Lecture 14

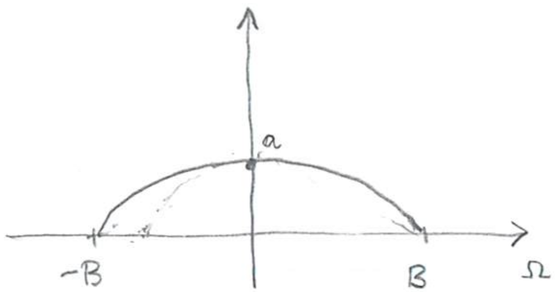
Effect of sampling in frequency domain

$$x_c(t) \xrightarrow[T]{\quad / \quad} x[n] = x_c(nT)$$

CTFT: $x_c(t) \longleftrightarrow X_c(\Omega)$

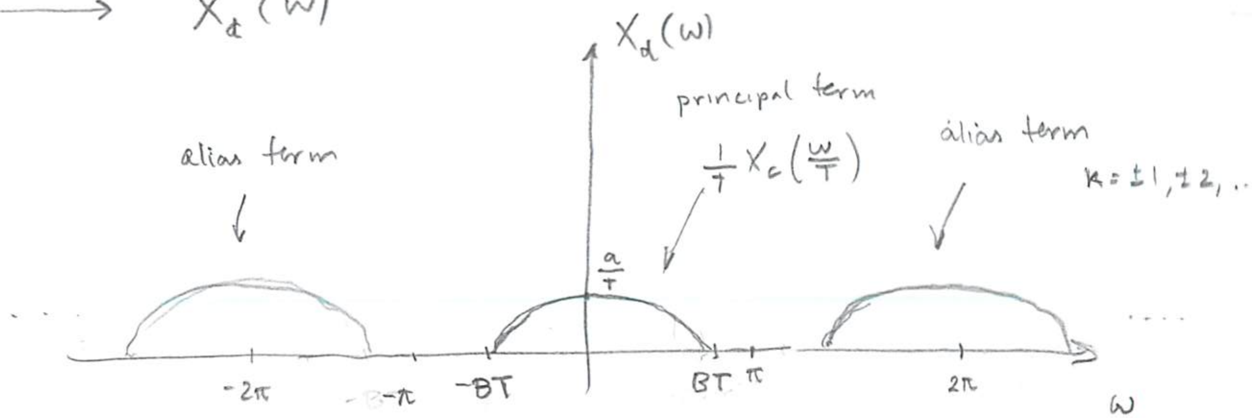
DTFT: $x[n] \longleftrightarrow X_d(\omega)$

$$X_c(\Omega) \xrightarrow{\ \boxed{X_d(\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(\frac{\omega + 2k\pi}{T}\right)}\ } X_d(\omega)$$



bandlimited

$X_c(\Omega) = 0$ for $|\Omega| > B$

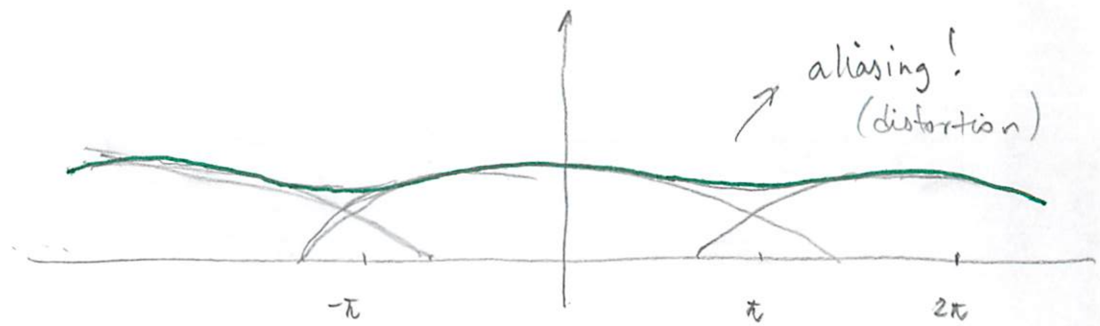


If $BT < \pi$, terms don't interfere

There is no <u>aliasing</u>.

If $BT > \pi$

(If T is too big; i.e., F_s is too small)



Proof of $X_d(\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(\frac{\omega + 2k\pi}{T}\right)$

① From CTFT: $x_c(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_c(\Omega)\, e^{j\Omega t}\, d\Omega$

$$x[n] = x_c(nT) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_c(\Omega)\, e^{j\Omega nT}\, d\Omega$$

must be equal

② From DTFT: $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_d(\omega)\, e^{j\omega n}\, d\omega$

change of variables: $\omega = \Omega T$

$$\int_{-\pi}^{\pi} X_d(\omega)\, e^{j\omega n}\, d\omega = \int_{-\infty}^{\infty} X_c(\Omega)\, e^{j\Omega T n}\, d\Omega = \frac{1}{T}\int_{-\infty}^{\infty} X_c\left(\frac{\omega}{T}\right) e^{j\omega n}\, d\omega$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty}\int_{-\pi+2k\pi}^{\pi+2k\pi} X_c\left(\frac{\omega}{T}\right) e^{j\omega n}\, d\omega =$$

change of variables

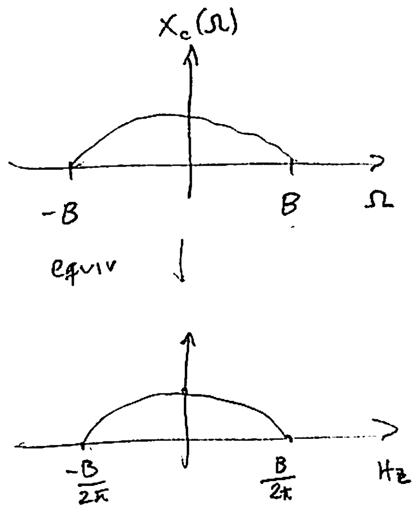
$\omega' = \omega - 2k\pi$

$$= \int_{-\pi}^{\pi} \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(\frac{\omega' + 2k\pi}{T}\right) e^{j(\omega' + \cancel{2k\pi})n}\, d\omega'$$

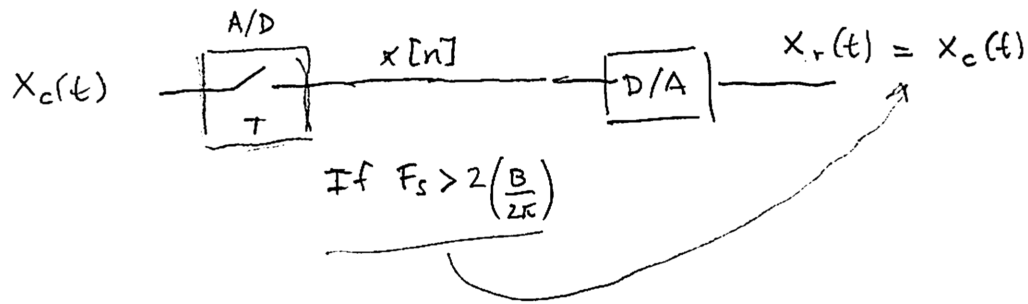
must be equal

If $BT < \pi$, no aliasing happens, and shape of FT is preserved

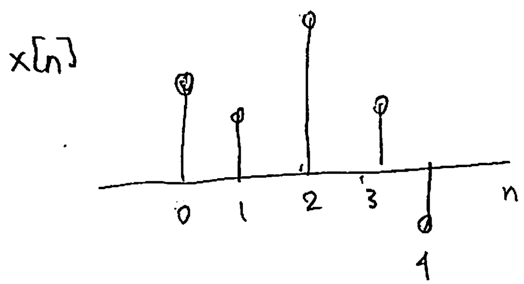
$$BT < \pi \iff F_s = \frac{1}{T} > \frac{B}{\pi} \iff F_s > \underbrace{2\Big(\overbrace{\frac{B}{2\pi}}^{\text{bandwidth of } x_c(t) \text{ in Hz}}\Big)}_{\text{Nyquist rate (2 times largest frequency in the signal)}}$$



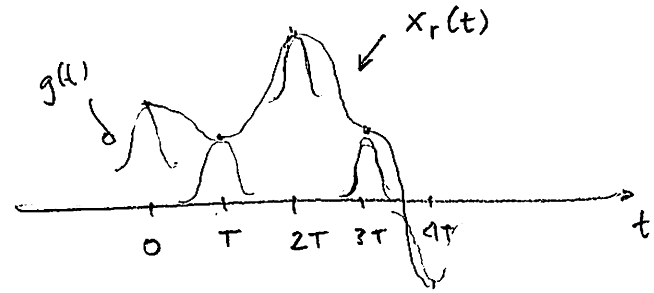
Theorem: If $x_c(t)$ is bandlimited with bandwidth B (rad/s) and if is sampled with $F_s > 2\left(\frac{B}{2\pi}\right)$, then $x_c(t)$ can be perfectly reconstructed from $x[n]$.

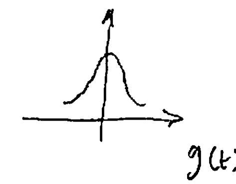


How do you reconstruct $x_c(t)$?



interpolation $\longrightarrow$



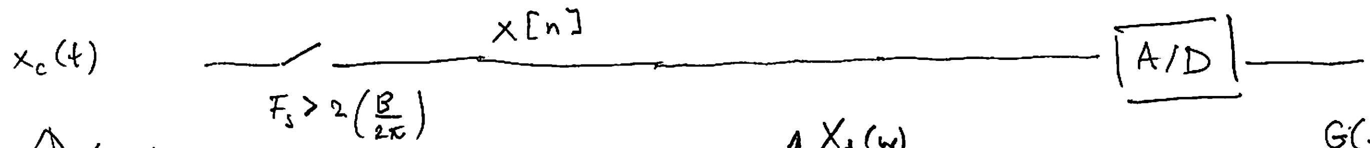


In general, interpolation is given by $x_r(t) = \sum_{n=-\infty}^{\infty} x[n]\, g(t-nT)$

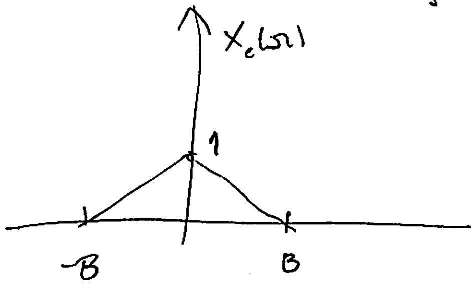
What should $g(t)$ be? Let's take CTFT.

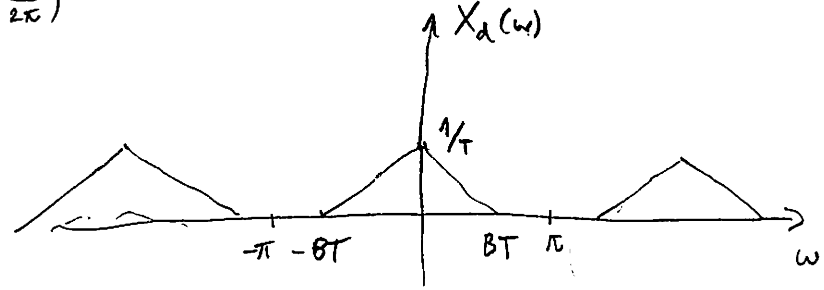
$$\text{CTFT}\{x_r(t)\} = \sum_{n=-\infty}^{\infty} x[n] \cdot \text{CTFT}\{g(t-nT)\} = \sum_{n=-\infty}^{\infty} x[n] \cdot G(\Omega)\, e^{-j\Omega n T}$$

$$= G(\Omega) \cdot \underbrace{\sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega T n}}_{\substack{\text{DTFT of } x[n] \\ \text{for } w = \Omega T}} := G(\Omega)\, X_d(\Omega T)$$

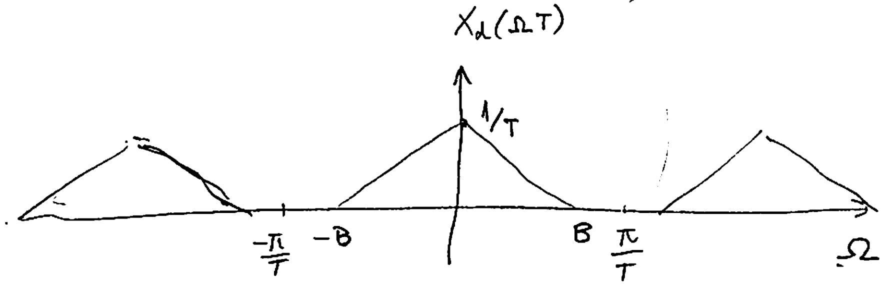


$$\underline{G(\Omega)\, X_d(\Omega T)} = X_c(\Omega)$$

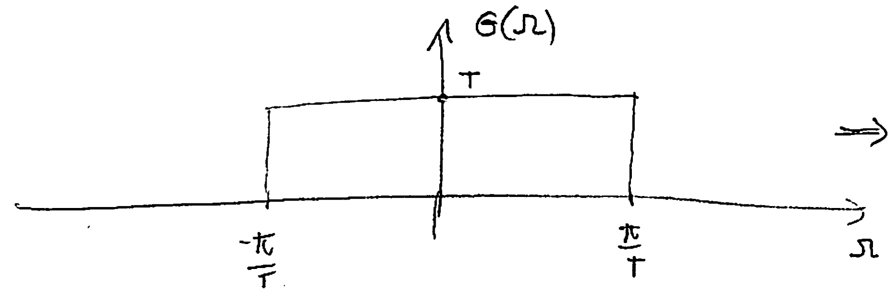




$$\omega = \Omega T, \quad \boxed{\Omega = \frac{\omega}{T}}$$



$\otimes$



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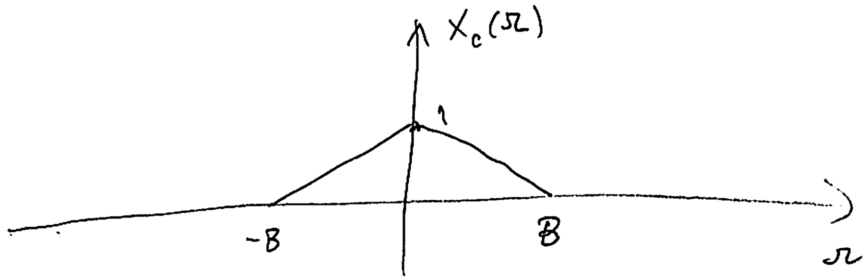

$\Longrightarrow$ Therefore,

$$g(t) = CTFT^{-1}\{G(\Omega)\}$$

$$= \underbrace{\mathrm{sinc}\left(\frac{\pi t}{T}\right)}_{\text{ideal interpolating function!}}$$