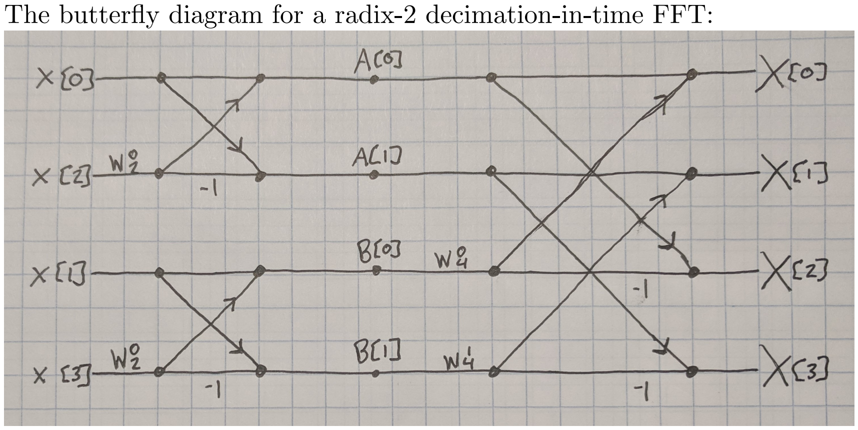
ECE 310: Quiz #8 Solution (3pm Section E) Fall 2018

November 7, 2018

1. (4pts) Consider the signal $\{x[n]\}_{n=0}^{3} = \{2, -2, 1, -1\}$. Use a decimation-in-time radix-2 FFT to compute the DFT of $x[n]$. i.e., calculate explicitly all the intermediate quantities that are computed in this FFT, and show how they are combined to produce the final output, which you should also give explicitly.

Solution

The butterfly diagram for a radix-2 decimation-in-time FFT:



The intermediate values:

$$
\begin{aligned}
A[0] &= x[0] + x[2] = 2 + 1 = 3 \\
A[1] &= x[0] - x[2] = 2 - 1 = 1 \\
B[0] &= x[1] + x[3] = -2 + (-1) = -3 \\
B[1] &= x[1] - x[3] = -2 - (-1) = -1
\end{aligned}
$$

Calculating the 4-point DFT:

$$
\begin{aligned}
X[0] &= A[0] + B[0] = 3 + (-3) = 0 \\
X[1] &= A[1] + W_4^1 B[1] = 1 + (-1)\, e^{-j\frac{2\pi}{4}} = 1 + j \\
X[2] &= A[0] - B[0] = 3 - (-3) = 6 \\
X[3] &= A[1] - W_4^1 B[1] = 1 - (-1)\, e^{-j\frac{2\pi}{4}} = 1 - j
\end{aligned}
$$

$$
\{X[k]\}_{k=0}^{3} = \{0, 1+j, 6, 1-j\}
$$

2. (3 pts) Given $\{x[n]\}_{n=0}^{N-1}$, with $x[n] = x_c(nT)$ and $T = 200\mu\text{sec}$, you compute the length-N FFT of $x[n]$ and plot the magnitude. Using this method, you wish to resolve analog sinusoidal signals that are separated by as little as 20 Hz in frequency. Assume that the frequency resolution for the windowed DFT-based spectral analysis is equal to the width of the main lobe of the DTFT of the window. Determine the minimum length $N = 2^\nu$ that will meet your resolution requirement. (**Hint:** The DTFT of the sequence $v[n] = u[n] - u[n-N]$ is $V_d(\omega) = e^{-j\omega\frac{N-1}{2}} \frac{\sin\left(\frac{N\omega}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}$.)

Solution

To determine the width of the main lobe, calculate the zeros of $V_d(\omega)$:

$$
\begin{aligned}
\frac{N\omega}{2} &= \pm\pi k \\
\omega &= \frac{\pm 2\pi k}{N}
\end{aligned}
$$

The peak of the main lobe is at $\omega = 0$, so the width of the main lobe will be defined by the zeros at $\omega = \pm\frac{2\pi}{N}$, leading to a width of $\frac{4\pi}{N}$.

Given a desired resolution of 20 Hz, find the required minimum width of the main lobe using the relationship $\omega = \Omega T$:

$$
\begin{aligned}
\frac{4\pi}{N} &= \Omega T \\
&= 2\pi F T \\
&= 2\pi \times 20 \times 200 \times 10^{-6} \\
N &= \frac{4\pi}{2\pi \times 20 \times 200 \times 10^{-6}} \\
&= 500
\end{aligned}
$$

To satisfy the restriction that N must be a power of 2:

$$
N = 512 = 2^9
$$

3. (3 pts) Let $\{x[n]\}_{n=0}^{2} = \{2, 4, 6\}$ and $\{v[n]\}_{n=0}^{2} = \{-1, 0, 1\}$. A new sequence $\{g[n]\}_{n=0}^{3}$ is generated as follows: $\{g[n]\}_{n=0}^{3} = \text{IFFT}\left(\{G[k]\}_{k=0}^{3}\right)$ where the IFFT is a 4-point inverse FFT, $G[k] = X[k]\,V[k]\,, k = 0, 1, 2, 3$, and the sequences $X[k]$ and $V[k]$ are generated each by a 4-point FFT of the sequences $\{x[n]\}$ and $\{v[n]\}$ respectively, after zero padding them to length 4. Determine $g[1]$ and $g[3]$.

Solution

Multiplication of FFTs/DFTs corresponds to circular convolution in the time domain. Therefore, $g[n]$ is the result of the circular convolution of $x[n]$ and $v[n]$ after zero padding each to length 4. This is represented by the following matrix multiplication:

$$\begin{bmatrix} 2 & 0 & 6 & 4 \\ 4 & 2 & 0 & 6 \\ 6 & 4 & 2 & 0 \\ 0 & 6 & 4 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} g[0] \\ g[1] \\ g[2] \\ g[3] \end{bmatrix}$$

Therefore:

$$g[1] = -4$$
$$g[3] = 4$$