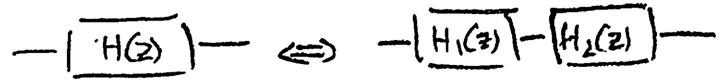
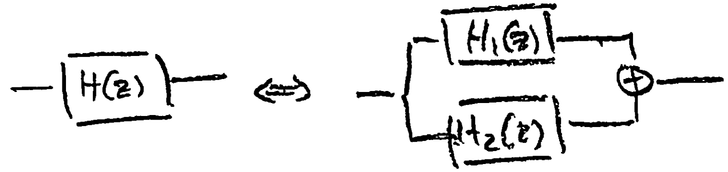
Additional structures:

- Cascade structures $H(z) = H_1(z) \cdot H_2(z)$



- Parallel structures $H(z) = H_1(z) + H_2(z)$

partial fraction expansion



Ex of cascade structure:

$$H(z) = \frac{b_0 + \dots + b_4 z^{-4}}{1 + \dots + a_4 z^{-4}}$$ where a_i, b_i are real

$$= \frac{b_0 \prod_{i=1}^{4} (1 - z_i z^{-1})}{\prod_{i=1}^{4} (1 - p_i z^{-1})}$$ complex zeros/poles come in conjugate pairs

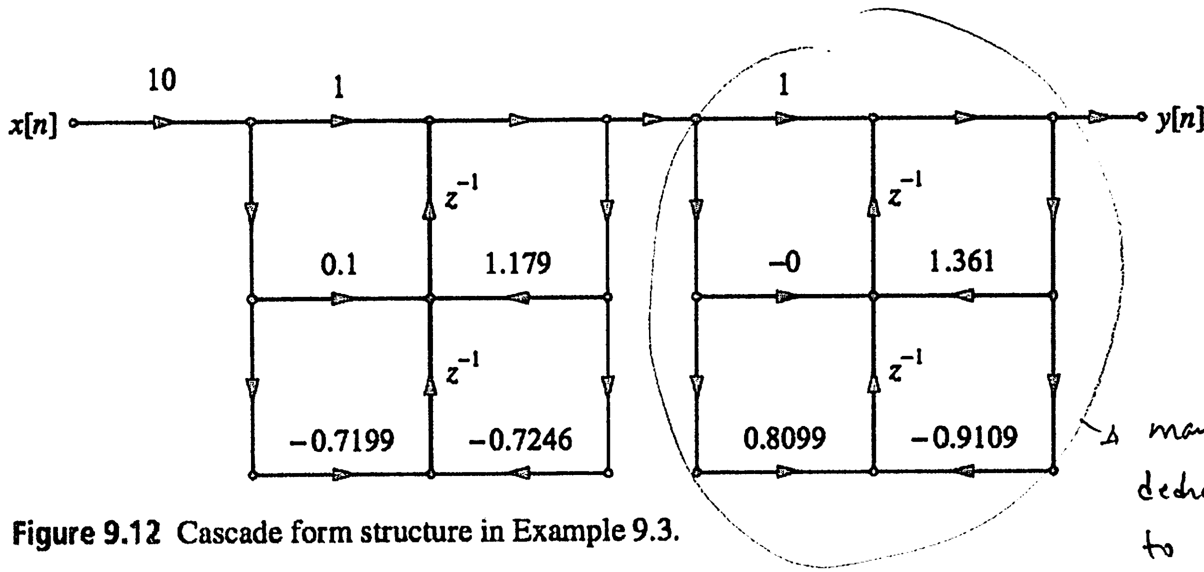
$$= b_0 \frac{(1 - z_1 z^{-1})(1 - z_2 z^{-1})}{(1 - p_1 z^{-1})(1 - p_2 z^{-1})} \cdot \frac{(1 - z_3 z^{-1})(1 - z_4 z^{-1})}{(1 - p_3 z^{-1})(1 - p_4 z^{-1})}$$ we pair complex conjugates

complex conjugates

$$= b_0 \cdot \frac{B_1(z)}{A_1(z)} \cdot \frac{B_2(z)}{A_2(z)}$$ all coefficients are real

$$H(z) = \frac{10 + z^{-1} + 0.9z^{-2} + 0.8z^{-3} - 5.8z^{-4}}{1 - 2.54z^{-1} + 3.24z^{-2} - 2.06z^{-3} + 0.66z^{-4}} = H_1(z) \cdot H_2(z)$$

with real coefficients



may have dedicated hardware to compute an order-2 filter

Figure 9.12 Cascade form structure in Example 9.3.

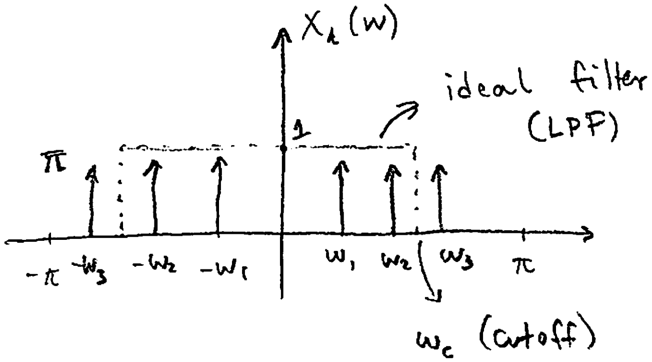
Lecture 20: From ideal to practical filters (Generalized Linear Phase)

$$\{x[n]\} \rightarrow \boxed{h} \rightarrow \{y[n]\}$$

Typical application: Filter out frequencies in <u>stop band</u> and let the frequencies in the <u>pass band</u> pass unaltered

Ex: $x[n] = \cos(\omega_1 n) + \cos(\omega_2 n) + \cos(\omega_3 n)$, $\omega_1 < \omega_2 < \omega_3$

Suppose I want to remove $\cos(\omega_3 n)$ (it's noise)

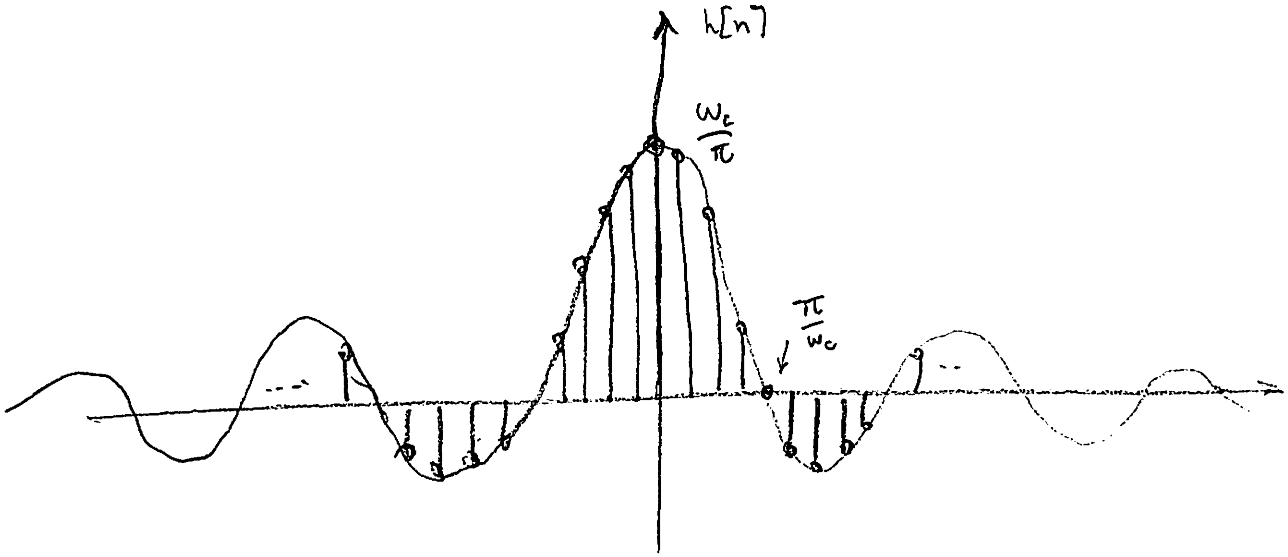


$$H_d(\omega) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \text{otherwise} \end{cases}$$

$\angle H_d(\omega) = 0$ for all ω

Impulse response of ideal LPF :

$$h[n] = DTFT^{-1}\{H_d(\omega)\} = \frac{\sin(\omega_c n)}{\pi n}$$



Issues :

- Extends to $\pm\infty$
- It's non-causal.

$\Rightarrow$ impractical

Possible solution : shift and truncate

What happens when we shift $h[n]$? Output is delayed.

Let us relax the requirement and accept delayed outputs :

$$x[n] = \cos(w_1 n) + \cos(w_2 n) + \underbrace{\cos(w_3 n)}_{\text{"noise"}} \rightarrow \overset{\text{(real)}}{\boxed{h}} \rightarrow y[n] = \cos(w_1(n-n_d)) + \cos(w_2(n-n_d))$$

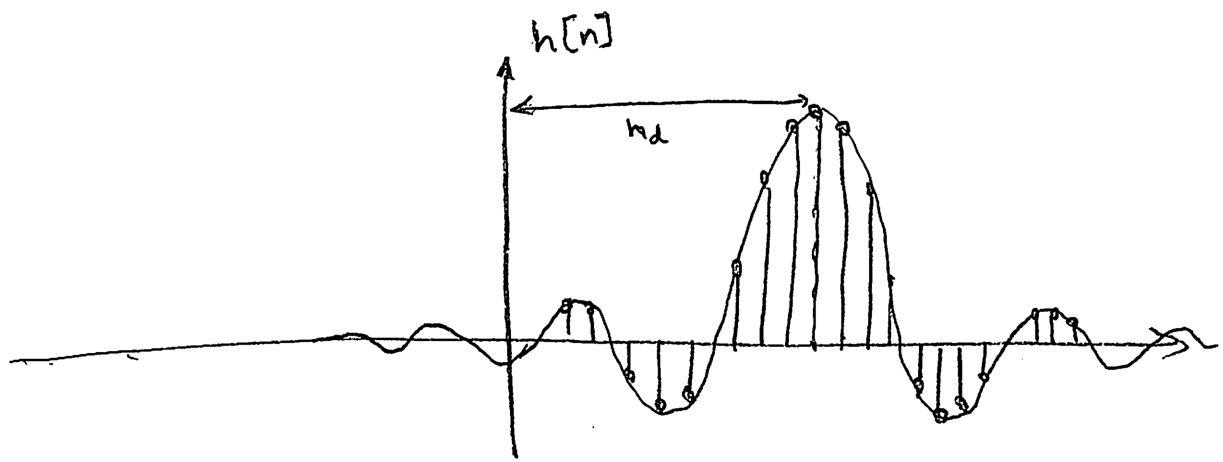
$$= \cos(w_1 n - w_1 n_d) + \cos(w_2 n - w_2 n_d)$$

$$\angle H_d(w_1) = -w_1 n_d$$

$$\angle H_d(w_2) = -w_2 n_d$$

In general, $\boxed{\angle H_d(w) = -w\, n_d}$ $\rightarrow$ linear phase (LP)

$$\Rightarrow \quad H_d(w) = \begin{cases} 1 \cdot e^{-jwn_d} & |w| < w_c \\ 0 & \text{otherwise} \end{cases}$$



In general, linear phase is desirable (for most filters)

<u>Linear Phase</u> $\quad H_d(\omega) = \underbrace{|H_d(\omega)|}_{\geq 0} e^{j(-\alpha\omega)}$

However, LP is hard to impose in filter design.

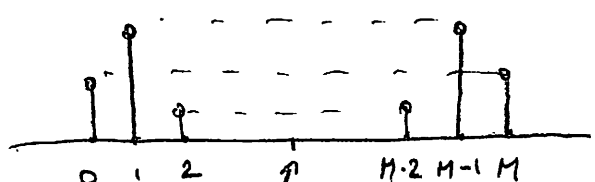
A weaker requirement:

Generalized linear phase: $\quad H_d(\omega) = \underbrace{A(\omega)}_{\text{real}} e^{j(\alpha\omega+\beta)} \quad \alpha, \beta$ are constants

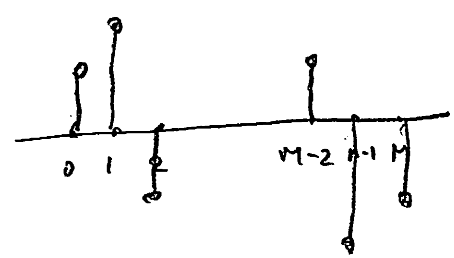
For FIR filters, GLP is attained if $\{h[n]\}_{n=0}^{M}$ is $\begin{cases} \text{symmetric, or} \\ \text{anti-symmetric} \end{cases}$

symmetric: $h[n] = h[M-n]$
(even symmetry)

anti-symmetric: $h[n] = -h[M-n]$
(odd symmetry)



midpoint $\frac{M}{2}$ (not necessarily a sample)



Ex : $M=2$: $\{h[n]\}_{n=0}^{2} = \{1, 3, 1\}$ (even symmetry)

$$H_d(w) = \sum_{n=0}^{2} h[n] e^{-jwn} = e^0 + 3e^{-jw} + 1e^{-j(2w)}$$

$$= e^{-jw}\left(e^{jw} + 3 + e^{-jw}\right) = \underbrace{(3+2\cos(w))}_{A(w)} e^{-jw}$$

linear phase and GLP

Ex 2 : Anti-symmetric : $\{h[n]\}_{n=0}^{3} = \{1, 2, -2, -1\}$

$$H_d(w) = e^0 + 2e^{-jw} - 2e^{-j(2w)} - e^{-j(3w)} = e^{-j\frac{3}{2}w}\left(e^{j\frac{3}{2}w} + 2e^{j\frac{w}{2}} - 2e^{-j\frac{w}{2}} - e^{-j\frac{3}{2}w}\right)$$

Type 4

$$= \left(2j\sin\left(\tfrac{3}{2}w\right) + 4j\sin\left(\tfrac{w}{2}\right)\right) e^{-j\frac{3}{2}w} = \underbrace{\left(2\sin\left(\tfrac{3}{2}w\right) + 4\sin\left(\tfrac{w}{2}\right)\right)}_{A(w)} e^{-j\left(-\frac{3}{2}w + \frac{\pi}{2}\right)}$$

GLP

In general, for
- symmetric, $H_d(w) = A(w)\, e^{-j\frac{M}{2}w}$
- anti-symmetric. $H_d(w) = A(w)\, e^{j\left(-\frac{M}{2}w + \frac{\pi}{2}\right)}$

We end up with 4 types of GLP FIR filters.

Table 10.1 Properties of impulse response sequence $h[n]$ and frequency response function $H(e^{j\omega}) = A(e^{j\omega})e^{j\Psi(e^{j\omega})}$ of FIR filters with linear phase.

Type	$h[k]$	M	$A(e^{j\omega})$	$A(e^{j\omega})$	$\Psi(e^{j\omega})$
I	even (symmetric)	even	$\sum_{k=0}^{M/2} a[k]\cos\omega k$	even–no restriction	$-\frac{\omega M}{2}$
II	even	odd	$\sum_{k=1}^{\frac{M+1}{2}} b[k]\cos\left[\omega\left(k-\frac{1}{2}\right)\right]$	even $A(e^{j\pi}) = 0$	$-\frac{\omega M}{2}$
III	odd (anti-symmetric)	even	$\sum_{k=1}^{M/2} c[k]\sin\omega k$	odd $A(e^{j0}) = 0$ $A(e^{j\pi}) = 0$	$\frac{\pi}{2} - \frac{\omega M}{2}$
IV	odd	odd	$\sum_{k=1}^{\frac{M+1}{2}} d[k]\sin\left[\omega\left(k-\frac{1}{2}\right)\right]$	odd $A(e^{j0}) = 0$ not suitable for LPF	$\frac{\pi}{2} - \frac{\omega M}{2}$