Lecture 9 (Fourier Transforms (FT), CT, DT)

Sinusoidal signals

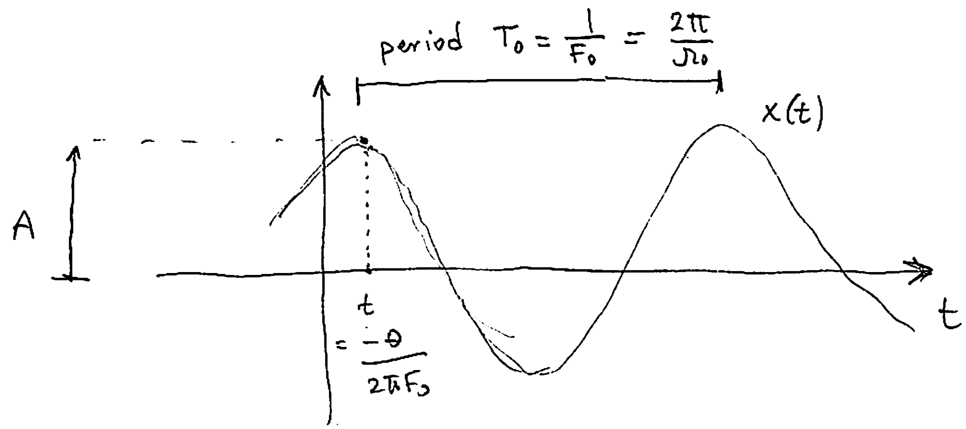
① continuous time

$$x(t) = \underbrace{\boxed{A}}_{\text{amplitude}} \cdot \cos\left(2\pi \underbrace{\boxed{F_0}}_{\text{frequency}} t + \underbrace{\boxed{\theta}}_{\text{phase}}\right)$$

$$= A \cos(\Omega_0 t + \theta)$$

$\Omega_0 = 2\pi F_0$ (angular frequency)

$$2\pi F_0 t + \theta = 0 \Rightarrow t = \frac{-\theta}{2\pi F_0}$$



- periodic: $x(t+T_0) = A\cos(2\pi F_0(t+T_0)+\theta) = A\cos(2\pi F_0 t + \cancel{2\pi} + \theta) = x(t)$

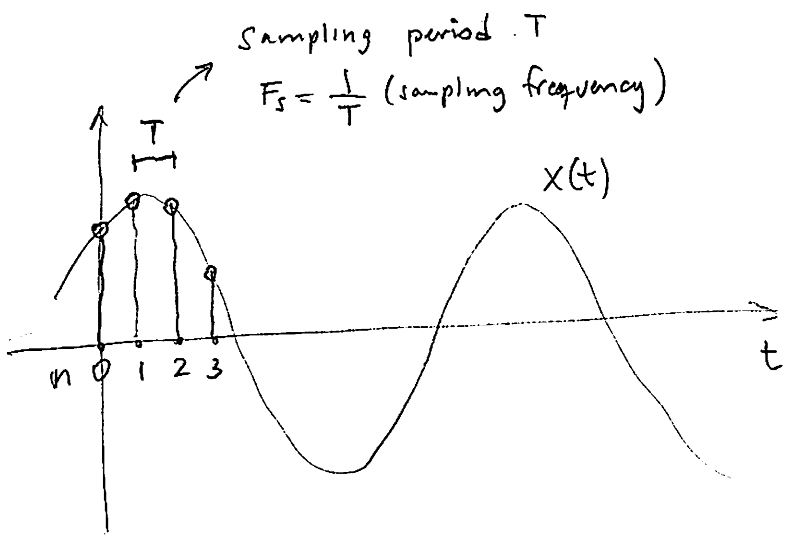
- Recall: Euler's identity: $e^{j\theta} = \cos\theta + j\sin\theta \Rightarrow \cos\theta = \dfrac{e^{j\theta} + e^{-j\theta}}{2}$

$$x(t) = A \cdot \frac{e^{j(\Omega_0 t + \theta)} + e^{-j(\Omega_0 t + \theta)}}{2}$$

$$= \frac{A}{2} e^{j\theta} \cdot e^{j\Omega_0 t} + \frac{A}{2} e^{-j\theta} \cdot e^{j(-\Omega_0)t}$$

(linear combination of complex sinusoids)

② Discrete - time : sampling continuous - time sinusoid $x(t) = A\cos(\Omega_0 t + \theta)$



$$x[n] = x(nT)$$
$$= A\cos(\Omega_0 nT + \theta)$$
$$= A\cos(\omega n + \theta)$$

↳ normalized angular frequency $\omega = \Omega_0 T$

- periodic ?

$x[n] = A\cos(\omega n + \theta)$ is periodic iff (if and only if) $x[n] = x[n+N]$ for some $N \in \mathbb{Z}_+$

$\Longleftrightarrow$ $A\cos(\omega n + \theta) = A\cos(\omega n + \omega N + \theta)$ $\Longleftrightarrow$ $\omega N = 2\pi \cdot k$ for some k

$\Longleftrightarrow$ ω must be a rational multiple of π

- A shift of 2π in the angular frequency, sinusoid doesn't change

$x[n] = A\cos((\omega + 2\pi)n + \theta) = A\cos(\omega n + 2\pi n + \theta) = A\cos(\omega n + \theta)$:

Continuous-time Fourier Transform

$$x(t) = \boxed{\int_{-\infty}^{\infty}} \boxed{\frac{X(\Omega)}{2\pi}} \cdot \boxed{e^{j\Omega t}} \, d\Omega$$

complex sinusoid with frequency Ω

limit of a sum

scaling / weight applied to frequency Ω

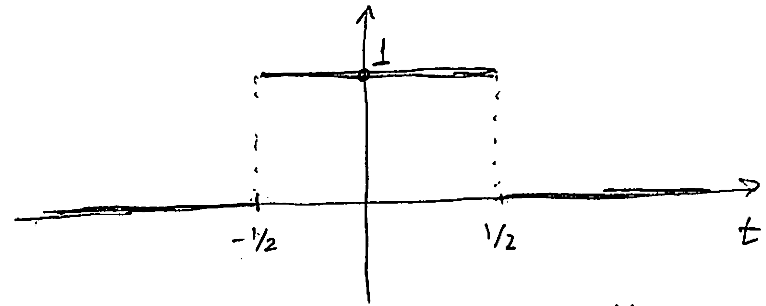
$\mathcal{F}$

$$X(\Omega) = \underbrace{\int_{-\infty}^{\infty} x(t) \, e^{-j\Omega t} \, dt}_{\text{CTFT of } x(t)}$$

In words: Any signal $x(t)$ can be expressed as a linear combination of complex sinusoids $e^{j\Omega t}$, each with $\frac{X(\Omega)}{2\pi}$ as a weight

Ex: Rectangular signal

$$x(t) = \begin{cases} 1, & |t| < 1/2 \\ 0, & \text{otherwise} \end{cases}$$



$$X(\Omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t}\, dt = \int_{-1/2}^{1/2} 1 \cdot e^{-j\Omega t}\, dt = \left.\frac{e^{-j\Omega t}}{-j\Omega}\right|_{t=-1/2}^{1/2}$$

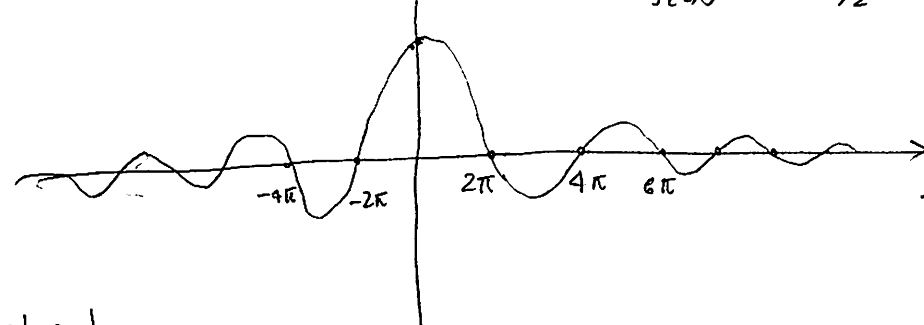
$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$= \frac{e^{-j\Omega/2} - e^{-j\Omega(-1/2)}}{-j\Omega} = 2\left(\frac{e^{j\Omega/2} - e^{-j\Omega/2}}{j\Omega \cdot 2}\right)$$

$$= 2\,\frac{\sin(\Omega/2)}{\Omega} = \frac{\sin(\Omega/2)}{\Omega/2}$$

In this case, $X(\Omega)$ is real-valued.

$$\lim_{\Omega \to 0} \frac{\sin(\Omega/2)}{\Omega/2} = 1$$

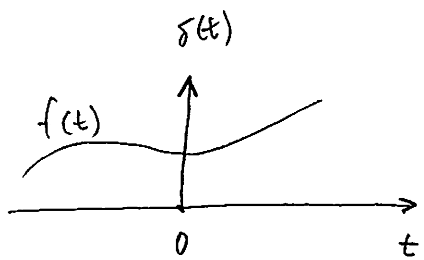


In general, $X(\Omega)$ is complex-valued

and we need to plot magnitude and phase

The Dirac delta function

$$\delta(t) = \begin{cases} \infty & \text{at } t=0 \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \int_{-\infty}^{\infty} \delta(t)\,dt = 1$$



key property: $\displaystyle\int_{-\infty}^{\infty} f(t)\,\delta(t)\,dt = f(t)\Big|_{t=0} = f(0)$

$$\int_{-\infty}^{\infty} f(t)\,\delta(t-t_0)\,dt = f(t_0)$$

Ex: $\displaystyle\int_{-\infty}^{\infty} \delta(5t)\,f(t)\,dt = \int_{-\infty}^{\infty} \delta(v)\,f\!\left(\tfrac{v}{5}\right)\cdot\tfrac{1}{5}\cdot dv = \frac{1}{5}\int_{-\infty}^{\infty} \delta(v)\,f\!\left(\tfrac{v}{5}\right)dv$

$v = 5t$

$dv = 5\,dt$

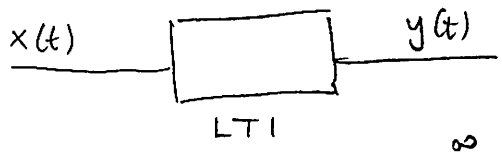
$$= \frac{1}{5}\cdot f\!\left(\frac{0}{5}\right) = \frac{1}{5}f(0)$$

Ex 2: $\displaystyle\int_{-\infty}^{\infty} \delta(5t-15)\,f(t)\,dt = \int_{-\infty}^{\infty} \delta\big(5(t-3)\big)\,f(t)\,dt = \frac{1}{5}f(3)$

CTFT of $\delta(t)$: $x(t) = \delta(t)$, $\displaystyle X(\Omega) = \int_{-\infty}^{\infty} \underset{\delta(t)}{x(t)}\, e^{-j\Omega t}\,dt = e^{-j\Omega\cdot 0} = 1$

CTFT of $\delta(t-t_0)$. $X(\Omega) = e^{-j\Omega t_0}$

Continuous-time convolution



$$y(t) = (x * h)(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

If $x(t) = \delta(t)$, then

$$y(t) = \int_{-\infty}^{\infty} \delta(\tau)\, h(t-\tau)\, d\tau = h(t-\tau)\Big|_{\tau=0} = \underbrace{h(t)}_{\text{impulse response}}$$

We notice that convolution with $\delta(t)$ doesn't change signal:

$$x(t) * \delta(t) = x(t)$$

Ex: $e^{-t}u(t) * \delta(5t-15)$, $\quad u(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$

$$\delta(5t-15) = \frac{1}{5}\delta(t-3)$$

$$\Rightarrow e^{-t}u(t) * \frac{1}{5}\cdot\delta(t-3) = \frac{1}{5} e^{-(t-3)} u(t-3)$$

CTFT

$$x(t) \xleftrightarrow{\text{CTFT}} X(\Omega)$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega)\, e^{j\Omega t}\, d\Omega \qquad X(\Omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t}\, dt$$

DTFT

continuous

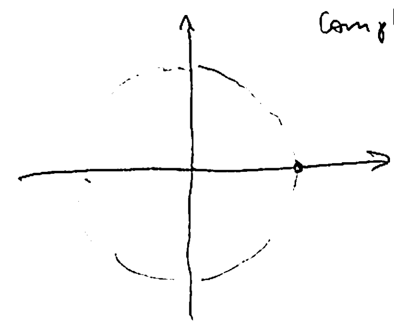
$$x[n] \xleftrightarrow{\text{DTFT}} X(e^{jw}) \quad (\text{function of } w)$$

equivalent notation: $X_d(w)$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{jw})\, e^{jwn}\, dw \qquad X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-jwn}$$

$$X\left(e^{j(w+2\pi\cdot k)}\right) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j(w+2\pi k)n} = \sum_{n=-\infty}^{\infty} x[n]\, e^{-jwn} \cdot \underbrace{e^{-j2\pi k n}}_{1}$$

$$= X(e^{jw})$$



Complex plane

Hence $X(e^{jw})$ is a periodic function of w with period 2π