UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
Department of Electrical and Computer Engineering

ECE 310 DIGITAL SIGNAL PROCESSING

Homework 10 Solutions

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Problem 1

(15 points) The nonzero part of the impulse response of a particular filter is given by $\{h[n]\}_{n=0}^{2} = \{0.5; -1; 0.5\}$. Your goal is to convolve $h[n]$ with the following signal:

$$\{x[n]\}_{n=0}^{27} = \{1; 2; 3; 4; 5; 6; 7; 8; -8; -7; -6; -5; -4; -3; -2; -1; 1; 2; 3; 4; 5; 6; 7; 8; 7; 6; 5; 4\}$$

You decide to use the overlap-add method with a length-8 FFT, dividing the input into frames $x_m[n]$, where m is the frame index, and each frame $x_m[n]$ is a signal of length $K = 8$. (A frame $x_m[n]$ may include zero-padding - but if so, use the minimum possible amount of zero-padding.)

(a) (5 points) Specify the contents of the first two frames, $\{x_0[n]\}_{n=0}^{7}$ and $\{x_1[n]\}_{n=0}^{7}$.

(b) (5 points) Specify the contents of the first two intermediate output frames, $\{y_0[n]\}$ and $\{y_1[n]\}$.

(c) (5 points) Let $y[n] = h[n] * x[n]$. Recall that in the overlap-add method, the values of the output $y[n]$ are formed by a combination of the values in the intermediate output frames. Give expressions for each of the outputs $\{y[n]\}_{n=4}^{9}$ in terms of the entries of the intermediate output frames, $\{y_0[n]\}$ and $\{y_1[n]\}$.

Solution

(a) Since we want to use a length-8 FFT, we're told that $K = 8$. We also know $M = 3 =$ the length of $h[n]$. So, to perform the overlap-and-add method, we split $x[n]$ up into "blocks" of length L, where $K = L + M - 1$. Then, the input frames are formed by zero-padding the blocks to length K.

In this case, since $M = 3$ and $K = 8$, we find that $L = 6$. Therefore, the first two blocks are $\{x[n]\}_{n=0}^{5}$ and $\{x[n]\}_{n=6}^{11}$, and the first two input frames are

$$\{x_0[n]\}_{n=0}^{7} = \boxed{\{1, 2, 3, 4, 5, 6, 0, 0\}}$$
$$\{x_1[n]\}_{n=0}^{7} = \boxed{\{7, 8, -8, -7, -6, -5, 0, 0\}}$$

(b) Once we have the input frames, we can obtain the output frames by circularly convolving $\{x_i[n]\}_{n=0}^{7}$ with the zero-padded version of the filter. Furthermore, this convolution can be performed using the FFT, since we've chosen K to be a power of two.

However, since the length of the *linear* convolution between $\{x_i[n]\}_{n=0}^{L-1}$ and $\{h[n]\}_{n=0}^{M-1}$ will be $L+M-1$, and both signals are zero-padded to length $L+M-1$, the circular convolution will be equivalent to the linear convolution of the block with the non-zero padded filter. We can calculate these convolutions either through the table method, or, more simply, as matrix multiplications:

$$y_0 = \begin{bmatrix} 0.5 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0.5 & 0 & 0 & 0 & 0 \\ 05 & -1 & 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & -1 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & -1 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 & -1 & 0.5 \\ 0 & 0 & 0 & 0 & 0.5 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -3.5 \\ 3 \end{bmatrix}$$

So,

$$\boxed{\{y_0[n]\}_{n=0}^{7} = \{0.5, 0, 0, 0, 0, 0, -3.5, 3\}}$$

Similarly, we can find

$$y_1 = \begin{bmatrix} 0.5 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0.5 & 0 & 0 & 0 & 0 \\ 05 & -1 & 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & -1 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & -1 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 & -1 & 0.5 \\ 0 & 0 & 0 & 0 & 0.5 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \\ -8 \\ -7 \\ -6 \\ -5 \end{bmatrix} = \begin{bmatrix} 3.5 \\ -3 \\ -8.5 \\ 8.5 \\ 0 \\ 0 \\ 2 \\ -2.5 \end{bmatrix}$$

So,

$$\boxed{\{y_1[n]\}_{n=0}^{7} = \{3.5, -3, -8.5, 8.5, 0, 0, 2, -2.5\}}$$

(c) Finally, we need to apply the overlap-and-add step; since "missed" $M-1=2$ input samples when performing the convolution, we need to *overlap* the two frames by $M-1$ to get the true output. So the first two samples of $y_1[n]$ are overlapped with the last two samples of $y_0[n]$:

$$\begin{array}{llllllllllllll} 0.5 & 0 & 0 & 0 & 0 & 0 & -3.5 & 3 & & & & & & \\ & & & & & & 3.5 & -3 & -8.5 & 8.5 & 0 & 0 & 2 & -2.5 \end{array}$$

We see that, for $0 \leq n \leq (L-1)$, $y[n] = y_0[n]$. However, for $L \leq n \leq (M-1)$, $y[n] = y_0[n] + y_1[n-L]$ - this is the overlap portion. Therefore, we find that

$$\boxed{\{y[n]\}_{n=4}^{9} = \{y_0[4], y_0[5], y_0[6] + y_1[0], y_0[7] + y_1[1], y_1[2], y_1[3]\}}$$

Problem 2

(20 points) We wish to digitally lowpass filter a real-valued analog signal. Although this signal has an infinite number of samples, we have already determined that truncating the signal to keep only 25,000 consecutive time samples is adequate to meet our error requirements in this application. We wish to filter this signal with a real-valued length-240 FIR low-pass filter.

(a) (6 points) Determine the number of real multiplications and additions required to implement this linear convolution directly.

(b) (7 points) Determine the number of real multiplications and additions required to implement this linear convolution using "fast convolution," for which two complex-valued radix-2 FFT and one inverse FFT (with transform length chosen as a power of two) are to be used. Assume in this sub-problem and the next one reduced butterflies (with one complex multiply per butterfly), and that 4 real multiplies and 2 real additions are needed to implement each complex multiply. Do not try to exclude from your count multiplies by "trivial" twiddle factors. By what factor is fast convolution faster compared to direct convolution?

(c) (7 points) Since the data vector is much longer than the filter length, the overlap-add algorithm yields a more efficient solution. Determine the size of the power-of-two length FFT that minimizes the total number of multiplies per output sample using the overlap-add method of fast convolution. Find the total computational cost (number of real multiplies and additions) using this algorithm. Assume that the appropriate DFT of the filter has been precomputed and stored. Compare to the costs of direct convolution, and to the (single block) fast convolution that you found in (a) and (b), respectively.

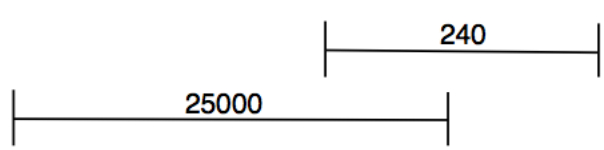
Solution

(a) In graphical convolution, every sample of the input is multiplied exactly once with every sample of the filter, as the flipped input slides past the filter. Therefore, there are LM required multiplications, where L is the length of $x[n]$ and M is the length of $h[n]$. To get the number of additions, realize that all of these products need to be added up to create the output, which is length $L+M-1$. In each output, the number of additions is 1 less than the number of multiplications. Therefore, we "lose" $L+M-1$ additions, and the total number of additions is $LM-(L+M-1)=(L-1)(M-1)$. Plugging in numbers, we see that linear convolution requires

$$\boxed{(25000)(240) = 6 \times 10^6 \text{ multiplications and } (24999)(239) = 5974761 \text{ additions.}}$$

Another way to prove this result is to break the convolution up into three stages. There will be two stages without full overlap (at the beginning and end of the convolution), and one stage with full overlap.

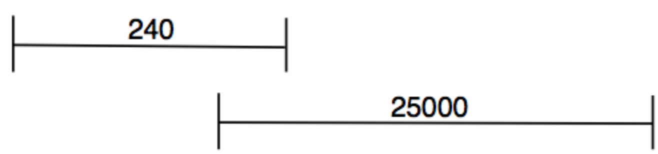
In the beginning of the convolution, we don't have full overlap because we haven't shifted over far enough; the situation looks as in the diagram below.



We don't have full overlap for $n = 0, 1, ..., M-2$; at $n = M-1$, we've shifted $M-1$ times, so the number of overlaps is M. For each index in this stage, n multiplications and $n-1$ additions are required. Therefore, we need

$$\sum_{n=1}^{M-1} n = \frac{M(M-1)}{2} \text{ multiplications.}$$
$$\sum_{n=0}^{M-2} n = \frac{(M-1)(M-2)}{2} \text{ additions.}$$

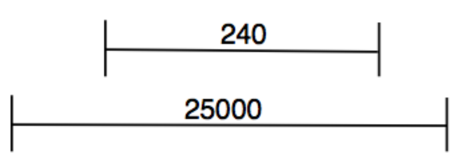
Similarly, at the end of the convolution, we don't have full overlap because we've shifted too far; the situation looks as in the diagram below.



By symmetry, this is identical to the beginning of the convolution. There will be $M-1$ indices without full overlap, and the number of overlapping terms decreases by one until only one term is overlapping. Therefore, we need

$$\sum_{n=1}^{M-1} n = \frac{M(M-1)}{2} \text{ multiplications.}$$
$$\sum_{n=0}^{M-2} n = \frac{(M-1)(M-2)}{2} \text{ additions.}$$

Finally, we need to consider the case where there is full overlap, as shown in the diagram below.



Since $h[n]$ is length M, every n where this occurs requires M multiplications and $M-1$ additions. Since the overall length of the convolution is $L+M-1$, and we know the first and last $M-1$ indices have incomplete overlap, there are $(L+M-1)-(M-1)-(M-1) = L-M+1$ indices with complete overlap. Therefore, this stage requires

$$(L-M+1)(M) \text{ multiplications.}$$
$$(L-M+1)(M-1) \text{ additions.}$$

Putting it all together, we indeed see that linear convolution requires

$$\frac{M(M-1)}{2} + \frac{M(M-1)}{2} + M(L-M+1) = M^2 - M + LM - M^2 + M = \boxed{LM \text{ multiplications}}$$

and

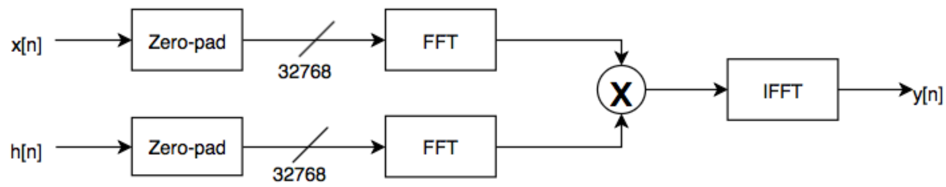
$$\frac{(M-1)(M-2)}{2} + \frac{(M-1)(M-2)}{2} + (M-1)(L-M+1) = (M-1)(L-M+1+M-2) = \boxed{(L-1)(M-1) \text{ additions}}$$

since each partial overlap stage requires $\sum_{n=1}^{M-1} n = \frac{M(M-1)}{2}$ multiplications and $\sum_{n=1}^{M-2} n = \frac{(M-1)(M-2)}{2}$ additions, and the full overlap stage requires $M(L-M+1)$ multiplications and $(M-1)(L-M+1)$ additions. Plugging in $L = 25000$ and $M = 240$, we find we require

$$\boxed{6 \times 10^6 \text{ multiplications and } 5974761 \text{ additions.}}$$

These multiplications and additions are all real, since we're told both $x[n]$ and $h[n]$ are real.

(b) If we want to implement the linear convolution using fast convolution, then we need to zero-pad to the closest power of two that's larger than 25239; that's $N = 2^{15} = 32768$. The fast convolution procedure can be summarized in the diagram below.



Each FFT/IFFT takes $\frac{N}{2}\log_2(N)$ complex multiplications and $N\log_2(N)$ complex additions. Therefore, to compute the fast convolution, we need:

$$\frac{3N}{2}\log_2(N) + N = 770048 \text{ complex multiplications.}$$
$$3N\log_2(N) = 1474560 \text{ complex additions.}$$

The extra factor of N multiplications comes from the fact that we need to multiply the FFTs pointwise. This requires N multiplications, and no additions. Finally, because each complex multiplication takes four real multiplications and two real additions, and each complex addition takes two real additions, the fast convolution approach requires

$$770048(4) = \boxed{3080192} \text{ real multiplications.}$$
$$770048(2) + 1474560(2) = \boxed{4489216} \text{ real additions.}$$

Compared to direct convolution, fast convolution requires 51.34% of the multiplications, and 75.14% of the additions.

(c) When performing overlap-and-add, we know that the frame length needs to be a power of 2. We also know that $K = L + M - 1$, where M is the length of $h[n]$, and L is the length of the block (or the number of input samples processed in each frame). Since we want to minimize the total number of multiplications, we need to find the number of multiplications for different values of K.

Since $K = L + M - 1 = L + 239$, we set $L = K - 239$. So, there will be

$$\left\lceil \frac{25000}{K-239} \right\rceil \text{ frames.}$$

Each frame processes L of the 25000 samples, and the ceiling is required in the case the last block is incomplete; it must be zero-padded to form a complete frame. We then need to examine the operations performed in each frame:

- We take the DFT of the input frame. This will require $\frac{K}{2}\log_2(K)$ complex multiplications and $K\log_2(K)$ complex additions using the FFT.
- We multiply the DFT of the input frame with the DFT of the filter. This takes K complex multiplications, and no additions, since the multiplication is performed pointwise. Since we're told that the DFT of the filter is already available, no multiplications or additions will be required to compute it.
- We take the inverse DFT of the block. This will again require $\frac{K}{2}\log_2(K)$ complex multiplications and $K\log_2(K)$ complex additions using the IFFT.

Therefore, each frame requires $K\log_2(K) + K$ complex multiplications, and $2K\log_2(K)$ complex additions. So the total number of complex multiplications is

$$\left\lceil \frac{25000}{K-239} \right\rceil (K\log_2(K) + K)$$

and the total number of complex additions is

$$\left\lceil \frac{25000}{K-239} \right\rceil (2K\log_2(K))$$

However, these calculations don't take into effect the additional additions required to overlap and add the frames. The last $M-1$ entries in the previous frame need to be added to the first $M-1$ entries in the current frame. We need to perform this operation for every frame except the last one, and these additions are all real, because the intermediate output frame will be real. Therefore, we need an additional

$$\left(\left\lceil \frac{25000}{K-239} \right\rceil - 1\right)(M-1)$$

real additions.

Since each complex multiplication requires four real multiplications and two real additions, and each complex addition requires two real additions, we find that the overlap-and-add approach requires

$$\left\lceil \frac{25000}{K-239} \right\rceil (4K\log_2(K) + 4K) \text{ multiplications}$$

$$\left\lceil \frac{25000}{K-239} \right\rceil (2K\log_2(K) + 2K) + 2\left\lceil \frac{25000}{K-239} \right\rceil (2K\log_2(K)) + \left(\left\lceil \frac{25000}{K-239} \right\rceil - 1\right)(M-1) \text{ additions}$$

To calculate the number of multiplications per output sample, we divide by 25239, or the length of the output. Note that this length is independent of the block size.

To optimize over K, we restrict our search to powers of two. We also start at $K = 256$, as that's the smallest power of two larger than the length of the filter. Performing the calculations gives:

K	Total Multiplications	Multiplications per Output Sample	Additions
256	13556736	537.134	19180130
512	1884160	74.653	2659573
1024	1441792	57.125	2039025
2048	1376256	54.529	1952803
4096	1490944	59.073	2123162
8192	1835008	72.705	2622157

Note that an alternate definition of the number of multiplications per sample given in lecture is

$$\frac{K \log_2(K) + K}{L} = \frac{K \log_2(K) + K}{K - 239}$$

which yields the same results. We see that the optimal value is $\boxed{K = 2048}$, in which case the overlap-and-add approach requires

$$\boxed{\text{1376256 multiplications and 1952803 additions.}}$$

Compared to the linear convolution, we only need 22.94% of the multiplications and 32.74% of the additions. Compared to fast convolution, we only need 44.68% of the multiplications and 43.57% of the additions.

Problem 3

(10 points) The transfer functions of three causal LSI systems are given below. For each system, determine whether it is an FIR or an IIR filter.

(a) (3 points) $\frac{z^2+3z+2}{2z^2+3z-1}$

(b) (3 points) $\frac{1+0.8z^{-1}}{1-0.64z^{-2}}$

(c) (4 points) $3 + z^{-1} - 4z^{-2}$

Solution

(a) Since any causal FIR system can be written as

$$H(z) = b_0 + b_1 z^{-1} + ... + b_n z^{-n}$$

the only possible pole location will be at $z = 0$. This is known as the *trivial* pole; any system with a nontrivial pole must be IIR.

In this case, the system is IIR, since the poles are at $-\frac{3}{4} \pm \frac{\sqrt{17}}{4}$, which are nontrivial.

Alternatively, this conclusion can be reached by noting that the system is *recursive*. Recursive systems (without pole-zero cancellations, which don't occur in this case) are always IIR.

(b) We can write

$$H(z) = \frac{1 + 0.8z^{-1}}{1 - 0.64z^{-2}} = \frac{1 + 0.8z^{-1}}{(1 + 0.8z^{-1})(1 - 0.8z^{-1})} = \frac{1}{1 - 0.8z^{-1}}, |z| > 0.8$$

Taking the inverse z-transform gives $h[n] = (0.8)^n u[n]$. Therefore, the system is **IIR**, since the impulse response is infinite-length.

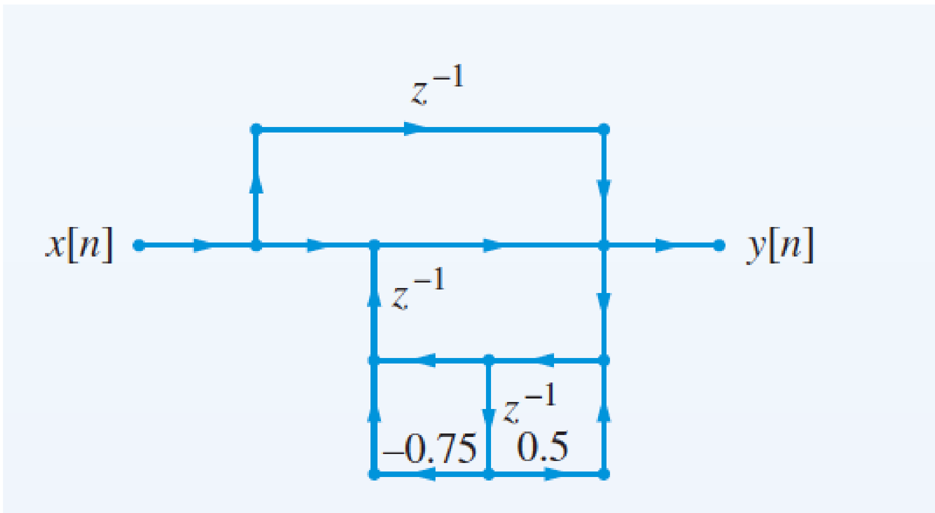
Alternatively, you could have come to this conclusion without performing any calculations by noting that the system is recursive, as in (a), or by noting that the system pole is 0.8, which is nontrivial.

(c) Taking the inverse z-transform gives $h[n] = 3\delta[n] + \delta[n-1] - 4\delta[n-2]$. Since $h[n]$ is only nonzero for three samples, the system is **FIR**.

Alternatively, you could have come to this conclusion by noting that the only pole is at $z = 0$, which is trivial.

Problem 4

(15 points) A discrete-time system is described by the following signal flow graph:

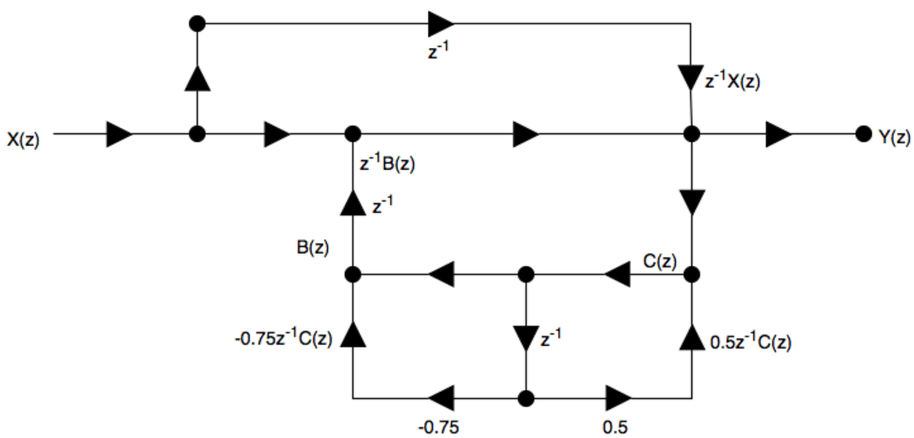


(a) (10 points) Determine the difference equation representation.

(b) (5 points) Determine the impulse response of the system.

Solution

(a) If given a block diagram that's not in any standard notation, it's often easiest to solve for the transfer function by using auxiliary outputs; working in the z-domain, label the output of each delay and adder, and combine until a relation between $X(z)$ and $Y(z)$ is obtained. The labeled flowgraph looks as follows:



Writing equations for the outputs of the adders gives:

$$\begin{aligned} Y(z) &= z^{-1}X(z) + z^{-1}B(z) + X(z) \\ B(z) &= C(z) - 0.75z^{-1}C(z) \\ C(z) &= Y(z) + 0.5z^{-1}C(z) \end{aligned}$$

Plugging in for $B(z)$ gives

$$Y(z) = (1 + z^{-1})X(z) + (z^{-1} - 0.75z^{-2})C(z)$$

But, since we know the relation between $C(z)$ and $Y(z)$ from the third equation, we can express $C(z) = \frac{Y(z)}{1-0.5z^{-1}}$. This gives

$$Y(z) = (1 + z^{-1})X(z) + \left(\frac{z^{-1} - 0.75z^{-2}}{1 - 0.5z^{-1}}\right) Y(z)$$

Therefore,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{(1 + z^{-1})(1 - 0.5z^{-1})}{1 - 1.5z^{-1} + 0.75z^{-2}} = \boxed{\frac{1 + 0.5z^{-1} - 0.5z^{-2}}{1 - 1.5z^{-1} + 0.75z^{-2}}}, |z| > \sqrt{\frac{3}{4}}$$

where the ROC comes from the fact that we assume the system is causal. So, we can obtain the difference equation by cross-multiplying and taking the inverse z-transform of both sides. This gives

$$\boxed{y[n] - 1.5y[n-1] + 0.75y[n-2] = x[n] + 0.5x[n-1] - 0.5x[n-2]}$$

(b) Note that the order of the numerator is the same as the order of the denominator; in this case, one cannot directly perform a partial fraction decomposition. In order to do so, we need to perform polynomial long division to break up the numerator. Doing so gives

$$H(z) = -\frac{2}{3} + \frac{\frac{5}{3} - 0.5z^{-1}}{1 - 1.5z^{-1} + 0.75z^{-2}}$$

Recognizing that the denominator factors into two complex roots, we can write

$$H(z) - \frac{2}{3} + \frac{5}{3}\left(\frac{1 - 0.3z^{-1}}{(1 - \sqrt{\frac{3}{4}}e^{j\frac{\pi}{6}}z^{-1})(1 - \sqrt{\frac{3}{4}}e^{-j\frac{\pi}{6}}z^{-1})}\right)$$

Therefore, we can focus our partial fraction expansion on the part inside the parentheses, writing

$$\frac{1 - 0.3z^{-1}}{(1 - \sqrt{\frac{3}{4}}e^{j\frac{\pi}{6}z^{-1}})(1 - \sqrt{\frac{3}{4}}e^{-j\frac{\pi}{6}z^{-1}})} = \frac{A}{1 - \sqrt{\frac{3}{4}}e^{j\frac{\pi}{6}}z^{-1}} + \frac{A^*}{1 - \sqrt{\frac{3}{4}}e^{-j\frac{\pi}{6}}z^{-1}}$$

where we know $B = A^*$ since the impulse response needs to be real (similar to Problem 7 on the midterm).

We can solve for A using the cover-up method:

$$\begin{aligned}
A = \frac{1 - 0.3z^{-1}}{1 - \sqrt{\frac{3}{4}}e^{-j\frac{\pi}{6}}z^{-1}}\Big|_{z=\sqrt{\frac{3}{4}}e^{j\frac{\pi}{6}}} &= \frac{1 - 0.3\sqrt{\frac{4}{3}}e^{-j\frac{\pi}{6}}}{1 - \sqrt{\frac{3}{4}}e^{-j\frac{\pi}{6}}(\sqrt{\frac{4}{3}}e^{-j\frac{\pi}{6}})} \\
&= \frac{(1 - 0.3\sqrt{\frac{4}{3}}e^{-j\frac{\pi}{6}})(1 - e^{-j\frac{\pi}{3}})}{(1 - e^{-j\frac{\pi}{3}})(1 - e^{j\frac{\pi}{3}})} \\
&= \frac{1 + 0.3\sqrt{\frac{4}{3}}(e^{j\frac{\pi}{6}} - e^{-j\frac{\pi}{6}}) - e^{j\frac{\pi}{3}}}{2 - 2\cos(\frac{\pi}{3})} \\
&= 1 + 0.3\sqrt{\frac{4}{3}}e^{j\frac{\pi}{2}} - e^{j\frac{\pi}{3}}
\end{aligned}$$

Therefore,

$$\frac{1 - 0.3z^{-1}}{(1 - \sqrt{\frac{3}{4}}e^{j\frac{\pi}{6}}z^{-1})(1 - \sqrt{\frac{3}{4}}e^{-j\frac{\pi}{6}}z^{-1})} = \frac{1 + 0.3\sqrt{\frac{4}{3}}e^{j\frac{\pi}{2}} - e^{j\frac{\pi}{3}}}{1 - \sqrt{\frac{3}{4}}e^{j\frac{\pi}{6}}z^{-1}} + \frac{1 + 0.3\sqrt{\frac{4}{3}}e^{-j\frac{\pi}{2}} - e^{-j\frac{\pi}{3}}}{1 - \sqrt{\frac{3}{4}}e^{-j\frac{\pi}{6}}z^{-1}}$$

which allows us to write the transfer function as

$$H(z) = -\frac{2}{3} + \frac{5}{3}\left(\frac{1 + 0.3\sqrt{\frac{4}{3}}e^{j\frac{\pi}{2}} - e^{j\frac{\pi}{3}}}{1 - \sqrt{\frac{3}{4}}e^{j\frac{\pi}{6}}z^{-1}} + \frac{1 + 0.3\sqrt{\frac{4}{3}}e^{-j\frac{\pi}{2}} - e^{-j\frac{\pi}{3}}}{1 - \sqrt{\frac{3}{4}}e^{-j\frac{\pi}{6}}z^{-1}}\right), |z| > \sqrt{\frac{3}{4}}$$

We can now invert the z-transform using the pair for $a^n u[n]$. This gives

$$h[n] = -\frac{2}{3}\delta[n] + \frac{5}{3}\left(1 + 0.3\sqrt{\frac{4}{3}}e^{j\frac{\pi}{2}} - e^{j\frac{\pi}{3}}\right)\left(\sqrt{\frac{3}{4}}e^{j\frac{\pi}{6}}\right)^n u[n] + \frac{5}{3}\left(1 + 0.3\sqrt{\frac{4}{3}}e^{-j\frac{\pi}{2}} - e^{-j\frac{\pi}{3}}\right)\left(\sqrt{\frac{3}{4}}e^{-j\frac{\pi}{6}}\right)^n u[n]$$

Combining like complex exponentials together, we can simplify:

$$h[n] = -\frac{2}{3}\delta[n] + \frac{5}{3}\left(\sqrt{\frac{3}{4}}\right)^n (e^{j\frac{\pi}{6}n} + e^{-j\frac{\pi}{6}n})u[n] + \frac{1}{2}\sqrt{\frac{4}{3}}\left(\sqrt{\frac{3}{4}}\right)^n (e^{j(\frac{\pi}{6}n+\frac{\pi}{2})} + e^{-j(\frac{\pi}{6}n+\frac{\pi}{2})})u[n]$$
$$- \frac{5}{3}\left(\sqrt{\frac{3}{4}}\right)^n (e^{j(\frac{\pi}{6}n+\frac{\pi}{3})} + e^{-j(\frac{\pi}{6}n+\frac{\pi}{3})})u[n]$$
$$= \boxed{-\frac{2}{3}\delta[n] + \left(\sqrt{\frac{3}{4}}\right)^n \left(\frac{10}{3}\cos\left(\frac{\pi}{6}n\right) + \sqrt{\frac{4}{3}}\cos\left(\frac{\pi}{6}n + \frac{\pi}{2}\right) - \frac{10}{3}\cos\left(\frac{\pi}{6}n + \frac{\pi}{3}\right)\right) u[n]}$$

Alternatively, one could realize that the denominator can also be written as

$$1 - 1.5z^{-1} + 0.75z^{-2} = \left(1 - \frac{\sqrt{3}}{2}e^{-j\frac{\pi}{6}}z^{-1}\right)\left(1 - \frac{\sqrt{3}}{2}e^{j\frac{\pi}{6}}z^{-1}\right) = 1 - 2\left(\frac{\sqrt{3}}{2}\cos\left(\frac{\pi}{6}\right)\right)z^{-1} + \frac{3}{4}z^{-2}$$

This is in the form $(1 - re^{-j\omega_0}z^{-1})(1 - re^{j\omega_0}z^{-1})$ - which is the denominator of the z-transform of a sine or cosine. Therefore, we need to manipulate the numerator to match those given in the z-transform table.

First, we split the numerator up into three portions:

$$H(z) = \frac{1}{1 - 2(\frac{\sqrt{3}}{2}\cos(\frac{\pi}{6}))z^{-1} + \frac{3}{4}z^{-2}} + \frac{0.5z^{-1}}{1 - 2(\frac{\sqrt{3}}{2}\cos(\frac{\pi}{6}))z^{-1} + \frac{3}{4}z^{-2}} - \frac{0.5z^{-2}}{1 - 2(\frac{\sqrt{3}}{2}\cos(\frac{\pi}{6}))z^{-1} + \frac{3}{4}z^{-2}}$$

We need each of these to match the z-transform for $r^n\sin(\omega_0 n)u[n]$:

$$x[n]r^n\sin(\omega_0 n)u[n] \xrightarrow{\mathcal{Z}} X(z) = \frac{r\sin(\omega_0)z^{-1}}{1 - 2r\cos(\omega_0)z^{-1} + r^2z^{-2}}, |z| > r$$

Since $\sin(\frac{\pi}{6}) = \frac{1}{2}$, we can do so by multiplying by clever choices of 1. This gives

$$H(z) = \sqrt{\frac{16}{3}}z\left(\frac{\frac{\sqrt{3}}{2}\sin(\frac{\pi}{6})z^{-1}}{1 - 2(\frac{\sqrt{3}}{2}\cos(\frac{\pi}{6}))z^{-1} + \frac{3}{4}z^{-2}}\right) + \sqrt{\frac{4}{3}}\left(\frac{\frac{\sqrt{3}}{2}\sin(\frac{\pi}{6})z^{-1}}{1 - 2(\frac{\sqrt{3}}{2}\cos(\frac{\pi}{6}))z^{-1} + \frac{3}{4}z^{-2}}\right)$$
$$- \sqrt{\frac{4}{3}}z^{-1}\left(\frac{\frac{\sqrt{3}}{2}\sin(\frac{\pi}{6})z^{-1}}{1 - 2(\frac{\sqrt{3}}{2}\cos(\frac{\pi}{6}))z^{-1} + \frac{3}{4}z^{-2}}\right), |z| > \frac{\sqrt{3}}{2}$$

Therefore, inverting the transforms gives

$$\boxed{\begin{aligned} h[n] = \sqrt{\frac{16}{3}}\left(\frac{\sqrt{3}}{2}\right)^{(n+1)}\sin\left(\frac{\pi}{6}(n+1)\right)u[n] + \sqrt{\frac{4}{3}}\left(\frac{\sqrt{3}}{2}\right)^n\sin\left(\frac{\pi}{6}n\right)u[n] \\ - \sqrt{\frac{4}{3}}\left(\frac{\sqrt{3}}{2}\right)^{(n-1)}\sin\left(\frac{\pi}{6}(n-1)\right)u[n-1] \end{aligned}}$$

Note that the $u[n+1]$ term was replaced with a $u[n]$ term, as $h[-1] = 0$. We know that since the system is implemented causally, $h[n]$ must be causal.

Problem 5

(20 points) Consider the following causal LSI system with the following transfer function:

$$H(z) = \frac{(1 - 3z^{-1})}{(1 - 1.2z^{-1} + 0.4z^{-2})(1 - 0.5z^{-1})}$$

Draw the filter structures for implementations of the system in each of the following forms:

(a) (2 points) Direct form I.

(b) (2 points) Direct form II.

(c) (3 points) Transpose form I.

(d) (3 points) Transpose form II.

(e) (3 points) Cascade form using first- and second-order direct form II sections.

(f) (4 points) Parallel form using first- and second-order direct form II sections.

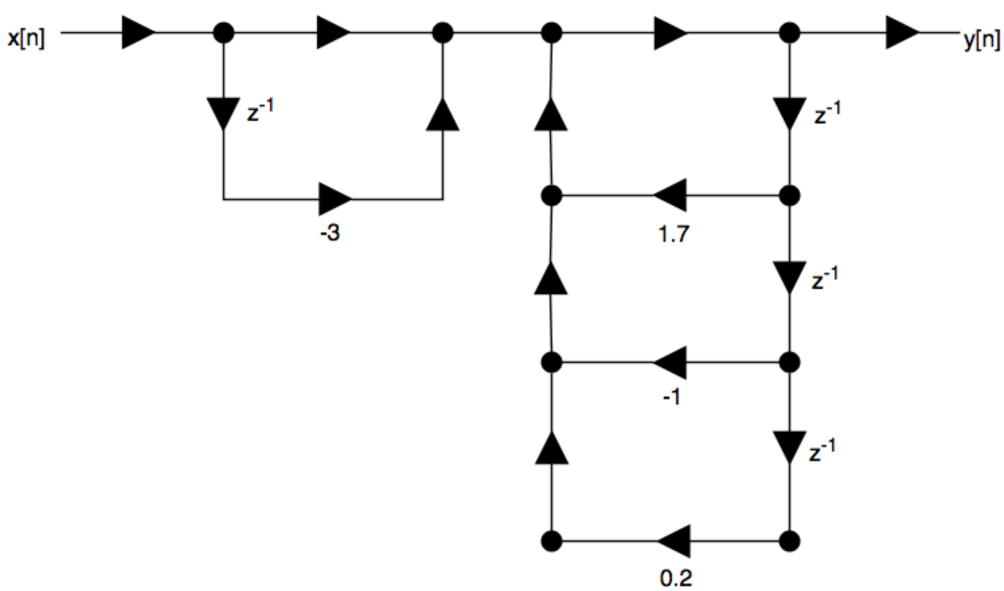
(g) (3 points) Cascade form using first- and second-order transpose form II sections.

Solution

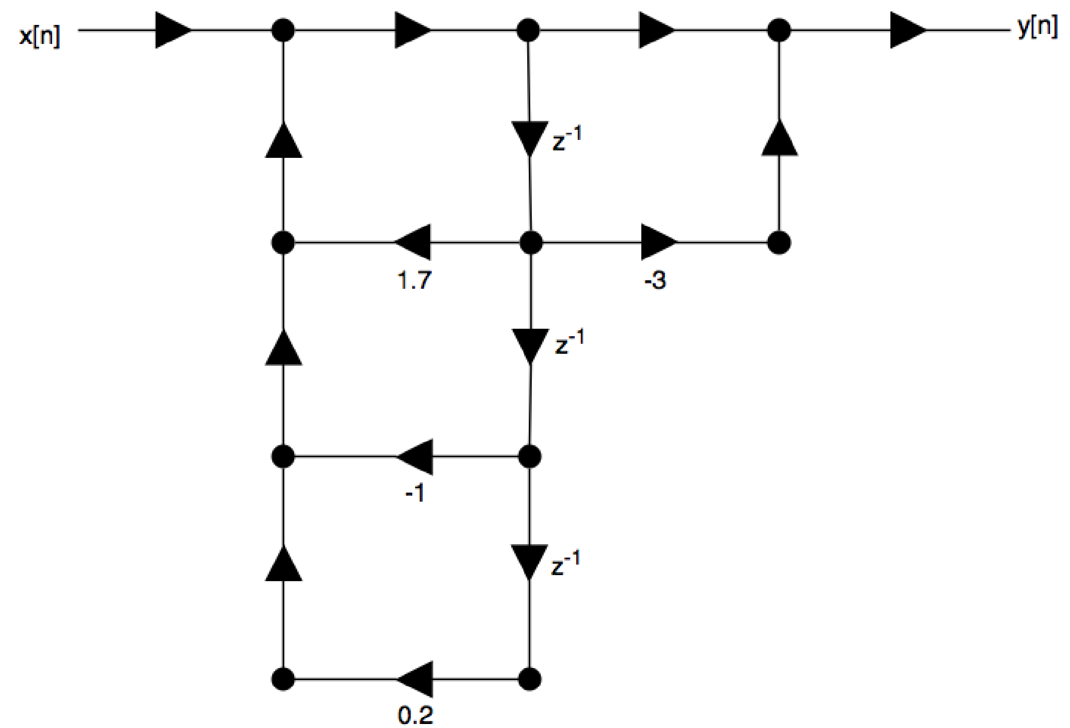
(a) To get the Direct Form I structure, we need to write $H(z)$ in the form of

$$\frac{B(z)}{A(z)} = \frac{1 - 3z^{-1}}{1 - 1.7z^{-1} + z^{-2} - 0.2z^{-3}}$$

By inspection (refer to Page 488 of the textbook), we immediately obtain the Direct Form I structure, which is seen below.

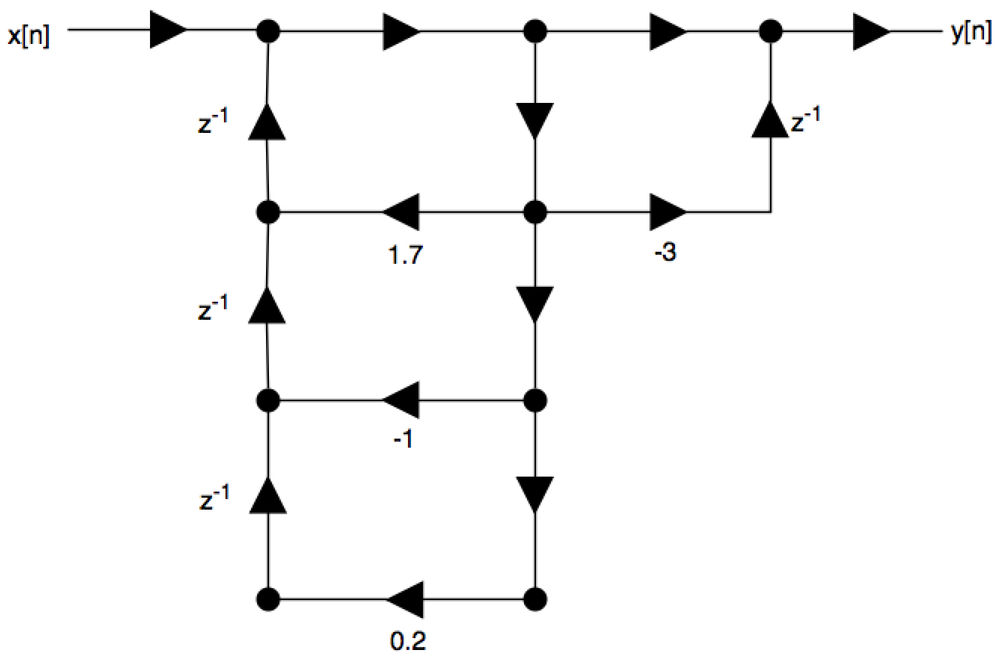


(b) We can derive the Direct Form II structure by inspection of $H(z)$, similar to Direct Form i. Alternatively, we could form the Direct Form I structure by "cutting" the Direct Form I structure in half at the adder, and flipping the two halves. Doing so gives the structure shown below.

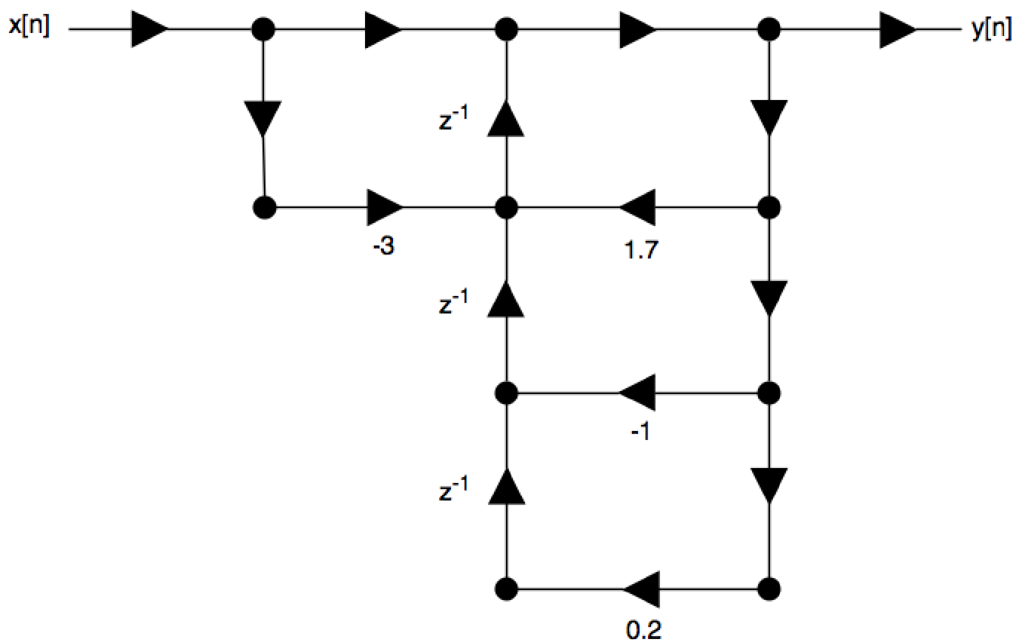


We can see the similarities; the adder was split into two adders, the delay blocks were combined together, and the gains switched sides. These are all consequences of switching the two halves.

(c) We get the Transpose Form I structure by "transposing" the Direct Form I structure. This is essentially just *reversing* the direction of all signal paths; in this process, branches become adders, and adders become branches. This can also be performed by inspection by comparing with the diagram on Page 490 of the textbook.



(d) We get the Transpose Form II structure by "transposing" the Direct Form II structure, following the same steps outlined in (c). The structure is shown below.



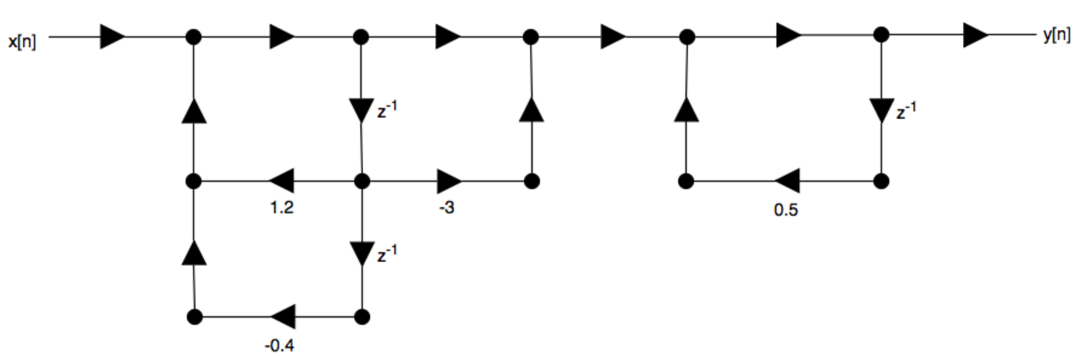
(e) We get the cascade form by "splitting" the transfer function up into two smaller transfer functions. That is, we write $H(z) = H_1(z)H_2(z)$, and get the output by feeding the input through $H_1(z)$, and then feeding the intermediate output through $H_2(z)$. The system looks as follows:

$$x[n] \rightarrow \boxed{h_1[n]} \xrightarrow{s[n]} \boxed{h_2[n]} \rightarrow y[n]$$

In this case, $H(z)$ is already in the correct form; we can write

$$H(z) = \underbrace{\left(\frac{1 - 3z^{-1}}{1 - 1.2z^{-1} + 0.4z^{-2}}\right)}_{H_1(z)} \underbrace{\left(\frac{1}{1 - 0.5z^{-1}}\right)}_{H_2(z)}$$

By inspection, this leads to the structure shown below. We implement $H_1(z)$ using Direct Form II, then pass the output into the implementation of $H_2(z)$ using Direct Form II.



(f) We get the parallel form by writing $H(z) = H_1(z) + H_2(z)$, then $y[n] = x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$. So, we feed the input through two separate transfer functions, and combine the individual outputs to get the true output.

For this problem, we see that a partial fraction expansion needs to be performed. We can write

$$H(z) = \frac{1 - 3z^{-1}}{(1 - 1.2z^{-1} + 0.4z^{-2})(1 - 0.5z^{-1})} = \frac{A + Bz^{-1}}{1 - 1.2z^{-1} + 0.4z^{-2}} + \frac{C}{1 - 0.5z^{-1}}$$

We can find C using the cover-up method:

$$C = \frac{1 - 3z^{-1}}{1 - 1.2z^{-1} + 0.4z^{-2}}\bigg|_{z=0.5} = -25$$

Then, we can cross-multiply to find equations for A and B:

$$1 - 3z^{-1} = (A + Bz^{-1})(1 - 0.5z^{-1}) + C(1 - 1.2z^{-1} + 0.4z^{-2}) = (A + C) + (-0.5A + B - 1.2C)z^{-1} + (-0.5B + 0.4C)z^{-2}$$

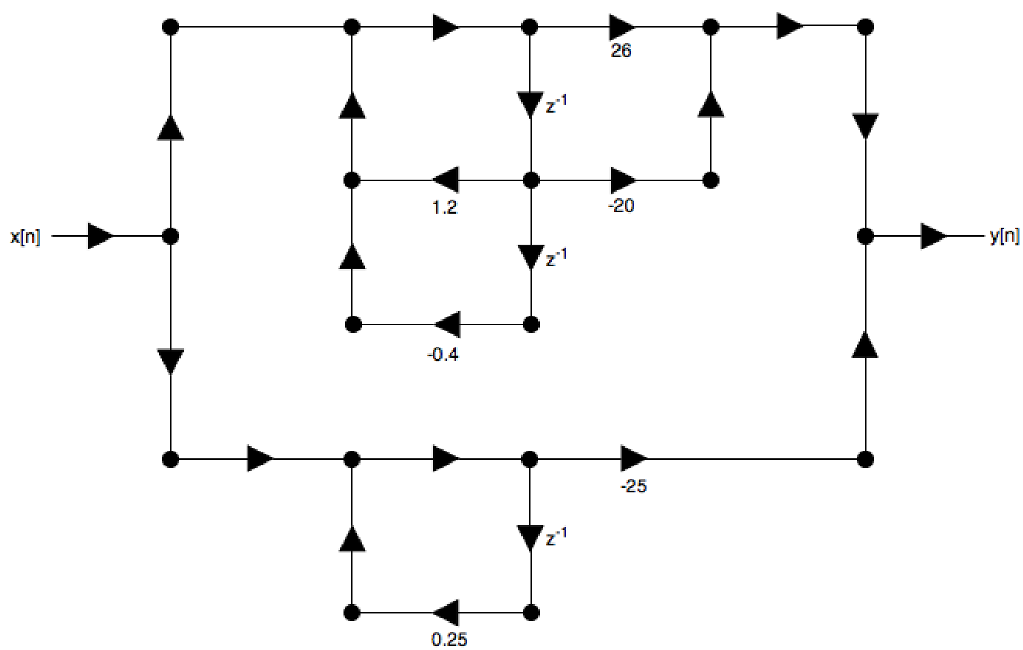
This gives:

$$C + A = 1 \to A = 26$$
$$-0.5B + 0.4C = 0 \to B = -20$$

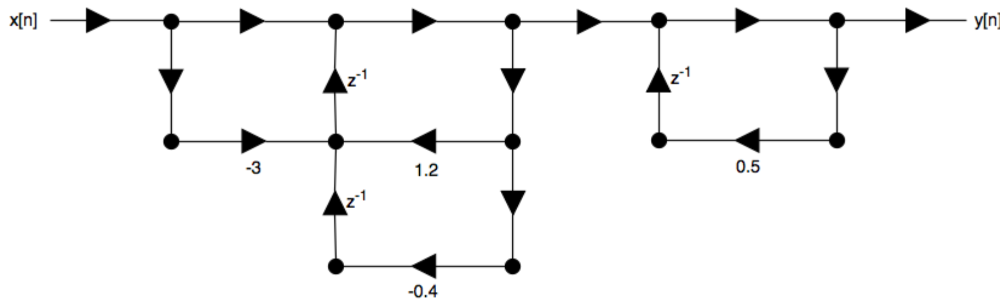
Another option would have been to let $z \to \infty$ on both sides. This gives $1 = A + C \to A = 26$, at which point we can solve for B. Therefore, we have

$$H(z) = \underbrace{\frac{26 - 20z^{-1}}{1 - 1.2z^{-1} + 0.4z^{-2}}}_{H_1(z)} + \underbrace{\frac{-25}{1 - 0.25z^{-1}}}_{H_2(z)}$$

By the canonical Direct Form II structure found on Page 492 of the textbook, this leads to the structure shown below.



(g) To get the transpose cascade form, we perform the same steps outlined in part (c) on the block diagram found in part (e). Note that this also requires switching the order of the systems, since the input and output switch sides when we take the transpose form. The resulting block diagram is shown below.



Problem 6

(20 points) The frequency response of a generalized linear phase (GLP) filter can be expressed as $H_d(\omega) = R(\omega)e^{j(\alpha-\beta\omega)}$ where $R(\omega)$ is a real function and α and β are constant. For each of the following LTI systems, described by their impulse response, transfer function, or difference equation, determine whether it is a GLP filter. If it is, determine $R(\omega)$, α, and β, and indicate whether it is also a strictly linear phase filter.

(a) (3 points) $y[n] = x[n] - x[n-2]$

(b) (3 points) $y[n] = 0.5y[n-1] + 0.5y[n-2] + x[n]$

(c) (3 points) $y[n] = 0.5x[n] + x[n-1] - x[n-2] - 0.5x[n-1]$

(d) (4 points) $\{h[n]\}_{n=0}^{2} = \{3, 7, 3\}$

(e) (4 points) $\{h[n]\}_{n=0}^{2} = \{3, 4, 3\}$

(f) (3 points) $H(z) = 1 - 3z^{-1} - z^{-2}$

Solution

(a) The filter **is GLP**, because we can write $\{h[n]\}_{n=0}^{2} = \{1, 0, -1\}$, which has odd symmetry. Taking the DTFT and splitting the phase gives

$$\begin{aligned} H_d(\omega) &= 1 - e^{-j2\omega} \\ &= e^{-j\omega}(e^{j\omega} - e^{-j\omega}) \\ &= e^{-j\omega}(2j\sin(\omega)) \\ &= 2\sin(\omega)e^{j(\frac{\pi}{2}-\omega)} \end{aligned}$$

Therefore,

$$\boxed{R(\omega) = 2\sin(\omega), \quad \alpha = \frac{\pi}{2}, \quad \beta = 1}$$

However, the filter does **not** have strict linear phase, since $R(\omega)$ has a zero-crossing at $\omega = 0$, which corresponds to a jump of π in the phase.

(b) This filter is **not GLP**. One simple way to see this is realize that $H(z)$ has a nontrivial pole. Therefore, it is IIR, and IIR filters cannot exhibit GLP.

(c) We can again find the impulse response by plugging in $x[n] = \delta[n]$. Then

$$\{h[n]\}_{n=0}^{3} = \{0.5, 1, -1, -0.5\}$$

This has odd symmetry, so the filter **is GLP**. As in (a), taking the DTFT and splitting the phase gives

$$\begin{aligned} H_d(\omega) &= 0.5 + e^{j\omega} - e^{-j2\omega} - 0.5e^{-j3\omega} \\ &= e^{-j\omega 1.5}(0.5e^{j\omega 1.5} + e^{j\omega 0.5} - e^{-j\omega 0.5} - 0.5e^{-j\omega 1.5}) \\ &= e^{-j\omega 1.5}(j\sin(1.5\omega) + 2j\sin(0.5\omega)) \\ &= (\sin(1.5\omega) + 2\sin(0.5\omega))e^{j(\frac{\pi}{2}-1.5\omega)} \end{aligned}$$

Therefore,

$$\boxed{R(\omega) = \sin(1.5\omega) + 2\sin(0.5\omega), \quad \alpha = \frac{\pi}{2}, \quad \beta = 1.5}$$

However, it does **not** exhibit strict linear phase, since there's a zero-crossing at $\omega = 0$. An easier way to see this is to observe that filters with odd symmetry always have $\alpha = \pm\frac{\pi}{2}$ - there's a jump of π at $\omega = 0$, as $R(\omega)$ is strictly composed of sines. Therefore, these filters can never be strictly LP.

(d) We observe that this filter has even symmetry; therefore, it **is GLP**. Taking the DTFT gives:

$$\begin{aligned} H_d(\omega) &= 3 + 7e^{-j\omega} + 3e^{-j2\omega} \\ &= e^{-j\omega}(3e^{j\omega} + 7 + 3e^{-j\omega}) \\ &= (7 + 6\cos(\omega))e^{-j\omega} \end{aligned}$$

We find that

$$\boxed{R(\omega) = 7 + 6\cos(\omega), \quad \alpha = 0, \quad \beta = 1}$$

It also **does** have strict linear phase, since there are no zero-crossings between $\omega = -\pi$ and $\omega = \pi$. This is straightforward to see, since the minimum value of $\cos(\omega) = -1$, the minimum value of $7 + 6\cos(\omega)$ is 1.

(e) As before, we observe that this filter has even symmetry; therefore, it **is GLP**. Taking the DTFT gives:

$$\begin{aligned} H_d(\omega) &= 3 + 4e^{-j\omega} + 3e^{-j2\omega} \\ &= e^{-j\omega}(3e^{j\omega} + 4 + 3e^{-j\omega}) \\ &= (4 + 6\cos(\omega))e^{-j\omega} \end{aligned}$$

Therefore, this filter **is GLP**, with

$$\boxed{R(\omega) = 4 + 6\cos(\omega), \alpha = 0, \beta = 1}$$

However, it does **not** have strict linear phase, since there are zero-crossings at $\omega = \pm 2.3$.

(f) In this case, $h[n]$ has no symmetry, since the middle term is not zero. Therefore, it is **not GLP**.