Lecture 6

Poles and zeros

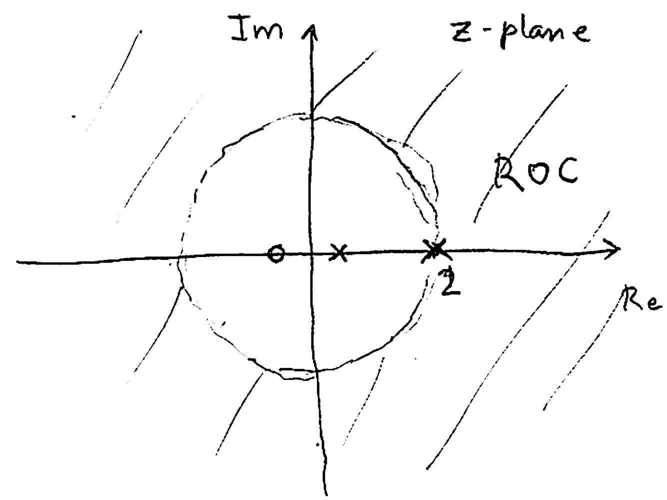
$x[n]$	$X(z)$, ROC
$a^n u[n]$	$\frac{1}{1-az^{-1}}$, $\lvert z\rvert > \lvert a\rvert$

Ex: $x[n] = \underbrace{2\,(0.5)^n u[n]}_{} - \underbrace{5 \cdot 2^n u[n]}_{}$

$$\downarrow \mathcal{Z} \qquad\qquad \downarrow \mathcal{Z}$$

$$X(z) = 2 \cdot \frac{1}{1-0.5z^{-1}} - 5 \cdot \frac{1}{1-2z^{-1}}$$

ROC: $|z| > 2$



$$X(z) = \frac{2(1-2z^{-1}) - 5(1-0.5z^{-1})}{(1-0.5z^{-1})(1-2z^{-1})}$$

rational function

$$= \frac{-3-1.5z^{-1}}{1-2.5z^{-1}+z^{-2}} = \frac{B(z)}{A(z)}$$

polynomials in z^{-1}

zero

$$= \frac{-3\,(1-(-0.5)z^{-1})}{(1-0.5z^{-1})(1-2z^{-1})}$$

poles

In general: $X(z) = \frac{B(z)}{A(z)}$

Recall: Fundamental Theorem of Algebra

For a complex polynomial

n roots of polynomial

deg

$$a_0 + a_1 x + a_2 x^2 + \dots + a_n x^{n} = a_n (x - r_1)(x - r_2) \dots (x - r_n)$$

$$X(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_N z^{-N}} = \frac{z^{-M}(b_0 z^M + b_1 z^{M-1} + b_2 z^{M-2} + \dots + b_M)}{z^{-N}(a_0 z^N + a_1 z^{N-1} + \dots + a_N)}$$

$$= \frac{z^{-M}}{z^{-N}} \cdot \frac{b_0 (z - z_1)(z - z_2) \cdots (z - z_M)}{a_0 (z - p_1)(z - p_2) \cdots (z - p_N)} = \frac{b_0 (1 - z_1 z^{-1})(1 - z_2 z^{-1}) \dots (1 - z_M z^{-1})}{a_0 (1 - p_1 z^{-1})(1 - p_2 z^{-1}) \dots (1 - p_N z^{-1})}$$

Def: $z_1, z_2, \dots, z_M$ (roots of $B(z)$): zeros of $X(z)$

$p_1, p_2, \dots, p_N$ (roots of $A(z)$): poles of $X(z)$

Ex: $X(z) = \dfrac{1+z^{-1}}{2+z^{-2}} = \dfrac{1-(-1)z^{-1}}{2\left(1-\left(j\frac{\sqrt{2}}{2}\right)z^{-1}\right)\left(1-\left(-j\frac{\sqrt{2}}{2}\right)z^{-1}\right)}$

roots of $2+z^{-2}$:

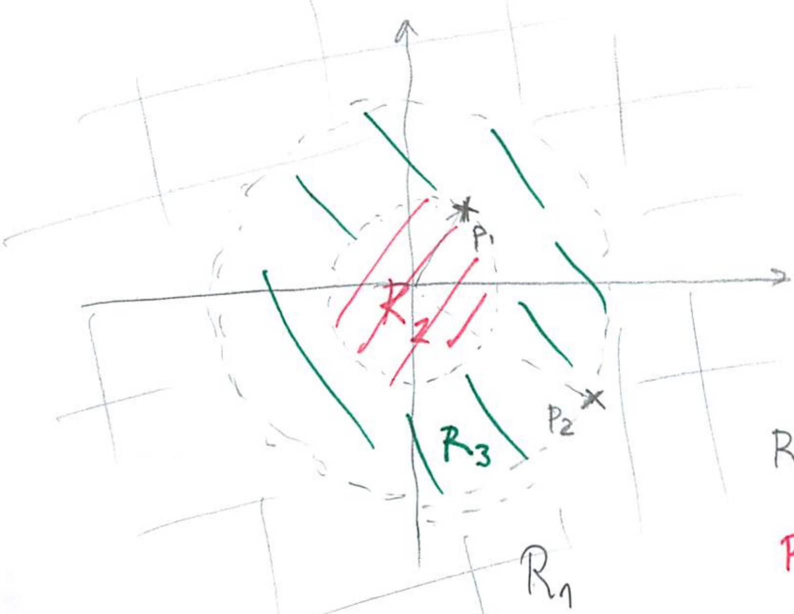
$2+z^{-2}=0 \Rightarrow z^{-2}=-2$

$\Rightarrow z^2 = -\frac{1}{2} \Rightarrow z = \pm j\sqrt{\frac{1}{2}} = \pm j\frac{\sqrt{2}}{2}$

Fact: Poles cannot be inside RoC

If p is a pole of $\dfrac{B(z)}{A(z)}$, $A(p)=0 \Rightarrow \dfrac{B(p)}{A(p)} = \dfrac{*}{0} = \pm\infty$

Suppose $X(z) = \dfrac{B(z)}{A(z)}$ has two poles p_1, p_2 s.t. $0 < |p_1| < |p_2|$



Fact 2: ROC of $X(z)$ can only be one of R_1, R_2 or R_3

R_1: $|z| > |p_2|$

R_2: $|z| < |p_1|$

R_3: $|p_1| < |z| < |p_2|$

Example of inverse z-transform

$x[n]$	$X(z)$, ROC
$a^n u[n]$	$\frac{1}{1-az^{-1}}$, $\lvert z\rvert > \lvert a\rvert$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$, $\lvert z\rvert < \lvert a\rvert$

Ex: $X(z) = \frac{2}{(1-0.5z^{-1})} + \frac{-5}{(1-2z^{-1})}$, ROC: $0.5 < |z| < 2$

$$\Big\downarrow \mathcal{Z}^{-1} \qquad \Big\downarrow \mathcal{Z}^{-1} \qquad \Big\downarrow \mathcal{Z}^{-1}$$

$x[n] = 2 \cdot 0.5^n u[n] + (-5)(-1)(2)^n u[-n-1]$

partial fraction expansion

Ex 2: $X(z) = \frac{-3-1.5z^{-1}}{(1-0.5z^{-1})(1-2z^{-1})} = \boxed{\frac{A_1}{1-0.5z^{-1}} + \frac{A_2}{1-2z^{-1}}}$ ROC: $|z| < 0.5$

$$= \frac{A_1(1-2z^{-1}) + A_2(1-0.5z^{-1})}{(1-0.5z^{-1})(1-2z^{-1})} = \frac{2}{1-0.5z^{-1}} + \frac{-5}{1-2z^{-1}}$$

$$A_1(1-2z^{-1}) + A_2(1-0.5z^{-1}) = -3-1.5z^{-1}$$

trick: set z to the pole values

$z = 0.5$: $A_1\left(1-\frac{2}{0.5}\right) + 0 = -3 - \frac{1.5}{0.5} = -6 \Rightarrow A_1 = 2$

$z = 2$: $0 + A_2\left(1-\frac{0.5}{2}\right) = -3 - \frac{1.5}{2} \Rightarrow A_2 = -5$

$\mathcal{Z}^{-1}$

$x[n] = 2(-1)(0.5)^n u[-n-1]$

$+ (-5)(-1)2^n u[-n-1]$