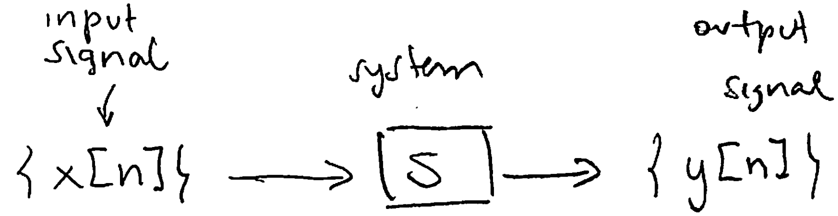
Lecture 3

Discrete-time system

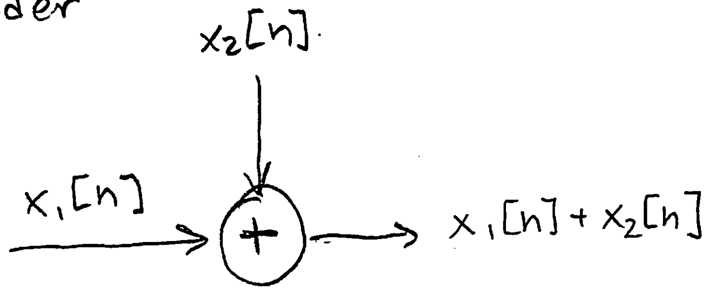


eg.: $y[n] = x[n-1] + \frac{1}{2}x[n-2]$

Block diagram representation

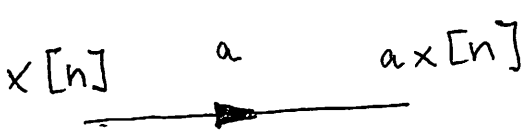
4 building blocks

• adder

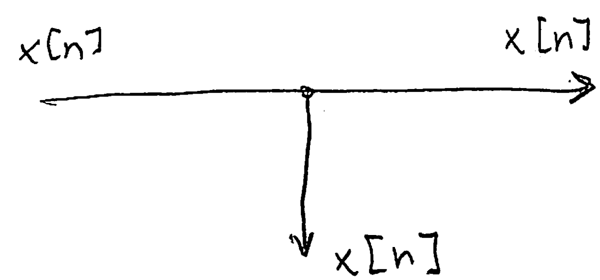


• unit delay

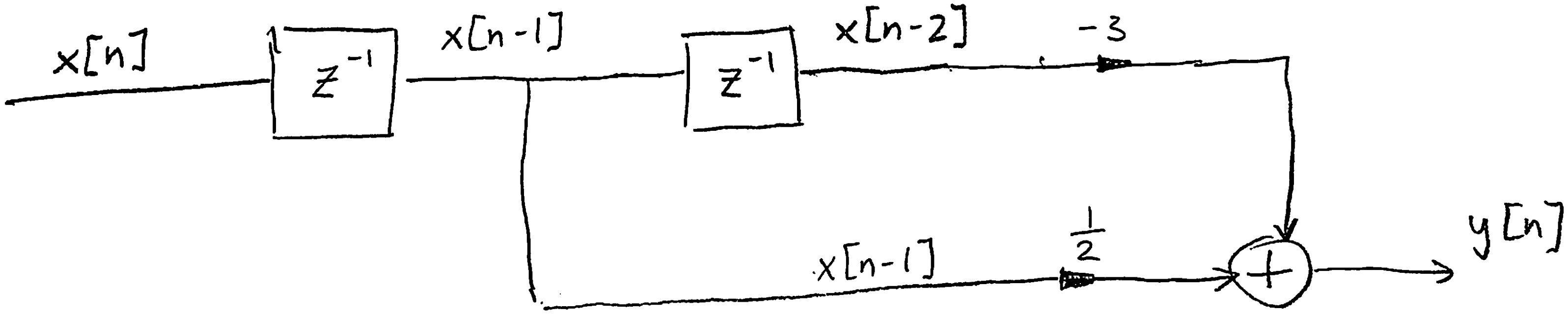
• multiplier



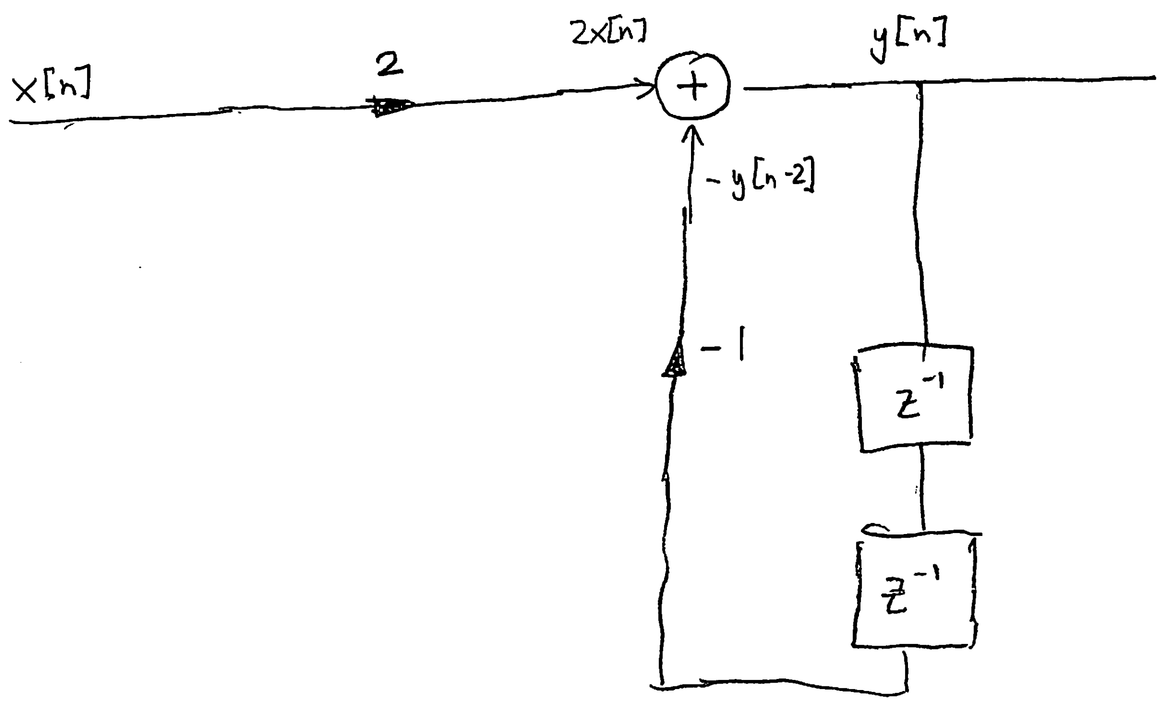
• Splitter



Ex : $y[n] = \frac{1}{2}x[n-1] - 3x[n-2]$



Ex 2 : $y[n] = 2x[n] - y[n-2]$



Properties of Discrete-time Systems

① Causal: present output doesn't depend on future inputs

$y[n_0]$ only depends on $\{\ldots, x[n_0-2], x[n_0-1], x[n_0]\}$

"future"

Ex: $y[n] = x[n+1] + x[n] + x[n-1]$ (non-causal)

$y[n] = \sum_{k=0}^{\infty} x[n-k]$ (causal)

bounded-input bounded-output

② Stable (BIBO sense)

$\{x[n]\} \rightarrow \boxed{S} \rightarrow \{y[n]\}$

If $|x[n]| \leq M_x < \infty$ for all n, then $|y[n]| \leq M_y$

Ex $y[n] = x[n+1] + x[n] + x[n-1]$

Suppose $|x[n]| \leq M_x$ for some M_x.

triangle inequality

$|y[n]| = |x[n+1] + x[n] + x[n-1]| \leq |x[n+1]| + |x[n]| + |x[n-1]|$

$\leq 3M_x = M_y$

Ex 2: $y[n] = \sum_{k=0}^{\infty} x[n-k]$

counter-example. $\{x[n]\} = \{u[n]\} = \{\ldots, 0, 0, \underset{\uparrow}{1}, 1, \ldots\}$ $n > 0$

$y[n] = \sum_{k=0}^{\infty} u[n-k] = u[n] + u[n-1] + u[n-2] + \ldots + u[0] + u[-1] + \ldots$

$= n + 1 \longrightarrow \infty$ as $n \to \infty$

unbounded $\Rightarrow$ system is unstable

③ Linear

Suppose $\begin{cases} \{x_1[n]\} \to \boxed{S} \to \{y_1[n]\} \\ \{x_2[n]\} \to \boxed{S} \to \{y_2[n]\} \end{cases}$

If S is linear, then: $\{a_1 x_1[n] + a_2 x_2[n]\} \longrightarrow \boxed{S} \longrightarrow \{a_1 y_1[n] + a_2 y_2[n]\}$

Ex: $y[n] = x[n] \cdot \cos(\pi n)$ (for all n)

Suppose $\begin{cases} \{x_1[n]\} \to \boxed{S} \to \{y_1[n]\} = \{x_1[n] \cos(\pi n)\} \\ \{x_2[n]\} \to \boxed{S} \to \{y_2[n]\} = \{x_2[n] \cos(\pi n)\} \end{cases}$

$\underbrace{a_1 x_1[n] + a_2 x_2[n]}_{x[n]} \longrightarrow \boxed{S} \longrightarrow y[n] = x[n] \cos(\pi n)$

$$y[n] = x[n]\cos(\pi n) = \left(a_1 x_1[n] + a_2 x_2[n]\right)\cos(\pi n)$$

$$= a_1 \underbrace{x_1[n]\cos(\pi n)}_{y_1[n]} + a_2 \underbrace{x_2[n]\cos(\pi n)}_{y_2[n]}$$

$$= a_1 y_1[n] + a_2 y_2[n].$$ System is linear

Ex 2: $y[n] = (x[n])^2$ for all n

Suppose $x_1[n] \xrightarrow{S} y_1[n] = (x_1[n])^2$, $x_2[n] \xrightarrow{S} y_2[n] = (x_2[n])^2$

$$\underbrace{a_1 x_1[n] + a_2 x_2[n]}_{x[n]} \xrightarrow{S} y[n] = (x[n])^2$$

$$y[n] = \left(a_1 x_1[n] + a_2 x_2[n]\right)^2 =$$

$$= a_1^2 x_1^2[n] + a_2^2 x_2^2[n] + 2a_1 a_2 x_1[n] x_2[n]$$

If system were linear:

$$y[n] = a_1 y_1[n] + a_2 y_2[n]$$

$$= a_1 x_1^2[n] + a_2 x_2^2[n]$$

$\neq$ System is not linear

④ Time - invariant

Suppose $x_1[n] \xrightarrow{S} y_1[n]$

Then, $x_2[n] = x_1[n-n_0] \longrightarrow \boxed{S} \longrightarrow y_2[n] = y_1[n-n_0]$

Ex. $x[n] \xrightarrow{S} y[n] = x[n]\cos(\pi n)$

Suppose $x_1[n] \xrightarrow{S} y_1[n] = x_1[n]\cos(\pi n)$

Let $x_2[n] = x_1[n-n_0]$

$x_2[n] \xrightarrow{S} y_2[n] = x_2[n]\cos(\pi n)$

$$y_2[n] = x_2[n]\cos(\pi n)$$
$$= x_1[n-n_0]\cos(\pi n)$$

If system were time-invariant

$$y_2[n] = y_1[n-n_0]$$
$$= x_1[n-n_0]\cos(\pi(n-n_0))$$

$\cos(\pi n) \neq \cos(\pi(n-n_0))$ for odd n_0

$\Longrightarrow$ system is not time-invariant

Ex: $x[n] \xrightarrow{S} y[n] = x[n \cdot M]$ (M is fixed)

Linearity

suppose $\begin{cases} x_1[n] \xrightarrow{S} y_1[n] = x_1[nM] \\ x_2[n] \xrightarrow{S} y_2[n] = x_2[nM] \end{cases}$

$$\underbrace{a_1 x_1[n] + a_2 x_2[n]}_{x[n]} \xrightarrow{S} y[n] = x[nM]$$

$$y[n] = x[nM] = a_1 \underbrace{x_1[nM]}_{y_1[n]} + a_2 \underbrace{x_2[nM]}_{y_2[n]} = a_1 y_1[n] + a_2 y_2[n]$$

Time-invariance

Suppose $x_1[n] \xrightarrow{S} y_1[n] = x_1[nM]$

Let $x_2[n] = x_1[n-n_0] \xrightarrow{S} y_2[n] = x_2[nM]$

$$y_2[n] = x_2[nM] = x_1[nM - n_0]$$

not equal $\Rightarrow$ not time invariant

If system were time-invariant:

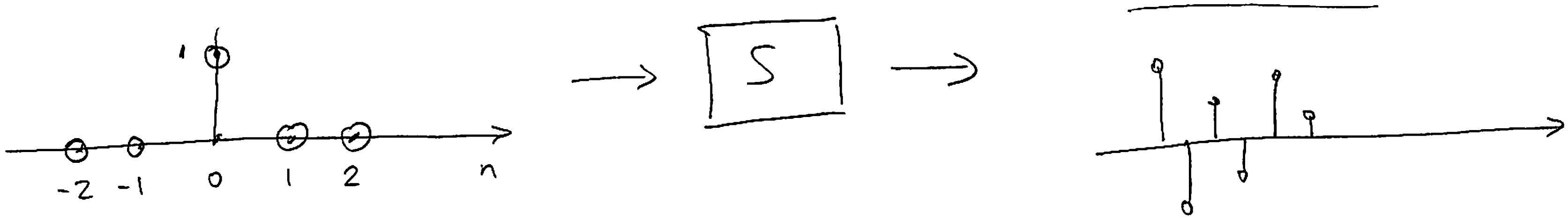
$$y_2[n] = y_1[n-n_0]$$
$$= x_1[(n-n_0)M] = x_1[nM - n_0 M]$$

LTI systems: Linear and time-invariant systems

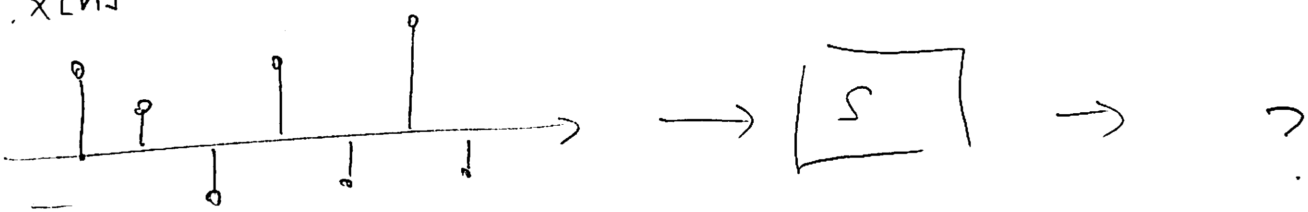
Why are they special?

Response of LTI system to any input can be determined from its impulse response

impulse $\{\delta[n]\}$ impulse response $\{h[n]\}$



$x[n]$



Can be written in terms of $h[n]$

linear combination of delayed impulses