‘Double Categories of Relations’

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Abstract

A ‘double category of relations’ is essentially a cartesian equipment in which every object is suitably discrete. The main result of this paper is a characterization theorem that a ‘double category of relations’ is equivalent a double category of relations on a regular category precisely when it has strong and monic tabulators and a double-categorical comprehension scheme. This result is based in part on the recent characterization of double categories of spans due to Aleiferi. The overall development can be viewed as a double-categorical version of that of the notion of a “tabular allegory” or that of a “functionally complete bicategory of relations.”

Contents

1 Introduction

A relation in a regular category $\mathscr{C}$ is a monomorphism $S \to A \times B$ in $\mathscr{C}$. Relations $S \to A \times B$ and $T \to B \times C$ compose by taking a pullback $S \times_B T$ and then the image of the resulting span $S \times_B C \to A \times C$. However, composition is not associative unless a suitable equivalence relation is imposed. In the former case, the structure ends up being that of a bicategory $\mathfrak{Rel}(\mathscr{C})$. In the latter case, it is an honest category $\mathbf{Rel}(\mathscr{C})$.

A **bicategory of relations** [CW87] is a cartesian bicategory in which every object is suitably discrete. Every functionally complete bicategory of relations is equivalent to one of the form $\mathfrak{Rel}(\mathscr{C})$. “Functionally complete” means that every arrow has a so-called “tabulation.” An **allegory** [FS90] is a locally-ordered 2-category equipped with local products and an anti-involution $(-)^\circ$ satisfying a modularity law $fg \wedge h \leq (f \wedge hg^\circ)g$ written in compositional order. Each “tabular” allegory is equivalent to $\mathbf{Rel}(\mathscr{C})$ for some $\mathscr{C}$. In this sense, functionally complete bicategories of relations and tabular allegories are axiomatizations of a “calculus of relations.”

Another approach to formalizing a calculus of relations is that of the double category $\mathbb{R}\mathbf{el}(\mathscr{C})$ for a regular category $\mathscr{C}$. Its objects and arrows are objects and morphisms of $\mathscr{C}$, while its proarrows and cells are relations and suitable morphisms between them. This paper presents an answer to the question of which double categories occur as $\mathbb{R}\mathbf{el}(\mathscr{C})$ for some regular category $\mathscr{C}$. Briefly, these are the functionally complete 'double categories of relations'. It is the task of this paper to explain this definition and how it comes about.

Bicategories of relations are first of all *cartesian* bicategories. Thus, a starting point for the present question is the recent definition [Ale18] of a "cartesian double category" and the accompanying characterization of cartesian double categories of the form $\mathbb{S}\mathbf{pan}(\mathscr{C})$ for some finitely-complete category $\mathscr{C}$. A double category $\mathbb{D}$ is equivalent $\mathbb{S}\mathbf{pan}(\mathscr{C})$ for some finitely-complete $\mathscr{C}$ if, and only if, $\mathbb{D}$ is a unit-pure cartesian equipment with strong tabulators and certain Eilenberg-Moore objects. "Unit-pure" is a sort of discreteness requirement whereas the latter two conditions are completeness requirements on top of that of being cartesian. That $\mathbb{D}$ is an equipment seems a minimal condition reproducing much of the conventional calculus of Kan extensions arising in the case of profunctors and enabling various constructions in formal category theory (Cf. [Roa15] and [Roa19]).

The goal of the paper is thus to characterize $\mathbb{R}\mathbf{el}(\mathscr{C})$ as a cartesian equipment satisfying further completeness conditions. What distinguishes the present approach from that of [Ale18] is the emphasis on tabulators. This is is simply because of their centrality in the characterization theorems for allegories and 'bicategories of relations'. The required conditions in each account are that tabulators are "strong" and suitably "monic." In an arbitrary double category, tabulators are given by a right adjoint $\top\colon \mathbb{D}_1 \to \mathbb{D}_0$ to the external identity $y\colon \mathbb{D}_0 \to \mathbb{D}_1$ [GP04]. The "strong" and "monic" conditions requires equipment structure. This was done for "strong" in [Ale18], but "monic" has required a good notion of "inclusion" which was first proposed in [Sch15]. These conditions lead to the present definition of a "functionally complete" cartesian equipment.

The approach to proving the main characterization theorem of the paper has two parts. In §3 conditions are given under which a double category $\mathbb{D}$ is equivalent to $\mathbb{R}\mathbf{el}(\mathbb{D}_0)$. This follows [Nie12] that gives conditions under which a double category $\mathbb{D}$ admits an oplax/lax adjunction to $\mathbb{S}\mathbf{pan}(\mathbb{D}_0)$. The corresponding adjointness in the relations case reveals conditions under which the constructed oplax/lax adjunction is a strong equivalence of double categories. These are presented in Theorem 3.13. The second part is §4 where conditions are given under which $\mathbb{D}_0$ is a regular category. Many familiar properties and constructions enter into this development. For example, a Modular Law is developed in §2.2 which is used in the proof of Theorem 4.4 showing that $\mathbb{D}_0$ admits a suitably rich factorization system for forming $\mathbb{R}\mathbf{el}(\mathbb{D}_0)$. Additionally, "unit-pure" is required to understand the relationship between monic arrows in $\mathbb{D}_0$ and inclusions in the equipment structure on $\mathbb{D}$.

However, these and other properties and conditions are embraced by a couple of innovations. It is shown in Proposition 2.12 that the horizontal bicategory of any cartesian double category is a cartesian bicategory. This leads to the definition of a 'double category of relations' as a cartesian double category satisfying the analogue of the Frobenius Law from [CW87]. As a consequence, in any 'double category of relations' the Modularity Law will hold, since this is true in its horizontal bicategory. Secondly, the exactness and functoriality conditions discussed in §3.3 are embraced by one concerning the existence of "subobject comprehension schema." These are discussed in §3.4. Briefly, interpreting tabulators as generalized elements constructions, the exactness and functoriality conditions are subsumed by asking that tabulators have a pseudo-inverse, or "fibers" construction, leading to a certain equivalence of categories. Well-definition includes the assumption that tabulators are strong and monic.

The main result of the paper, namely, Theorem 4.7 is that these two conditions are sufficient for $\mathbb{D}$ to be of the form $\mathbb{R}\mathbf{el}(\mathscr{C})$. This is

Theorem 1.1. *If $\mathbb{D}$ is a 'double category of relations' with a subobject comprehension scheme, then $\mathbb{D}_0$ is regular and the identity functor $1\colon \mathbb{D}_0 \to \mathbb{D}_0$ extends to an adjoint equivalence $\mathbb{R}\mathbf{el}(\mathscr{C}) \simeq \mathbb{D}$.*

Once this result is completely proved, some consequences and extensions of the theory are discussed in the remainder of §4. It is shown that double categories of relations are regular equipments in the sense of [Sch15]. A concluding subsection presents definitions of "division" and "powers," giving analogues of those in [FS90].

The theory presented here has many potential applications. Some of these are discussed in a prospectus in §5. The presentation is only in outline since this paper is a theoretical one. Briefly, several of these applications have to do with using 'double categories of relations' as a meeting place, or single axiomatic forum, for "functional" and "relational" approaches to various topics in (applied) category theory. For example, the "functional ologs" of [SK12] have their counterpart in the "relational ologs" of [Pat17]. Traditional algebraic theories have their relational version in the "relational" and perhaps "partial" theories of for example [Nes21] and [LLNS21]. Other potential applications concern interpretation of various type theories and logics [Jac99], double categorical models of databases [Cod72], [RW92] and of asynchronous communication in distributed systems [Sel99], and finally with classifications of monoidal bifibrations [Shu08]. The last important point worth mentioning is that the "subobject classification schema" discussed here ought to be a central ingredient in some hypothetical notion of a "double topos" which is yet to be fully formulated, but should generalize the notion of a 2-topos [Web07].

Details on double categories can be found in [GP99] and [GP04]. Throughout double categories will always be "pseudo," that is, pseudo-categories in **Cat** viewed as a 2-category. Overall conventions and notations essentially follow those in [CS10] and [Sch15] and are summarized in a previous paper [Lam21]. An **oplax/lax adjunction** between double categories is a conjoint pair of arrows in the strict double category of lax and oplax functors [GP04]. An oplax/lax adjunction is **strong** if both functors are pseudo. §2 presents a review of $\mathbb{R}\mathbf{el}(\mathscr{C})$ and its salient properties. It is shown to be a cartesian equipment whose local products satisfy the Modular Law. The theorem above is proved in two parts. §3 is dedicated to proving the first part, namely, that under suitable conditions on $\mathbb{D}$, the identity functor on $\mathbb{D}_0$ extends to an oplax/lax adjunction $\mathbb{R}\mathbf{el}(\mathbb{D}_0) \rightleftarrows \mathbb{D}$. It is shown that under certain further conditions this is in fact a strong adjoint equivalence. In §4 it is seen that related conditions on $\mathbb{D}$ imply that $\mathbb{D}_0$ admits a suitable factorization system for forming a double category of relations $\mathbb{R}\mathbf{el}(\mathbb{D}_0)$. The main theorem above is then basically a corollary to the two parts of the development.

2 Relations as Cartesian Equipments

A calculus of relations is supported by any regular category $\mathscr{C}$. This leads to the formation of a double category $\mathbb{R}\mathbf{el}(\mathscr{C})$ whose salient structures are analyzed in this section. In particular, $\mathbb{R}\mathbf{el}(\mathscr{C})$ is cartesian as a double category and an equipment, meaning that it has all extensions and restrictions, as explained below. These structures provide minimal expectations of any 'double category of relations'. They should also satisfy a modular law. These basic topics are explained in §2.1. It turns out that the horizontal bicategory of every locally posetal cartesian equipment is a 'bicategory of relations'. This leads to the main definition of a 'double category of relations' in §4.2. Finally, the crucial ingredient of tabulators is discussed in 2.3.

2.1 Double Category Structure of Relations

Recall (§A1.3 [Joh01], §I,1.5 [FS90]) that the **image** of a morphism $f\colon A \to B$ is the smallest subobject of B through which f factors, if it exists. Such a factorization $f = me$ is an **image factorization**. A **cover** is a morphism $f\colon A \to B$ whose image is all of B. Denote covers using '$\twoheadrightarrow$' and monics by '$\rightarrowtail$'.

Definition 2.1. A cartesian category $\mathscr{C}$ is **regular** if

1. every morphism has an image factorization;
2. covers are pullback-stable.

These conditions imply the familiar ones, namely, that every kernel has a coequalizer and that regular epimorphisms are pullback-stable (cf. §I,1.566 [FS90]). A **relation** is a monic arrow $R \rightarrowtail A \times B$. Denote these by '$R\colon A \nrightarrow B$'. For any regular category $\mathscr{C}$, take $\mathbb{R}\mathbf{el}(\mathscr{C})$ to denote the double category of relations in $\mathscr{C}$. Its ordinary underlying category is $\mathbb{R}\mathbf{el}(\mathscr{C})_0 = \mathscr{C}$. Its proarrows are relations and its cells θ, as at left below, are morphisms $\theta\colon R \to S$ making the square on the right commute:

$$\begin{tikzcd} A \arrow[r, "R", "|"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\theta"] & B \arrow[d, "g"] \\ C \arrow[r, "S"', "|"{marking}] & D \end{tikzcd} \qquad\qquad \begin{tikzcd} R \arrow[r] \arrow[d, "\theta"'] & A \times B \arrow[d, "f\times g"] \\ S \arrow[r] & C \times D \end{tikzcd}$$

Take $y\colon \mathscr{C} \to \mathbb{R}\mathbf{el}(\mathscr{C})$ to denote the functor sending an object X to the span consisting of identity arrows on X. This is fully faithful by construction. The external source and target functors $\mathrm{src}, \mathrm{tgt}\colon \mathbb{R}\mathbf{el}(\mathscr{C}; \mathcal{F}) \rightrightarrows \mathscr{C}$ are given by taking a cell θ as above to f and to g, respectively. What will be external composition is given by pullback in $\mathscr{C}$ and taking images. That is, given two relations $R \to A \times B$ and $S \to B \times C$ take the pullback $R \times_B S$ in $\mathscr{C}$ and define the composite $R \otimes S$ to be given by the image factorization of $R \times_B S \to A \times C$; the arrow assignment is then induced by minimality of image factorizations. Up-to-iso associativity follows by pullback-stability of covers. With these constructions $\mathbb{R}\mathbf{el}(\mathscr{C})$ is a double category. It is rich with structure summarized in the rest of the subsection. In particular, $\mathbb{R}\mathbf{el}(\mathscr{C})$ is an "equipment," in the following sense.

Definition 2.2 (§4 [Shu08])**.** A double category $\mathbb{D}$ is an **equipment** if $\langle \mathrm{src}, \mathrm{tgt}\rangle\colon \mathbb{D}_1 \to \mathbb{D}_0 \times \mathbb{D}_0$ is a bifibration.

A bit of definitional exegesis is in order. A cell θ in $\mathbb{D}$ is **cartesian** if, and only if, it is a cartesian arrow for the functor $\langle \mathrm{src}, \mathrm{tgt}\rangle \colon \mathbb{D}_1 \to \mathbb{D}_0 \times \mathbb{D}_0$, that is, if given any other cell δ

$$\begin{tikzcd} A \arrow[r, "m", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\theta"] & B \arrow[d, "g"] \\ C \arrow[r, "n"', "|" marking] & D \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "p", "|" marking] \arrow[d, "h"'] \arrow[dr, phantom, "\delta"] & Y \arrow[d, "k"] \\ C \arrow[r, "n"', "|" marking] & D \end{tikzcd}$$

together with arrows u and v such that $f = hu$ and $g = kv$, there is a unique cell $\gamma \colon p \Rightarrow m$ with source h and target k such that $\theta\gamma = \delta$ holds. A **restriction** of a niche as at left below is a cartesian cell

$$\begin{tikzcd} A \arrow[d, "f"'] & B \arrow[d, "g"] \\ C \arrow[r, "n"', "|" marking] & D \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f_! \otimes n \otimes g^*", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\rho"] & B \arrow[d, "g"] \\ C \arrow[r, "n"', "|" marking] & D \end{tikzcd}$$

as on the right. Restrictions of this kind are so-called because they provide the reindexing/substitution/restriction functors for the fibration $\langle \mathrm{src}, \mathrm{tgt}\rangle \colon \mathbb{D}_1 \to \mathbb{D}_0 \times \mathbb{D}_0$. Thus, dually, an opcartesian cell is an opcartesian arrow for the same functor. An **extension** of a "coniche" as on the left below is an opcartesian cell

$$\begin{tikzcd} A \arrow[r, "m", "|" marking] \arrow[d, "f"'] & B \arrow[d, "g"] \\ C & D \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "m", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\xi"] & B \arrow[d, "g"] \\ C \arrow[r, "|" marking] & D \end{tikzcd}$$

as on the right. Put another way, a double category $\mathbb{D}$ is an equipment if and only if $\langle \mathrm{src}, \mathrm{tgt}\rangle \colon \mathbb{D}_1 \to \mathbb{D}_0 \times \mathbb{D}_0$ has all restrictions and extensions and extensions for given pairs of ordinary arrows are left adjoint to the corresponding restrictions (cf. Proposition 9.1.2 [Jac99]). Special cases of these restrictions and extensions are of importance. Recall (cf. [GP04]) that an arrow $f \colon A \to B$ and proarrow $f_! \colon A \nrightarrow B$ are **companions** with unit and counit

$$\begin{tikzcd} A \arrow[r, "y_A", "|" marking] \arrow[d, "1"'] \arrow[dr, phantom, "\eta"] & A \arrow[d, "f"] \\ A \arrow[r, "f_!"', "|" marking] & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f_!", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\epsilon"] & B \arrow[d, "1"] \\ B \arrow[r, "y_B"', "|" marking] & B \end{tikzcd}$$

if $\eta \otimes \epsilon = 1$ and $\epsilon\eta = y_f$ both hold. Dually, an arrow $f \colon A \to B$ and proarrow $f^* \colon B \nrightarrow A$ are **conjoint** with unit and counit

$$\begin{tikzcd} B \arrow[r, "f^*", "|" marking] \arrow[d, "1"'] \arrow[dr, phantom, "\epsilon"] & A \arrow[d, "f"] \\ B \arrow[r, "y_B"', "|" marking] & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "y_A", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\eta"] & A \arrow[d, "1"] \\ B \arrow[r, "f^*"', "|" marking] & A \end{tikzcd}$$

if $\epsilon \otimes \eta = 1$ and $\epsilon\eta = y_f$ hold. In the former case $(f, f_!)$ is said to form a companion pair; in the latter case (f, f^*) is a conjoint pair. $f_!$ is a companion of f and f^* is a conjoint of f. Companions and conjoints completely describe equipment structure in the sense that $\mathbb{D}$ is an equipment if, and only if, $\mathbb{D}$ has all companions and conjoints. A detailed proof is given in [Shu08]. In particular, the manner in which restrictions and extensions can be built from companions and conjoints is suggested in the notation '$f_! \otimes n \otimes g^*$'and '$f^* \otimes m \otimes g_!$'. Restrictions and extensions of external identities will be denoted as '$f_! \otimes g^*$' and '$f^* \otimes g_!$' without the 'y' to reduce notational clutter.

Example 2.3. $\mathbb{S}\mathbf{et}$, $\mathbb{P}\mathbf{rof}$, $\mathbb{R}\mathbf{el}(\mathscr{C})$ are equipments. Companions in $\mathbb{R}\mathbf{el}(\mathscr{C})$ are given by graphs; conjoints by opgraphs. Extensions in $\mathbb{R}\mathbf{el}(\mathscr{C})$ are computed by images; restrictions are given by pullback. In particular, any morphism $e \colon E \to B$ in $\mathscr{C}$ is a cover in $\mathscr{C}$ if, and only if, the corresponding cell y_e coming from the external identity is an extension. For the extension

$$\begin{tikzcd} A \arrow[r, "y_A", "|" marking] \arrow[d, "e"'] \arrow[dr, phantom, "\xi"] & A \arrow[d, "e"] \\ E \arrow[r, "e^* \otimes e_!"', "|" marking] & E \end{tikzcd}$$

results in a globular cell $\gamma\colon e^*\otimes e_! \Rightarrow y_E$ such that $\gamma\xi = y_e$. But this is computed by an image in $\mathscr{C}$. So, e is a cover if, and only if, the unique globular cell γ above is an iso $e^*\otimes e_! \cong y_E$, if and only if, y_e is an extension. □

Definition 2.4 (Cf. §4.2 in [Sch15])**.** The **kernel** of a morphism $f\colon A\to B$ is the restriction ρ of the unit on B along f. Dually, the **cokernel** of f is the extension cell ξ

$$\begin{tikzcd} A \arrow[r, "f_!\otimes f^*", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\rho"] & A \arrow[d, "f"] \\ B \arrow[r, "y_B"', "\shortmid"{marking}] & B \end{tikzcd} \qquad\qquad \begin{tikzcd} A \arrow[r, "y_A", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\xi"] & A \arrow[d, "f"] \\ B \arrow[r, "f^*\otimes f_!"', "\shortmid"{marking}] & B \end{tikzcd}$$

A morphism $e\colon A\to E$ in an equipment is a **cover** if the canonical globular cell is an iso $e^*\otimes e_!\cong y_E$. Dually, a morphim $m\colon E\to B$ is an **inclusion** if the canonical globular cell is an iso $m_!\otimes m^*\cong y_E$.

A pithy or sloganish formulation is that a cover is a morphism with trivial cokernel; an inclusion is one with trivial kernel. Notice that these recall the definitions of simple and entire maps in an allegory (§II,2.13 of [FS90]). This partly explains the choice of notation for companions and conjoints. Below it will be seen that at least for certain 'double categories of relations' $f\mapsto f^*$ is a kind of involution operator similar to the axiomatized in the definition of an allegory.

Preservation of extensions and restrictions by oplax or lax functors will be important throughout. The main result in this connection is the following.

Proposition 2.5. *An oplax double functor preserves extensions; a lax one preserves restrictions. Thus, a pseudo-double functor preserves both.*

Proof. See Proposition 6.4 and its proof in [Shu08]. □

Now, turn to the cartesian structure of relations. Let **Dbl** denote the 2-category of double categories, pseudo-double functors and (vertical) transformations. The definition and properties of cartesian double categories are given in §4 of [Ale18].

Definition 2.6 (Cf. §4.2 of [Ale18])**.** A double category $\mathbb{D}$ is **cartesian** if the double functors $\Delta\colon\mathbb{D}\to\mathbb{D}\times\mathbb{D}$ and $\mathbb{D}\to\mathbf{1}$ have right adjoints in **Dbl**.

If $\mathbb{D}$ is cartesian, then $\mathbb{D}_0$ and $\mathbb{D}_1$ both have finite products (Proposition 4.2.2 of [Ale18]). Additionally $\mathbb{D}$ has "local products" in the following sense.

Lemma 2.7 (See Prop. 4.3.2 of [Ale18])**.** *If $\mathbb{D}$ is a cartesian equipment, then every category $\mathbb{D}(A,B)$ has products.*

Proof. Given two proarrows $m\colon A\nrightarrow B$ and $n\colon A\nrightarrow B$, the product in $\mathbb{D}(A,B)$ is given by taking the restriction

$$\begin{tikzcd} A \arrow[r, dashed, "m\wedge n", "\shortmid"{marking}] \arrow[d, "\Delta"'] \arrow[dr, phantom, "\rho"] & B \arrow[d, "\Delta"] \\ A\times A \arrow[r, "m\times n"', "\shortmid"{marking}] & B\times B \end{tikzcd}$$

along the diagonals. Note that by the construction of restrictions, this is the composite $\Delta_!\otimes(m\times n)\otimes\Delta^*$ which is essentially the formula given in the proof of Theorem 1.6 (ii) of [CW87]. The terminal object is the restriction of the identity on 1 along the unique morphisms from A and B to 1. □

Double categories such as $\mathbb{S}\mathbf{pan}$ and $\mathbb{P}\mathbf{rof}$ are cartesian equipments. The main example however is relations on a regular category.

Lemma 2.8. $\mathbb{R}\mathbf{el}(\mathscr{C})$ *is cartesian and thus has local products which are computed as intersections of relations.*

Proof. The underlying category is $\mathscr{C}$, which has a terminal object and products by assumption. The product of relations $R\colon A\nrightarrow B$ and $S\colon C\nrightarrow D$ is the product

$$R\times S\to A\times B\times C\times D\cong A\times C\times B\times D$$

which is again a relation. This defines the required right adjoint to the diagonal double functor. The required adjoint for the double functor $\mathbb{R}\mathbf{el}(\mathscr{C})\to\mathbf{1}$ uses the fact that $\mathscr{C}$ has a terminal object. The terminal relation is the evident morphism $1\to 1\times 1$. Since restrictions are computed as pullbacks, the formula in the proof of Lemma 2.7 implies that local products of relations are intersections. □

Proposition 2.9 (Separability)**.** *In a locally posetal cartesian equipment, the diagonals are inclusions.*

Proof. Fix an object A. The goal is to prove that $\Delta_! \otimes \Delta^* = y$ via the canonical map. If $\epsilon_! \colon A \nrightarrow 1$ is the counit of A viewed as a comonoid in the horizontal bicategory, the unit $1 \leq \epsilon_! \otimes \epsilon^*$ yields an inequality

$$\Delta_! \otimes \Delta^* \leq \Delta_! \otimes (y \times e_!) \otimes (y \times \epsilon^*) \otimes \Delta^* \tag{2.1}$$

However, the right side is y. For the composite

$$\begin{tikzcd}
A \arrow[r, "\Delta_!", "\shortmid"{marking}] \arrow[d, "\Delta"'] \arrow[dr, phantom, "\rho"] & A \times A \arrow[r, "y \times e_!", "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "1"] & A \times 1 \arrow[d] \\
A \times A \arrow[r, "y"', "\shortmid"{marking}] \arrow[d] & A \times A \arrow[r, "y \times e_!"', "\shortmid"{marking}] \arrow[d, phantom, "1 \times \rho"] & A \times 1 \arrow[d] \\
A \times 1 \arrow[rr, "y"', "\shortmid"{marking}] & & A \times 1
\end{tikzcd}$$

is a restriction cell, implying that $\Delta_! \otimes (y \times e_!) \cong y$. Similarly for $(y \times \epsilon^*) \otimes \Delta^*$. Since $y \leq \Delta_! \otimes \Delta^*$ always holds, this proves the result. Notice that "locally posetal" is not really necessary in the proof. □

The last general property of relations is the interaction between composition, base change and local products. This is the so-called "Modular Law," which is closely related to the Frobenius Law. In an allegory, the former is the assertion that there is a valid inequality

$$RT \wedge S \leq (R \wedge ST^\circ)T$$

written in the usual compositional order. Here $(-)^\circ$ is the involution coming with the allegory structure which gives the right adjoints for maps. Since $(-)^\circ$ satisfies $((-)^\circ)^\circ = 1$, there is the special case

$$RT \wedge S \leq (R \wedge ST)T^\circ.$$

Insofar as an opgraph is a right adjoint to a graph in a double category $\mathbb{R}\mathbf{el}(\mathscr{C})$, it might be expected that there is some such relationship governing their interaction with local products. And indeed this is the case. The relationship is the so-called **modular law**, formulable for any locally posetal cartesian equipment, stating that

$$f^* \otimes R \wedge S \leq f^* \otimes (R \wedge f_! \otimes S) \tag{2.2}$$

holds for any morphism $f \colon A \to B$ and proarrows $R \colon A \nrightarrow X$ and $S \colon B \nrightarrow X$. This statement looks ahead to the further developments of §4.2 where it will be seen in Proposition 4.8 that any double category of relations has an involution $(-)^\circ$ with in particular $(f_!)^\circ = f^*$.

Lemma 2.10. *Local products in* $\mathbb{R}\mathbf{el}(\mathscr{C})$ *satisfy the Modular Law.*

Proof. It suffices to look at the bicategorical fragment of $\mathbb{R}\mathbf{el}(\mathscr{C})$ and prove the modular law in this context since all the relevant structure is the same. A complete proof for this case is that of Proposition A3.1.5 in [Joh01]. □

Remark 2.11*.* In fact the inequality is an equality using a dual formulation of the law. This is equivalent to the **Frobenius Law** for the hyperdoctrine [Law70] given by

$$\mathbb{R}\mathbf{el}(\mathscr{C})(-,-) \colon (\mathbb{D}_0 \times \mathbb{D}_0)^{op} \to \mathbf{Cat} \qquad (A, B) \mapsto \mathbb{R}\mathbf{el}(\mathscr{C})(A, B)$$

stating that extension as a left adjoint to restriction as a right adjoint partially distributes over local products. □

2.2 'Double Categories of Relations'

The connection to cartesian bicategories can be made explicit. This leads to a definition of 'double categories of relations'. The single quotations follow the conventions of [CW87] to distinguish the axiomatic notion presented below from double categories defined to be $\mathbb{R}\mathbf{el}(\mathscr{C})$ for some regular category $\mathscr{C}$.

Recall more precisely that a cartesian bicategory as in [CW87] is a locally posetal bicategory with a monoidal tensor for which every object is a commutative comonoid, every morphism $X \to Y$ is a lax comonoid homomorphism, and every comonoid structure map and counit has a right adjoint. It is further required that this comonoid structure on a given object is the unique one having these right adjoints. Recall [GP99] that any double category $\mathbb{D}$ has a **horizontal bicategory** $\mathcal{H}(\mathbb{D})$ formed by restricting to the globular cells, that is, the cells whose external source and target morphisms are identities.

Proposition 2.12. *The horizontal bicategory of a locally posetal cartesian equipment is a cartesian bicategory.*

Proof. First note that the pseudo-double functor $\times\colon \mathbb{D}\times\mathbb{D}\to\mathbb{D}$ induces a suitable homomorphism of horizontal bicategories. Now, fix an object B. The comultiplication and counit on B are defined to be the proarrows arising in the extension cells

$$\begin{tikzcd} B \arrow[r, "y", "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "\xi"] & B \arrow[d] \\ B \arrow[r, "\Delta_!"', "\shortmid"{marking}] & B\times B \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "y", "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "\xi"] & B \arrow[d] \\ B \arrow[r, "\tau_!"', "\shortmid"{marking}] & 1. \end{tikzcd}$$

Note that these could equally well be given by restriction cells. In any case, both have right adjoints Δ^* and τ^* given by the equipment structure. The double functor $B\times -\colon \mathbb{D}\to\mathbb{D}$ preserves extensions. So, the two composites

$$\begin{tikzcd} B \arrow[r, "\Delta_!", "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "\xi"] & B\times B \arrow[r, "y", "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "\xi"] & B\times B \arrow[d, "1\times\Delta"] \\ B \arrow[r, "\Delta_!"', "\shortmid"{marking}] & B\times B \arrow[r, "y\times\Delta_!"', "\shortmid"{marking}] & B\times(B\times B) \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "\Delta_!", "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "\xi"] & B\times B \arrow[r, "y", "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "\xi"] & B\times B \arrow[d, "\Delta\times 1"] \\ B \arrow[r, "\Delta_!"', "\shortmid"{marking}] & B\times B \arrow[r, "\Delta_!\times y"', "\shortmid"{marking}] & (B\times B)\times B \end{tikzcd}$$

compute the same extension, meaning that the comultiplication law $(y\times\Delta_!)\Delta_! = (\Delta_!\times y)\Delta_!$ must hold. Similarly for the unit law. It remains to see that each morphism $p\colon B\nrightarrow C$ of $\mathcal{H}(\mathbb{D})$ is a lax comonoid homomorphism. But this is straightforward by the construction of the comultiplication and counit morphisms. For example, given the two composite cells

$$\begin{tikzcd} B \arrow[r, "y", "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "\xi"] & B \arrow[r, "p", "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "\Downarrow"] & C \arrow[d, "\Delta"] \\ B \arrow[r, "\Delta_!"', "\shortmid"{marking}] & B\times B \arrow[r, "p\times p"', "\shortmid"{marking}] & C\times C \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "p", "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "1"] & C \arrow[r, "y", "\shortmid"{marking}] \arrow[d] & C \arrow[d, "\Delta"] \\ B \arrow[r, "p"', "\shortmid"{marking}] & C \arrow[r, "\Delta_!"', "\shortmid"{marking}] & C\times C \end{tikzcd}$$

the rightmost is an extension, meaning that

$$p\otimes\Delta_! \le \Delta_!\otimes(p\times p) \tag{2.3}$$

holds as required. The identity rule is similar. Finally, this comonoid structure needs to be seen to be unique. The argument is a hybrid of those proving Lemma A3.2.3 in [Joh01] and Corollary 2.1.6 in [Ale18]. Suppose that (B, d_B, e_B) is another comonoid structure on B for which d and e have right adjoints d^* and e^*. Since y_1 is terminal in $\mathbb{D}_1$, there is a unique cell

$$\begin{tikzcd} X \arrow[r, "e", "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "\exists\,!"] & 1 \arrow[d] \\ 1 \arrow[r, "\shortmid"{marking}] & 1 \end{tikzcd}$$

meaning that $e\le\tau_!$ holds since $\tau_!$ is equivalently given by a cartesian cell over y_1. Using the unit and counit of the adjunctions, it follows that

$$\tau_! \le \tau_! e^* e^* \le \tau_!\tau^* e \le e$$

proving that $e=\epsilon$ holds. Now, use the argument of Corollary 2.1.6 in [Ale18] which shows that d and $\Delta_!$ are coequalized by the projections from $B\times B$ and are thus equal. $\square$

What makes a cartesian bicategory a 'bicategory of relations' is a further discreteness condition, namely, that for each object $B\in\mathbb{B}$, the corresponding comonoid structure morphism Δ and its right adjoint Δ^* satisfy the equation

$$\Delta^*\otimes\Delta = (1\times\Delta)\otimes(\Delta^*\times 1).$$

This is called the **Frobenius Law** in [CW87]. Inasmuch as a locally posetal cartesian equipment has a cartesian bicategory as its bicategorical fragment, it now makes sense to define a 'double category of relations' as a locally posetal cartesian equipment satisfying the analogous discreteness condition.

Definition 2.13. A '**double category of relations**' is a locally posetal cartesian equipment in which every object is **discrete** in the sense that the equation

$$\Delta^* \otimes \Delta_! = (1 \times \Delta_!) \otimes (\Delta^* \times 1) \tag{2.4}$$

holds. Following [CW87] call this the **Frobenius Law**. The single quotation marks in 'double category of relations' will distinguish this axiomatic notion from double categories equal to $\mathbb{R}\mathbf{el}(\mathscr{C})$ for some $\mathscr{C}$.

Consequently, the horizontal bicategory of any 'double category of relations' is indeed a 'bicategory of relations'. It could be viewed as a multi-object compact closed category [KL80]. See Theorem 2.4 in [CW87] for more on that point. For the present purposes, the main consequence of making this connection with 'bicategories of relations' is that the modular law now follows from the 'double category of relations' axioms. This will be used in the proof of Theorem 4.4 below.

Corollary 2.14. *In any 'double category of relations', the modular law 2.2 holds.*

Proof. Remark 2.9(ii) of [CW87] shows that any bicategory of relations satisfies the modular law. Thus, the modular law certainly holds for the horizontal bicategory $\mathcal{H}(\mathbb{D})$ of any bicategory of relations. However, since these adjoints and products are inherited from $\mathbb{D}$, this means that the modular law holds in $\mathbb{D}$ too. □

Corollary 2.15. *In any 'double category of relations', local products of covariant representables $p_!$ are idempotent in the sense that $p_! \cong p_! \wedge p_!$ holds canonically. Equivalently, each diagonal cell*

$$\begin{tikzcd} A \arrow[r, "p_!"] \arrow[d, "\Delta_A"'] \arrow[dr, phantom, "\Delta_{p_!}"] & B \arrow[d, "\Delta_B"] \\ A \times A \arrow[r, "p_! \times p_!"'] & B \times B \end{tikzcd}$$

is cartesian. Dually, local products of contravariant representables p^* are idempotent.

Proof. There are canonical isos

$$p_! \otimes \Delta_! \cong \Delta_! \otimes (p \times p)_! \cong \Delta_! \otimes (p_! \times p_!) \tag{2.5}$$

The leftmost iso is from the fact that two sides compute the same restriction, namely, that of the ordinary morphism $\Delta_B p = (p \times p)\Delta_A$. The rightmost is from the fact that the product functor $\times\colon \mathbb{D} \times \mathbb{D} \to \mathbb{D}$ is pseudo, hence preserves restrictions. As a result, the isomorphism

$$\begin{tikzcd} A \arrow[r, "p_!"] \arrow[d, equal] \arrow[dr, phantom, "1"] & B \arrow[rr, "y"] \arrow[d, equal] \arrow[drr, phantom, "\cong"] & & B \arrow[d, equal] \\ A \arrow[r, "p_!"'] \arrow[d, equal] & B \arrow[r, "\Delta_!"] \arrow[dr, phantom, "\cong"] & B \times B \arrow[r, "\Delta^*"] \arrow[d, equal] \arrow[dr, phantom, "1"] & B \arrow[d, equal] \\ A \arrow[r, "\Delta_!"'] & A \times A \arrow[r, "p_! \times p_!"'] & B \times B \arrow[r, "\Delta^*"'] & B \end{tikzcd}$$

establishes the result by the construction of local products (cf. proof of Lemma 2.7 above). The topmost iso is the fact that Δ_B is an inclusion by Separability in Proposition 2.9. The dual case is analogous. □

Remark 2.16. Equation 2.5 is saying that each $p_!$ is a comonoid homomorphism. Since $p_!$ has a right adjoint p^* this is basically the characterization of so-called "maps" in Lemma 2.5 of [CW87]. □

2.3 Tabulators and Functional Completeness

Allegories and cartesian bicategories have their respective notions of a "tabulation" of a given arrow. In the context of double categories, the tabulator of a proarrow is a kind of finite limit.

Definition 2.17 (See [GP99])**.** A double category $\mathbb{D}$ has **tabulators** if $y\colon \mathbb{D}_0 \to \mathbb{D}_1$ has a right adjoint $\top\colon \mathbb{D}_1 \to \mathbb{D}_0$ in $\mathbf{Dbl}$. In this case, the **tabulator** of a proarrow $m\colon A \to B$ is the object $\top m$ together with the counit cell $\top m \Rightarrow m$. Denote the external source and target by l and r, respectively.

Lemma 2.18. *$\mathbb{R}\mathbf{el}(\mathscr{C})$ has tabulators. The unit of the adjunction $y \dashv \top$ is iso. Equivalently y is fully-faithful.*

Proof. Define $\top \colon \mathbb{R}\mathbf{el}(\mathscr{C})_1 \to \mathbb{R}\mathbf{el}(\mathscr{C})_0$ by sending $R \to A \times B$ to R with the evident assignment on arrows. In other words, $\top$ takes the apex of spans and morphisms between them. The component of the counit at $R \to A \times B$ is the cell given by the morphism of relations

$$\begin{tikzcd} R \arrow[r, "\Delta"] \arrow[d, "1"'] & R \times R \arrow[d, "d\times c"] \\ R \arrow[r, "\langle d,c\rangle"'] & A \times B \end{tikzcd}$$

On the other hand, the unit is up to iso the identity map on a given object A. That is, y takes the diagonal $A \to A \times A$ and then $\top$ takes the apex A, meaning that $1 \cong \top y$ canonically. By a general result on adjoint functors (IV.3.1 in [Mac98]) this is equivalent to the statement that y is fully faithful. □

Definition 2.19 (Cf. §4.3.7 [Ale18]). A double category $\mathbb{D}$ is **unit-pure** if $y \colon \mathbb{D}_0 \to \mathbb{D}_1$ is fully faithful. □

Example 2.20. $\mathbb{S}\mathbf{et}$, $\mathbb{S}\mathbf{pan}(\mathscr{C})$ and $\mathbb{R}\mathbf{el}(\mathscr{C})$ are all unit-pure whereas $\mathbb{P}\mathbf{rof}$ is not.

Since y is always faithful, technically all that is required in the definition is "full." Owing to the lemma above, keep "faithful." Tabulators in $\mathbb{R}\mathbf{el}(\mathscr{C})$ are additionally "strong" in the following sense since extensions are given by taking image factorizations..

Definition 2.21. The tabulator $\langle l, r\rangle \colon \top m \to A \times B$ of a proarrow $m \colon A \nrightarrow B$ is **strong** if m is the cokernel of its tabulator in the sense that $m \cong l^* \otimes r_!$ holds canonically.

In a unit-pure equipment with strong tabulators, inclusions are precisely the monic arrows.

Lemma 2.22. *If $y \colon \mathbb{D}_0 \to \mathbb{D}_1$ is fully faithful, then*

1. *inclusions are monic;*
2. *if tabulators are strong, then monic arrows are inclusions.*

Proof. (1) Assume that $f \colon A \to B$ is an inclusion and take arrows $u, v \colon X \rightrightarrows A$ such that $fu = fv$. Using the fact that f is an inclusion and that $y_{fu} = y_{fv}$, it follows that $y_u = y_v$ holds. From this $u = v$ since y is faithful.

(2) On the other hand, given a monic $f \colon A \to B$, take the tabulator of its kernel. Denote the legs by l and r. Since B is the tabulator of y_B, it follows that $fl = fr$ must hold. But then $l = r$ follows since f is monic. Now, since tabulators are strong, the kernel of f is isomorphic to the cokernel of its tabulator. Thus, there are unique (globular) cells

$$y_A \Rightarrow f_! \otimes f^* \cong l^* \otimes l_! \Rightarrow y_A.$$

Since y is fully faithful, these must compose to the identity. The other composite is also the identity by uniqueness of lifts for cartesian cells. Thus, $f_! \otimes f^* \cong y_A$ holds proving that f is an inclusion. □

Tabulators in $\mathbb{R}\mathbf{el}(\mathscr{C})$ are then monic in the following sense.

Proposition 2.23 (Tabulators are Monic). *The legs of the tabulator of a proarrow $R \to A \times B$ in $\mathbb{R}\mathbf{el}(\mathscr{C})$ are jointly monic and the iso*

$$l^* \otimes l_! \wedge r^* \otimes r_! \cong y \tag{2.6}$$

holds canonically.

Proof. The proof of Lemma 2.18 shows that any relation is its own tabulator. Thus, the first statement is trivial owing to the fact that a proarrow is just a monic arrow $R \to A \times B$. For the second statement, the restriction gives the kernel of the relation:

$$\begin{tikzcd} R \arrow[r, "\langle l,r\rangle_! \otimes \langle l,r\rangle^*", "\shortmid"{marking}] \arrow[d, "\langle l,r\rangle"'] \arrow[dr, phantom, "\rho"] & R \arrow[d, "\langle l,r\rangle"] \\ A \times B \arrow[r, "y"', "\shortmid"{marking}] & A \times B \end{tikzcd}$$

But the composite

$$\begin{tikzcd}
R \arrow[r, "l^*\otimes l_!\wedge r^*\otimes r_!", "|"'] \arrow[d, "\Delta"'] \arrow[dr, phantom, "\rho"] & R \arrow[d, "\Delta"] \\
R\times R \arrow[r, "l^*\otimes l_!\times r^*\otimes r_!"', "|"] \arrow[d, "l\times r"'] \arrow[dr, phantom, "\rho"] & R\times R \arrow[d, "l\times r"] \\
A\times B \arrow[r, "y\times y"', "|"] & A\times B
\end{tikzcd}$$

computes the same restriction. So, if one is y, then the other is too and conversely. ☐

Definition 2.24 (Cf. §3 of [CW87])**.** A cartesian equipment is **functionally complete** if it has tabulators and these are strong and monic in the sense that if $R\colon A \nrightarrow B$ is any proarrow with tabulator $\top R \to A\times B$ with legs l and r, then the equations

$$m \cong l^*\otimes r_! \qquad \text{and} \qquad l^*\otimes l_! \wedge r^*\otimes r_! \cong y$$

both hold.

Consequently, in any unit-pure functionally complete cartesian equipment, the legs of any tabulator are genuinely jointly monic by Lemma 2.22. The main characterization result of the paper, namely, Theorem 4.7, deals precisely with functionally complete 'double categories of relations' satisfying an additional technical condition. It shows that they are equivalent to double categories of the form $\mathbb{R}\mathbf{el}(\mathscr{C})$ for $\mathscr{C}$ regular. The technical condition has to do with the existence of certain "subobject comprehension schema" and is explained in §3.4. As a prelude to this main result, the next section examines conditions under which a cartesian

3 Preliminary Characterization

This section is devoted to the first part of the main result. This is an answer to the question of the conditions under which a double category $\mathbb{D}$ is equivalent to $\mathbb{R}\mathbf{el}(\mathbb{D}_0)$ where $\mathbb{D}_0$ is a regular category in the sense of Definition 2.1. The question is answered in Theorem 3.13 below. The improvement in term of subobject comprehension schema is then Theorem 3.16. The starting point is the now well-established treatment of spans in a cartesian category.

3.1 The Case of Spans

The starting point is a result of [Nie12]. Namely, a double category admits a normalized oplax/lax adjunction if, and only if, it is an equipment with tabulators.

Theorem 3.1 (Theorems 5.5/5.6 of [Nie12])**.** *Let $\mathbb{D}$ denote a double category with pullbacks. The following are equivalent:*

1. *There is an oplax/lax adjunction $F\colon \mathbb{S}\mathbf{pan}(\mathbb{D}_0) \rightleftarrows \mathbb{D}_0 \colon G$ where F is normal and identity on $\mathbb{D}_0$.*
2. *$\mathbb{D}$ has all companions, conjoints and tabulators.*

Proof. That $\mathbb{D}$ has tabulators and is an equipment allows construction of the oplax and lax functors. G is defined on proarrows by taking a tabulator; F is defined on proarrows by taking an extension of a coniche given by a span. ☐

The main result of [Ale18] concerning spans gives equivalent conditions under which such a normalized oplax/lax adjunction is a strong equivalence of double categories. These extra conditions are just that $\mathbb{D}$ is cartesian and possess certain internal Eilenberg-Moore objects defined in the following way.

Definition 3.2 (Cf. §5.3 of [Ale18])**.** A **copoint** of an proarrow $m\colon A\nrightarrow A$ in a double category $\mathbb{D}$ is a cell

$$\begin{tikzcd}
A \arrow[r, "m", "|"'] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow"] & A \arrow[d, equal] \\
A \arrow[r, "y_A"', "|"] & A.
\end{tikzcd}$$

Let $\mathbf{Copt}(\mathbb{D})$ denote the category of pairs (m,γ) where γ is a copoint of the endoproarrow m. The morphisms $(m,\gamma)\to(n,\epsilon)$ are cells $\theta\colon m\Rightarrow n$ of $\mathbb{D}$ such that $\epsilon\theta=\gamma$ holds. A double category $\mathbb{D}$ **admits Eilenberg-Moore objects for copointed endomorphisms** if the inclusion $\mathbb{D}_0\to\mathbf{Copt}(\mathbb{D})$ has a right adjoint.

The characterization of spans is then the following. Its proof is the topic of §5 of the reference and so will not be reproduced here. The Beck-Chevalley condition appearing in the third equivalent condition has not yet been discussed but is stated for an equipment in Definition 3.9 below. It discussion is postponed only because it fits better later on and is not part of the template for the present results.

Theorem 3.3 (Theorem 5.3.2 of [Ale18]). *For a double category $\mathbb{D}$ the following are equivalent:*

1. *$\mathbb{D}$ is equivalent to $\mathbb{S}\mathbf{pan}(\mathscr{C})$ for some finitely-complete category $\mathscr{C}$.*
2. *$\mathbb{D}$ is a unit-pure cartesian equipment admitting Eilenberg-Moore objects for copointed endoproarrows.*
3. *$\mathbb{D}_0$ has pullbacks satisfying the strong Beck-Chevalley condition and the canonical functor*

$$\mathbb{S}\mathbf{pan}(\mathbb{D}_0) \to \mathbb{D}$$

is an equivalence of double categories.

Proof. This is stated and proved completely in §5.3 of the reference. $\square$

The development for relations will follow this pattern. Namely, start with conditions equivalent to the existence of an oplax/lax adjunction and isolate the further conditions under which such an adjunction is a strong equivalence of double categories.

3.2 Equivalent Conditions for Oplax/Lax Adjunction

First develop the relation version of Niefield's Theorem 3.1 quoted above. This appears as Theorem 3.8 below. Consider first some necessary conditions.

Lemma 3.4. *Let $\mathbb{D}$ denote a double category with $\mathbb{D}_0$ regular. Suppose that $F\colon \mathbb{R}\mathbf{el}(\mathbb{D}_0) \rightleftarrows \mathbb{D}\colon G$ is a normalized oplax/lax adjunction that is the identity on $\mathbb{D}_0$. It then follows that $\mathbb{D}$*

1. *is an equipment;*
2. *has monic tabulators;*
3. *the unit $1 \Rightarrow \top y$ is an iso;*
4. *and for each cover, the cell*

$$\begin{tikzcd} A \arrow[r, "y_A", "\shortmid"{marking}] \arrow[d, two heads, "e"'] \arrow[dr, phantom, "y_e"] & A \arrow[d, two heads, "e"] \\ E \arrow[r, "y_E"', "\shortmid"{marking}] & E \end{tikzcd}$$

is an extension in $\mathbb{D}$.

Proof. Take $f\colon A \to B$ in $\mathbb{D}_0$. The graph and opgraph give the companion and conjoint in in $\mathbb{R}\mathbf{el}(\mathbb{D}_0)$. The images under F given the corresponding companion and conjoint in $\mathbb{D}$ making it an equipment. Since oplax functors preserve extensions and every cell

$$\begin{tikzcd} A \arrow[r, "\Delta", "\shortmid"{marking}] \arrow[d, two heads, "e"'] \arrow[dr, phantom, "e"] & A \arrow[d, two heads, "e"] \\ E \arrow[r, "\Delta"', "\shortmid"{marking}] & E \end{tikzcd}$$

is one in $\mathbb{R}\mathbf{el}(\mathbb{D}_0)$, the corresponding image under F is an extension, making y_e an extension in $\mathbb{D}$ since it is isomorphic to Fe by normalization. Existence of tabulators results from the fact that the composite

$$\mathbb{D}_1 \xrightarrow{G_1} \mathbb{R}\mathbf{el}(\mathbb{D}_0)_1 \xrightarrow{apex} \mathbb{D}_0$$

is a right adjoint for $y\colon \mathbb{D}_0 \to \mathbb{D}_1$. By normalization of G, its unit is an isomorphism. $\square$

Now, it can be seen that for the most part these conditions are also sufficient. Do this in a series of lemmas: one developing F, another for G, and a final one putting together the adjunction. It is worth doing the details of these constructions since they shows precisely what is required for the resulting adjunction to be a strong adjoint equivalence. Assume throughout that $\mathbb{D}_0$ is regular.

Lemma 3.5. *If $\mathbb{D}$ has companions and conjoints and y_e is an extension for each cover e, then the identity functor $1\colon \mathbb{D}_0 \to \mathbb{D}_0$ extends to an opnormal oplax functor $F\colon \mathbb{R}\mathbf{el}(\mathbb{D}_0) \to \mathbb{D}$.*

Proof. For an relation $R \to A \times B$, take the image FR in $\mathbb{D}$ to be the proarrow $A \nrightarrow B$ arising in the canonical extension

$$\begin{tikzcd}
R \arrow[r, "y", "|" marking] \arrow[d, "d"'] \arrow[dr, phantom, "\xi_R"] & R \arrow[d, "c"] \\
A \arrow[r, "d^*\otimes c_!"', "|" marking] & B
\end{tikzcd}$$

That the cell is opcartesian gives the arrow assignment, yielding a functor $F_1\colon \mathbb{R}\mathbf{el}(\mathbb{D}_0)_1 \to \mathbb{D}_1$ by uniqueness properties. Comparison cells for composition are given using the extension property of the composite cell ξy_e. That is, they arise as in the lower-left corner of the diagram:

$$\begin{tikzcd}
R\times_B S \arrow[rrrr, "R\times_B S", "|" marking] \arrow[d, equal] & & & & R\times_B S \arrow[d, equal] \\
R\times_B S \arrow[rr, "R\times_B S", "|" marking] \arrow[d, "p"'] \arrow[dr, phantom, "y_p"] & & R\times_B S \arrow[rr, "R\times_B S", "|" marking] \arrow[dl, "p"'] \arrow[dr, "q"] & & R\times_B S \arrow[d, "q"] \arrow[dl, phantom, "y_q"] \\
R \arrow[r, "R", "|" marking] \arrow[d, "d"'] \arrow[dr, phantom, "\xi_R"] & R \arrow[d, "c"] & & S \arrow[r, "S"', "|" marking] \arrow[d, "d"'] \arrow[dr, phantom, "\xi_S"] & S \arrow[d, "c"] \\
A \arrow[r, "d^*\otimes_R c_!"', "|" marking] & B \arrow[rr, equal] & & B \arrow[r, "d^*\otimes_S c_!"', "|" marking] & C
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
R\times_B S \arrow[rr, "R\times_B S", "|" marking] \arrow[d, two heads, "e"'] \arrow[drr, phantom, "y_e"] & & R\times_B S \arrow[d, two heads, "e"] \\
R\otimes S \arrow[rr, "R\otimes S"', "|" marking] \arrow[d, "d"'] \arrow[drr, phantom, "\xi"] & & R\otimes S \arrow[d, "c"] \\
A \arrow[rr, "d^*\otimes c_!"', "|" marking] \arrow[d, equal] \arrow[drr, phantom, "\exists!\,\phi_{R,S}"] & & C \arrow[d, equal] \\
A \arrow[r, "d^*\otimes_R c_!"', "|" marking] & B \arrow[r, "d^*\otimes_S c_!"', "|" marking] & C
\end{tikzcd}$$

In general these are not invertible. The coherence laws for an oplax functor follow by the fact that all the cells are defined using the uniqueness clause of the lifting property of opcartesian cells. ☐

Lemma 3.6. *If $\mathbb{D}$ has monic tabulators, then $1\colon \mathbb{D}_0 \to \mathbb{D}_0$ extends to a normal lax functor $G\colon \mathbb{D} \to \mathbb{R}\mathbf{el}(\mathbb{D}_0)$.*

Proof. Write $\top\colon \mathbb{D}_1 \to \mathbb{D}_0$ for the right adjoint to $y\colon \mathbb{D}_0 \to \mathbb{D}_1$. For a proarrow $p\colon A \nrightarrow B$ in $\mathbb{D}$, define the image Gp in $\mathbb{R}\mathbf{el}(\mathbb{D}_0)$ to be the inclusion $Tp \to A \times B$ given by the tabulator of p. By the universal property of tabulators this induces a functorial arrow assignment yielding the required functor $G_1\colon \mathbb{D}_1 \to \mathbb{R}\mathbf{el}(\mathscr{C})$. Externally this is lax-functorial. For composable proarrows $p\colon A \nrightarrow B$ and $q\colon B \nrightarrow C$, the morphism

$$\begin{tikzcd}
\top p\otimes \top q \arrow[r] \arrow[d, dashed, "\gamma_{p,q}"'] & A\times C \arrow[d] \\
\top(p\otimes q) \arrow[r] & A\times C
\end{tikzcd}$$

gives the required laxity cell $\gamma\colon Tp\otimes Tq \Rightarrow T(p\otimes q)$. This arrow exists by orthogonality of the factorization system on $\mathbb{D}_0$. The naturality and associativity conditions for a lax functor follow by the uniqueness of image factorizations, the fact that tabulators are jointly inclusions, and the fact that $\mathcal{E}$ is pullback-stable. Unit comparison cells are induced again from the universal property of tabulators; given an object

$$\begin{tikzcd}
A \arrow[r, "\Delta"] \arrow[d, dashed, "\gamma_A"'] & A\times A \arrow[d, "1\times 1"] \\
\top y_A \arrow[r] & A\times A
\end{tikzcd}$$

defines the required cell $\gamma_A\colon y_A \Rightarrow \top y_A$. Note that G is normalized if, and only if, $y\colon \mathbb{D}_1 \to \mathbb{D}_0$ is fully faithful. ☐

Now put everything together. It remains to check a few details from the definition of an oplax/lax adjunction.

Proposition 3.7. *If $\mathbb{D}$ is an equipment with tabulators in which for each cover e the cell y_e is an extension, the functors $F_1\colon \mathbb{R}\mathbf{el}(\mathbb{D}_0)_1 \rightleftarrows \mathbb{D}_1\colon G_1$ of the previous lemmas form an adjunction $F_1 \dashv G_1$.*

Proof. Develop the unit $\eta\colon 1 \Rightarrow G_1F_1$. Starting with a relation $R \rightarrowtail A \times B$, take the canonical extension and then its tabulator. By the universal property of the tabulator, there is a unique morphism $R \to \top(d^* \otimes_R c_!)$ fitting into

$$\begin{tikzcd}
R \arrow[r] \arrow[d, "\eta_R"'] & A \times B \arrow[d, equal] \\
\top(d^* \otimes_R c_!) \arrow[r] & A \times B
\end{tikzcd}$$

making a morphism of relations. Take this to be the component η_R. These are natural in R by the uniqueness aspect of the universal property of tabulators. On the other hand, components of the counit $\epsilon\colon F_1G_1 \Rightarrow 1$ are given in the following way. For a given proarrow $p\colon A \nrightarrow B$, the proarrow F_1G_1p is the extension of the image of the tabulator $\top p$. The counit component ϵ_p arises as in the the left bottom corner of the diagram

$$\begin{tikzcd}
\top p \arrow[r, "y", "\shortmid"{marking}] \arrow[d, "l"'] \arrow[dr, phantom, "\xi_{\top p}"] & \top p \arrow[d, "r"] \\
A \arrow[r, "d^* \otimes_{\top p} c_!"', "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\exists !"] & B \arrow[d, equal] \\
A \arrow[r, "p"', "\shortmid"{marking}] & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\top p \arrow[r, "y", "\shortmid"{marking}] \arrow[dd, "l"'] \arrow[ddr, phantom, "\tau_p"] & \top p \arrow[dd, "r"] \\
 & \\
A \arrow[r, "p"', "\shortmid"{marking}] & B
\end{tikzcd}$$

since the extension $\xi_{\top p}$ is opcartesian. Again this is natural in p by functoriality of tabulators and uniqueness clauses of universal properties. Triangle identities follow by construction. For example, given a relation $R \rightarrowtail A \times B$, verify that $F_1\eta_R\epsilon_{F_1R} = 1$ holds. There are equalities of cells

$$\begin{tikzcd}
R \arrow[r, "y", "\shortmid"{marking}] \arrow[d, "d"'] \arrow[dr, phantom, "\xi_R"] & R \arrow[d, "c"] \\
A \arrow[r, "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "F_1\eta_R"] & B \arrow[d, equal] \\
A \arrow[r, "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\epsilon"] & B \arrow[d, equal] \\
A \arrow[r, "d^* \otimes_R c_!"', "\shortmid"{marking}] & B
\end{tikzcd}
=
\begin{tikzcd}
R \arrow[r, "y", "\shortmid"{marking}] \arrow[d, "\eta_R"'] \arrow[dr, phantom, "y_{\eta_R}"] & R \arrow[d, "\eta_R"] \\
\top(d^* \otimes c_!) \arrow[r, "\shortmid"{marking}] \arrow[d, "d"'] \arrow[dr, phantom, "\xi_{\top(d^* \otimes c_!)}"] & \top(d^* \otimes c_!) \arrow[d, "c"] \\
A \arrow[r, "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\epsilon"] & B \arrow[d, equal] \\
A \arrow[r, "d^* \otimes_R c_!"', "\shortmid"{marking}] & B
\end{tikzcd}
=
\begin{tikzcd}
R \arrow[r, "y", "\shortmid"{marking}] \arrow[ddd, "d"'] \arrow[dddr, phantom, "\xi_R"] & R \arrow[ddd, "c"] \\
 & \\
 & \\
A \arrow[r, "d^* \otimes_R c_!"', "\shortmid"{marking}] & B
\end{tikzcd}$$

by construction. The leftmost holds by the definition of $F_1\eta_R = d_!\eta_R c^*$; the right holds by construction of $\epsilon_{d_!Rc^*}$. Now, the composite in the lower left is $F_1\eta_R\epsilon_{F_1R}$. It must be an identity since ξ_R occurring on both sides is an extension. Verifying the other triangle identity is a similar kind of argument but more straightforward. $\square$

Theorem 3.8. *For a double category $\mathbb{D}$ where $\mathbb{D}_0$ is regular, the identity functor $\mathbb{D}_0 \to \mathbb{D}_0$ extends to an oplax/lax adjunction $F\colon \mathbb{R}\mathbf{el}(\mathbb{D}_0) \rightleftarrows \mathbb{D}\colon G$ if, and only if,*

1. *$\mathbb{D}$ is a unit-pure equipment;*
2. *has monic tabulators;*
3. *y_e is an extension for each cover e.*

Proof. There remains only to verify the remaining conditions of an oplax/lax adjunction. These are those of (d) in §3.2 of [GP04]. Given composable relations $R \to A \times B$ and $S \to B \times C$, the components of η need to be coherent with external composition and laxity cells. But the morphisms of relations on each side of

$$
\begin{tikzcd}
R \otimes S \arrow[r] \arrow[d, "\eta_{R\otimes S}"'] & A \times C \arrow[d, equal] \\
\top(d^* \otimes_{R\otimes S} c_!) \arrow[r] \arrow[d, "\top\phi"'] & A \times C \arrow[d, equal] \\
\top(d^* \otimes_{R\otimes S} c_!) \arrow[r] & A \times C
\end{tikzcd}
\qquad
\begin{tikzcd}
R \otimes S \arrow[r] \arrow[d, "\eta_R \otimes \eta_S"'] & A \times C \arrow[d, equal] \\
\top(d^* R c_!) \otimes \top(d^* S c_!) \arrow[r] \arrow[d, "\gamma"'] & A \times C \arrow[d, equal] \\
\top(d^* \otimes_{R\otimes S} c_!) \arrow[r] & A \times C
\end{tikzcd}
$$

are the same by the uniqueness of image factorizations. Similarly, that components of ϵ are coherent with external composition follows by the construction of ϕ and the uniqueness property of cells induced by opcartesian cells. ☐

3.3 Conditions for Adjoint Equivalences

The constructions from the previous subsection lead to the main result characterizing the existence of an adjoint equivalence $\mathbb{D} \simeq \mathbb{R}\mathbf{el}(\mathscr{C})$. The proofs of the preliminaries to Theorem 3.8 and some extra streamlining reveal two further conditions guaranteeing an adjoint equivalence. Namely, these are that tabulators are strong and that every relation is a tabulator of its cokernel.

To start, recall that a "strong" adjoint equivalence of double categories is an oplax/lax adjoint equivalence where both functors are pseudo. In the present development, the proof above show that this amounts to a Beck-Chevalley condition and the requirement that tabulators are in a particular sense "functorial." In logic, Beck-Chevalley is the condition, roughly speaking, that substitution commutes with quantification (cf. §1.8 of [Jac99]). Categorically, this is to ask that certain adjoints partially commute. Double categorically this is expressed by the following.

Definition 3.9 (Cf. §13 of [Shu08] and §5.2 of [Ale18])**.** An equipment $\mathbb{D}$ satisfies the **Beck-Chevalley condition** if for any pullback square

$$
\begin{tikzcd}
\cdot \arrow[r, "q"] \arrow[d, "p"'] & \cdot \arrow[d, "g"] \\
\cdot \arrow[r, "f"'] & \cdot
\end{tikzcd}
$$

the associated composite cell

$$
\begin{tikzcd}
\cdot \arrow[r, "p^*", "|" marking] \arrow[d, "1"'] \arrow[dr, phantom, "\Downarrow"] & \cdot \arrow[r, "y", "|" marking] \arrow[d, "p"] & \cdot \arrow[r, "q_!", "|" marking] \arrow[d, "q"'] \arrow[dr, phantom, "\Downarrow"] & \cdot \arrow[d, "1"] \\
\cdot \arrow[r, "|" marking] \arrow[d, "1"'] \arrow[dr, phantom, "\Downarrow"] & \cdot \arrow[d, "f"] & \cdot \arrow[r, "|" marking] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow"] & \cdot \arrow[d, "1"] \\
\cdot \arrow[r, "f^*"', "|" marking] & \cdot \arrow[r, "y"', "|" marking] & \cdot \arrow[r, "g_!"', "|" marking] & \cdot
\end{tikzcd}
$$

is invertible.

Note that precisely this composite cell appears in the proof of Lemma 3.5 where the oplax comparison cells were induced. However, Beck-Chevalley is implied by another condition involved in the definition of "functionally complete" (Definition 2.24) which will end up being supposed in the characterization theorem anyway.

Lemma 3.10. *If $\mathbb{D}$ is an equipment with strong tabulators, then $\mathbb{D}_0$ has pullbacks that satisfy Beck-Chevalley. In particular, the oplax functor F as in Lemma 3.5 is pseudo.*

Proof. That $\mathbb{D}_0$ has pullbacks is proved under similar conditions in the next subsection. Proposition 5.2.3 of [Ale18] proves the entire result in detail. ☐

Now turn to the lax functor G. As in the proof of Lemma 3.6 composable proarrows p and q induce a morphism between tabulators

$$
\begin{tikzcd}
\top p \otimes \top q \arrow[r] \arrow[d, dashed, "\gamma_{p,q}"'] & A \times C \arrow[d] \\
\top(p \otimes q) \arrow[r] & A \times C
\end{tikzcd}
$$

making a commutative square. If $\mathbb{D}_0$ is regular, then γ is an iso if, and only if, it is a cover.

Lemma 3.11. *If every relation tabulates its cokernel, then each γ as above is an iso. In particular, the lax functor G of Lemma 3.6 is pseudo.*

Proof. The composite of tabulators in the top row of the diagram above is a relation, hence a tabulator under the assumption. Thus, the unique morphism $\gamma_{p,q}$ is an iso. □

The condition that every relation tabulates is cokernel is a powerful one. It will be discussed more in §3.4. For now it is worth noting that it implies "unit-pure."

Lemma 3.12. *If every relation of $\mathbb{D}$ tabulates its cokernel, then $\mathbb{D}$ is unit pure.*

Proof. Given morphisms $f, g\colon A \rightrightarrows B$, since the identity spans on A and B tabulate the corresponding identity proarrows, $f = g$ must hold by uniqueness. □

Now, the first part of the characterization result can be given.

Theorem 3.13. *Suppose that $\mathbb{D}_0$ is regular. The identity functor $1\colon \mathbb{D}_0 \to \mathbb{D}_0$ extends to an adjoint equivalence of pseudo-functors*

$$F\colon \mathbb{R}\mathbf{el}(\mathbb{D}_0) \rightleftarrows \mathbb{D}\colon G$$

if, and only if,

1. *y_e is an extension cell for each cover e;*
2. *$\mathbb{D}$ is functionally complete;*
3. *every relation tabulates its cokernel.*

Proof. The conditions are sufficient. For necessity, note that "relations tabulate" and each y_e is an extension will imply the existence of the oplax/lax adjunction by Theorem 3.8. This uses Lemma 3.12 showing that $\mathbb{D}$ must be unit-pure and Lemma 2.22 showing that inclusions are then precisely the monics. So, it needs only to be seen that the lax and oplax functors are pseudo; and that these induce an adjoint equivalence.

Strong tabulators implies Beck-Chevalley by Lemma 3.10. The composite cell on the left-hand side of the equation in the proof of Lemma 3.5 contains the Beck-Chevalley cell and is thus opcartesian, making the comparison cell $\phi_{R,S}$ induced there invertible, meaning that F is pseudo. For G, that relations tabulate means that each comparison γ is iso by Lemma 3.11, hence that each laxity comparison cells $\gamma_{p,q}$ is iso.

Now, the unit and counit of the induced adjunction are invertible. On the one hand, by construction, the unit $\eta_R\colon R \to \top(d^* \otimes c_!)$ is an isomorphism since R is isomorphic to the tabulator of its cokernel $d^* \otimes c_!$. On the other hand, concerning the counit, ϵ_p is iso if, and only if, p is the cokernel of its tabulator. But this is precisely the condition that tabulators are "strong" in "functionally complete." □

3.4 Comprehension Schema

The goal of this subsection is to eliminate or at least explain the unnatural condition of the previous result that every relation tabulates its cokernel. This is a sort of regularity condition on monomorphisms. It can be explained, or put in a more natural setting, by looking at generalized comprehension schema. In fact this condition solves a problem that has not appeared yet, but will arise in the construction of a factorization system on $\mathbb{D}_0$ in §4.1 below. This is the issue of whether tabulators provide an image factorization for $\mathbb{D}_0$. Specifically the issue is that of minimality, which appears to be equivalent to the statement that every relation tabulates its cokernel.

Suppose that $\mathbb{D}$ is a functionally complete cartesian equipment. Taking tabulators induces a functor to subobjects

$$\top\colon \mathbb{D}(A, B) \longrightarrow \mathrm{Sub}(A \times B) \tag{3.1}$$

by the universal property of tabulators. It is well-defined by the "monic" condition in "functionally complete." A couple of instructive examples are worth keeping in mind. The most immediately relevant is that or an ordinary topos $\mathscr{E}$. Look as this as a double category with a proarrow $p\colon A \nrightarrow B$ being a morphism $p\colon A \times B \to \Omega$. Reducing to the case where $B = 1$, the tabulator is simply the subobject classified by p and that the function $\top$ is a bijection is precisely the statement of the universal property of the subobject classifier. Another example is less relevant and requires a slight stretch of the imagination. Owing to the existence of the so-called "comprehensive factorization" [SW73], one can think of a discrete opfibration as a kind of subobject. This is precisely the approach taken in the development of 2-toposes [Web07]. In the case of $\mathbb{D} = \mathbb{P}\mathbf{rof}$, the tabulator is precisely the elements construction associated to a set-valued functor. The "Representation Theorem" for discrete opfibrations is that this has a pseudo-inverse, yielding an equivalence of categories

$$\top\colon \mathbb{P}\mathbf{rof}(\mathscr{A}, 1) \longrightarrow \mathbf{DOpf}(\mathscr{A}, 1) \tag{3.2}$$

on the pattern of 3.1 above. In each case, there can be constructed a sort of inverse, or pseudo-inverse, to the tabulator/elements construction, yielding what one might think of as a **comprehension scheme**. The type varies depending upon a choice of a certain monad. More on this below in §5.3 In any case, think of the inverse as a **fibers construction**. In the context of topos theory, the fibers construction is the characteristic function of a subset. In the present context, this fibers construction takes the following form.

Lemma 3.14. *$\top$ is an equivalence if, and only if, every relation $A \nrightarrow B$ is a tabulator of its cokernel.*

Proof. $\top$ is fully faithful by uniqueness of the arrows induced between tabulators. That relations tabulate cokernels is equivalent to the statement that $\top$ is essentially surjective. A choice of cokernels makes a strong equivalence. □

Definition 3.15. A functionally complete double category admits a **subobject comprehension scheme** if each $\top$ as above is a (strong) equivalence of categories.

Theorem 3.16. *Suppose that $\mathbb{D}_0$ is regular. The identity functor $1\colon \mathbb{D}_0 \to \mathbb{D}_0$ extends to an adjoint equivalence of pseudo-functors*

$$F\colon \mathbb{R}\mathbf{el}(\mathbb{D}_0) \rightleftarrows \mathbb{D}\colon G$$

if, and only if,

1. *y_e is an extension cell for each cover e;*
2. *$\mathbb{D}$ has a subobject comprehension scheme.*

Proof. This is just Theorem 3.13 in light of the definition above. □

4 Cartesian Equipments as 'Double Categories of Relations'

Theorem 3.13 prompts consideration of conditions under which $\mathbb{D}_0$ is regular. This question is answered in Theorem 4.4 where it is shown that if $\mathbb{D}$ is a 'double category of relations' with a subobject comprehension scheme, then $\mathbb{D}_0$ is regular. This leads to the main characterization result in Theorem 4.7. Turning to some auxiliary points of interest, it is shown in Corollary 4.10 that any double category of relations is regular as an equipment in the sense of [Sch15]. A final subsection discusses the analogoue of "division allegories" and gives conditions under which a double category of relation is a double category of relations in a topos.

4.1 Conditions for Regularity

The point of this subsection is to prove that $\mathbb{D}_0$ is regular under the assumption that $\mathbb{D}$ is a 'double category of relations' with a subobject comprehension scheme. It was shown in Theorem 3.5 of [CW87] that $\mathbf{Maps}(\mathbb{B})$, the category of maps in any 'bicategory of relations', is regular and moreover that $\mathfrak{Rel}(\mathbf{Maps}(\mathbb{B})) \simeq \mathbb{B}$ as bicategories. Maps are those morphisms p having a right adjoint. Here, however, following the set-up from Theorem 3.16, focus on $\mathbb{D}_0$. This is almost a category of maps in $\mathbb{D}$, since any ordinary morphism f induces a companion and conjoint $f_! \dashv f^*$. However, given an arbitrary proarrow with a right adjoint, there is not necessarily a way of recovering an ordinary arrow in $\mathbb{D}_0$. The point of the development is that given the other assumptions, $\mathbb{D}_0$ nonetheless works as a base for taking relations.

Assume that $\mathbb{D}$ is a 'double category of relations' with a subobject comprehension scheme as in Definition 3.15. In particular, $\mathbb{D}$ is functionally complete and every relation tabulates it cokernel. The next two lemma construct an image factorization for any morphism of $\mathbb{D}_0$.

Lemma 4.1. *Any $f\colon A \to B$ in $\mathbb{D}_0$ has an image factorization $f = le$ as a cover e followed by a monic l.*

Proof. By the universal property of the tabulator there is a factorization of the cokernel of f

$$\begin{tikzcd} A \arrow[r, "y", "\shortmid"{marking}] \arrow[d, "e"'] \arrow[dr, phantom, "\exists !"] & A \arrow[d, "e"] \\ \top(f^* \otimes f_!) \arrow[r, "y", "\shortmid"{marking}] \arrow[d, "l"'] \arrow[dr, phantom, "\tau"] & \top(f^* \otimes f_!) \arrow[d, "r"] \\ B \arrow[r, "f^* \otimes f_!"', "\shortmid"{marking}] & B \end{tikzcd} \quad = \quad \begin{tikzcd} A \arrow[r, "y", "\shortmid"{marking}] \arrow[dd, "f"'] \arrow[ddr, phantom, "\xi"] & A \arrow[dd, "f"] \\ & \\ B \arrow[r, "f^* \otimes f_!"', "\shortmid"{marking}] & B \end{tikzcd}$$

Since y_B is the tabulator of y_B, it follows that $l = r$. Now, $\langle l, l \rangle$ is an inclusion, but by Lemmas 3.12 and 2.22, it is therefore monic. Thus, l itself must be monic. It needs to be seen that e is a cover. For this, take any other monic $m\colon M \to B$ through which f factors by say $p\colon A \to M$. The cokernel of f, since it is an opcartesian cell, factors through the cokernel of m. Therefore, since M tabulates the cokernel of $\langle m, m \rangle$, l factors through m uniquely. In other words, l is the smallest subobject through which f factors, meaning that $f = le$ is an image factorization. ☐

Remark 4.2. It is worth pausing here to explain the importance of the assumption that relation tabulate. This was used above to prove essentially that e is extremal. In each of [CW87], [FS90], and [Joh01], there is only one level of arrow, so the corresponding map is extremal is a consequence of the fact that it turns out to be both simple and entire. However, this same move cannot be made in the present context of double categories. For the induced arrow to the tabulator does not provably have an inverse from properties of its companion and conjoint. Precisely what is needed is the “fibers construction” coming with the subobject comprehension scheme. ☐

Now that $\mathbb{D}_0$ has its factorization system, for regularity, it must be shown that covers are pullback-stable. First it needs to be seen that pullbacks exist. The proof follows closely that of Proposition 3.2.7 in [Joh01]. See also Proposition 5.2.3 of [Ale18].

Lemma 4.3. *$\mathbb{D}_0$ has all pullbacks.*

Proof. Take a corner diagram $h\colon A \to C \leftarrow B\colon e$. Take the tabulator of the restriction as in the diagram

$$\begin{tikzcd} \top(h_! \otimes e^*) \arrow[r, "y", "\shortmid"{marking}] \arrow[d, "d"'] \arrow[dr, phantom, "\tau"] & \top(h_! \otimes e^*) \arrow[d, "c"] \\ A \arrow[r, "h_! \otimes e^*", "\shortmid"{marking}] \arrow[d, "h"'] \arrow[dr, phantom, "\xi"] & B \arrow[d, "e"] \\ C \arrow[r, "y"', "\shortmid"{marking}] & C \end{tikzcd}$$

The arrows d and c now complete the pullback square

$$\begin{tikzcd} P \arrow[r, "c"] \arrow[d, "d"'] & B \arrow[d, "e"] \\ A \arrow[r, "h"'] & C \end{tikzcd}$$

The square commutes because C is the tabulator of y_C. The universal property for the pullback follows by the universal property of the tabulator. ☐

Now it can be seen that $\mathbb{D}_0$ is regular. Pullback-stability of covers is all that remains. Recall that “functionally complete” implies the Beck-Chevalley condition by Lemma 3.10.

Theorem 4.4. *If $\mathbb{D}$ is cartesian, then $\mathbb{D}_0$ is a regular category, hence $\mathbb{R}\mathbf{el}(\mathbb{D}_0)$ is well-defined.*

Proof. Since $\mathbb{D}$ is cartesian, $\mathbb{D}_0$ has finite products. Existence of pullbacks was proved above in Lemma 4.3. It needs only to be seen that covers are stable under pullback. For this let $e\colon B \to C$ denote a cover. The pullback square

$$\begin{tikzcd} P \arrow[r, "c"] \arrow[d, "d"'] & B \arrow[d, "e", two heads] \\ A \arrow[r, "h"'] & C \end{tikzcd}$$

is formed using the tabulator of the restriction $h_! \otimes e^*$ as in Lemma 4.3. On the one hand, since $h_! \otimes h^*$ is a restriction, there is a canonical cell $y_A \Rightarrow h_! \otimes h^*$. To see that d is a cover, calculate that

$$\begin{aligned} y_A &= y \wedge h_! \otimes h^* \\ &= y \wedge h_! \otimes e^* \otimes e_! \otimes h^* && (e \text{ is a cover}) \\ &= y \wedge d^* \otimes c_! \otimes c^* \otimes d_! && (\text{Beck-Chevalley 3.9}) \\ &\leq d^* \otimes (d_! \wedge c_! \otimes c^* \otimes d_!) && (\text{Modular Law 2.2}) \\ &\leq d^* \otimes d_!. \end{aligned}$$

Since there is always a canonical map $d^* \otimes d_! \Rightarrow y$, this shows by the hypothesis that y is fully faithful that $d^* \otimes d_! \cong y$ holds, meaning that d is indeed a cover, proving that $\mathbb{D}_0$ is regular. □

Remark 4.5. A shorter proof of Theorem 4.4 might simply quote the criteria of Theorem 4.4.4 in [Jac99]. This result says that a finitely-complete category $\mathscr{C}$ is regular if, and only if, the subobject fibration $\mathrm{Sub}(\mathscr{C}) \to \mathscr{C}$ has coproducts satisfying a corresponding Frobenius identity. This seems a rather easy criterion to use since under the hypotheses of the subsection, inclusions are monic and the Modular Law 2.2 is very close to what is meant by "Frobenius" in the reference. However, without proving the equivalence of the conditions in the quoted result, it is not clear that this approach would be of considerable benefit. The space-saving otherwise would result in a loss of explicitness in the constructions whereas on the other hand proving the theorem would involve for the most part reproducing the work done here already. □

4.2 Characterization Theorem

Now, the main result of the paper can be given. The goal is to use Theorem 3.16. For this note the following result, saying, essentially, that "covers are covers."

Lemma 4.6. *Let $\mathbb{D}$ denote a 'double category of relations' with a subobject comprehension scheme. Any cover in the regular category $\mathbb{D}_0$ is then one in $\mathbb{D}$.*

Proof. First show that e as in the proof of Lemma 4.1 is a cover in the equipment structure. Note that $e_! = f_! \otimes l^*$ holds and dually $e^* = l_! \otimes f^*$ holds too, since in each case either side computes the same restriction. Now, f and l have the same cokernel because tabulators are strong. This means that

$$e^* \otimes e_! = l_! \otimes f^* \otimes f_! \otimes l^* = l_! \otimes l^* \otimes l_! \otimes l^* = y \otimes y = y$$

holds since l is monic, hence an inclusion, proving that e is a cover in $\mathbb{D}$. This means that every morphism factors uniquely as a cover in the equipment structure followed by a monic arrow in $\mathbb{D}_0$. In particular, every cover in $\mathbb{D}_0$ is one in $\mathbb{D}$. For in this case l is invertible, hence a cover too, meaning that $y \cong l^* \otimes l_! \cong f^* \otimes f_!$ must hold since f and l have the same cokernel. □

Now, the main result of the paper, characterizing 'double categories of relations.'

Theorem 4.7. *If $\mathbb{D}$ is a 'double category of relations' with a subobject comprehension scheme then the identity functor $1\colon \mathbb{D}_0 \to \mathbb{D}_0$ extends to an adjoint equivalence*

$$\mathbb{R}\mathbf{el}(\mathbb{D}_0) \simeq \mathbb{D}.$$

In short, any 'double category of relations' with a subobject comprehension scheme is equivalent to a double category $\mathbb{R}\mathbf{el}(\mathscr{C})$ for some regular category $\mathscr{C}$.

Proof. Theorem 4.4 shows that $\mathbb{D}_0$ is regular. That the identity functor on $\mathbb{D}_0$ then extends to an equivalence is then Theorem 3.16 by way of Lemma 4.6. □

Now that the characterization has been given, there are several immediate results. Recall that allegories are defined in such a way as to possess an **anti-involution operator**. This is integral to the definition of the Modular Law. In relations, the involution is just to take the opposite relation. Double categories of relations have a derived operation coming from the existence of tabulators and other exactness conditions. For the second statement of the following cf. A3.2.3 of [Joh01].

Proposition 4.8. *A double category of relations has an involution operator, that is, an operation on proarrows $R \mapsto R^\circ$ such that $y^\circ = y$ and $(R^\circ)^\circ = R$ both hold. For a proarrow of the form $p_!$, the involution is the same as the right adjoint $(p_!)^\circ = p^*$.*

Proof. Given a proarrow $R\colon A \nrightarrow B$, take R° to be the cokernel of the opposite of the tabulator of R. For the second statement, if the tabulator of $p_!$ has legs l and r, then by lifting and extension properties, the cokernel $r^* \otimes l_!$ is right adjoint to $p_!$ in the proarrow bicategory of $\mathbb{D}$ and so must be isomorphic to p^* canonically. □

Double categories of relations are also regular as equipments. This is an immediate corollary since every double category $\mathbb{R}\mathbf{el}(\mathscr{C})$ is regular [Sch15]. It is, however, worth spelling out in somewhat more detail what this means.

Recall first a few preliminary definitions. For those of a monoid, bimodule and their homomorphisms one can see the reference, §2 of [CS10], or [Lam21]. A monoid in $\mathbb{D}$ is **effective** if it is the kernel of some ordinary morphism. An **embedding** of a monoid $m\colon A \nrightarrow A$ into an object X is a monoid homomorphism $m \to X$ from m to the trivial monoid on X. The **collapse** of a monoid is a universal embedding. Likewise a **bimodule collapse** is a universal bimodule embedding. The collapse of a monoid is **normal** if it presents the bimodule collapse of the trivial bimodule on A. A morphism f is a **regular cover** if its kernel is a normal collapse cell.

Definition 4.9 (Cf. Definition 4.7 of [Sch15])**.** A double category $\mathbb{D}$ is **regular** if

1. every effective monoid has a normal collapse;
2. every restriction cell

$$\begin{tikzcd} \cdot \arrow[r, "|" marking] \arrow[d, two heads] \arrow[dr, phantom, "\Downarrow"] & \cdot \arrow[d, two heads] \\ \cdot \arrow[r, "|" marking] & \cdot \end{tikzcd}$$

   is a bimodule collapse cell.

The first condition is the analogue of the condition that every kernel has a coequalizer; the second is the condition that regular epimorphisms are pullback-stable.

Corollary 4.10. *Any double category of relations is regular as an equipment.*

Proof. Any double category of relations is up to equivalence of the form $\mathbb{R}\mathbf{el}(\mathscr{C})$ for some regular category $\mathscr{C}$ by Theorem 4.7, which is regular by Proposition 4.8 in the reference. □

In a regular equipment, the canonical factorization of a morphism $f\colon A \to B$ developed in §4.2 of [Sch15] is given by taking the collapse of the kernel of f. This is meant to mimic the factorization system in ordinary regular categories given by taking the coequalizer of the kernel of a given morphism. The approach in the development of double categories of relations here has been a dual construction, namely, taking the tabulator of the cokernel of a morphism. So, on the one hand, for regular equipments, the factorization is a quotient of a kernel, for double categories of relations the factorization is a subobject of a cokernel. However, owing to the further exactness conditions in a double category of relations, these two factorizations coincide.

Proposition 4.11. *The factorization of a morphism in a double category of relations viewed as a regular double category coincides with the factorization produced in Lemma 4.1 above.*

Proof. The factorization as a regular double category is as an regular cover followed by an inclusion. However, inclusions here have the same definition as in the reference. The condition that there is a fibers construction as in Definition 3.9 means that any inclusion is a tabulator, hence that the two factorizations coincide. □

4.3 Division and Powers

Recall (§I,7 [FS90] or §A1.4 [Joh01]) that a **logos** is a regular category $\mathscr{C}$ such that each $\mathrm{Sub}(A)$ is a lattice and every pullback functor $f^*\colon \mathrm{Sub}(B) \to \mathrm{Sub}(A)$ has a right adjoint $\forall_f$. Thus, for a logos, each pullback functor f^* has both a left and a right adjoint with the left adjoint $\exists_f$ given by taking images. The following is the double-categorical version of a division allegory, that is, an allegory equipped locally with certain division operators (cf. §II,2.31 [FS90] or §A3.4 [Joh01]).

Definition 4.12. A double category of relations $\mathbb{D}$ has **division** if for any proarrow $p\colon A \to B$ and any object C, the functor

$$p \otimes (-)\colon \mathbb{D}(B,C) \to \mathbb{D}_1(A,C)$$

given by precomposition with p has a right adjoint. Denote the right adjoint by $(-)/p$.

Double categories of relations with division are related to logoses in the following way. This gives an appropriate version of §II,2.32 of [FS90] which says that an allegory is a division allegory if, and only if, its underlying category of maps is a logos.

Theorem 4.13. *A double category of relations $\mathbb{D}$ has division if, and only if, each pullback functor f^* between subobject posets has a right adjoint. Consequently, a double category of relations with division is of the form $\mathbb{R}\mathbf{el}(\mathscr{C})$ with $\mathscr{C}$ a logos if, and only if, each subobject poset is a lattice.*

Proof. Theorem 4.7 shows that any double category of relations is of the form $\mathbb{R}\mathbf{el}(\mathscr{C})$ for some regular category $\mathscr{C}$. So, on the one hand, if $\mathbb{R}\mathbf{el}(\mathscr{C})$ has division, then identifying $\mathbb{R}\mathbf{el}(C)_1(X,1) \simeq \mathrm{Sub}(X)$ for any object X, each pullback functor between posets has a right adjoint. On the other hand, suppose that each pullback functor between subobject posets has a right adjoint. Fix a relation $R\colon A \nrightarrow B$ with legs l and r. Both functors on the bottom row of the commutative diagram

$$\begin{tikzcd}
\mathbb{R}\mathbf{el}(C)_1(B,C) \arrow[r] \arrow[d, "\simeq"'] & \mathbb{R}\mathbf{el}(C)_1(R,C) \arrow[r] \arrow[d, "\simeq"'] & \mathbb{R}\mathbf{el}(C)_1(A,C) \arrow[d, "\simeq"] \\
\mathrm{Sub}(B\times C) \arrow[r, "(r\times 1)^*"'] & \mathrm{Sub}(R\times C) \arrow[r, "\exists_{l\times 1}"'] & \mathrm{Sub}(A\times C)
\end{tikzcd}$$

have right adjoints. The top row is first restriction along r and then extension along l. Thus, up to the equivalences above, the right adjoint is $\forall_{r\times 1} \circ (l\times 1)^*$. The last statement follows by the foregoing result and the definition of a logos recalled above. $\square$

Remark 4.14. Notice that in the background the identification $\mathbb{D}_1(A,B) \simeq \mathrm{Sub}(A\times B)$ given by the tabulator in Definition 3.15 is actually doing most of the work in the proof of the theorem. $\square$

Now recall that a **topos** is a finitely complete category $\mathscr{E}$ with **power objects**, namely, special objects PA for each object A and special monomorphisms $\in_A \rightarrowtail PA \times A$ that are suitably universal among subobjects of the form $S \rightarrowtail X \times A$. Universality is expressed by a pullback condition. This is equivalent to the usual standard definition (§A2.1 of [Joh01]) as a cartesian closed category with a subobject classifier. An allegory is a **power allegory** if each object A has a map $PA \to A$ satisfying a couple of technical conditions, namely, equations stating that the morphism $PA \to A$ satisfies extensionality and comprehension. The result of §II,2.414 of [FS90] is that any tabular power allegory is of the form $\mathbf{Rel}(\mathscr{E})$ where $\mathscr{E}$ is a topos. See also Corollary A3.4.7 of [Joh01]. In the present context, the universal property of power objects is especially easy to state using just the equipment axioms. Recall that restrictions in an equipment should be thought of as pullbacks.

Definition 4.15. An equipment $\mathbb{D}$ has **powers** if each object A is equipped with a special proarrow $\in_A\colon PA \nrightarrow A$ such that for any $R\colon X \nrightarrow A$ there is a unique morphism $X \to PA$ such making a restriction cell

$$\begin{tikzcd}
X \arrow[r, "\shortmid" marking] \arrow[d] \arrow[dr, phantom, "\Downarrow"] & A \arrow[d, equal] \\
PA \arrow[r, "\shortmid" marking, "\in_A"'] & A
\end{tikzcd}$$

Equivalently, $\mathbb{D}$ has powers if $\mathbb{D}(-,A)\colon \mathbb{D}_0^{op} \to \mathbf{Set}$ is representable.

Theorem 4.16. *Any double category of relations with powers is of the form $\mathbb{R}\mathbf{el}(\mathscr{E})$ for $\mathscr{E}$ a topos.*

Proof. Since restrictions in $\mathbb{R}\mathbf{el}(\mathscr{E})$ are computed by pullback, the universal property of $\in_A$ above is precisely that of the universal subobject $\in_A \rightarrowtail PA \times A$ in the equivalent definition of a topos. $\square$

5 Prospectus

Let us end with some comments on future and ongoing work. In particular, there are many potential applications, mostly within category theory, but possibly in other areas. Several were mentioned in the introduction. Here a few can be discussed in somewhat more detail.

5.1 Double-Categorical Semantics

It was remarked in §2.2 that the horizontal bicategory $\mathcal{H}(\mathbb{D})$ of any cartesian equipment $\mathbb{D}$ might be regarded as a generalized compact closed category. This is essentially a consequence of Proposition 2.12 showing that $\mathcal{H}(\mathbb{D})$ is a cartesian bicategory. More precisely, a compact closed category [KL80] is a symmetric monoidal category with a certain dualization operation. As such, compact closed categories are "$*$-autonomous" in the sense of [Bar79]. It is well-known that $*$-autonomous categories provide a categorical semantics for linear logic [See89], and it seems *prima facie* possible to import much of the development to the context of cartesian equipments. In some respects this is a bit trivial, since. But since resources are interpreted as objects in a $*$-autonomous category, in the double categorical context, they would be interpreted as proarrows, with cells providing interpretations of deductions. This allows the possibility of introducing context-dependency for deployment of resources and of an underlying type theory that is modeled by the objects and ordinary arrows of the cartesian double category. The "logic" part of linear logic would then be interpreted by whatever extra structure and local connectives were asked for in $\mathbb{D}_1$.

This is part of a broader program, that owing to the fact that any equipment $\mathbb{D}$, being at least a fibration, has both an internal type theory given by $\mathbb{D}_0$ and an internal logic of types and terms given by $\mathbb{D}_1$. This makes sense as proarrows should be interpreted as relations or their generalizations such as spans or honest profunctors. In ordinary first-order logic, predicate symbols and thus formulas are interpreted categorically by subobjects, which technically speaking live in a separate poset. The virtue of modeling type theories and their logics in suitably structured double categories is that the proarrows and various local connectives (i.e. local products) have enough structure to deal with the type-theoretic and logical aspects in the same structure. For example, the interpretation of regular logic in bicategories of relations [Pat17], which is limited to predicate symbols only, can be done for full regular logic with predicate and function symbols by working in a cartesian double category instead.

5.2 Monoidal Fibrations

In §14 of [Shu08], it is shown that under certain conditions, every monoidal bifibration gives rise to an equipment with some extra structure. However, it appears that the definition of a cartesian equipment, as presented in [Ale18], is needed for a sort of inverse construction taking a cartesian equipment to a monoidal bifibration. Under such a hypothetical correspondence, it is of interest to see which monoidal bifibrations correspond to double categories of relations. The conjecture is that these will be closely related to regular fibrations and subobject fibrations as in §4.2 and §4.4 of [Jac99]. If this is the case, it is another point in favor of the view of the close connection between type theories and suitably structured double categories.

5.3 Classification Schema for Double Toposes

As justification for the name "subobject comprehension schema" adopted in Definition 3.15, recall some of the development of hyperdoctrines from [Law70]. An **elementary existential doctrine** is a pseudo-functor on a cartesian category $P\colon \mathscr{C}^{op} \to \mathbf{Cat}$ such that each substitution functor $f^*\colon PB \to PA$ has a left adjoint Σ_f. Suppose that each category PA has a terminal object. There is then a natural functor $\mathscr{C}/B \to PB$ taking a morphism $f\colon A \to B$ to $\Sigma_f(1)$. If each such functor has a right adjoint $\{-\}$, then P is a **comprehension scheme**. In the case that $\mathbb{D}$ is a cartesian equipment with tabulators, the hyperdoctrine

$$\mathbb{D}(-,1)\colon \mathbb{D}_0^{op} \to \mathbf{Cat}$$

is an elementary existential doctrine with extension providing the left adjoint to restriction as substitution. The comprehension scheme is then given by tabulators

$$\Sigma_{(-)}1\colon \mathbb{D}_0/B \leftrightarrows \mathbb{D}(B,1)\colon \top.$$

If $\mathbb{D}_0$ is regular and tabulators are monic, tabulators factor through the subobject poset as in

$$\begin{tikzcd}
\mathbb{D}_0/B \arrow[rr, shift left, leftarrow, "\top"] \arrow[rr, shift right, "\Sigma"'] \arrow[dr, shift right, "\sigma"'] & & \mathbb{D}(B,1) \arrow[dl, "\top"] \\
& \mathrm{Sub}(B) \arrow[ul, shift right, "i"'] &
\end{tikzcd}$$

where σ is left adjoint to the inclusion of subobjects. What the "subobject comprehension schema" of §3.4 axiomatizes is that the left adjoint Σ, when restricted to subobjects, results in an equivalence making the other triangle above commute and identifying $\mathbb{D}(B,1)$ as a reflective subcategory of the slice.

Although strictly speaking [Law70] *defines* the comprehension schema as being the mere presence of the right adjoint in the top row of the diagram above, the present view is that the structure given by this right adjoint should be taken into account. In the example under discussion, the tabulator gives a monic arrow which represents a subobject. In the other example, namely, that of $\mathbb{D} = \mathbb{P}\mathbf{rof}$, tabulators are not monic, but instead are discrete opfibrations. There results a situation somewhat like that above

$$\begin{tikzcd}
\mathbf{Cat}/\mathscr{B} \arrow[dr, shift right, "\sigma"'] & & \mathbb{P}\mathbf{rof}(\mathscr{B},1) \arrow[ll, shift right, "\mathbf{Elt}"'] \arrow[dl, "\mathbf{Elt}"] \\
& \mathbf{DOpf}(\mathscr{B}) \arrow[ul, shift right, "i"'] &
\end{tikzcd}
\qquad \mathbf{Cat}/\mathscr{B} \xrightarrow{\Sigma} \mathbb{P}\mathbf{rof}(\mathscr{B},1)$$

where now σ gives the free discrete opfibration on a given functor over $\mathscr{B}$. Again there is a fibers construction assigning to every discrete opfibration a set-valued functor, that is, a profunctor $\mathscr{B} \to 1$. This results in the well-established equivalence between discrete opfibrations and set-valued functors. This fibers construction gives the analogue of the characteristic function from ordinary topos theory. The reason for treating this part of the triangle as the distinctive aspect of the comprehension scheme is simply that fibers do not vary functorially unless the construction starts with a discrete opfibration. Hence the language "subobject comprhension schema" or "discrete opfibration comprehension schema" clearly identifying the properties of the projection morphism coming with the right adjoint.

Now, discrete opfibrations are not reflective in $\mathbf{Cat}/\mathscr{B}$, but there is a commonality with the previous example in that both categories $\mathrm{Sub}(B)$ and $\mathbf{DOpf}(\mathscr{B})$ are monadic over the respective slice categories. In particular, discrete opfibrations are algebras for a "pull-push monad" on $\mathbf{Cat}/\mathscr{B}$ described for example in §2.2-2.3 of [Joh77]. What is envisioned is that $\mathbb{D}$ is a cartesian equipment with tabulators, each slice $\mathbb{D}_0/B$ is equipped with a monad T and tabulators factor through T-algebras as in

$$\begin{tikzcd}
\mathbb{D}_0/B \arrow[rr, shift right, "\Sigma"'] \arrow[dr, shift right, "\sigma"'] & & \mathbb{D}(B,1). \arrow[ll, shift right, "\top"'] \arrow[dl, "\top"] \\
& T\mathbf{Alg} \arrow[ul, shift right, "i"'] &
\end{tikzcd}$$

For this to be a T-**comprehension schema** required would be a fibers construction from T-algebras back to proarrows resulting in an equivalence of categories. Such a template recovers the examples discussed so far. Toposes would be a special case where there is a representing object, namely, the subobject classifier Ω for the original hyperdoctrine. Such structures could figure prominently in double-categorical interpretations of higher-order type theory (cf. the "comprehension categories" in Ch. 10 of [Jac99]).

This approach could avoid the issue of "admissibility" in the development of 2-toposes [Web07] and possibly Yoneda structures generally [SW73] where size issues prevent the development of a genuine pseudo-inverse to the elements construction (cf. Definition 4.1 of [Web07] where the functor that might be expected to be an equivalence is merely fully faithful). This is because of the requirement that **Set** lives in an enlarged **Cat** and so there is no guarantee that an arbitrary discrete opfibrations has small fibers. The approach being suggested here is that **Set** is *supposed to be* the representing object for the hyperdoctrine, but **Set** is "hidden behind the proarrows" coming with the double category structure. In this way, the fibers construction makes sense yielding the genuine equivalence without requiring the existence of a representing object for the hyperdoctrine. So, in this sense, it is plausible that 2-toposes are rather 2-categorical fragments of "double toposes," which should be at least cartesian equipments with T-comprehension schema for certain well-chosen monads T. This is further supported by the fact that a nice enough 2-topos ends up supporting a Yoneda structure [Web07] which in any case behaves very much like the proarrows of an equipment [Shu08].

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