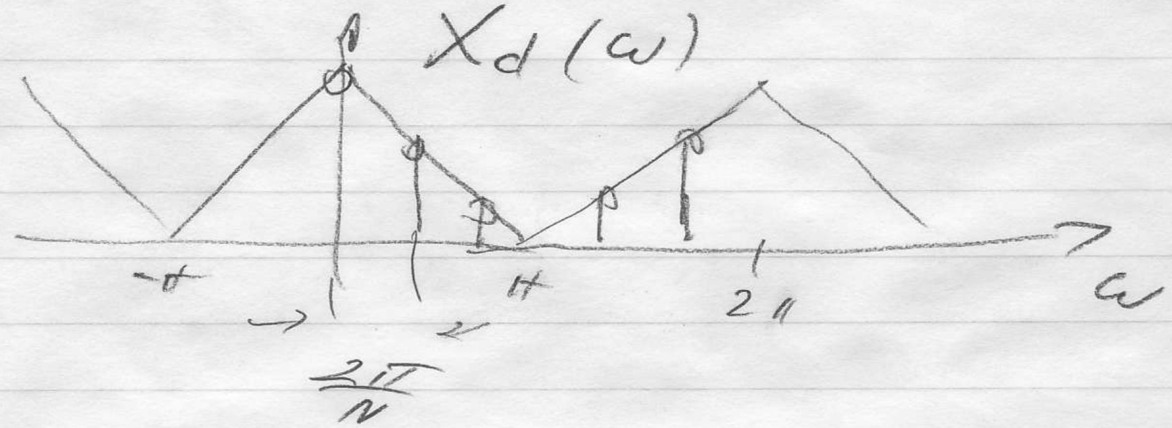
Properties of the DFT

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}, \quad k = 0, 1, \cdots N-1$$

$$\{x[n]\}_{n=0}^{N-1} \xrightarrow{DFT} \{X[k]\}_{k=0}^{N-1}$$



$$X[k] = X_d(\omega)\Big|_{\omega = \frac{2\pi}{N}\cdot k}$$

1 linearity

0. Samples of the DTFT

2. Zero Padding
3. (Circular) Shifting

Recall DTFT: $\quad x[n] \xrightarrow{DTFT} X_d(\omega)$

$$y[n] = x[n - m] \longleftrightarrow e^{-jm\omega} X_d(\omega)$$

eg

Modulo operation

$\langle m \rangle_N$ $\quad$ m mod N

Ex $\quad \langle 12 \rangle_7 = 5$

$0 \le \langle m \rangle_N \le N-1$

Ex: $\quad \langle -3 \rangle_7 = \langle -3+7 \rangle_7 = 4$

Circular shift modulo N

$x[n] \longrightarrow x[\langle n-m \rangle_N]$

Why circular shift?

Inverse DFT

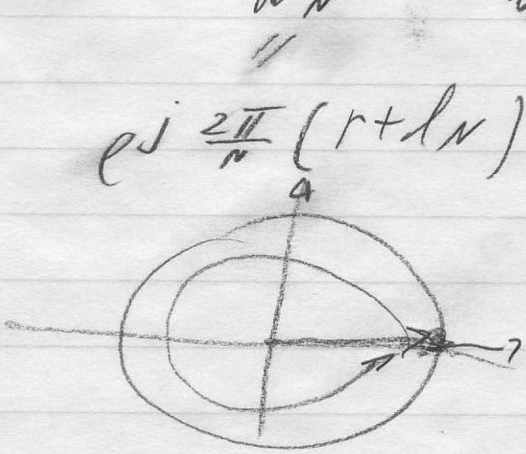
$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{+j\frac{2\pi}{N}nk} = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, W_N^{-kn}$$

$W_N \triangleq e^{-j\frac{2\pi}{N}}$ $\quad$ $W_N^{r+lN} = W_N^{r}$ $\quad$ $r, l \in \mathbb{Z}$

$$\parallel$$

$$e^{j\frac{2\pi}{N}(r+lN)} = e^{j\frac{2\pi}{N}\cdot r}\cdot e^{j\frac{2\pi lN}{N}}$$

$e^{j2\pi} = 1$

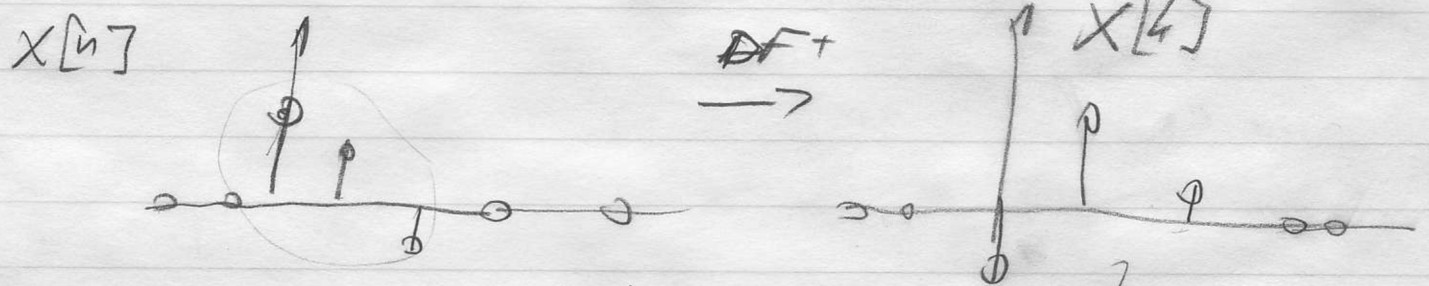


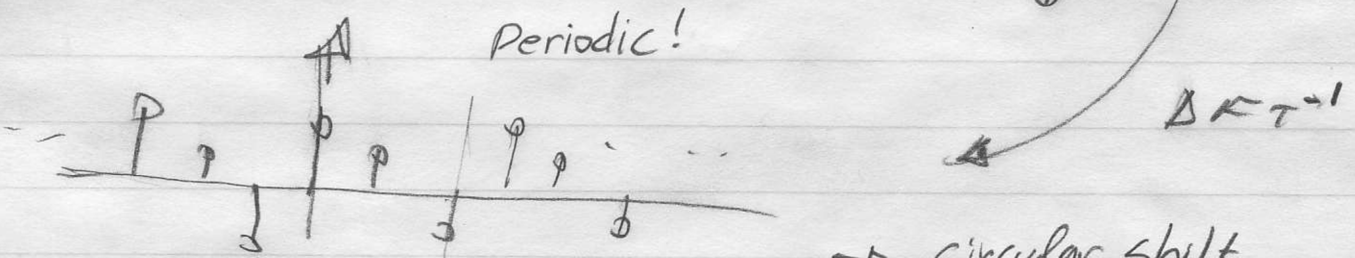
$$W_N^{r+\ell N} = W_N^{r}$$

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, W_N^{-kn}$$

$$x[n+\ell N] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, W_N^{-k(n+\ell N)}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, W_N^{-kn} = x[n]$$





$\Rightarrow$ circular shift

$$y[n] \triangleq x[\langle n-m\rangle_N]$$

$$Y[k] = \mathrm{DFT}\{y[n]\} =$$

$$\sum_{n=0}^{N-1} x[\underbrace{\langle n-m\rangle_N}_{\ell}]\, W_N^{kn}$$

$$\ell = \langle n-m\rangle_N$$
$$n = \langle \ell+m\rangle_N$$

$$Y[k] = \sum_{\ell=0}^{N-1} x[\ell]\cdot W_N^{kn}$$

$$\langle \ell+m\rangle_N$$

$$W_N^{k\langle \ell+m\rangle_N} = W_N^{k(\ell+m+tN)} = W_N^{k(\ell+m)}$$

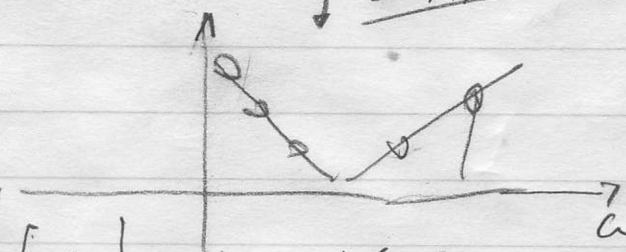
$$Y[k] = \sum_{\ell=0}^{N-1} x[\ell]\, W_N^{k\ell}\, \underbrace{W_N^{km}}$$

$$= W_N^{km} \underbrace{\sum_{\ell=0}^{N-1} x[\ell]\, W_N^{k\ell}}_{X[k]} = W_N^{km} X[k]$$

$$\boxed{\begin{array}{l} Y[k] = \underbrace{e^{-j\frac{2\pi}{N}km}}\, X[k] \\ \big\updownarrow \text{DFT} \\ x[\langle n-m\rangle_N] \end{array}}$$

Inverse DFT

$x[n] \longleftrightarrow X[k]$ samples of the DTFT $\cancel{X}_d(\omega)$



$$\{X[k]\} \xrightarrow{DFT^{-1}} \{x[n]\} \xrightarrow[DTFT]{} X_d(\omega)$$

* Not obvious that DFT is invertible, because this implies recovery of $X_d(\omega)$ from its samples!

$$X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{nk}$$

Proof of Inversion Formula

$$\frac{1}{N} \sum_{k=0}^{N-1} X[k]\, W_N^{-nk} =$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \underbrace{\left(\sum_{l=0}^{N-1} x[l]\, W_N^{kl} \right)}_{= X[k]} W_N^{-kn}$$

$$= \frac{1}{N} \sum_{l=0}^{N-1} \sum_{k=0}^{N-1} x[l]\, W_N^{k(l-n)} \qquad \phi(l-n)$$

$$= \frac{1}{N} \sum_{l=0}^{N-1} x[l] \overbrace{\left(\sum_{k=0}^{N-1} W_N^{k(l-n)} \right)}^{\phi(l-n)}$$

$$\sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a}$$

$$\phi(l-n) = \frac{1 - W_N^{k(l-n)N}}{1 - W_N^{k(l-n)}}$$

$$W_N^{k(l-n)N} = e^{j\frac{2\pi}{\cancel{N}}\cdot\cancel{N}k(l-n)} = 1$$

$$\phi(l-n) = \begin{cases} N & n=l \\ 0 & n \neq l \end{cases} = N \cdot \delta[n-l]$$

$$\frac{1}{N}\sum_{k=0}^{N-1} X[k]\, W_N^{-kn} = \frac{1}{\cancel{N}}\sum_{l=0}^{N-1} x[l] \cdot \cancel{N}\, \delta[n-l]$$

$$= x[n] \quad \checkmark$$

$$\boxed{x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, W_N^{-kn}}$$

Recall - Inverse DTFT

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_d(\omega)\, e^{j\omega n}\, d\omega$$

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Switching order of summation - why can you do that?

$$\sum_{k=0}^{2} \sum_{m=0}^{2} V(m,k)$$

$$= \sum_{k=0}^{2} \left(V(0,k) + V(1,k) + V(2,k) \right)$$

$$= \begin{array}{l} V(0,0) + V(1,0) + V(2,0) \\ V(0,1) + V(1,1) + V(2,1) \\ V(0,2) \quad V(1,2) + V(2,2) \end{array}$$

Properties

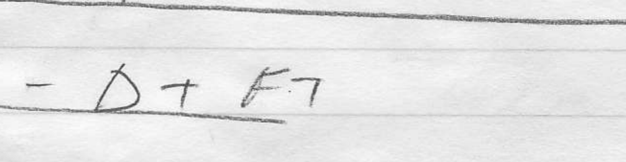
- ✓ Sampling of DTFT
- ✓ Linearity
- ✓ Zero padding
- ✓ Circular Shift
- ✓ Inverse
- $\boxed{\text{DFT of real-valued signals}}$

Recall - DTFT

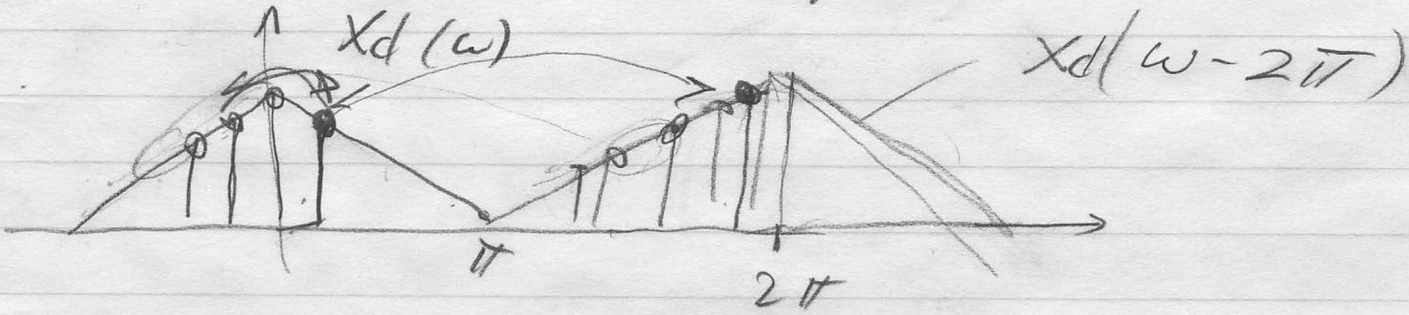
$x[n]$ - real $\qquad x[n] = x^*[n]$

$$\Longleftrightarrow \quad X_d^*(\omega) = X_d(-\omega)$$ Hermitian Symmetry

$$|X_d^*(\omega)| = |X_d(\omega)| = |X_d(-\omega)|$$



Symmetry of the DFT of a <u>real valued sequence</u> $x[n] = x^*[n]$



$$X_d(-\omega) = X_d^*(\omega) \Leftarrow \text{Real } \{x[n]\}$$

$$\| $$

$$X_d(2\pi - \omega) \Leftarrow \text{Periodicity of } X_d(\omega)$$

$$X[k] = X_d\left(\frac{2\pi}{N}k\right), \quad k = 0, 1, \cdots, N-1$$

$$X^*[k] = X_d^*\left(\frac{2\pi}{N}k\right) = X_d\left(2\pi - \frac{2\pi}{N}k\right) = X_d\left(\frac{2\pi}{N}(N-k)\right)$$

$$= \begin{cases} X_d\left(\frac{2\pi}{N}(N-k)\right) = X[N-k], & k = 1, 2, \cdots N-1 \\ X_d(2\pi) = X[0] & k = 0 \end{cases}$$

$$= X[\langle N-k \rangle_N]$$

$$\boxed{X^*[k] = X[\langle N-k \rangle_N] \quad k = 0, 1, 2, \cdots N-1}$$

Need the mod only for $k = 0$

Symmetry of the DFT of Real $\{x[n]\}$ - Cont.

$$\boxed{X[k] = X^*[\langle N-k\rangle_N]}, \quad k = 0, 1, \ldots N-1$$

Ex $N = 8$

$$X[0] = X^*[\langle 8\rangle_8] = X^*[0] = \text{Real!}$$

Why?

$$X[0] = \sum_{n=0}^{N-1} x[n] \underbrace{W_N^{n\cdot 0}}_{=1 = \text{real}}$$

$$X[1] = X^*[\langle 8-1\rangle_8] = X^*[7]$$

$$X[4] = X^*[8-4] = X^*[4] \text{ - Real!}$$

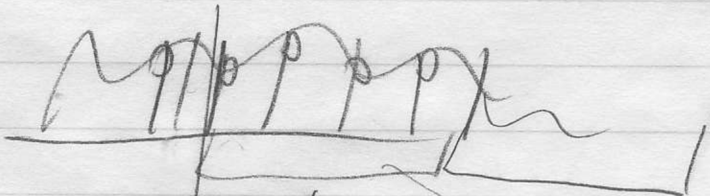
Why?

$$W_N^{n\frac{N}{2}} = e^{-j\frac{2\pi}{N}\cdot n\frac{N}{2}} = e^{-j\pi n} = \underbrace{(-1)^n}_{\text{real}}$$

Application of the DFT

Compression

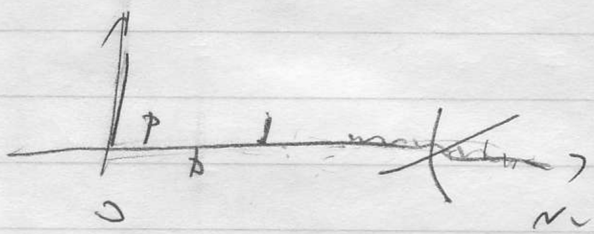
speech, images, video



Sampling @ Nyquist

"Block"

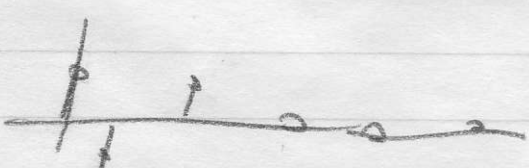
DFT

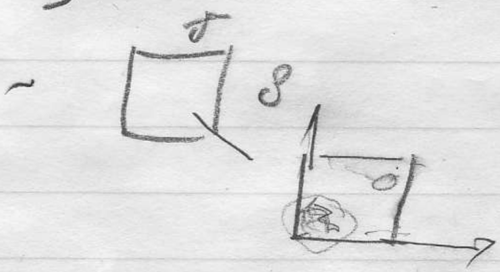


Truncation

Z.P

JPEG





DFT^{-1}

→ D/A