ECE 310: Recitation 6

October 7, 2018

Recall the **eigensequence** property of LSI systems (characterized by $h[n]$):

$$x[n] = e^{j\omega_0 n} \rightarrow \boxed{H_d(\omega)} \rightarrow y[n] = x[n]*h[n] = \sum_{k=-\infty}^{\infty} e^{j\omega_0(n-k)}h[k] = e^{j\omega_0 n}\sum_{k=-\infty}^{\infty} e^{-j\omega_0 k}h[k] = x[n]H_d(\omega_0)$$

This tells us that, for **any** LSI system, if the input is a complex exponential with a certain frequency, the output will be the same complex exponential multiplied by the frequency response evaluated at that frequency.

Real Systems

Now, we also learned that if we have a real system, it satisfies another property:

$$x[n] = \cos(\omega_0 n + \phi) \rightarrow \boxed{H_d(\omega)} \rightarrow y[n] = |H_d(\omega_0)|\cos(\omega_0 n + \phi + \angle H_d(\omega_0))$$

Why is this true? Recall that if we have a real system, it must satisfy the following property:

$$H_d(\omega) = H_d^*(-\omega)$$

This is equivalent to the magnitude being symmetric, and the phase being antisymmetric; that is

$$|H_d(\omega)| = |H_d(-\omega)|$$

$$\angle H_d(\omega) = -H_d(-\omega)$$

How does this relate to the sinusoidal input? We know that, using Euler's formula, we can break the cosine up into two compmlex exponentials and put them both through the system. Doing so gives

$$x[n] = \frac{1}{2}e^{j\phi}e^{j\omega_0 n} + \frac{1}{2}e^{-j\phi}e^{-j\omega_0 n} \rightarrow \boxed{H_d(\omega)} \rightarrow \frac{1}{2}H_d(\omega_0)e^{j\phi}e^{j\omega_0 n} + \frac{1}{2}H_d(-\omega_0)e^{-j\phi}e^{-j\omega_0 n}$$

But $H_d(\omega_0)$ and $H_d(-\omega_0)$ will just be complex numbers. Therefore, they can be written in terms of their magnitude and phase. Furthermore, if the system is real, we can apply the symmetry properties and write

$$H_d(\omega_0) = |H_d(\omega_0)|e^{j\angle H_d(\omega_0)}$$
$$H_d(-\omega_0) = |H_d(\omega_0)|e^{-j\angle H_d(\omega_0)}$$

Therefore, the output becomes

$$y[n] = \frac{1}{2}|H_d(\omega_0)|e^{j\angle H_d(\omega_0)}e^{j\phi}e^{j\omega_0 n} + \frac{1}{2}|H_d(\omega_0)|e^{-j\angle H_d(\omega_0)}e^{-j\phi}e^{-j\omega_0 n} = \boxed{|H_d(\omega_0)|\cos(\omega_0 n + \phi + \angle H_d(\omega_0))}$$

We see that if we put a cosine into the system, we get a cosine back, scaled by the magnitude and with a phase addition. This **only holds if we have a real system** because we exploited the phase antisymmetry and magnitude symmetry. If the system wasn't real, we would be unable to combine the complex exponentials back into our cosine.

Practice Problems

1. The difference equation of a causal LSI system is given by

$$y[n] + \frac{1}{2}y[n-1] = x[n], \qquad -\infty < n < \infty$$

Determine $y[n]$ for input $x[n] = 1 + 2(-1)^n, \qquad -\infty < n < \infty.$

Solution: We need to start by findin gthe impulse response of the system. Taking the DTFT of both sides and applying the shifting property gives:

$$\begin{aligned} Y_d(\omega) + \frac{1}{2}e^{-j\omega}Y_d(\omega) &= X_d(\omega) \\ Y_d(\omega)(1 + \frac{1}{2}e^{-j\omega}) &= X_d(\omega) \\ \rightarrow H_d(\omega) &= \frac{1}{1 + \frac{1}{2}e^{-j\omega}} \end{aligned}$$

It can be seen that the system is real (its described by an LCCDE with soley real coefficients), so for the given input, we can write the output as:

$$y[n] = H_d(0) + 2e^{j\pi n}H_d(\pi)$$

Calculating the DTFT values gives:

$$H_d(0) = \frac{1}{1 + \frac{1}{2}} = \frac{2}{3}$$

$$H_d(\pi) = \frac{1}{1 - \frac{1}{2}} = 2$$

Therefore:

$$\boxed{y[n] = \frac{2}{3} + 4e^{j\pi n} = \frac{2}{3} + 4(-1)^n}$$

2. The response of a real LSI system for input

$$x[n] = 2 + \cos\left(\frac{\pi}{3}n + 15^\circ\right)$$

is

$$y[n] = 1 + 2\cos\left(\frac{\pi}{3}n + 10^\circ\right).$$

Determine the system response $\tilde{y}[n]$ for input

$$\tilde{x}[n] = 1 + \cos\left(\frac{\pi}{3}n + 25^\circ\right)$$

Solution: We are given that the system is real, and we know the output for some input. We can use this to determine specific values of $H_d(\omega)$:

$$2 \to 1, \text{ so } H_d(0) = \frac{1}{2}$$

$$\cos(\frac{\pi}{3}n + 15^\circ) \to 2\cos(\frac{\pi}{3}n + 10), \text{ so } |H_d(\frac{\pi}{3})| = 2, \quad \angle H_d(\frac{\pi}{3}) = -5^\circ$$

Given the new input sequence, we know that the output must be:

$$\tilde{y}[n] = H_d(0) + |H_d(\frac{\pi}{3})| \cos(\frac{\pi}{3}n + 25^\circ + \angle H_d(\frac{\pi}{3}))$$

Therefore:

$$\boxed{\tilde{y}[n] = \frac{1}{2} + 2\cos(\frac{\pi}{3}n + 20^\circ)}$$

3. Please bring up any question regarding the **midterm**.