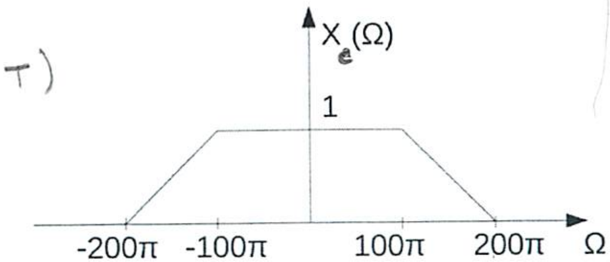
The continuous-time signal $x_a(t)$ $x_c(t)$ has the continuous-time Fourier transform shown in the figure below. The signal $x_a(t)$ is sampled with sampling interval T to get the discrete-time signal $x[n] = x_a(nT)$. Sketch $X_d(\omega)$ (the DTFT of $x[n]$) for the sampling intervals $T = 1/100, 1/200$ sec.

$x_c(t)$ ——/—— $x[n] = x_c(nT)$



$$\boxed{X_d(\omega) = \frac{1}{T} \sum_k X_c\left(\frac{\omega + 2k\pi}{T}\right)}$$

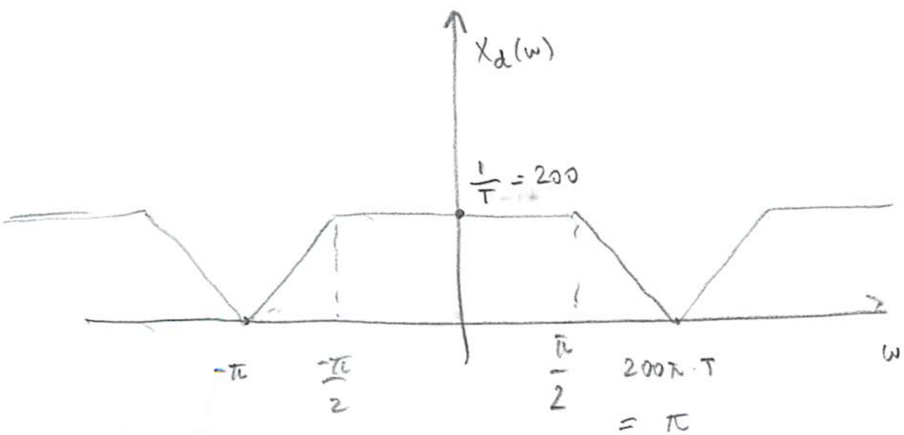
$\Omega_{max} = 200\pi = 2\pi \cdot \underbrace{100}_{F_{max}}$

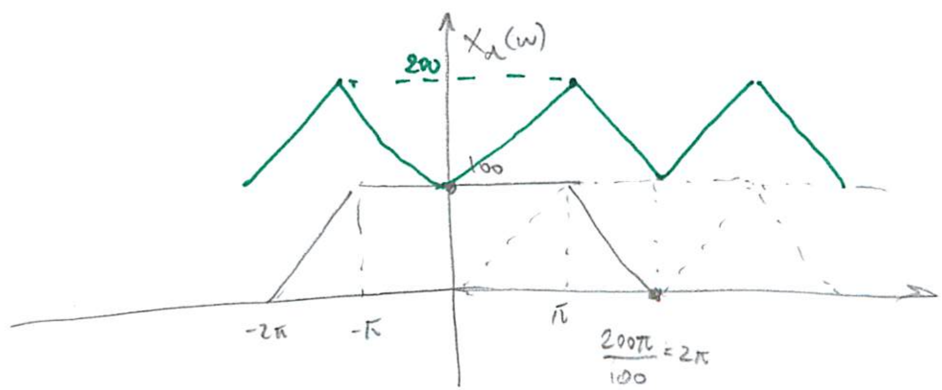
Nyquist sampling rate : $\frac{1}{T} = 200$

$T = \frac{1}{200}$

$T = \frac{1}{200}$

$T = \frac{1}{100}$





$z = re^{j\theta} \Rightarrow \angle z = \theta$

$\angle(z^*) = -\theta$

DFT

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}nk}$$

- sampling the DTFT of $\tilde{x}[n] = \begin{cases} x[n] & \text{for } 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$

- $X[k] = \tilde{X}_d\left(\frac{2\pi}{N}k\right)$

Let $(X[k])_{k=0}^{99}$ be the 100−point **DFT** of a **real-valued** sequence $(x[n])_{n=0}^{99}$ and $\tilde{X}_d(\omega)$ be the **DTFT** of $x[n]$ zero-padded to infinite length. Circle all correct equations in the following list.

(a) $X[70] = X_d\left(-\frac{6\pi}{10}\right)$ ✓

(b) $X[70] = X_d\left(\frac{70\pi}{50}\right)$ ✓

(c) $|X[70]| = |X_d\left(\frac{70\pi}{100}\right)|$ ✗

(d) $\angle X[70] = -\angle X_d\left(\frac{3\pi}{5}\right)$ ✓

(e) $|X[70]| = |X[30]|$ ✓

$$X[k] = X_d\left(\frac{2\pi}{N}k\right) = X_d\left(\frac{2\pi k}{100}\right)$$

a) $X[70] = X_d\left(\frac{2\pi \cdot 70}{100}\right) = X_d\left(\frac{14\pi}{10}\right) = X_d\left(\frac{14\pi}{10} - 2\pi\right) = X_d\left(\frac{-6\pi}{10}\right)$

$x[n]$ is real-valued $\Rightarrow \begin{cases} |X_d(\omega)| = |X_d(-\omega)| \\ \angle X_d(\omega) = -\angle X_d(-\omega) \end{cases}$

$|X[70]| = \left|X_d\left(\frac{14\pi}{10}\right)\right| = \left|X_d\left(\frac{6\pi}{10}\right)\right| = \left|X_d\left(\frac{3\pi}{5}\right)\right|$

$\angle X[70] = \angle X_d\left(-\frac{3\pi}{5}\right) = -\angle X_d\left(\frac{3\pi}{5}\right)$

$\Rightarrow X^*[k] = X[\langle N-k \rangle_N]$

e) $X^*[70] = X[\langle 100-70 \rangle_{100}] = X[30] \Rightarrow |X[70]| = |X[30]|$

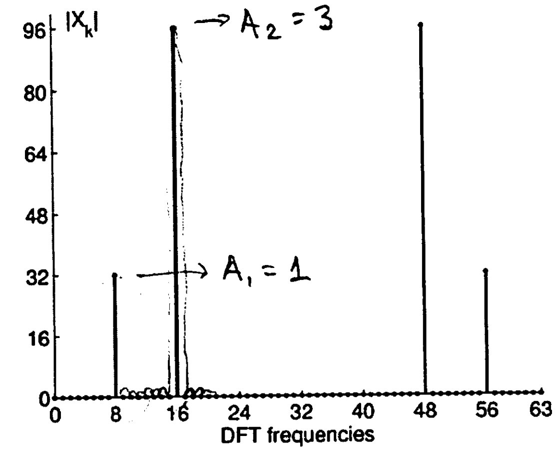
$-\angle X[70] = \angle(X^*[70]) = \angle X[30]$

$$x[n] = \sum_{\ell=0}^{1} A_\ell \cos(\overbrace{\Omega_\ell T}^{w_\ell} n) \quad \xleftrightarrow{\text{DTFT}} \quad X_d(w) = \sum_{\ell=0}^{1} A_\ell \pi \left(\delta(w - w_\ell) + \delta(w + w_\ell) \right)$$

Assume that $x_a(t) = \sum_{\ell=1}^{L} A_\ell \cos(\Omega_\ell t)$, where the A_ℓ have positive values. We further assume that $x_a(t)$ is measured at $t = nT$ for $T = 1/8$ second and $n = 0, 1, \ldots, 63$ to obtain $\{x_n\}_{n=0}^{63} = \{x_a(nT)\}_{n=0}^{63}$. The 64-point DFT of $\{x_n\}_{n=0}^{63}$ is represented by $\{X_k\}_{k=0}^{63}$, whose magnitude is shown in the figure below. Find L, A_i's, Ω_i's

$\frac{A_1 N}{2} = 32$

$A_1 = \frac{2 \cdot 32}{N}$



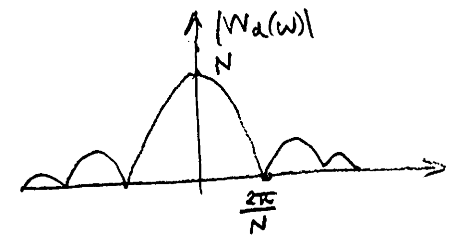
We can guess that $L = 2$.

DFT spectral analysis.

$y[n] = x[n] \cdot w[n]$, $\quad w[n] = \begin{cases} 1 & 0 \le n \le 63 \\ 0 & \text{otherwise} \end{cases}$

$Y_d(w) = \frac{1}{2\pi} X_d(w) \circledast W_d(w)$

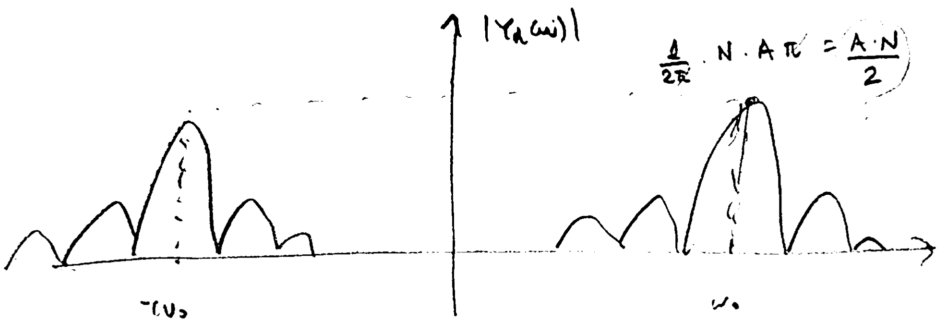
$W_d(w) = e^{-jw\frac{N-1}{2}} \cdot \frac{\sin(\frac{N}{2} w)}{\sin(\frac{w}{2})}$



In this case, we got "lucky" because $\frac{2\pi}{N} k = w_i$ for some k.

For a single sinusoid.

$A\cos(w_0 n) \xleftrightarrow{\text{DTFT}} A\pi\left(\delta(w - w_0) + \delta(w + w_0)\right)$



For first sinusoid:

$\frac{2\pi \cdot 8}{64} = w_1 \Rightarrow w_1 = \frac{\pi}{4} \Rightarrow \Omega_1 = \frac{w_1}{T} = 2\pi$

For second sinusoid:

$\frac{2\pi \cdot 16}{64} = w_2 \Rightarrow w_2 = \frac{\pi}{2} \Rightarrow \Omega_2 = \frac{w_2}{T} = 4\pi$

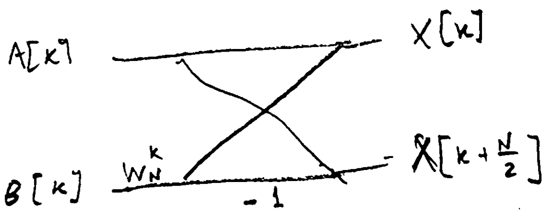
$$W_N = e^{-j\frac{2\pi}{N}}$$

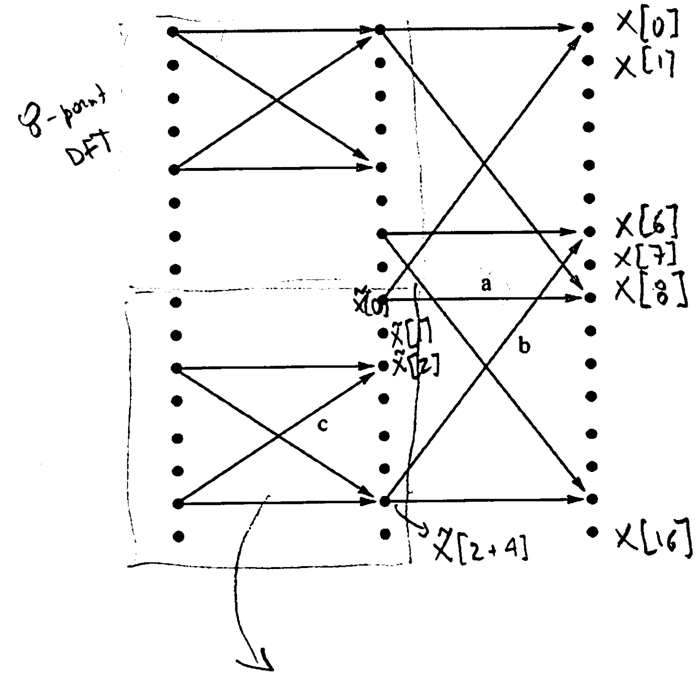
The diagram below represents a part of the computation in a 16-point decimation-in-time radix-2 FFT. Indicate the values of the three requested branch weights a, b and c.

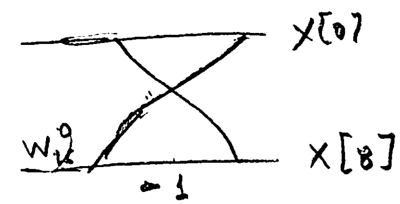
key thing to remember :

$$X[k] = A[k] + W_N^k B[k]$$

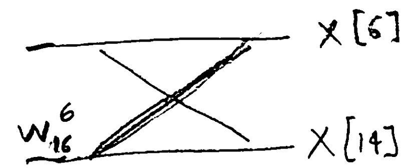
$$X[k+\tfrac{N}{2}] = A[k] - W_N^k B[k]$$





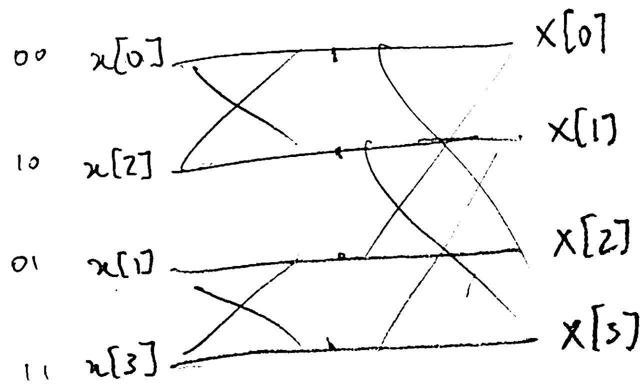


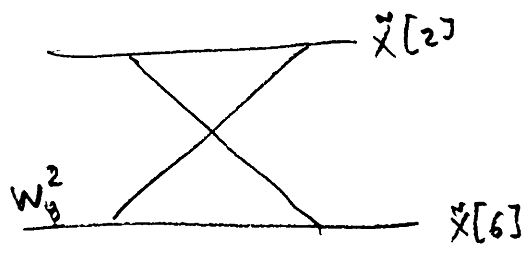
$$a = -W_{16}^0 = -1$$



$$b = W_{16}^6$$

4 point DFT





$$c = e^{-j\frac{2\pi \cdot 2}{8}} = -j$$

00 —	00	→ 0
01 —	10	→ 2
10 —	01	→ 1
11		→ 3

The following linear convolution

$$\{x_n\}_{n=0}^{46} * \{h_n\}_{n=0}^{32}$$

is to be evaluated using the DFT method. Namely,

$$\{x_n\}_{n=0}^{46} * \{h_n\}_{n=0}^{32} = \text{DFT}^{-1}\{\text{DFT}\{x_n\} \cdot \text{DFT}\{h_n\}\}$$

(a) Determine the minimum number of zeros that should be padded to $\{x_n\}$ and $\{h_n\}$ respectively before the DFTs are applied.

(b) If the DFTs are to be calculated with a radix-2 FFT algorithm, how many zeros should now be padded to $\{x_n\}$ and$\{h_n\}$?

(c) How many complex multiplications are needed if we use a radix-2 FFT?

a) Vectors should have length $M+L-1 = 47+33-1 = 79$

Add 32 zeros to x, 46 to h

b) $2^v = 128 > 79$

c) N-point FFT : $\frac{N}{2}\log_2 N$ complex multiplications, $N\log_2 N$ complex additions

Need to compute $\text{DFT}^{-1}\{\text{DFT}\{x_{zp}\} \cdot \text{DFT}\{h_{zp}\}\}$

$3 \cdot \frac{N}{2}\log_2 N + N$

GLP FIR Filters

For each of the following impulse responses, determine whether the system is a GLP filter. If so, determine the type and whether it is also a strictly LP filter.

(a) $\{h[n]\}_{n=0}^{1} = \{1,1\}$

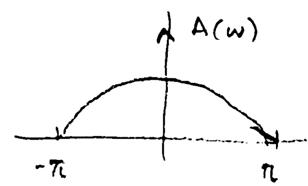
(b) $\{h[n]\}_{n=0}^{2} = \{2,0,-2\}$

(c) $\{h[n]\}_{n=0}^{2} = \{2,-1,-2\}$

a) $H_a(\omega) = 1 + e^{-j\omega} = e^{-j\omega/2}\left(e^{j\omega/2} + e^{-j\omega/2}\right)$

$$= e^{-j\omega/2} \cdot \underbrace{2\cos\left(\frac{\omega}{2}\right)}_{A(\omega)} \Rightarrow \text{GLP}.$$

Even symmetry, M odd, $\Rightarrow$ Type II



No zero crossing. $\Rightarrow$ strict LP

b) odd-symmetry, $M = 2 \Rightarrow$ Type III. Not strict LP (odd symmetry never strict LP)

c) No symmetry. Not GLP.

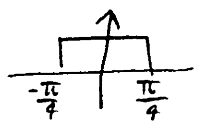
Using the windowing method, design a GLP FIR lowpass filter to approximate a filter with the following specifications:

- Pass band: $[0, 0.2\pi]$; Passband ripple: at most 1dB;
- Stop band: $[0.3\pi, \pi]$, attenuation of at least 45dB.

Assume you are only given four choices of windows: rectangular, Hann, Hamming, and Blackman. Choose the window to obtain the shortest filter length.

cutoff frequency $\omega_c = 0.25\pi$

Ideal response



For LPF, can use Type I or II (depending on length)

Shifted filter response $D_d(\omega) = \begin{cases} e^{-j\omega M/2} & |\omega| \le \pi/4 \\ 0 & \text{otherwise} \end{cases}$

$$d[n] = DTFT^{-1}\{D_d(\omega)\} = \frac{\sin(\omega_c(n - M/2))}{\pi(n - M/2)}$$

$h[n] = d[n] \cdot w[n]$. Hamming satisfies requirements. $\Delta\omega = 0.1\pi = \frac{6.6\pi}{L} \Rightarrow L = 66$

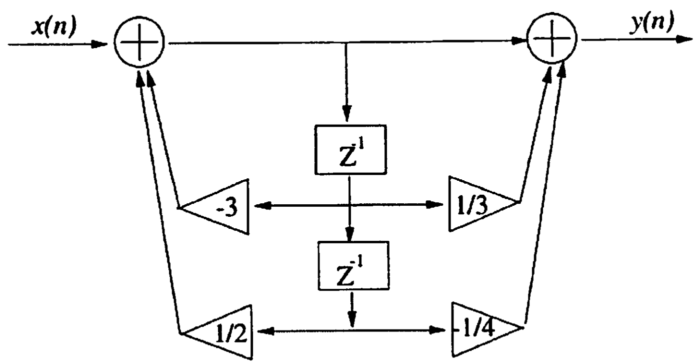
$\Downarrow$

Type II

$$= \begin{cases} \frac{\sin\left(\frac{\pi}{4}\left(n - \frac{65}{2}\right)\right)}{\pi\left(n - \frac{65}{2}\right)} \cdot \left(0.54 - 0.46\cos\left(\frac{2\pi n}{65}\right)\right), & 0 \le n \le 65 \\ 0 & \text{otherwise} \end{cases}$$

Digital filter structures:

Derive the transfer function and the corresponding difference equation for the following block diagram



Direct form II.

Denominator to the left.

$a_1 = 3$, $a_2 = -1/2$

$b_0 = 1$, $b_1 = 1/3$, $b_2 = -1/4$

$$H(z) = \frac{1 + \frac{1}{3} z^{-1} - \frac{1}{4} z^{-2}}{1 + 3 z^{-1} - \frac{1}{2} z^{-2}}$$

Draw a cascade structure with first-order sections in Direct Form I for the transfer function:

$$\frac{0.5 - z^{-1}}{(1 - 0.25z^{-1})(1 + 0.5z^{-1})}$$

Cascade : $\boxed{H_1(z)}$ — $\boxed{H_2(z)}$ $\quad H(z) = H_1(z)\, H_2(z)$

$\frac{0.5 - z^{-1}}{1 - 0.25 z^{-1}}$ $\quad \frac{1}{1 + 0.5 z^{-1}}$

Parallel:

$H(z) = H_1(z) + H_2(z)$

(usually need PFE)