Tpd approximation for the MWN-OU process

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The pdf of a MWN $(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ is $f_{\mathrm{MWN}}(\boldsymbol{\theta}; \boldsymbol{\mu}, \boldsymbol{\Sigma}) = \sum_{\mathbf{k}\in\mathbb{Z}^p} \phi_{\boldsymbol{\Sigma}}(\boldsymbol{\theta} - \boldsymbol{\mu} + 2\mathbf{k}\pi)$. Let $\mathbf{A}$ be an invertible matrix such that $\mathbf{A}^{-1}\boldsymbol{\Sigma}$ is symmetric and positive-definite. The Langevin diffusion with stationary distribution MWN $\left(\boldsymbol{\mu}, \frac{1}{2}\mathbf{A}^{-1}\boldsymbol{\Sigma}\right)$ is given by

$$\mathrm{d}\boldsymbol{\Theta}_t = \mathbf{A}\left(\boldsymbol{\mu} - \boldsymbol{\Theta}_t - 2\pi\frac{\sum_{\mathbf{k}\in\mathbb{Z}^p} \mathbf{k}\phi_{\frac{1}{2}\mathbf{A}^{-1}\boldsymbol{\Sigma}}(\boldsymbol{\Theta}_t - \boldsymbol{\mu} + 2\mathbf{k}\pi)}{\sum_{\mathbf{k}\in\mathbb{Z}^p} \phi_{\frac{1}{2}\mathbf{A}^{-1}\boldsymbol{\Sigma}}(\boldsymbol{\Theta}_t - \boldsymbol{\mu} + 2\mathbf{k}\pi)}\right)\mathrm{d}t + \boldsymbol{\Sigma}^{\frac{1}{2}}\mathrm{d}\mathbf{W}_t.$$

The tpd pf the MWN-OU can be approximated by the conditional density of the wrapped multivariate OU process, that is, the wrapping of

$$\mathrm{d}\mathbf{X}_t = \mathbf{A}(\boldsymbol{\mu} - \mathbf{X}_t)\mathrm{d}t + \boldsymbol{\Sigma}^{\frac{1}{2}}\mathrm{d}\mathbf{W}_t.$$

This process has also stationary distribution MWN $\left(\boldsymbol{\mu}, \frac{1}{2}\mathbf{A}^{-1}\boldsymbol{\Sigma}\right)$ and its *conditional* density (is not Markovian), assuming that the process is stationary, is given by

$$\tilde{p}_t(\boldsymbol{\theta} \,|\, \boldsymbol{\theta}_0) = \sum_{\mathbf{m}\in\mathbb{Z}^p}\sum_{\mathbf{k}\in\mathbb{Z}^p} \phi_{\boldsymbol{\Gamma}_t}(\boldsymbol{\theta} + 2\mathbf{k}\pi - \boldsymbol{\mu}_t^{\mathbf{m}}) w_{\mathbf{m}}(\boldsymbol{\theta}_0),$$

with $\phi_{\boldsymbol{\Gamma}_t}(\cdot - \boldsymbol{\mu}_t^{\mathbf{m}})$ denoting the density of a multivariate normal with covariance matrix $\boldsymbol{\Gamma}_t = \int_0^t e^{-s\mathbf{A}}\boldsymbol{\Sigma}e^{-s\mathbf{A}^T}\,\mathrm{d}s$ and mean $\boldsymbol{\mu}_t^{\mathbf{m}} = \boldsymbol{\mu} + e^{-t\mathbf{A}}(\boldsymbol{\theta}_0 - \boldsymbol{\mu} + 2\mathbf{m}\pi)$ and

$$w_{\mathbf{m}}(\mathbf{x}) = \frac{\phi_{\frac{1}{2}\mathbf{A}^{-1}\boldsymbol{\Sigma}}(\mathbf{x} - \boldsymbol{\mu} + 2\mathbf{m}\pi)}{\sum_{\mathbf{k}\in\mathbb{Z}^p} \phi_{\frac{1}{2}\mathbf{A}^{-1}\boldsymbol{\Sigma}}(\mathbf{x} - \boldsymbol{\mu} + 2\mathbf{k}\pi)}.$$

Recall that the MWN-OU process can be seen as a *weighting of drifts* to achieve periodicity:

$$\mathrm{d}\boldsymbol{\Theta}_t = \sum_{\mathbf{k}\in\mathbb{Z}^p} \mathbf{A}\left(\boldsymbol{\mu} - \boldsymbol{\Theta}_t - 2\pi\mathbf{k}\right) w_{\mathbf{k}}(\boldsymbol{\Theta}_t)\mathrm{d}t + \boldsymbol{\Sigma}^{\frac{1}{2}}\mathrm{d}\mathbf{W}_t.$$

$\tilde{p}_t$ approximates p_t, the true tpd of the MWN-OU, due to several facts:

i. $\lim_{t\to 0} |p_t(\boldsymbol{\theta} \,|\, \boldsymbol{\theta}_0) - \tilde{p}_t(\boldsymbol{\theta} \,|\, \boldsymbol{\theta}_0)| = 0$, $\forall \boldsymbol{\theta}, \boldsymbol{\theta}_0 \in \mathbb{T}^p$.

ii. $\lim_{t\to \infty} |p_t(\boldsymbol{\theta} \,|\, \boldsymbol{\theta}_0) - \tilde{p}_t(\boldsymbol{\theta} \,|\, \boldsymbol{\theta}_0)| = 0$, $\forall \boldsymbol{\theta}, \boldsymbol{\theta}_0 \in \mathbb{T}^p$.

iii. $\lim_{|\mathbf{A}^{-1}\boldsymbol{\Sigma}|\to 0} |p_t(\boldsymbol{\theta} \,|\, \boldsymbol{\theta}_0) - \tilde{p}_t(\boldsymbol{\theta} \,|\, \boldsymbol{\theta}_0)| = 0$, $\forall \boldsymbol{\theta}, \boldsymbol{\theta}_0 \in \mathbb{T}^p$, $\forall t > 0$.

iv. $\tilde{p}_t$ exhibits similar modes to p_t.

v. As p_t, $\tilde{p}_t$ is time-reversible wrt the stationary density $f_{\mathrm{MWN}}(\boldsymbol{\theta}; \boldsymbol{\mu}, \frac{1}{2}\mathbf{A}^{-1}\boldsymbol{\Sigma})$:

$$\tilde{p}_t(\boldsymbol{\theta}_1 \,|\, \boldsymbol{\theta}_2) f_{\mathrm{MWN}}(\boldsymbol{\theta}_2; \boldsymbol{\mu}, \tfrac{1}{2}\mathbf{A}^{-1}\boldsymbol{\Sigma}) = \tilde{p}_t(\boldsymbol{\theta}_2 \,|\, \boldsymbol{\theta}_1) f_{\mathrm{MWN}}(\boldsymbol{\theta}_1; \boldsymbol{\mu}, \tfrac{1}{2}\mathbf{A}^{-1}\boldsymbol{\Sigma}).$$

It is worth to discuss what happens in the bivariate case and what parametrizations of $\mathbf{A}$ and $\boldsymbol{\Sigma}$ lead to $\mathbf{A}^{-1}\boldsymbol{\Sigma}$ symmetric and positive definite, which is required for an ergodic time-reversible multivariate OU (Lemma 2 of Bladt et al. (2015)). These parametrizations are collected in the following lemma:

Lemma 1. *Let $\mathbf{\Sigma}$ be a 2×2 positive-definite matrix and $\mathbf{A}$ a 2×2 matrix. The next statements characterize the matrices $\mathbf{A}$ such that $\mathbf{A}^{-1}\mathbf{\Sigma}$ is symmetric and positive-definite, under different forms of $\mathbf{\Sigma}$:*

i. $\mathbf{\Sigma} = \sigma^2\mathbf{I}_2$, $\sigma > 0$: $\mathbf{A} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ *with* $ac > b^2$.

ii. $\mathbf{\Sigma} = \mathrm{diag}\left(\sigma_1^2, \sigma_2^2\right)$, $\sigma_1, \sigma_2 > 0$: $\mathbf{A} = \begin{pmatrix} a & \frac{\sigma_1}{\sigma_2}b \\ \frac{\sigma_2}{\sigma_1}b & c \end{pmatrix}$ *with* $ac > b^2$.

iii. $\mathbf{\Sigma} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}$, $\sigma_1, \sigma_2, \sigma_{12} > 0$, $\sigma_1^2\sigma_2^2 > \sigma_{12}^2$: $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *with* $ac > b^2$ *and* $\sigma_2^2 b - \sigma_{12} d = -\sigma_{12} a + \sigma_1^2 c$.

We use *ii* as a parametrization of $\mathbf{A}$ to provide a compromise between flexibility (the diffusions are linked by $b \in \mathbb{R}$) and tractability (only the constraint $ac > b^2$ has to be satisfied). In other words, the components of the diffusion are only dependent through the drift and not by the noise. The stationary densities associated to *ii* have covariance matrices:

$$\frac{1}{2(ac-b^2)}\begin{pmatrix} c\sigma_1^2 & -b\sigma_1\sigma_2 \\ -b\sigma_1\sigma_2 & a\sigma_2^2 \end{pmatrix}, \quad \sigma_1, \sigma_2 > 0, \quad ac > b^2.$$

Both $\mathbf{\Gamma}_t$ and $e^{-t\mathbf{A}}$ deserve some attention. In virtue of Corollary 2.4 of Bernstein and So (1993), the exponential matrix of any two dimensional matrix $\mathbf{A}$ has the analytical expression

$$e^{t\mathbf{A}} = s_1(t)\mathbf{I} + s_2(t)\mathbf{A},$$

with

$$s_1(t) = e^{s(A)t}\left(\cosh(q(A)t) - s(A)\frac{\sinh(q(A)t)}{q(A)}\right), \quad s_2(t) = e^{s(A)t}\frac{\sinh(q(A)t)}{q(A)}$$

where $s \equiv s(A) = \frac{\mathrm{tr}[A]}{2}$ and $q \equiv q(A) = \sqrt{|\det(\mathbf{A} - s\mathbf{I})|}$. If $q(A) = 0$, then, by continuity, $\frac{\sinh(q(A)t)}{q(A)} = t$. Trivially, $s(A) = s(A^T)$ and $q(A) = q(A^T)$.

Using these facts, it can be seen that:

$$\mathbf{\Gamma}_t = \left(\int_0^t s_1(-u)^2\,\mathrm{d}u\right)\mathbf{\Sigma} + \left(\int_0^t s_1(-u)s_2(-u)\,\mathrm{d}u\right)(\mathbf{A\Sigma} + \mathbf{\Sigma A}^T) + \left(\int_0^t s_2(-u)^2\,\mathrm{d}u\right)\mathbf{A\Sigma A}^T,$$

where:

$$\begin{aligned}
\int_0^t s_1(-u)^2\,\mathrm{d}u &= \frac{e^{-2st}}{4q^2 s(s^2-q^2)}\Big(-s^2(3q^2+s^2)\cosh(2qt) - qs(q^2+3s^2)\sinh(2qt) \\
&\quad - q^2(q^2-5s^2)e^{2st} + (q^2-s^2)^2\Big), \\
\int_0^t s_1(-u)s_2(-u)\,\mathrm{d}u &= \frac{e^{-2st}}{4q^2(s^2-q^2)}\Big((q^2+s^2)\cosh(2qt) + 2qs\sinh(2qt) - 2q^2e^{2st} + q^2 - s^2\Big), \\
\int_0^t s_2(-u)^2\,\mathrm{d}u &= \frac{e^{-2st}}{4q^2 s(s^2-q^2)}\Big(-s(s\cosh(2qt) + q\sinh(2qt)) + q^2(e^{2st}-1) + s^2\Big).
\end{aligned}$$

This allows the direct computation of $\mathbf{\Gamma}_t$ and avoids the numerical integration of $\mathbf{\Gamma}_t$ by a trapezoidal-like formula:

$$\mathbf{\Gamma}_t \approx \sum_{i=1}^N e^{s_i\mathbf{A}}\mathbf{\Sigma}(e^{s_i\mathbf{A}})^T w_i, \quad s_i \in (0,t),\ w_i \geq 0,\ i = 1, \ldots, N.$$

References

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