ECE310: Quiz#4 (6pm Section CSS) Fall 2018 Solutions

1. (6 pts) A causal linear time invariant system has the transfer function $H(z) = \frac{1-z^{-1}}{(1+z^{-1})(1-2z^{-1})}$.

   (a) Determine a difference equation relating the input $x[n]$ to the output $y[n]$ for this system.

   (b) Find the impulse response of this system.

Solution

(a)

$$
\begin{aligned}
H(z) = \frac{1-z^{-1}}{(1+z^{-1})(1-2z^{-1})} &= \frac{Y(z)}{X(z)} \\
Y(z)(1+z^{-1})(1-2z^{-1}) &= X(z)(1-z^{-1}) \\
Y(z) - z^{-1}Y(z) - 2z^{-2}Y(z) &= X(z) - z^{-1}X(z)
\end{aligned}
$$

This gives

$$\boxed{y[n] - y[n-1] - 2y[n-2] = x[n] - x[n-1]}$$

(b)

$$
\begin{aligned}
H(z) &= \frac{1-z^{-1}}{(1+z^{-1})(1-2z^{-1})} \\
&= \frac{A}{1+z^{-1}} + \frac{B}{1-2z^{-1}} \\
A &= \left.\frac{1-z^{-1}}{1-2z^{-1}}\right|_{z=-1} = \frac{1-(-1)}{1-(-2)} = \frac{2}{3} \\
B &= \left.\frac{1-z^{-1}}{1+z^{-1}}\right|_{z=2} = \frac{1-\frac{1}{2}}{1+\frac{1}{2}} = \frac{1}{3} \\
H(z) &= \frac{2}{3}\frac{1}{1+z^{-1}} + \frac{1}{3}\frac{1}{1-2z^{-1}}, \quad \text{ROC:} z > 2
\end{aligned}
$$

This gives

$$\boxed{h[n] = \frac{2}{3}(-1)^n u[n] + \frac{1}{3}2^n u[n]}$$

2. (4 pts) Determine in each case whether or not the system is BIBO stable. Justify your answer.

   (a) A causal system described by the difference equation $y[n]+y[n-2] = x[n]-x[n-1]$. (2 pts)

   (b) A system described by the equation $y[n] = h[n] * x[n]$, with $h[n] = n(0.9)^n u[n]$. (2 pts)

Solution

(a) Start by finding the transfer function.

$$\begin{aligned} y[n] + y[n-2] &= x[n] - x[n-1] \\ Y(z) + z^{-2}Y(z) &= X(z) - z^{-1}X(z) \\ Y(z)(1+z^{-2}) &= X(z)(1+z^{-1}) \\ H(z) &= \frac{Y(z)}{X(z)} \\ &= \frac{1+z^{-1}}{1+z^{-2}} \\ &= \frac{1+z^{-1}}{(1+jz^{-1})(1-jz^{-1})} \end{aligned}$$

We see that the poles are located at $z = \pm j$. Because the system is causal, the ROC must lie outside the pole with the greatest magnitude, so the ROC is $|z| > 1$¿ Because the ROC does not include the unit circle (equivalently, for the causal system, the poles are not in the interior of the unit cirlce) the system is $\boxed{\textbf{not BIBO stable.}}$

(b) The system is characterized by a convolution, so we know that it is LTI.

<u>Method 1:</u>

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{\infty} |n(0.9)^n u[n]| = \sum_{n=0}^{\infty} n(0.9)^n = \frac{0.9}{(1-0.9)^2} = 90$$

Because the impulse response is absolutely summable, the system is $\boxed{\textbf{BIBO stable.}}$

Comment: In this case, it was easy to determine the actual sum of the infinite series, but this is not always the case, nor is it required. Instead, if suffices to determine whether or not the infinite series converges (i.e., is finite). So, one can use one of the standard tests for convergence of an infinite series (which you learned in Calculus) - for example the ratio test. For problem 2b-i: $\lim_{n\to\infty} \frac{|h[n+1]|}{|h[n]|} = 0.9 < 1$, hence $h[n]$ is absolutely summable, and therefore the system is BIBO stable. By the same method, you should be able to determine that $h[n] = n\log^2(n+1)a^n u[n]$ is absolutely summable for $|a| < 1$ and not summable for $|a| > 1$.

Method 2: Use the z-transform pair $na^n u[n] \leftrightarrow \frac{az^{-1}}{(1-az^{-1})^2}$, ROC: $|z| > |a|$, to find the transfer function of the system. Note that the presence of the unit step tells us that the pulse response is causal and that the ROC will lie outside the pole of the greatest magnitude.

$$H(z) = \frac{0.9z^{-1}}{(1-0.9z^{-1})^2}, \text{ ROC} : |z| > 0.9$$

Because the ROC contains the unit circle, the system is $\boxed{\textbf{BIBO stable.}}$