ECE310 Fall 2018 Section E Quiz 4 Solutions

1. (6 pts) A causal linear time invariant system has transfer function $H(z) = \frac{1-z^{-1}}{(1+3z^{-1})(1-2z^{-1})}$

   (a) Determine a difference equation relating the input $x[n]$ to the output $y[n]$ for this system.

$$H(z) = \frac{Y(z)}{X(z)}$$
$$\frac{Y(z)}{X(z)} = \frac{1-z^{-1}}{(1+3z^{-1})(1-2z^{-1})}$$
$$Y(z)\left(1+3z^{-1}\right)\left(1-2z^{-1}\right) = X(z)\left(1-z^{-1}\right)$$
$$Y(z)\left(1+z^{-1}-6z^{-2}\right) = X(z)\left(1-z^{-1}\right)$$
$$Y(z)+z^{-1}Y(z)-6z^{-2}Y(z) = X(z)-z^{-1}X(z)$$
$$y[n]+y[n-1]-6y[n-2] = x[n]-x[n-1]$$

   (b) Find the impulse response of this system.

$$H(z) = \frac{1-z^{-1}}{(1+3z^{-1})(1-2z^{-1})}$$
$$= \frac{A}{1+3z^{-1}} + \frac{B}{1-2z^{-1}}$$
$$1-z^{-1} = A\left(1-2z^{-1}\right) + B\left(1+3z^{-1}\right)$$

   For $z^{-1} = \frac{1}{2}$:

$$1-\frac{1}{2} = A\left(1-2\left(\frac{1}{2}\right)\right) + B\left(1+3\left(\frac{1}{2}\right)\right)$$
$$\frac{1}{2} = A(0) + B\left(\frac{5}{2}\right)$$
$$B = \frac{1}{5}$$

For $z^{-1} = -\frac{1}{3}$:

$$\begin{aligned}
1 - \left(-\frac{1}{3}\right) &= A\left(1 - 2\left(-\frac{1}{3}\right)\right) + B\left(1 + 3\left(-\frac{1}{3}\right)\right) \\
\frac{4}{3} &= A\left(\frac{5}{3}\right) + B(0) \\
A &= \frac{4}{5}
\end{aligned}$$

Returning to $H(z)$:

$$\begin{aligned}
H(z) &= \frac{4}{5}\frac{1}{1 + 3z^{-1}} + \frac{1}{5}\frac{1}{1 - 2z^{-1}} \\
h[n] &= \frac{4}{5}(-3)^n u[n] + \frac{1}{5}(2)^n u[n]
\end{aligned}$$

2. (4 pts) Determine in each case whether or not the system is BIBO stable. Justify your answer.

   (a) A causal system described by the difference equation $y[n] - 0.9y[n-2] = 90x[n] + x[n-1]$.

   $$\begin{aligned}
   Y(z) - 0.9z^{-2}Y(z) &= 90X(z) + z^{-1}X(z) \\
   Y(z)\left(1 - 0.9z^{-2}\right) &= X(z)\left(90 + z^{-1}\right) \\
   H(z) = \frac{Y(z)}{X(x)} &= \frac{90 + z^{-1}}{1 - 0.9z^{-2}}
   \end{aligned}$$

   The poles are at:

   $$\begin{aligned}
   z^2 &= 0.9 \\
   z &= \pm\sqrt{0.9}
   \end{aligned}$$

   Because the system is causal, the ROC is: $|z| > \sqrt{0.9}$. The ROC includes the unit circle, so the system is stable.

   (b) A system described by the equation $y[n] = h[n] * x[n]$, with $h[n] = (-1)^n u[n]$.
   From the definition, it clear this is an LTI system with impulse response $h[n]$. Therefore, BIBO stability can determined by the summability of $h[n]$ or the location of the poles of $H(z)$.

   Summability of $h[n]$:

   $$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{\infty} |(-1)^n u[n]| = \sum_{n=0}^{n=\infty} |(-1)^n| = \sum_{n=0}^{n=\infty} 1 = \infty$$

   The system is unstable because the impulse response is not absolutely summable.

Poles of $H(z)$:

$$H(z) = \frac{1}{1 + z^{-1}}$$

The pole is at $z = -1$. Because $h[n] = 0$ for $n < 0$, the system is causal and the ROC is: $|z| > 1$. Because the ROC does not include the unit circle, the system is not stable.