ECE 310: Problem Set 4 Solution

Due: 5 pm, September 28, 2018

Problem 1 (20 pts)

$x[n] = 2^{-n}u[n-2]$ and $v[n] = (1+3^{-n})u[n]$

$$\begin{aligned}
x[n] &= 2^{-n}u[n-2] \\
&= \left(\frac{1}{2}\right)^n u[n-2] \\
&= \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{n-2} u[n-2] \\
&\text{Recall } a^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z}{z-a} \quad \text{ROC} : |z| > |a| \quad \text{and} \quad x[n-k] \overset{\mathcal{Z}}{\longleftrightarrow} z^{-k}X(z) \\
X(z) &= \frac{1}{4}z^{-2}\frac{z}{z-\frac{1}{2}} \quad \text{ROC} : |z| > \frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
v[n] &= (1+3^{-n})u[n] \\
&= u[n] + \left(\frac{1}{3}\right)^n u[n] \\
V(z) &= \frac{z}{z-1} + \frac{z}{z-\frac{1}{3}} \quad \text{ROC} : |z| > 1
\end{aligned}$$

$$\begin{aligned}
X(z)V(z) &= \left(\frac{1}{4}z^{-2}\frac{z}{z-\frac{1}{2}}\right)\left(\frac{z}{z-1} + \frac{z}{z-\frac{1}{3}}\right) \\
&= \frac{1}{4}\left(\frac{1}{z-\frac{1}{2}}\right)\left(\frac{1}{z-1} + \frac{1}{z-\frac{1}{3}}\right) \\
&= \frac{1}{4}\left(\frac{4}{z-\frac{1}{2}} + \frac{2}{z-1} - \frac{6}{z-\frac{1}{3}}\right) \quad \text{by Part. Fraction Expansion} \\
&= \frac{1}{4}z^{-1}\left(\frac{z}{z-\frac{1}{2}} + \frac{1}{2}\frac{z}{z-1} - \frac{3}{2}\frac{z}{z-\frac{1}{3}}\right) \quad \text{ROC} : |z| > 1
\end{aligned}$$

$$x[n] * v[n] = \boxed{\left(\frac{1}{2}\right)^{n-1} u[n-1] + \frac{1}{2}u[n-1] - \frac{3}{2}\left(\frac{1}{3}\right)^{n-1} u[n-1]}$$

Grading:
+5 for correct $X(z)$
+5 for correct $H(z)$
+6 for valid PFD for $X(z)V(z)$
+4 for correct solution $x[n] * v[n]$
Half credit for any of the above criteria if the student had the right approach with small errors.

Problem 2 (20 pts)

$$y[n] - \frac{5}{6}y[n-1] + \frac{1}{6}y[n-2] = x[n]$$

(a) Use the z-transform to find the impulse response, h[n], of this system. *(10 pts)*

Transform the causal system into the z-domain:

$$Y(z) - \frac{5}{6}z^{-1}Y(z) + \frac{1}{6}z^{-2}Y(z) = X(z)$$

$$\begin{aligned}
H(z) &= \frac{Y(z)}{X(z)} \\
&= \frac{1}{1 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}} \\
&= \frac{z^2}{z^2 - \frac{5}{6}z + \frac{1}{6}} \\
&= \frac{z^2}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{3}\right)} \\
\frac{H(z)}{z} &= \frac{z}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{3}\right)} \\
&= \frac{3}{z - \frac{1}{2}} - \frac{2}{z - \frac{1}{3}} \text{by Part. Fraction Expansion} \\
H(z) &= \frac{3z}{z - \frac{1}{2}} - \frac{2z}{z - \frac{1}{3}} \quad \text{ROC} : |z| > \frac{1}{2}
\end{aligned}$$

Note that the ROC is $|z| > \frac{1}{2}$ because the system is causal. If the system is causal, the ROC must lie on the external region of the circle.

$$h[n] = \boxed{3\left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{3}\right)^n u[n]}$$

Grading:
+2 for correct transformation into z-domain
+4 for correct $H(z)$
+2 for correct ROC
+2 for correct $h[n]$
Half credit for any of the above criteria if the student had the right approach with small errors.

(b) Use the z-transform to find the output of this system for the input $x[n] = 3^{-n}u[n]$ when the system is initially at rest, i.e. zero initial conditions. *(10 pts)*

$x[n] = \left(\frac{1}{3}\right)^n u[n] \xleftrightarrow{\mathcal{Z}} X(z) = \frac{z}{z-\frac{1}{3}} \quad \text{ROC} : |z| > \frac{1}{3}$

$$\begin{aligned}
X(z)H(z) &= \frac{z}{z-\frac{1}{3}}\left(\frac{3z}{z-\frac{1}{2}} - \frac{2z}{z-\frac{1}{3}}\right) \\
&= 3z\frac{z}{\left(z-\frac{1}{3}\right)\left(z-\frac{1}{2}\right)} - 2z\frac{z}{\left(z-\frac{1}{3}\right)^2} \\
&= 3z\left(\frac{3}{z-\frac{1}{2}} - \frac{2}{z-\frac{1}{3}}\right) - 2z\frac{z}{\left(z-\frac{1}{3}\right)^2} \quad \text{by Part. Fraction Expansion} \\
&= \frac{9z}{z-\frac{1}{2}} - \frac{6z}{z-\frac{1}{3}} - 6z\frac{\frac{1}{3}z}{\left(z-\frac{1}{3}\right)^2} \quad \text{ROC} : |z| > \frac{1}{2}
\end{aligned}$$

Recall $\frac{az}{(z-a)^2} \xleftrightarrow{\mathcal{Z}} na^n u[n] \quad \text{ROC} : |z| > |a|$

$$y[n] = x[n] * h[n] = \boxed{9\left(\frac{1}{2}\right)^n u[n] - 6\left(\frac{1}{3}\right)^n u[n] - 6(n+1)\left(\frac{1}{3}\right)^{n+1} u[n+1]}$$

Grading:
+2 for correct $X(z)$
+5 for correct $Y(z)$
+1 for correct ROC
+2 for correct $y[n]$
Half credit for any of the above criteria if the student had the right approach with small errors.

Problem 3 (20 pts)

(a) $y[n] = x^3[n] + n^2 e^{-0.01n}$ *(5 pts)*

If $x[n]$ is bounded, then $|x[n]| \leq M_x < \infty$
If $y[n]$ is bounded, then $|y[n]| \leq M_y < \infty$

Given a bounded input $x[n]$ where $|x[n]| \leq M_x$, then $|y[n]| \leq |M_x^3 + n^2 e^{-0.01n}|$
The output is not bounded for all n, particularly as n approaches $-\infty$. The second term will approach infinity. Therefore, the system is **not BIBO stable**.

Grading:
+1 for representing $x[n]$ as a bounded input
+1 for representing $y[n]$ in terms of the bounded input
+2 for showing that second term will not be bounded as $n \to -\infty$.
+1 for putting down not BIBO stable
Half credit for the last criterion if they answer BIBO stable for only looking at the case where $n \to \infty$.

Written explanation of any criteria above is acceptable.

(b) $y[n] = \tan(x[n])$ *(5 pts)*

Let $x[n] = \frac{\pi}{2}$ which is a bounded input for all n. This will cause $y[n]$ to be unbounded since:

$$\lim_{n \to \frac{\pi}{2}} \tan(n) = \infty$$

Therefore, the system is $\boxed{\textbf{not BIBO stable}}$.

Grading:
+1 for representing $x[n]$ as a bounded input
+2 for showing $y[n]$ will approach ∞ given bounded input
+2 for putting down BIBO stable
Written explanation of any criteria above is acceptable.

(c) $y[n] = n\cos(x[n])$ *(5 pts)*

Let $x[n] = 0$ which is a bounded input for all n. Then $y[n] = n\cos(0) = n$. The output is unbounded since $y[n]$ will approach infinity as n approaches infinity. Therefore, the system is $\boxed{\textbf{not BIBO stable}}$.

Grading:
+1 for representing $x[n]$ as a bounded input
+2 for showing $y[n]$ is not bounded given bounded input
+2 for putting down not BIBO stable
Written explanation of any criteria above is acceptable.

(d) $y[n] = h[n] * x[n]$ where $h[n] = \begin{cases} 0 & \text{if } n < 0 \\ 100^{100} & 0 \le n \le 10^{10} \\ n(0.99)^n & 10^{10} < n < \infty \end{cases}$ *(5 pts)*

A convolution is an LTI system. An LTI system with impulse response $h[n]$ is BIBO stable if and only if the impulse response is absolutely summable, that is, if:

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

It is clear that for cases $n < 0$ and $0 \le n \le 10^{10}$, $h[n]$ is absolutely summable. The same holds for $10^{10} < n < \infty$ since:

$$\sum_{n=10^{10}+1}^{\infty} n(0.99)^n \quad \text{converges}$$

This is because $(0.99)^n$ goes to 0 faster than n goes to infinity. Therefore, the system is $\boxed{\textbf{BIBO stable}}$.

Grading:
+1 for explicitly or intuitively recognizing system is LTI or $h[n]$ is absolutely summable
+1 for expressing that $h[n]$ is absolutely summable in the first two cases of piecewise.
+2 for recognizing that the last case converges as $n \to \infty$
+1 for putting down BIBO stable
Written explanation of any criteria above is acceptable.

Problem 4 (20 pts)

(a) $H(z) = \frac{z+100}{z^2+1/2}$ *(5 pts)*

$H(z) = \frac{z+100}{\left(z-\frac{j}{\sqrt{2}}\right)\left(z+\frac{j}{\sqrt{2}}\right)} \quad \text{ROC} : |z| > \frac{1}{\sqrt{2}}$

Because the ROC contains the unit circle ($|z|$ contains 1), the system is $\boxed{\textbf{BIBO stable}}$.

Grading:
+2 for finding ROC
+3 for BIBO stable because ROC contains unit circle

(b) $H(z) = \frac{z+100}{z^2-1.8z+0.8}$ *(5 pts)*

$H(z) = \frac{z+100}{\left(z-\frac{4}{5}\right)\left(z-1\right)} \quad \text{ROC} : |z| > 1$

Because the ROC does not contain the unit circle, the system is $\boxed{\textbf{not BIBO stable}}$.

Let $x[n] = u[n] \xleftrightarrow{\mathcal{Z}} X(z) = \frac{z}{z-1}$
$Y(z) = X(z)H(z) = \frac{z}{z-1}\frac{z+100}{\left(z-\frac{4}{5}\right)\left(z-1\right)} = z\frac{z+100}{\left(z-\frac{4}{5}\right)\left(z-1\right)^2} = z(z+100)\left(\frac{25}{z-\frac{4}{5}} - \frac{25}{z-1} + \frac{5}{(z-1)^2}\right)$

With $Y(z)$ above, we can see that one of the terms will be $\frac{500z}{(z-1)^2}$ after expanding the equation. the inverse z-transform of this term will be $500nu[n]$ which is unbounded.

Grading:
+1 for finding ROC
+1 for not BIBO stable because ROC does not contain unit circle
+2 for example of bounded input that will give unbounded output
+1 for explaining that the input will give an unbounded output (explicitly solving or by stating "double pole" property)

(c) $H(z) = \frac{z+100}{z-4}$ *(5 pts)*

ROC: $|z| > 4$. Because the ROC does not contain the unit circle, the system is $\boxed{\textbf{not BIBO stable}}$.

Let $x[n] = \delta[n]$.
$Y(z) = X(z)H(z) = H(z) = \frac{z+100}{z-4} = \frac{z}{z-4} + \frac{100}{z-4}$

$y[n] = 4^n u[n] + (100)4^{n-1}u[n-1]$ which is unbounded.

Grading:
+1 for finding ROC
+1 for not BIBO stable because ROC does not contain unit circle
+2 for example of bounded input that will give unbounded output
+1 for explaining that the input will give an unbounded output (explicitly solving or by stating "double pole" property)

(d) $H(z) = \frac{z-0.5}{z^3+j}$ *(5 pts)*

Looking at the denominator of $H(z)$, it can be seen that there will be 3 roots that make up the poles of the system. One of these roots must be j. This tells us that the lower bound of the ROC is at least $|z| > 1$. Because the ROC does not contain the unit circle, the system is $\boxed{\textbf{not BIBO stable}}$.

Because there is a pole at $z = j$ for $H(z)$, a bounded input $x[n] = \cos\left(\frac{\pi}{2}n\right)u[n]$ will give an unbounded output. This is because:

Recall $\cos\left(\omega_0 n\right)u[n] \xleftrightarrow{\mathcal{Z}} \frac{z^2-\cos(\omega_0)z}{z^2-2cos(\omega_0 n)z+1}$ ROC : $|z| > 1$

$$\begin{aligned}
X(z) &= \frac{z^2 - \cos\left(\frac{\pi}{2}\right)z}{z^2 - 2\cos\left(\frac{\pi}{2}\right)z + 1} \\
&= \frac{z^2 - \cos\left(\frac{\pi}{2}\right)z}{z^2+1} \\
&= \frac{z^2 - \cos\left(\frac{\pi}{2}\right)z}{(z+j)(z-j)}
\end{aligned}$$

Because $X(z)$ and $H(z)$ both have poles at $z = j$, the output $y[n]$ will be unbounded.

Grading:
+1 for finding ROC
+1 for not BIBO stable because ROC does not contain unit circle
+2 for example of bounded input that will give unbounded output
+1 for explaining that the input will give an unbounded output (explicitly solving or by stating "double pole" property)

Problem 5 (20 pts)

$y[n] = 2^{-n}u[n]$ and $x[n] = 3^{-n}(0.5u[n] - u[n-1])$

(a) Find the impulse response $h[n]$ of the system. Is the impulse response unique? *(8 pts)*

$x[n] = \left(\frac{1}{3}\right)^n (0.5u[n] - u[n-1]) = 0.5\left(\frac{1}{3}\right)^n u[n] - \frac{1}{3}\left(\frac{1}{3}\right)^{n-1} u[n-1]$
$y[n] = \left(\frac{1}{2}\right)^n u[n]$

$$\begin{aligned} X(z) &= \frac{1}{2}\frac{z}{z-\frac{1}{3}} - \frac{1}{3}z^{-1}\frac{z}{z-\frac{1}{3}} \\ &= \frac{1}{2}\frac{z-\frac{2}{3}}{z-\frac{1}{3}} \quad \text{ROC} : |z| > \frac{1}{3} \end{aligned}$$

$$Y(z) = \frac{z}{z-\frac{1}{2}} \quad \text{ROC} : |z| > \frac{1}{2}$$

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} \\ &= 2\left(\frac{z}{z-\frac{1}{2}}\right)\left(\frac{z-\frac{1}{3}}{z-\frac{2}{3}}\right) \\ &= 2\frac{z\left(z-\frac{1}{3}\right)}{\left(z-\frac{1}{2}\right)\left(z-\frac{2}{3}\right)} \\ &= \frac{4z}{z-\frac{2}{3}} - \frac{2z}{z-\frac{1}{2}} \quad \text{ROC} : |z| > \frac{2}{3} \end{aligned}$$

$$h[n] = \boxed{4\left(\frac{2}{3}\right)^n u[n] - 2\left(\frac{1}{2}\right)^n u[n]}$$

Because the system is given as causal, the ROC of the z-transform is definitive (ROC: $|z| > \frac{2}{3}$), making the impulse response $\boxed{\textbf{is unique}}$. If it was not given as causal, the ROC for the system could also be anti-causal (ROC: $|z| < \frac{1}{2}$), or have both causal and anti-causal components (ROC: $\frac{1}{3} < |z| < \frac{1}{2}$), making the impulse response not unique.

Grading:
+1 for correct $X(z)$
+1 for correct $Y(z)$
+2 for valid PFD for $H(z)$
+2 for correct $h[n]$
+2 for unique
Half credit for any of the above criteria if the student had the right approach with small errors.

(b) Determine whether or not the system is BIBO stable. Where are the poles of the transfer function located? *(6 pts)*

Because the ROC of $H(z)$ contains the unit circle, the system is $\boxed{\textbf{BIBO stable}}$

Poles of the transfer function are located at $\boxed{z = \frac{1}{2}, z = \frac{2}{3}}$

Grading:
+2 for BIBO stable
+2 for correct pole at $\frac{1}{2}$
+2 for correct pole at $\frac{2}{3}$

(c) Find the difference equation that characterizes this system. *(6 pts)*

$$
\begin{aligned}
H(z) &= 2\frac{z\left(z-\frac{1}{3}\right)}{\left(z-\frac{1}{2}\right)\left(z-\frac{2}{3}\right)} \\
&= 2\frac{z^2-\frac{1}{3}z}{z^2-\frac{7}{6}z+\frac{1}{3}} \\
&= 2\frac{1-\frac{1}{3}z^{-1}}{1-\frac{7}{6}z^{-1}+\frac{1}{3}z^{-2}} \\
&= \frac{Y(z)}{X(z)}
\end{aligned}
$$

$$Y(z)\left(1-\frac{7}{6}z^{-1}+\frac{1}{3}z^{-2}\right) = 2X(z)\left(1-\frac{1}{3}z^{-1}\right)$$

$$\boxed{y[n]-\frac{7}{6}y[n-1]+\frac{1}{3}y[n-2] = 2x[n]-\frac{2}{3}x[n-1]}$$

Grading:
+2 for representing $H(z)$ as $\frac{Y(z)}{X(z)}$
+2 for separating $Y(z)$ and $X(z)$ onto two sides
+2 for correct solution