROC will be a "donut", or the union of the individual ROCs.

We denote the *transfer function* of an LTI system to be the z-transform of its impulse response. This is oftentimes much easier to calculate than attempting to use linearity and shift-invariance; given an input-output relation $x[n] \to y[n]$, we can simply take $H(z) = \frac{Y(z)}{X(z)}$, and find the impulse response $h[n]$ as the inverse z-transform. If the LCCDE is given instead, assuming zero initial conditions, we can find the transfer function by taking z-transforms of both sides, applying the shifting property. For example, if we're given a system represented by

$$y[n] + y[n-1] = 2x[n] - 3x[n-2]$$

then we could find the impulse response as follows:

$$\begin{aligned} Y(z) + z^{-1}Y(z) &= 2X(z) - 3z^{-2}X(z) \\ Y(z)(1 + z^{-1}) &= X(z)(2 - 3z^{-2}) \\ H(z) = \frac{Y(z)}{X(z)} &= \frac{2 - 3z^{-2}}{1 + z^{-1}} \end{aligned}$$

Inversion: While there is a formula for inverting the z-transform, it involves integration over the complex plane, and is beyond the scope of the course. However, since a transfer function will be a ratio of two polynomials, we can use *partial fractions* to simplify the result, such that simple z-transform pairs and properties can be used. Note that partial fractions can only be used if:

(a) If working with positive powers of z, the order of the numerator is less than the order of the denominator.

(b) If working with z^{-1} terms, the "order" of the denominator (or the absolute value of the largest exponent) is *greater than* the "order" of the numerator.

For example, partial fractions could not be applied to $H(z) = \frac{z^2}{(z+3)(z+4)}$ because the order of the numerator is equal to the order of the denominator. However, partial fractions could be applied to $H(z) = \frac{1}{(1+3z^{-1})(1+4z^{-1})}$, because now the "order" of the denominator is two.

The simplest way to perform partial fraction expansion is through the "cover-up" method. For example, suppose $H(z) = \frac{z}{(z-\alpha)(z-\beta)}$. We would write

$$H(z) = \frac{z}{(z-\alpha)(z-\beta)} = \frac{A}{z-\alpha} + \frac{B}{z-\beta}$$

Then, to find A and B, we "cover up" the multiplicative factor in the original fraction and evaluate:

$$A = \left.\frac{z}{z-\beta}\right|_{z=\alpha}$$
$$B = \left.\frac{z}{z-\alpha}\right|_{z=\beta}$$

If working with squared terms in the denominator, additional terms need to be added to the expansion:

$$H(z) = \frac{z}{(z-\alpha)(z-\beta)^n} = \frac{A}{z-\alpha} + \frac{B_1}{z-\beta} + \frac{B_2}{(z-\beta)^2} + ... + \frac{B_n}{(z-\beta)^n}$$

Note that the cover-up method no longer works, because we'd be evaluating at the pole. However, if we can solve for all of the other terms, we can solve for the B_i terms one at a time, starting with B_n. For example, suppose $H(z) = \frac{z}{(z-2)(z-3)^2}$. This would give

$$H(z) = \frac{z}{(z-2)(z-3)^2} = \frac{A}{z-2} + \frac{B_1}{z-3} + \frac{B_2}{(z-3)^2}$$

We can solve for A and B_2:

$$A = \frac{z}{(z-3)^2}\bigg|_{z=2}$$
$$B_2 = \frac{z}{z-2}\bigg|_{z=3}$$

However, since the only variable is B_1, and the equation has to hold for all z, we can simply plug in any value of z not corresponding to a pole and solve the corresponding equation.

Practice Problems and Solutions

Problem 1

Find the one-sided z-transform, if it exists, and the corresponding region of convergence of the following signals.

(a) $x[n] = 3^n u[n] + 0.5^n u[n-3]$

(b) $x[n] = 2^n (u[n] - u[n-30])$

Solution

(a) Notice that the unit step function is shifted to the right by 3; therefore, we will need to apply the shifting property. We apply the same shift to the exponent term by writing $n = (n-3)+3$:

$$x[n] = 3^n u[n] + 0.5^{n-3+3} u[n-3] = 3^n u[n] + 0.5^3 0.5^{n-3} u[n-3]$$

making it evident that

$$\boxed{X(z) = \frac{1}{1-3z^{-1}} + (0.5^3 z^{-3}) \frac{1}{1-0.5z^{-1}}}$$

(b) There are two approaches to solving this problem. The simpler approach is to realize that $u[n]-u[n-30]$ can be represented as a *finite* summation of delta functions, and take the z-transform using the definition:

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n] z^{-n} \\ &= \sum_{n=-\infty}^{\infty} 2^n (u[n] - u[n-30]) z^{-n} \\ &= \sum_{n=0}^{29} 2^n z^{-n} \\ &= \sum_{n=0}^{29} (2z^{-1})^n \\ &= \boxed{\frac{1-(2z^{-1})^{30}}{1-2z^{-1}}, |z| > 0} \end{aligned}$$

The ROC is $|z| > 0$ because the summation is over a *finite* number of terms; a finite sum of finite values will always converge, as long as we exclude the obvious exception $z = 0$.

Alternatively, we could try an approach similar to the one shown in (a), factoring the 2^n term and applying the shifting property:

$$\begin{aligned} x[n] &= 2^n (u[n] - u[n-30]) \\ &= 2^n u[n] - 2^{n-30+30} u[n-30] \\ &= 2^n u[n] - 2^{30} 2^{n-30} u[n-30] \end{aligned}$$

This gives

$$X(z) = \frac{1}{1-2z^{-1}} - (2z^{-1})^{30}\frac{1}{1-2z^{-1}} = \frac{1-(2z^{-1})^{30}}{1-2z^{-1}}$$

However, what should the ROC be? Even though each *individual* transform has ROC $|z| > 2$, it turns out that $z = 2$ is *not* a pole in the overall response. We can prove this through application of L'Hospital's Rule:

$$\begin{aligned}
\lim_{z\to 2} \frac{1-(2z^{-1})^{30}}{1-2z^{-1}} &= \lim_{z\to 2} \frac{1-2^30z^{-30}}{1-2z^{-1}} \\
&= \lim_{z\to 2} \frac{30\cdot 2^{30}z^{-31}}{2z^{-2}} \\
&= \frac{15}{0.5} = 30
\end{aligned}$$

Therefore, 2 is not a pole, and the ROC is once again $|z| > 0$. So we can write the z-transform as

$$\boxed{X(z) = \frac{1-(2z^{-1})^{30}}{1-2z^{-1}}, |z| > 0}$$

Problem 2

The input $x[n] = 2^n(u[n] - 3u[n-1])$ to an unknown **causal** LSI system produces the output $y[n] = (3^n - 2^n)u[n]$. Use the z-transform approach to determine the unit pulse response $h[n]$ of the system.

Solution

Note how difficult this problem would be using the system properties; it isn't at all evident how we would create a delta function out of shifted and scaled versions of the input. Thankfully, we can just take the z-transforms of the input and the output, and use them to ascertain the impulse response. As before, we rewrite $x[n]$ as

$$x[n] = 2^n u[n] - 3\cdot 2^n u[n-1] = 2^n u[n] - 6\cdot 2^{n-1}u[n-1]$$

which gives

$$X(z) = \frac{z}{z-2} - 6z^{-1}\left(\frac{z}{z-2}\right) = \frac{z-6}{z-2}, |z| > 2$$

Similarly, the z-transform of the output can be calculated as

$$Y(z) = \frac{z}{z-3} - \frac{z}{z-2} = \frac{z(z-2)-z(z-3)}{(z-3)(z-2)} = \frac{z}{(z-3)(z-2)}, |z| > 3$$

So, the transfer function is calculated to be

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z}{(z-3)(z-6)}, |z| > 6$$

Note that the ROC changed to $|z| > 6$; because the system is causal, the ROC will always be the outside of the circle formed by the largest pole. Once we have $H(z)$, we can find the impulse response, $h[n]$, by taking the inverse z-transform, which involves partial fractions; we write

$$H(z) = \frac{z}{(z-3)(z-6)} = \frac{A}{z-3} + \frac{B}{z-6}$$

Using the cover-up method gives

$$A = \frac{z}{z-6}\bigg|_{z=3} = -1, \quad B = \frac{z}{z-3}\bigg|_{z=6} = 2$$

So, using the shifting property, we have

$$\begin{aligned} H(z) &= -\left(\frac{1}{z-3}\right) + 2\left(\frac{1}{z-6}\right) \\ &= -z^{-1}\left(\frac{z}{z-3}\right) + 2z^{-1}\left(\frac{z}{z-6}\right) \end{aligned}$$

Taking the inverse z-transform gives the final result,

$$\boxed{h[n] = -(3)^{n-1}u[n-1] + 2(6)^{n-1}u[n-1]}$$

Problem 3

The z-transform of $x[n]$ is given below:

$$X(z) = \frac{1 - \frac{1}{2}z^{-1}}{1 - \frac{1}{3}z^{-1}}$$

(a) Determine all valid ROCs for $X(z)$.

(b) Assuming that $x[n]$ is a right-sided sequence, determine $x[n]$.

(c) Assuming that $x[n]$ is a left-sided sequence, determine $x[n]$.

Solution

(a) Note that $X(z)$ has a single pole at $z = \frac{1}{3}$. So, if the signal is causal/right-sided, the ROC will be the outside of the circle formed by the pole. If the signal is anticausal/left-sided, the ROC will be the inside of the circle formed by the pole. That is, the two possible ROCs are

$$\boxed{|z| < \frac{1}{3}, \quad |z| > \frac{1}{3}}$$

(b) If the signal is right-sided, then the ROC is $|z| > \frac{1}{3}$. If we write

$$X(z) = \frac{1}{1 - \frac{1}{3}z^{-1}} - \frac{1}{2}z^{-1}\left(\frac{1}{1 - \frac{1}{3}z^{-1}}\right)$$

then it becomes obvious that

$$\boxed{x[n] = \left(\frac{1}{3}\right)^n u[n] - \frac{1}{2}\left(\frac{1}{3}\right)^{n-1} u[n-1]}$$

(c) If the signal is left-sided, then the ROC is $|z| < \frac{1}{3}$. Using the same representation of $X(z)$ as before, we now find

$$\boxed{x[n] = -\left(\frac{1}{3}\right)^n u[-n-1] + \frac{1}{2}\left(\frac{1}{3}\right)^{n-1} u[-n]}$$

Without knowledge of the ROC, we would have no way to ascertain which input produced the z-transform.

Problem 4

Consider a causal linear shift-invariant system that is specified by the following LCCDE:

$$y[n] = \frac{5}{2}y[n-1] - y[n-2] + x[n] + 2x[n-1]$$

(a) Assuming zero initial conditions, find the transfer function of the system, $H(z)$.

(b) Find the impulse response of the system, $h[n]$.

(c) Sketch the pole-zero plot of the system.

(d) Draw a block diagram representation of the system.

(e) Is the system BIBO stable?

Solution

(a) We find the transfer function by taking the z-transform of both sides of the LCCDE:

$$Y(z) = \frac{5}{2}z^{-1}Y(z) + z^{-2}Y(z) + X(z) + 2z^{-1}X(z)$$

This gives

$$Y(z)(1 - \frac{5}{2}z^{-1} - z^{-2}) = X(z)(1 + 2z^{-1}) \rightarrow \boxed{H(z) = \frac{Y(z)}{X(z)} = \frac{1+2z^{-1}}{1-\frac{5}{2}z^{-1}+z^{-2}}}, |z| > 2$$

(b) To find the impulse response, or the inverse z-transform of the transfer function, we again apply partial fractions:

$$H(z) = \frac{1+2z^{-1}}{(1-2z^{-1})(1-\frac{1}{2}z^{-1})} = \frac{A}{1-2z^{-1}} + \frac{B}{1-\frac{1}{2}z^{-1}}$$

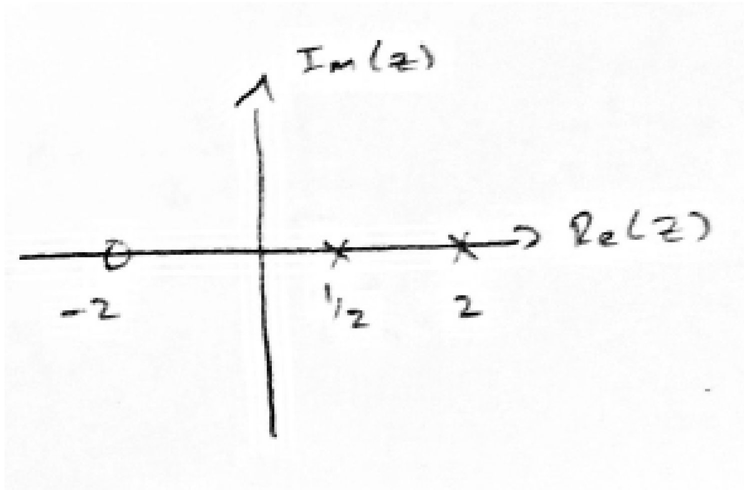
Using the cover-up method gives

$$A = \left.\frac{1+2z^{-1}}{1-\frac{1}{2}z^{-1}}\right|_{z=2} = \frac{8}{3}, \quad B = \left.\frac{1+2z^{-1}}{1-2z^{-1}}\right|_{z=\frac{1}{2}} = -\frac{5}{3}$$

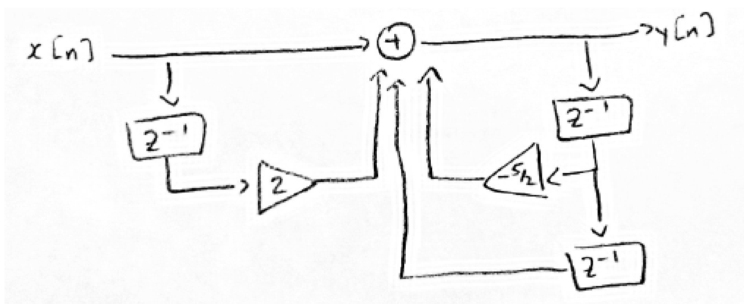
So, we can write

$$H(z) = \frac{8}{3}\left(\frac{1}{1-2z^{-1}}\right) - \frac{5}{3}\left(\frac{1}{1-\frac{1}{2}z^{-1}}\right) \rightarrow \boxed{h[n] = \frac{8}{3}2^n u[n] - \frac{5}{3}\left(\frac{1}{2}\right)^n u[n]}$$

(c) The poles, or z values where the denominator of $H(z)$ is equal to zero, are at $z = 2$ and $z = \frac{1}{2}$. The zeros, or z values where the numerator of $H(z)$ is equal to zero, are at $z = -2$. The corresponding pole-zero plot is shown below.



(d) An example block diagram is shown below.



(e) In this case, the system is **not stable**. There are a few ways to see this. First and foremost, the ROC of $H(z)$ is $|z| > 2$, which does not contain the unit circle. Alternatively, thanks to the $2^n u[n]$ term, the impulse response will not be absolutely summable.