University of Illinois Fall 2018

ECE 310: Problem Set 9

Due: 5 pm, November 2, 2018

1. In this problem we consider windowed DFT spectral analysis. A continuous-time signal $x_a(t)$ is sampled with period T to produce the sequence $x[n] = x_a(nT)$. An N-point window $w[n]$ is applied to $x[n]$ for $n = 0, 1, ..., N-1$ and $X[k], k = 0, 1, ..., N-1$ is the N-point DFT of the resulting sequence.

   (a) You are given that $x_a(t) = \cos(\Omega_0 t)$. Assuming that $w[n]$ is a rectangular window, and Ω_0, N, and k_0 are fixed, how should T be chosen so that $X[k_0]$ and $X[N-k_0]$ are non-zero and $X[k] = 0$ for all other values of k? Is your answer unique? If not, give another value of T that satisfies the conditions given.

   (b) Suppose now that $X_a(\Omega) = 0$ for $|\Omega| \geq 2\pi(5000)$, $w[n]$ is a rectangular window, and the window length N is constrained to be a integer power of 2. Recall that the $X[k]$ correspond to samples of the Fourier transform of a windowed version of $x_a(t)$. Determine T and the minimum value of $N = 2^\nu$ such that the following conditions are satisfied: (i) no aliasing occurs when you sample $x_a(t)$ to obtain $x[n]$; and (ii) the spacing between the $X(k)$ corresponds to analog frequency spacing of no more than 5 Hz.

   (c) Suppose instead that $w[n]$ is a length-N Hamming window, and $T = 50\mu$sec. A conservative rule of thumb for the frequency resolution of windowed DFT analysis is that the frequency resolution for the DTFT is equal to the width of the main lobe of $W_d(\omega)$. You wish to be able to resolve analog sinusoidal signals that are separated by as little as 15 Hz in frequency. In addition, your window length N is constrained to be a integer power of 2. What is the minimum length $N = 2^\mu$ that will meet your resolution requirement?

2. Let $x[n] = \cos(\pi n/4)$ and $v[n] = \begin{cases} x[n], & n \text{ even}, 0 \leq n \leq 27 \\ 0, & \text{otherwise.} \end{cases}$

   Sketch $|V_d(\omega)|$ for $-\pi \leq \omega \leq \pi$, labeling the frequencies of the peaks and the first nulls on either side of the peak. In addition, label the amplitudes of the peaks and the strongest side lobe of each peak.

   Hint 1: Let $w[n]$ be a window function and $v[n] = x[n]w[n]$.

   Hint 2: You can use the approximation $|X_1(\omega) + X_2(\omega)| \approx |X_1(\omega)| + |X_2(\omega)|$ if for $\omega < 0$, $|X_1(\omega)| \approx 0$ and for $\omega > 0$, $|X_2(\omega)| \approx 0$.

3. Given two sequences $\{x[n]\}_{n=0}^{514}$ and $\{h[n]\}_{n=0}^{127}$ you are asked to compute their linear convolution $y[n] = x[n] * h[n]$. You decide to use the DFT to speed up the computation.

   (a) What is the length of the sequence $y[n]$?

   (b) Find the smallest number of zeros that should be padded to each sequence so that the linear convolution can be computed using the DFT.

   (c) To further speed computation, you decide to use a radix-2 FFT to compute the DFT. How should the sequences be padded so that their linear convolution can be computed using the smallest possible radix-2 FFT?

4. You are to compute by hand the DFT of the following sequence:

$$x[n] = \{1, 4, 3, 6\}$$

(a) Compute the DFT of $x[n]$ using the definition of the DFT. Show your work and give exact answers.

(b) Use a decimation-in-time radix-2 FFT to compute the DFT of $x[n]$. Draw the corresponding flow diagram.

(c) Repeat part (b) using decimation-in-frequency.

5. Suppose that you wish to design a circuit to perform a length-8 DFT. You are given a pair of chips that compute the DFT of a length-4 complex input sequence. The inputs and outputs of this chip are all complex numbers. You also have access to complex multiplication and addition circuits, each of which has two complex inputs and one complex output.

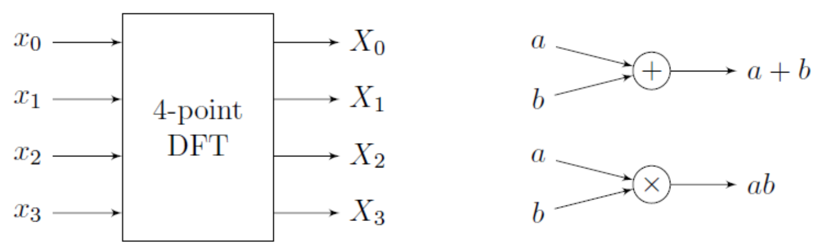

Figure 1: DFT chip and complex arithmetic circuits

(a) Your goal is to use as few complex multiplication circuits as possible. Fortunately, you don't need to use a multiplier circuit to multiply by $+1, -1, +\jmath$, or $-\jmath$. Explain why these are trivial multiplications.

(b) Show how you would connect two chips and the multiplication and addition circuits to compute a length-8 decimation-in-time FFT. Explain your solution. How many nontrivial multiplications are required?

(c) Repeat part (b) using decimation-in-frequency.

6. Determine $y[n]$, the cyclic convolution of $x[n]$ and $h[n]$ for the following cases:

(a) $\{x[n]\}_{n=0}^{4} = \{2, 4, 6, 8, 10\}$ and $\{h[n]\}_{n=0}^{4} = \{1, 0, 0, 0, 1\}$

(b) $\{x[n]\}_{n=0}^{7} = \{1, 2, 3, 4, 5, 0, 0, 0\}$ and $\{h[n]\}_{n=0}^{7} = \{1, 0, 0, 1, 0, 0, 0, 1\}$

7. Let $\{x[n]\}_{n=0}^{N-1}$ be a finite-length signal and $\{X[k]\}_{k=0}^{N-1}$ be the corresponding N-pont DFT. Define the following two sequences:

$$s[n] = \{x[0], x[1], \ldots, x[N-1], x[0], x[1], \ldots, x[N-1]\}$$
$$y[n] = \{x[0], x[1], \ldots, x[N-1], \underbrace{0, 0, \ldots, 0}_{N \text{ zeros}}\}$$

Let $\{S[k]\}_{k=0}^{2N-1}$ and $\{Y[k]\}_{k=0}^{2N-1}$ be the corresponding $2N$-point DFTs.

(a) Show that $S[2k] = 2X[k]$ and $S[2k+1] = 0$ for $k = 0, 1, \ldots, N-1$

(b) Show that $Y[2k] = X[k]$ for $k = 0, 1, \ldots, N-1$

(c) Find an expression for $Y[2k+1], k = 0, 1, \ldots, N-1$ in terms of $X[k]$ using the results in part (a)

Hint: Use the fact that $y[n] = s[n]w[n]$, where $w[n] = \{\underbrace{1, 1, \ldots, 1}_{N \text{ ones}}, \underbrace{0, 0, \ldots, 0}_{N \text{ zeros}}\}$ is an N-point rectangular window zero-padded to length $2N$