Problem 1

Billy recorded an amazing saxophone solo using a sampling rate of 24 kHz. Unfortunately, there was broken equipment in the room emitting an annoying 8 kHz tone. You can save Billy's musical career by designing a digital filter to remove the tone, while preserving the low-frequency saxophone music. Your filter will be given by:

$$y[n] = b_0 x[n] + b_1 x[n-1] + b_1 x[n-2] + b_0 x[n-3]$$

(a) Find an expression for the frequency response $H_d(\omega)$ of the filter in terms of b_0 and b_1. Your answer should be in GLP form; i.e., write $H_d(\omega) = R(\omega)e^{j(\alpha-\beta\omega)}$, where $R(\omega)$ is *real-valued.*

(b) The given filter has even symmetry. Could odd symmetry have been used instead?

(c) Find the values of the filter coefficients such that the frequency response is 1 at $\omega = 0$, and 0 at the frequency of the tone. Does the resulting filter have strict linear phase?

(d) Now, suppose that the maximum frequency of the saxophone music is 6 kHz. We wish to filter the recording using a digital filter sandwiched between an ideal A/D and an ideal D/A. The filter specifications are as follows:

- Frequencies below 6.5 kHz are passed through with less than 1 dB of error.
- Frequencies above 7.5 kHz are attenuated by at least 45 dB.
- The filter has the shortest possible length.

Give a closed-form expression for $h[n]$ as your answer.

Solution

(a) To find the impulse response, we simply have to set $x[n] = \delta[n]$, since the system is FIR. This gives

$$\{h[n]\}_{n=0}^{3} = \{b_0, b_1, b_1, b_0\}.$$

Taking the DTFT gives

$$\begin{aligned} H_d(\omega) &= b_0 + b_1 e^{-j\omega} + b_1 e^{-j\omega 2} + b_0 e^{-j\omega 3} \\ &= e^{-j\omega 1.5}(b_0 e^{j\omega 1.5} + b_1 e^{j\omega 0.5} + b_1 e^{-j\omega 0.5} + b_0 e^{-j\omega 1.5}) \\ &= \boxed{(2b_0 \cos(1.5\omega) + 2b_1 \cos(0.5\omega))e^{-j\omega 1.5}} \end{aligned}$$

This is indeed GLP, with $\alpha = 0$, $\beta = 1.5$, and $R(\omega) = 2b_0 \cos(1.5\omega) + 2b_1 \cos(0.5\omega)$.

(b) Since we're told that the saxophone music is low-frequency, we probably wish to design a low-pass filter. We know the most basic requirement for a low-pass filter is that everything at DC is passed through; that is, $H_d(0) = 1$.

Suppose we tried to implement this filter using odd symmetry, using

$$\{h[n]\}_{n=0}^{3} = \{b_0, b_1, -b_1, -b_0\}.$$

Then, taking the DTFT gives

$$\begin{aligned} H_d(\omega) &= b_0 + b_1 e^{-j\omega} - b_1 e^{-j\omega 2} - b_0 e^{-j\omega 3} \\ &= e^{-j\omega 1.5}(b_0 e^{j\omega 1.5} + b_1 e^{j\omega 0.5} - b_1 e^{-j\omega 0.5} - b_0 e^{-j\omega 1.5}) \\ &= e^{-j\omega 1.5}(2jb_0 \sin(1.5\omega) + 2jb_1 \sin(0.5\omega)) \\ &= e^{j(\frac{\pi}{2} - 1.5\omega)}(2b_0 \sin(1.5\omega) + 2b_1 \sin(0.5\omega)) \end{aligned}$$

This is again in GLP form; however, we observe that $H_d(0) = 0$. Therefore, we can **not** use odd symmetry to implement this filter; even symmetry is required.

(c) Using $\omega = \Omega T$, and given $T = \frac{1}{24000}$ s, we can convert the tone frequency of 8 kHz to $\omega = \frac{2\pi}{3}$. Therefore, we need to choose b_0 and b_1 such that $H_d(0) = 1$ and $H_d(\frac{2\pi}{3}) = 0$. Solving for $H_d(0)$ gives:

$$H_d(0) = 2b_0 + 2b_1 = 1.$$

Similarly, solving for $H_d(\frac{2\pi}{3})$ gives

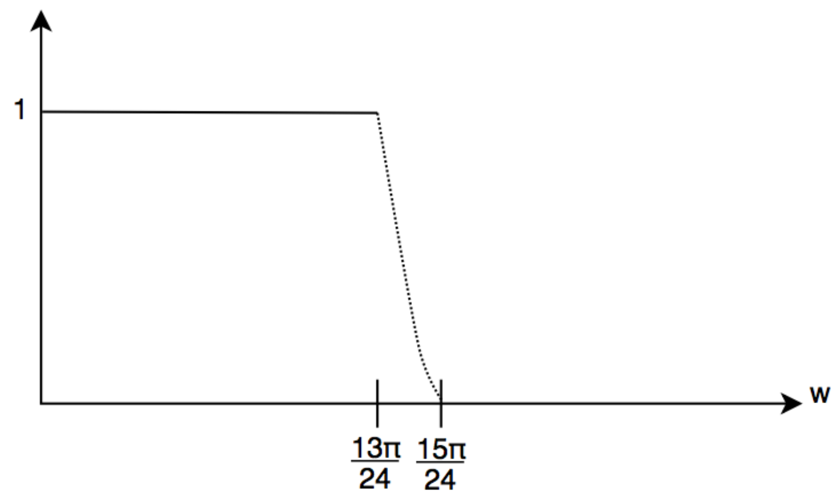
$$\begin{aligned} H_d\left(\frac{2\pi}{3}\right) &= e^{-j\frac{2\pi}{3}(\frac{3}{2})}\left(2b_0 \cos\left(\frac{3}{2}\left(\frac{2\pi}{3}\right)\right) + 2b_1 \cos\left(\frac{1}{2}\left(\frac{2\pi}{3}\right)\right)\right) \\ &= 2b_0 - b_1 = 0 \end{aligned}$$

Solving the system of equations gives $b_0 = \frac{1}{6}$ and $b_1 = \frac{1}{3}$. Therefore, the desired filter is

$$\boxed{\{h[n]\}_{n=0}^{3} = \left\{\frac{1}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{6}\right\}}$$

This gives $H_d(\omega) = e^{-j\omega 1.5}\left(\frac{1}{3}\cos(1.5\omega) + \frac{2}{3}\cos(0.5\omega)\right)$. To determine whether or not the filter exhibits strict linear phase, we need to determine whether $R(\omega) = \frac{1}{3}\cos(0.5\omega) + \frac{2}{3}\cos(0.5\omega)$ has any zero-crossings between $\omega = -\pi$ and $\omega = \pi$. If it does, then $\angle H_d(\omega)$ will have jumps of π, and the filter will be GLP. If not, the filter will be strict LP. In this case, there's a zero-crossing at $\omega = \frac{2\pi}{3}$, so the filter will be **GLP**.

(d) We start by converting the analog frequencies to digital. We're told that the end of the passband is 6.5 kHz, which corresponds to $\omega = \frac{13\pi}{24}$. We're also told that the beginning of the stopband is 7.5 kHz, which corresponds to $\omega = \frac{15\pi}{24}$. Therefore, the desired filter looks as follows:



Since we require at least 45 dB of attenuation in the stopband, we can only use the Hamming or Blackman windows. We choose the Hamming, as it allows us to design the filter using the minimum length. Since $\Delta\omega = \frac{15\pi}{24} - \frac{13\pi}{24} = \frac{\pi}{12}$, we solve for the required length as follows:

$$\frac{\pi}{12} = \frac{6.6\pi}{L} \rightarrow L = 79.2 \rightarrow 80$$

where we round up since a non-integer length is impossible. Finally, we choose the cutoff frequency to be the midpoint of the transition region, which is $\frac{7\pi}{12}$. Therefore, our ideal GLP response is

$$G_d(\omega) = \begin{cases} e^{-j\omega 39.5} & |\omega| < \frac{7\pi}{12} \\ 0 & \text{else} \end{cases}$$

Using the results from Homework 11 gives

$$g[n] = \frac{7}{12}\text{sinc}\left(\frac{7\pi}{12}(n-39.5)\right)$$

and applying the length-80 Hamming window gives the expression for the filter coefficients as

$$\boxed{h[n] = \begin{cases} \frac{7}{12}\text{sinc}\left(\frac{7\pi}{12}(n-39.5)\right)\left(0.54 - 0.46\cos\left(\frac{2\pi n}{79}\right)\right) & 0 \le n \le 79 \\ 0 & \text{else} \end{cases}}$$

Problem 2

Consider a system composed of a digital filter sandwiched between an ideal A/D and an ideal D/A operating at a common rate of 3 kHz. The input $x[n]$ and output $y[n]$ of the digital filter are related through a difference equation:

$$y[n] = x[n] + x[n-1] - 0.5y[n-1].$$

Find the analog output of the system, $y_a(t)$, given an analog input $x_a(t) = \cos(1500\pi t) + \cos(3000\pi t) + \cos(3600\pi t)$.

Solution

The output of the ideal D/A is given by

$$\begin{aligned} x[n] = x_a(nT) &= \cos\left(\frac{\pi}{2}n\right) + \cos(\pi n) + \cos\left(\frac{6\pi}{5}n\right) \\ &= \cos\left(\frac{\pi}{2}n\right) + \cos(\pi n) + \underbrace{\cos\left(\frac{4\pi}{5}n\right)}_{\text{aliasing}} \end{aligned}$$

To find the output of the filter, we recall the eigensequence property. Since the system is real (it's a difference equation with completely real coefficients), we can use the fact that

$$\cos(\omega_0 n + \phi) \to \boxed{H_d(\omega)} \to |H_d(\omega_0)|\cos(\omega_0 n + \phi + \angle H_d(\omega_0))$$

to get the output as

$$y[n] = \left|H_d\left(\frac{\pi}{2}\right)\right|\cos\left(\frac{\pi}{2}n + \angle H_d\left(\frac{\pi}{2}\right)\right) + |H_d(\pi)|\cos(\pi n + \angle H_d(\pi)) + \left|H_d\left(\frac{4\pi}{5}\right)\right|\cos\left(\frac{4\pi}{5}n + \angle H_d\left(\frac{4\pi}{5}\right)\right)$$

We take the DTFT of both sides of the difference equation to find the frequency response:

$$Y_d(\omega) = X_d(\omega) + e^{-j\omega}X_d(\omega) - 0.5e^{-j\omega}Y_d(\omega) \to H_d(\omega) = \frac{1+e^{-j\omega}}{1+0.5e^{-j\omega}}$$

Then, calculating the DTFT at the desired values gives

$$\begin{aligned} H_d\left(\frac{\pi}{2}\right) &= \frac{1-j}{1-j0.5} = 1.265\angle-18.435^\circ \\ H_d\left(\frac{4\pi}{5}\right) &= 0.931\angle-45.73^\circ \\ H_d(\pi) &= 0 \end{aligned}$$

Therefore, the discrete-time output is

$$y[n] = 1.265\cos\left(\frac{\pi}{2}n - 18.435^\circ\right) + 0.931\left(\frac{4\pi}{5}n - 45.73^\circ\right)$$

and the analog output is found by setting $\Omega = \frac{\omega}{T}$, since the D/A converter is ideal.This gives

$$\boxed{y_a(t) = 1.265\cos(1500\pi t - 18.435°) + 0.931\cos(2400\pi t - 45.73°)}$$

Note that, because aliasing occurred, the frequencies in the output no longer match the frequencies in the input.

Problem 3

You are given the task of implementing an analog HPF using a system composed of a digital filter sandwiched between an ideal A/D and D/A operating at a common rate of 40 kHz. The specifications for the analog HPF are as follows:

- Stopband: 0-8 kHz, with attenuation of at least 45 dB.
- Passband: Starts at 9 kHz, with passband error no larger than 1 dB.

(a) Design the shortest GLP FIR digital filter that meets these specifications using the window design method. Give an expression for the filter coefficients.

(b) What is the highest input frequency for which the system will perform its desired function?

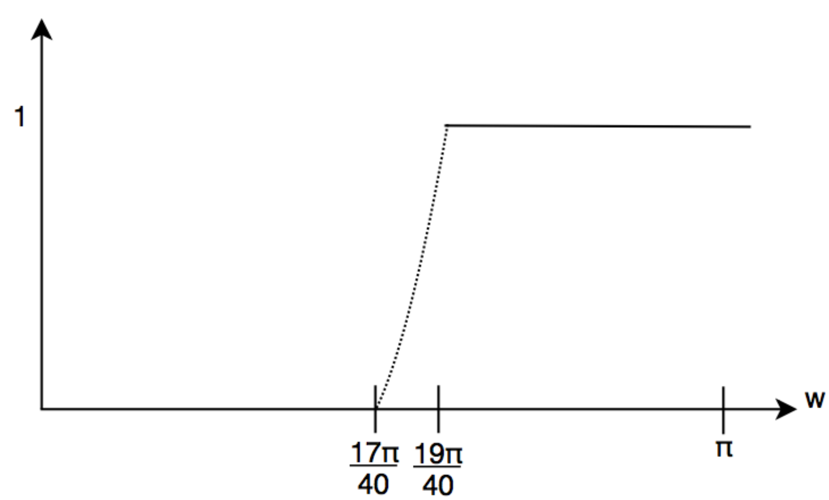
(c) Will the system (from analog input to analog output) perform as an LTI generalized linear phase system?

(d) Sketch the *approximate* magnitude of the analog frequency response of the system, $|H_a(\Omega)|$. Perform the sketch on a dB scale. Identify the end of the stopband, beginning of the passband, and the size of the largest passband ripple and stopband sidelobe. Perform the sketch over the range of frequencies for which the system acts as an LTI system.

(e) What is the minimum number of multiplications per second required to implement the digital system?

Solution

(a) Once again, we start by converting the analog frequencies to digital. In this case, we know that the stopband stops at 8 kHz, which corresponds to $\omega = \frac{8\pi}{20}$. We also know that the passband starts at 9 kHz, which corresponds to $\omega = \frac{9\pi}{20}$. Therefore, the designed filter looks as follows:



We again use the Hamming window, since it will provide the desired attenuation in the shortest length. Since the width of the transition region is $\frac{9\pi}{20} - \frac{8\pi}{20} = \frac{\pi}{20}$, we calculate the length as follows:

$$\frac{6.6\pi}{L} = \frac{\pi}{20} \rightarrow L = 132$$

We also calculate the cutoff frequency as the midpoint of the transition region - this is $\omega_c = \frac{17\pi}{40}$. We have two choices to actually design the filter; we can treat the high-pass filter as our desired response, or we could start with a low-pass prototype and use the modulation property to create the high-pass filter; shifting by π in the frequency domain corresponds to multiplying by $e^{j\pi n} = (-1)^n$ in the time domain. Therefore, we design a low-pass filter with cutoff frequency $\pi - \omega_c = \frac{23\pi}{40}$. Using the results from Problem 1, the filter coefficients are given by

$$\boxed{h[n] = \begin{cases} (-1)^n \frac{23}{40}\text{sinc}\left(\frac{23\pi}{40}(n-65.5)\right)\left(0.54 - 0.46\cos\left(\frac{2\pi n}{131}\right)\right) & 0 \leq n \leq 131 \\ 0 & \text{else} \end{cases}}$$

(b) The system will perform its desired function for input frequencies up to 20 kHz. Since we sample at $T = \frac{1}{40000}$ s, aliasing will occur for any higher input frequency.

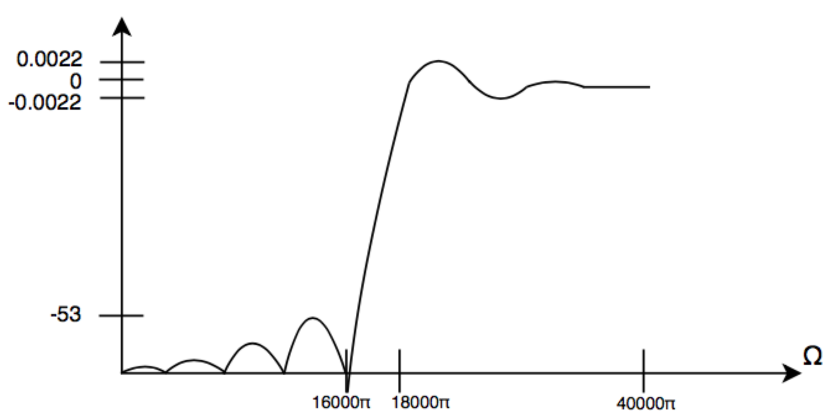
(c) The system will *only* perform as an LTI GLP system if aliasing does not occur; that is, whenever the input frequencies are all lower than 20 kHz. If no aliasing occurs, we can simply use $\omega = \Omega T$; since the digital response is $H_d(\omega) = R(\omega)e^{j(\alpha - \beta\omega)}$, the analog response will be

$$H_a(\Omega) = H_d(\Omega T) = R(\Omega T)e^{j(\alpha - \beta T \omega)}$$

which is still in GLP form.

However, if aliasing were to occur, the system will not behave as an LTI system; let alone a GLP filter. This is because aliasing has the potential to create new frequencies, which is impossible for LTI systems. For example, suppose $X_d(\Omega_0) = 0$. If this is passed through an analog LTI system, then it must be the case that $Y_d(\Omega_0) = 0$, since $Y_d(\Omega_0) = X_d(\Omega_0)H_d(\Omega_0)$. However, if we implement the filter digitally, it's possible that some input frequency aliases to $\omega_0 = \Omega_0 T$, creating a nonzero component at Ω_0 in the output.

(d) The approximate plot is given below. We sketch $|H_d(\Omega)|$ for $|\Omega| < 40000\pi$, since the system will behave as an LTI system for input frequencies up to 20 kHz. Since the Hamming window was used, the magnitude of the largest sidelobe in the stopband is -53 dB, and the largest passband ripple is 0.0022 dB. The end of the stopband is at $\Omega = 16000\pi$, and the beginning of the passband is at $\Omega = 18000\pi$.



(e) Since we sample at T, the system must process $\frac{1}{T}$ samples every second. Furthermore, since the system is a length-132 FIR filter, the output is a weighted sum of the last 132 input samples; this means that we need at most 132 multiplications per output sample. However, because the filter is symmetric, we can exploit that symmetry to reduce the number of multiplications to $\frac{132}{2} = 66$, since there are only 66 different gain values. Since $y[n] = b_0x[n] + b_1x[n-1] + b_2x[n-2] + ... + b_2x[n-131+2] + b_1x[n-131+1] + b_0x[n-131]$, we can "pre-add" $x[n-\alpha]$ and $x[n-N+\alpha]$ before multiplying by b_α. Therefore, we need

$$\left(40000\frac{\text{samples}}{\text{second}}\right)\left(66\frac{\text{multiplications}}{\text{sample}}\right) = \boxed{2.64 \times 10^6 \text{ multiplications/second}}$$