Lecture 5

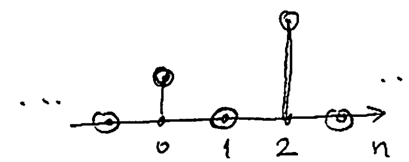
<u>z - transform</u>

$$\{x[n]\}_{n=-\infty}^{\infty} \xrightarrow{\mathcal{Z}} X(z) \quad (z \text{ is a complex number})$$

Definition: Given a signal $\{x[n]\}_{n=-\infty}^{\infty}$, its z-transform is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] \cdot z^{-n}$$

Ex 1: $\{x[n]\} = \{\underset{\uparrow}{1}, 0, 2\}$



$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n} = x[0] \cdot z^{-0} + \underbrace{x[1]}_{\to 0} z^{-1} + x[2]\, z^{-2}$$

$$= 1 + 2z^{-2}$$

Ex 2:

$\{x[n]\}$		$X(z)$
$\{\underset{\uparrow}{1}, 0, 2\}$	$\xrightarrow{\mathcal{Z}}$	$1 + 2z^{-2}$
$\{\frac{1}{2}, \underset{\uparrow}{1}, \frac{1}{2}\}$	$\xrightarrow{\mathcal{Z}}$	$\frac{1}{2} z + 1 + \frac{1}{2} \cdot z^{-1}$
$\{\underset{\uparrow}{2}, 1\}$	$\xleftarrow{\mathcal{Z}^{-1}}$	$2 + z^{-1}$

Review of geometric series $(1, b, b^2, b^3, \dots)$

$$S_N = \sum_{n=0}^{N} b^n = 1 + \underbrace{\cancel{b} + \cancel{b^2} + \cancel{b^3} + \dots + \cancel{b^N}}$$

$$b\,S_N = \underbrace{\cancel{b} + \cancel{b^2} + \cancel{b^3} + \cancel{b^4} + \dots} + b^{N+1}$$

$$(1-b)\,S_N = 1 - b^{N+1}$$

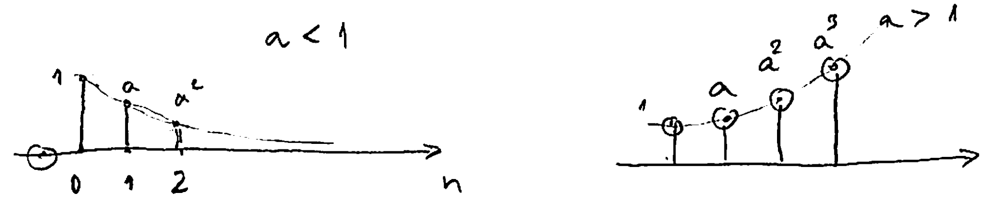
as $n \to \infty$ $\begin{cases} 0 & \text{if } |b| < 1 \\ \text{diverges} & \text{otherwise} \end{cases}$

If $b \neq 1$, $\boxed{S_N = \frac{1 - b^{N+1}}{1-b}}$

condition for convergence

$$\sum_{n=0}^{\infty} b^N = \lim_{N \to \infty} S_N = \boxed{\frac{1}{1-b}} \quad \text{if } \boxed{|b| < 1}$$

Ex 3: $x[n] = a^n u[n]$

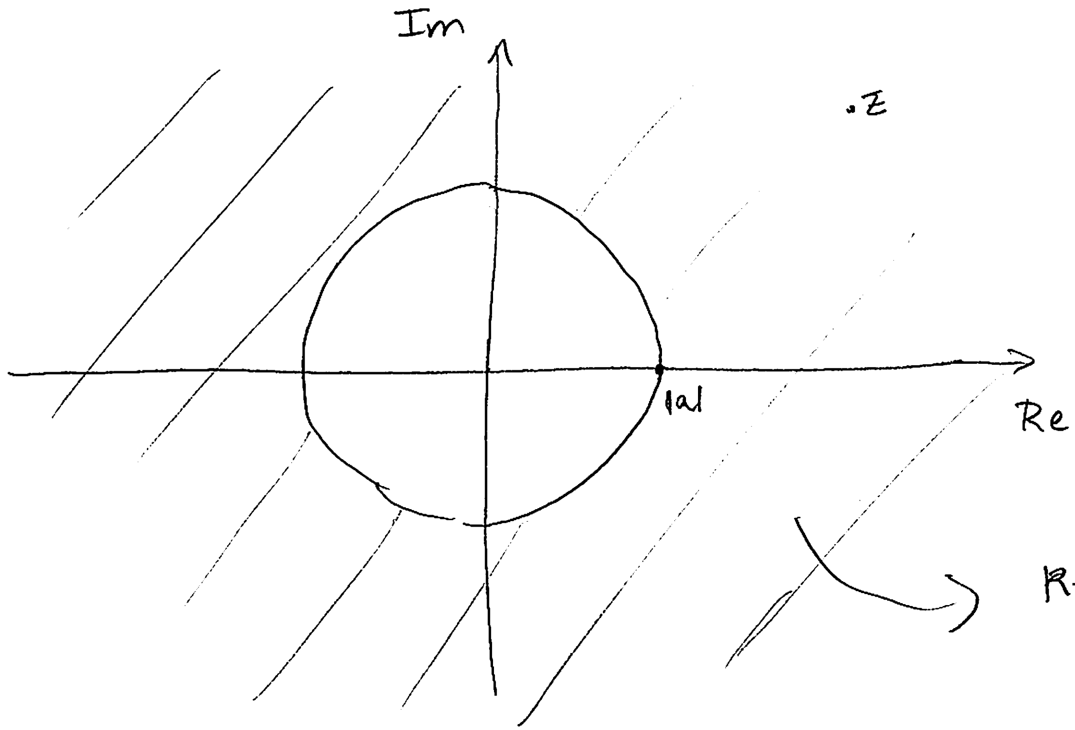


$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n} = \sum_{n=-\infty}^{\infty} a^n u[n]\, z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} \quad \text{if} \quad |az^{-1}| < 1$$

$$\iff \boxed{|z| > |a|}$$

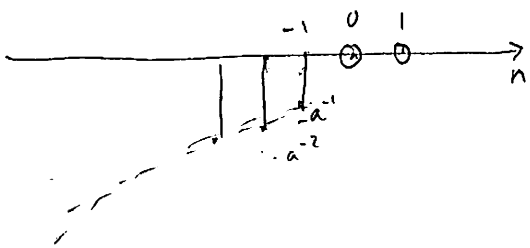
condition for convergence



Region of convergence (ROC)

Ex 4: $y[n] = -a^n \underbrace{u[-n-1]}_{\begin{cases} 1, & \text{if } -n-1 \geq 0 \Rightarrow n \leq -1 \\ 0, & \text{if } n \geq 0 \end{cases}}$

$0<a<1$



$$Y(z) = \sum_{n=-\infty}^{\infty} y[n] \cdot z^{-n} = \sum_{n=-\infty}^{-1} -a^n z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} (az^{-1})^n \underset{\substack{\uparrow \\ m=-n}}{=} -\sum_{m=1}^{\infty} (az^{-1})^{-m}$$

$$= 1 - \sum_{m=0}^{\infty} (az^{-1})^{-m} = 1 - \sum_{m=0}^{\infty} (a^{-1}z)^m$$

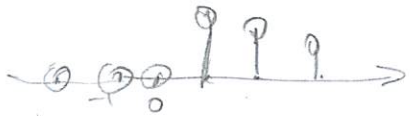
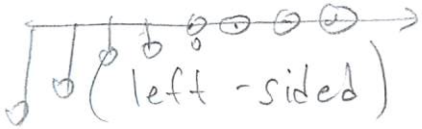
$$= 1 - \frac{1}{1-a^{-1}z} \qquad \text{if } |a^{-1}z| < 1$$

$$\Rightarrow |z| < |a|$$

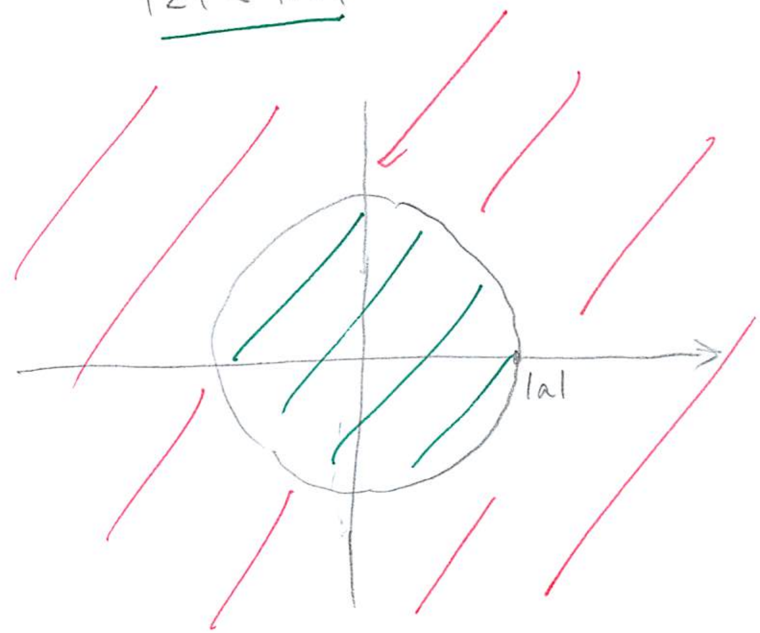
$$= \frac{\cancel{1} - a^{-1}z - \cancel{1}}{1-a^{-1}z} \frac{(-z^{-1}a)}{(-z^{-1}a)}$$

$$= \frac{1}{1-az^{-1}}$$

In summary

	$\{x[n]\}$	$X(z)$,	ROC
 (right-sided)	$a^n u[n]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
 (left-sided)	$-a^n u[-n-1]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$

(z-transform: $X(z)$, ROC)



$$\{x[n]\} \xleftrightarrow{\text{z-transform}} \left(X(z),\ \text{ROC}\right)$$

two ways to compute z-transform:

1. use definition
2. use table of standard z-transforms + properties of z-transforms

Properties of z - transform

① Linearity

Suppose $\begin{cases} \{x_1[n]\} \xrightarrow{z} X_1(z), & \text{ROC } R_1 \\ \{x_2[n]\} \xrightarrow{z} X_2(z), & \text{ROC } R_2 \end{cases}$

Then $\{\underbrace{a_1 x_1[n] + a_2 x_2[n]}_{x[n]}\} \xrightarrow{z} a_1 X_1(z) + a_2 X_2(z)$, ROC : $R_1 \cap R_2$ (at least)

Proof :

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=-\infty}^{\infty} \left(a_1 x_1[n] + a_2 x_2[n]\right) z^{-n}$$

$$= a_1 \underbrace{\sum_{n=-\infty}^{\infty} x_1[n] z^{-n}}_{X_1(z)} + a_2 \underbrace{\sum_{n=-\infty}^{\infty} x_2[n] z^{-n}}_{X_2(z)} = a_1 X_1(z) + a_2 X_2(z).$$

② Shifting

$\{x[n]\} \xrightarrow{z} X(z)$, ROC : R

Let $\{y[n]\} = \{x[n-n_0]\} \xrightarrow{z} z^{-n_0} X(z)$ ROC : $\begin{cases} R \setminus \{0\} & \text{if } n_0 \geq 1 \\ R \setminus \{\pm\infty\} & \text{if } n_0 \leq -1 \end{cases}$

Proof : $Y(z) = \sum_{n=-\infty}^{\infty} y[n] z^{-n} = \sum_{n=-\infty}^{\infty} x[n-n_0] z^{-(n-n_0)} \cdot z^{-n_0}$

$$= z^{-n_0} \sum_{n=-\infty}^{\infty} x[n-n_0] z^{-(n-n_0)} \underset{m=n-n_0}{=} z^{-n_0} \sum_{m=-\infty}^{\infty} x[m] z^{-m} = z^{-n_0} X(z)$$

Ex: $x[n] = \underbrace{2^n u[n]}_{} + \underbrace{0.5^n u[n-2]}_{}$

$a^n u[n]$	$\frac{1}{1-az^{-1}}$, $\|z\|>\|a\|$

$$2^n u[n] \xrightarrow{\mathcal{Z}} \frac{1}{1-2z^{-1}}, \quad \boxed{|z|>2}$$

$$\underbrace{(0.5^2)}_{\alpha_2}\underbrace{(0.5^{n-2})u[n-2]}_{}$$

$\mathcal{Z}$ (shifting)

$$z^{-2} \cdot \mathcal{Z}\{0.5^n u[n]\} = z^{-2} \cdot \left(\frac{1}{1-0.5z^{-1}}\right)$$

$$\boxed{|z|>0.5}$$

By linearity,

$$X(z) = \frac{1}{1-2z^{-1}} + (0.5)^2 \cdot \frac{z^{-2}}{1-0.5z^{-1}}$$

RoC: $|z| > 2$

③ Convolution

suppose $\begin{cases} \{x_1[n]\} \xrightarrow{Z} X_1(z), & \text{ROC } R_1 \\ \{x_2[n]\} \xrightarrow{Z} X_2(z), & \text{ROC } R_2 \end{cases}$

Then : $(x_1 * x_2)[n] \xrightarrow{Z} X_1(z)\,X_2(z)$, ROC: $R_1 \cap R_2$ (at least)

Proof follows from linearity + shifting properties.

④ Differentiation of z-transform

suppose $\{x[n]\} \xrightarrow{Z} X(z)$ ROC R_x

Then $\{n\,x[n]\} \xrightarrow{Z} -z \cdot \frac{d}{dz} X(z)$ ROC R_x.

See book for more properties