Lecture 21

Last week: Generalized Linear Phase (GLP)

$$\{h[n]\}_{n=0}^{M} \Longrightarrow H_d(\omega) = \underbrace{A(\omega)}_{\text{real}} \cdot e^{j(\alpha - \omega M/2)}$$

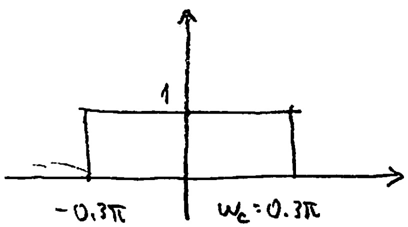
symmetric or anti-symmetric about $M/2$

- Four types of FIR GLP filters with different properties (see table)

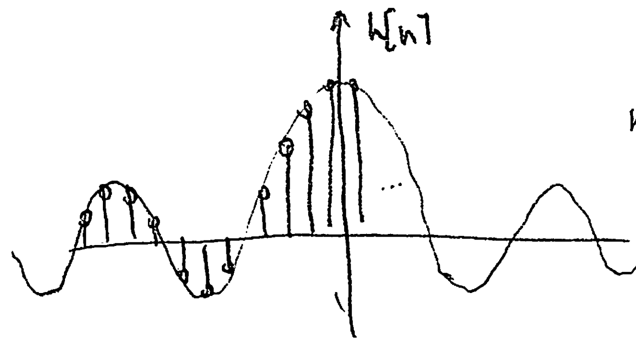
FIR Filter Design (by shifting and windowing the ideal response)

Ex. Design FIR Lowpass filter with cutoff $\omega_c = 0.3\pi$ and length $N = M+1$

Recall:



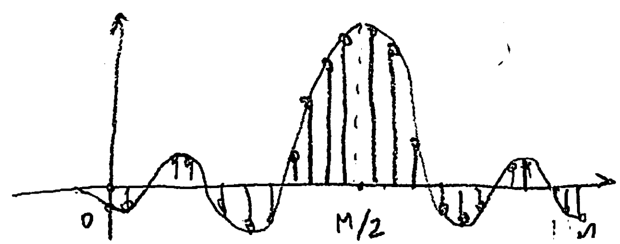
$\xrightarrow{DTFT^{-1}}$



non-causal

Proposed solution: shift and truncate (i.e. set $h[n]=0$ for $n<0$ and $n>M$)

How much do we shift by? $M/2$, to get GLP!



What is the effect of truncation/windowing?

Effect of windowing :

$$h[n] = h^{(ideal)}[n - M/2] \cdot w[n] \quad , \quad \text{where} \quad w[n] = \begin{cases} 1 & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$$

DTFT

(periodic convolution)

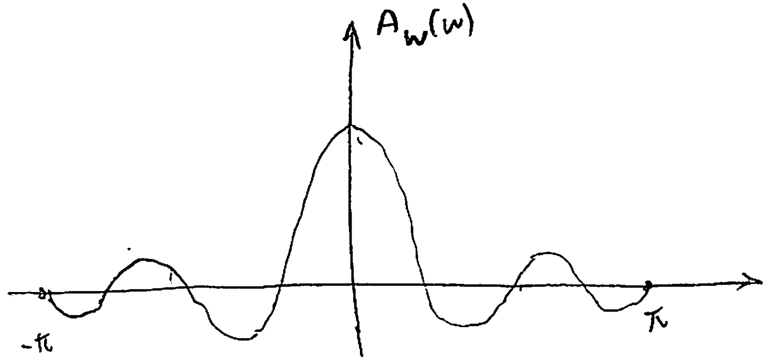
$$H_d(\omega) = \frac{1}{2\pi} \Big(\underbrace{e^{-j\omega \frac{M}{2}} H_d^{(ideal)}(\omega)}_{\tilde{H}_d(\omega)} \Big) \circledast W_d(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{H}_d(\theta) \, W_d(\omega-\theta) \, d\theta$$

$$W_d(\omega) = e^{-j\omega\frac{M}{2}} \cdot \underbrace{\frac{\sin\left(\frac{\omega(M+1)}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}}_{A_w(\omega)}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-j\theta\frac{M}{2}} H_d^{(ideal)}(\theta) \cdot e^{-j(\omega-\theta)\frac{M}{2}} \cdot A_w(\omega-\theta) \, d\theta$$

$$= \frac{1}{2\pi} e^{-j\omega\frac{M}{2}} \int_{-\pi}^{\pi} H_d^{(ideal)}(\theta) \, A_w(\omega-\theta) \, d\theta$$

$$= \frac{1}{2\pi} e^{-j\omega M/2} \, H_d^{(ideal)}(\omega) \circledast A_w(\omega)$$



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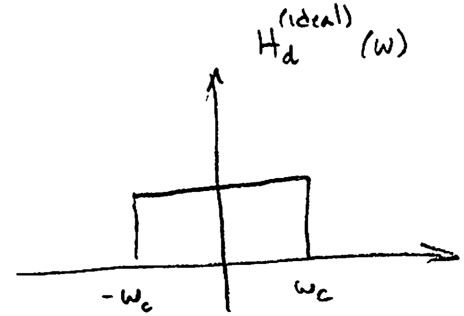

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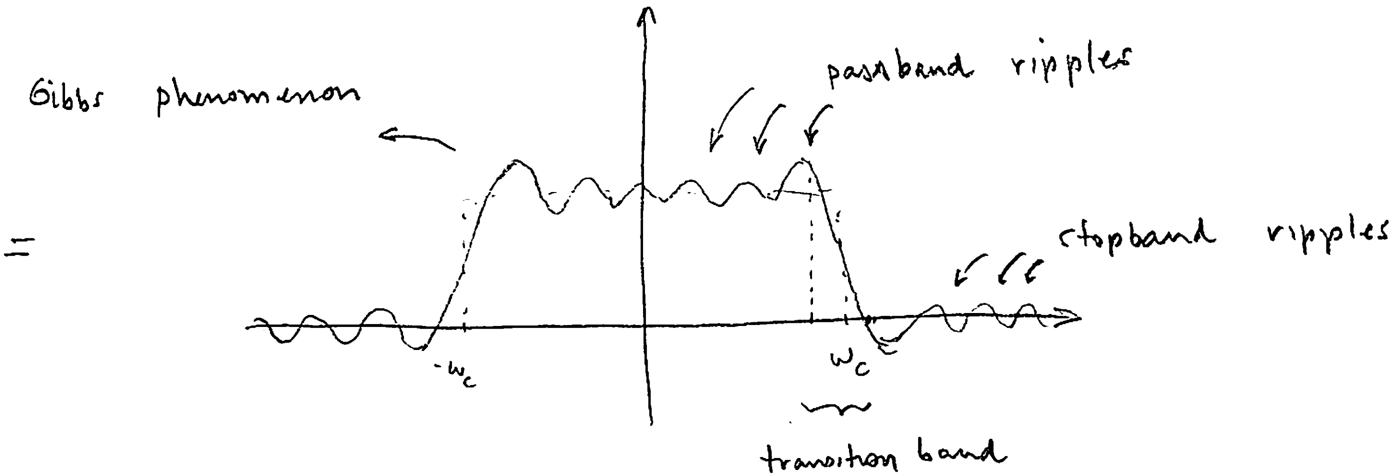


Gibbs phenomenon: At a sharp transition, we get tall ripples.

As N increases, ripples and transition band get narrower,
but the height of the tallest ripple remains the same

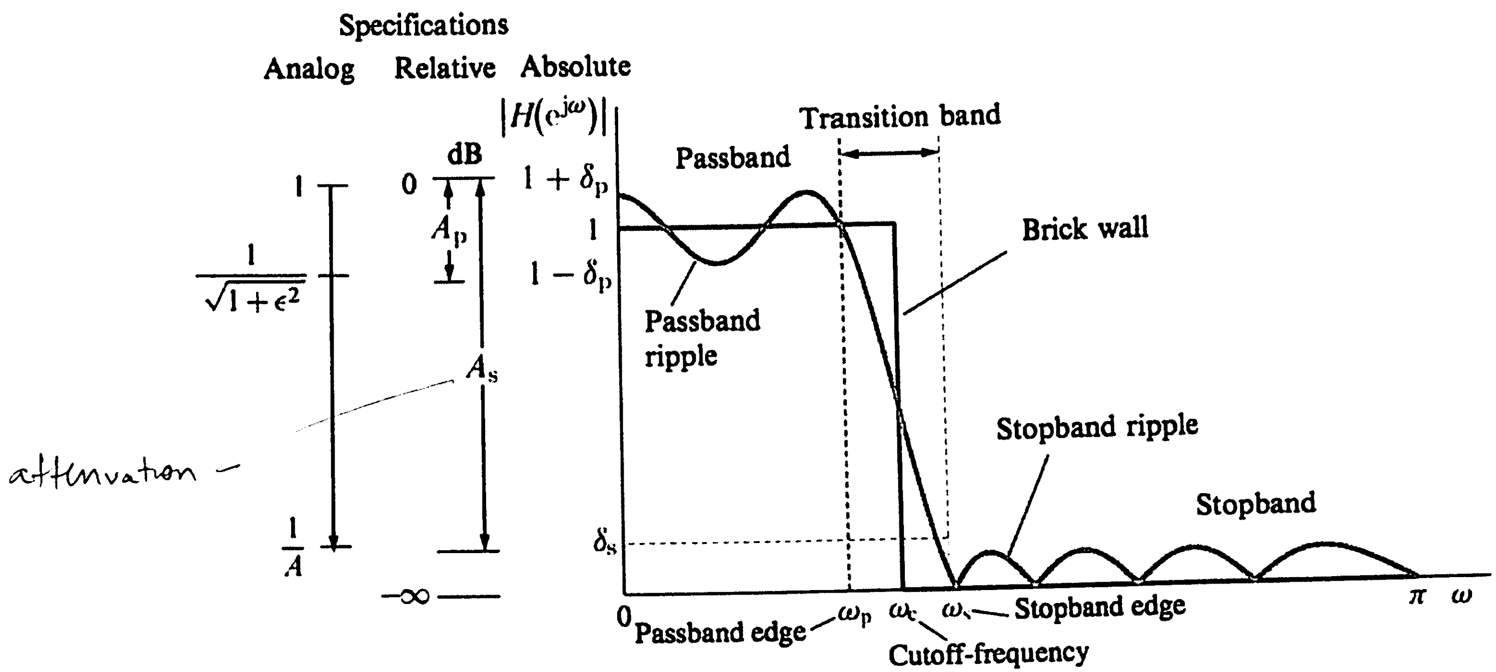
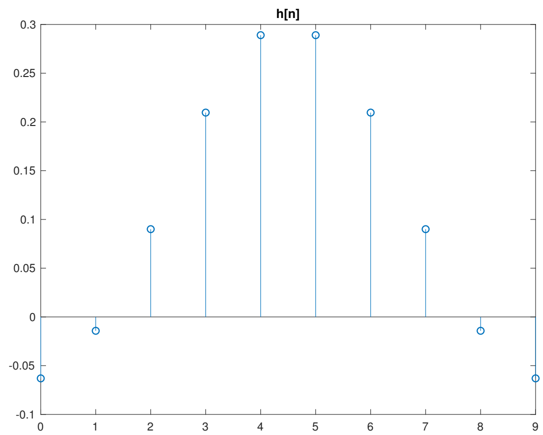


Figure 10.1 Example of tolerance diagram for a lowpass filter.

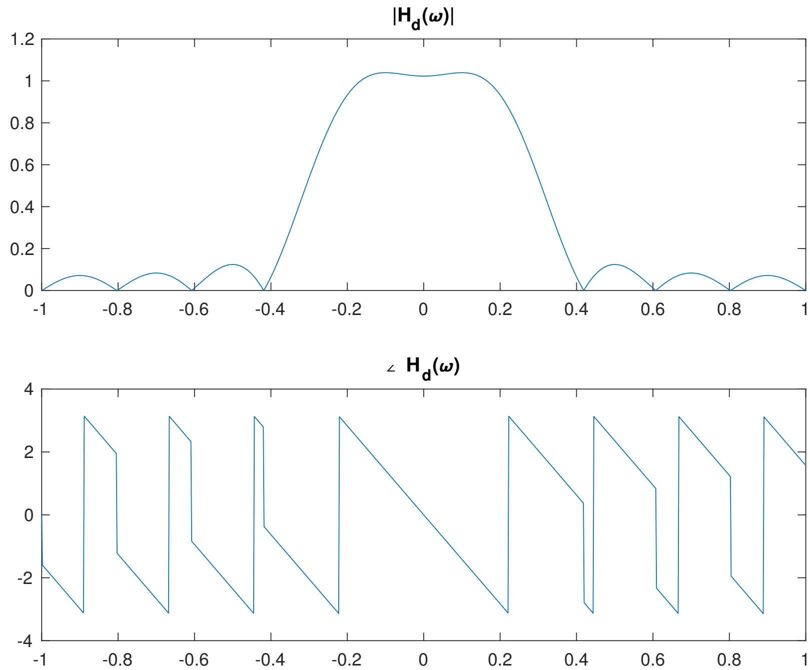
MATLAB example: LPF with $w_c = 0.3\pi$

$$h[n] = \begin{cases} \dfrac{\sin(0.3\pi(n - M/2))}{\pi(n - M/2)} & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$$

```
M = 9;
n = 0:M;
w_c = 0.3*pi;
h = sin(w_c*(n-M/2))./(pi*(n-M/2));

subplot(1,1,1); stem(n,h); title('h[n]')
```

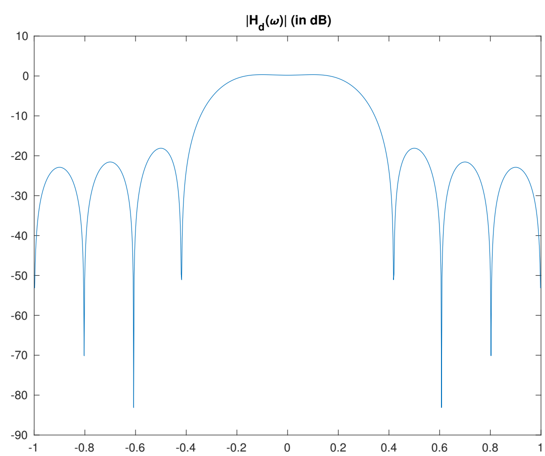



```
% computing frequency response (i.e., DTFT of h[n] (via fft)):
N_dft = 1024;
h_zp = [h, zeros(1,N_dft-length(h))];
H = fft(h_zp);
H = fftshift(H); % just moving it to the interval -pi,pi
w = (2*pi/N_dft*(0:N_dft-1) - pi)/pi;

subplot(2,1,1); plot(w,abs(H)); title('|H_d(\omega)|'); xlim([-1,1]);
subplot(2,1,2); plot(w,angle(H)); title('\angle H_d(\omega)'); ...
    xlim([-1,1]);
```



Notice that 2pi-jumps are actually just wraps from -pi to pi. The pi-jumps occur when amplitude changes sign. Because we have pi-jumps, this is GLP

```
subplot(1,1,1);
plot(w,mag2db(abs(H))); title('|H_d(\omega)| (in dB)');
```



Design of FIR filter with windowing :

① Determine ideal filter response

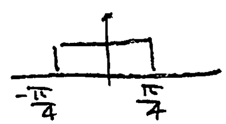
② Determin FIR filter type

③ Determine shifted filter response (usually multiplying by $e^{-jwM/2}$. Need to
D(w) be careful with Types 3, 4)

④ Compute $d[n] = DTFT^{-1}(D(w))$

⑤ Apply window of choice : $h[n] = d[n] \cdot w[n]$

Ex: Design FIR LPF with Length 30, cutoff $w_c = \frac{\pi}{4}$, $A_p \leq 1$ dB, $A_s \geq 40$ dB

① ideal response 

② $M = 29 \Rightarrow$ Filter of type II

③ Shifted filter response
$$D(w) = \begin{cases} 1 e^{-jw\frac{29}{2}} & |w| \leq \pi/4 \\ 0 & \end{cases}$$

④
$$d[n] = \begin{cases} \dfrac{\sin\left(\frac{\pi}{4}\left(n - \frac{29}{2}\right)\right)}{\pi(n - 29/2)} & 0 \leq n \leq 29 \\ 0 & \text{otherwise} \end{cases}$$

⑤ Hann window meets requirements $\Rightarrow$
$$h[n] = \begin{cases} d[n] \cdot \left(0.5 - 0.5\cos\left(\frac{2\pi n}{29}\right)\right) & 0 \leq n \leq 29 \\ 0 & \text{otherwise} \end{cases}$$

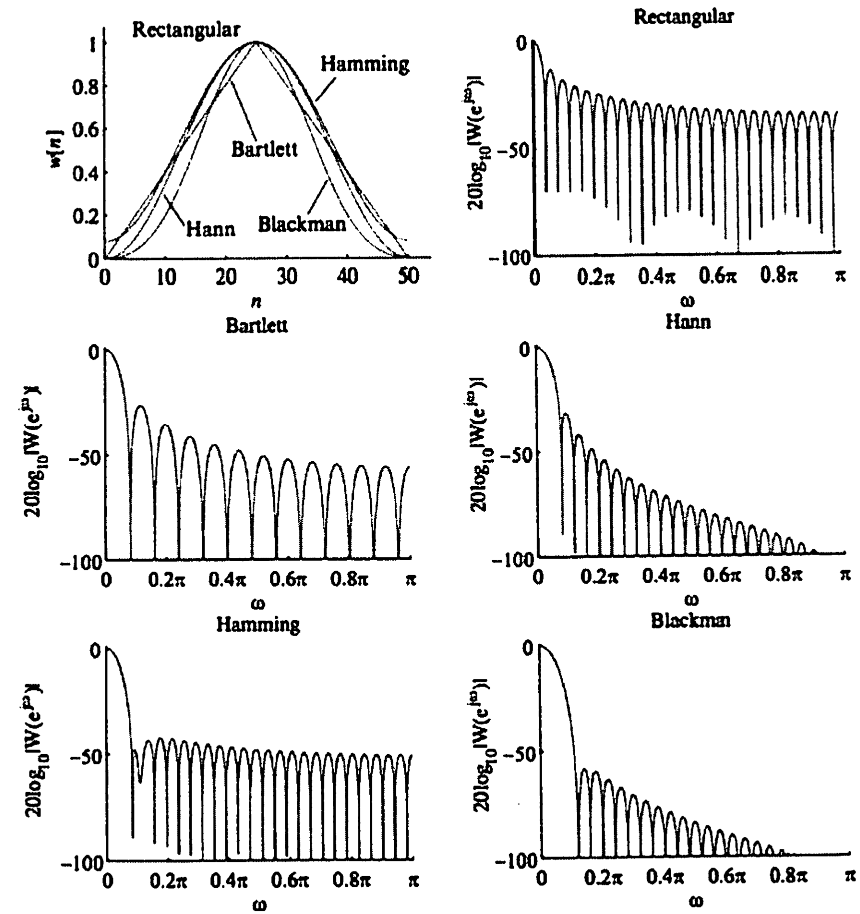


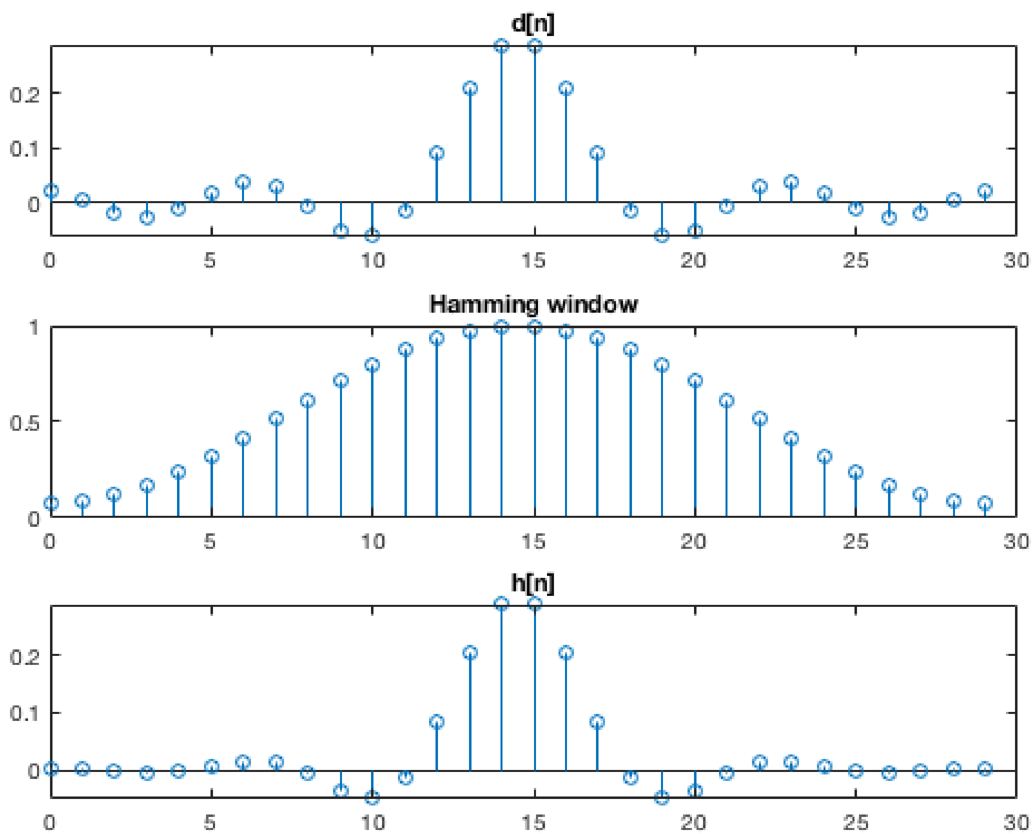
Figure 10.10 Time-domain and frequency-domain characteristics of some commonly used windows.

transition band   Gibbs phenomenon

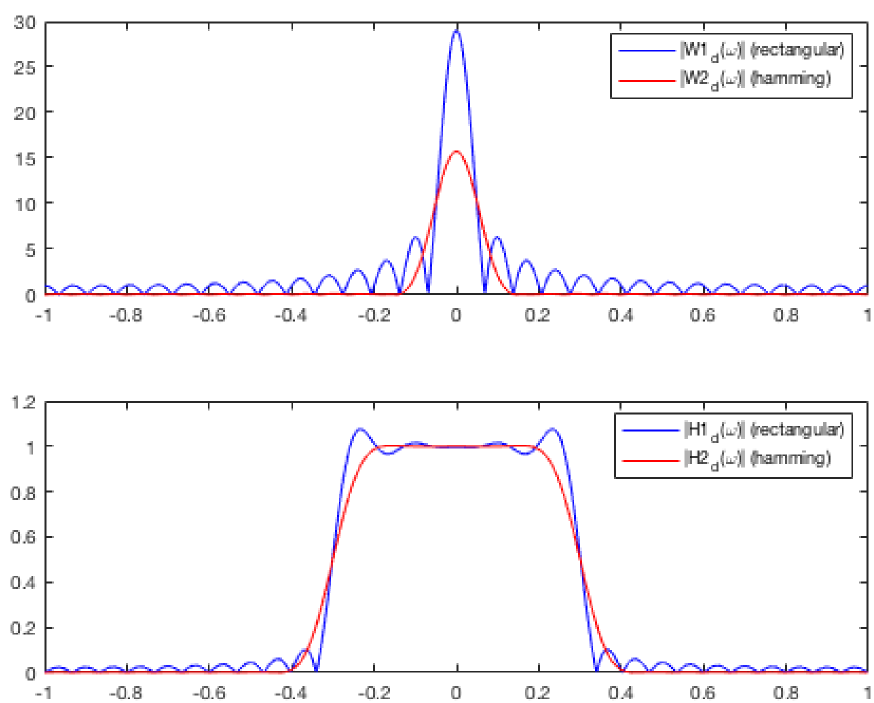
Table 10.3 Properties of commonly used windows ($L = M + 1$).

Window name	Side lobe level (dB)	Approx. $\Delta\hat{\omega}$	Exact $\Delta\omega$	$\delta_p \approx \delta_s$	A_p (dB)	A_s (dB)
Rectangular	−13	$4\pi/L$	$1.8\pi/L$	0.09	0.75	21
Bartlett	−25	$8\pi/L$	$6.1\pi/L$	0.05	0.45	26
Hann	−31	$8\pi/L$	$6.2\pi/L$	0.0063	0.055	44
Hamming	−41	$8\pi/L$	$6.6\pi/L$	0.0022	0.019	53
Blackman	−57	$12\pi/L$	$11\pi/L$	0.0002	0.0017	74

```
M = 29;
n = 0:M;
w_c = 0.3*pi;

d = sin(w_c*(n-M/2))./(pi*(n-M/2));

w_H = 0.54 - 0.46*cos(2*pi*n/M);

h = d.*w_H;

fig1 = figure(1);
subplot(3,1,1); stem(n,d); title('d[n]')
subplot(3,1,2); stem(n,w_H); title('Hamming window')
subplot(3,1,3); stem(n,h); title('h[n]')
```

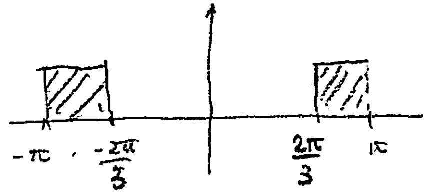


```
% First we will plot the DTFT of the windows:

[W_rect,omega] = DTFT(ones(1,29));
[W_hamming,omega] = DTFT(w_H);

figure(2);
subplot(2,1,1); plot(omega,abs(W_rect),'b'); %title('|H_d(\omega)|'); ...
    xlim([-1,1]);
hold on
subplot(2,1,1); plot(omega,abs(W_hamming),'r');
legend('|W1_d(\omega)| (rectangular)', '|W2_d(\omega)| (hamming)')
hold off

[D,omega] = DTFT(d);
[H,omega] = DTFT(h);

subplot(2,1,2); plot(omega,abs(D),'b'); %title('|H_d(\omega)|'); ...
    xlim([-1,1]);
hold on
subplot(2,1,2); plot(omega,abs(H),'r');
legend('|H1_d(\omega)| (rectangular)', '|H2_d(\omega)| (hamming)')

hold off
```



Ex 2: Design HPF, length 62, $w_c = \frac{2\pi}{3}$

① Ideal response:



② $M = 61 \Rightarrow$ Type IV because Type II has $A(\pi) = 0$

③ Type IV has response $D(w) = A(w) \cdot e^{j\left(\frac{\pi}{2} - w\frac{M}{2}\right)} = A(w) \cdot j \cdot e^{-jwM/2}$

$$\Rightarrow D(w) = \begin{cases} je^{-jwM/2} & \frac{2\pi}{3} \le w \le \pi \\ 0, & -\frac{2\pi}{3} \le w \le \frac{2\pi}{3} \\ -je^{-jwM/2} & -\pi \le w \le -\frac{2\pi}{3} \end{cases}$$

For $d[n]$ to be real, we need $D(w) = D^*(-w)$