# SparseRecovery

September 18, 2024

## 0.1 Sparse Signal Recovery

Interested in recovering a sparse signal that has been corrupted by noise that does not follow a (sub)-Gaussian distribution, i.e.

$$y = X\beta + \epsilon,$$

where $\beta$ is the ground truth which is sparse and $\epsilon$ is a noise vector.

Our model is:

$$\min_{\beta} \quad \frac{1}{m}\sum_{i=1}^{m} |x_i^T\beta - y_i| + \lambda \sum_{j=1}^{n} |\beta_j|.$$

The first term promotes fitting the data without overfitting to the noisy outliers and the second term induces sparsity in the solution. The parameter $\lambda > 0$ trades off betweent these two goals. In general the "best" $\lambda$ is unknown, so multiple options need to be tried.

The functions in the problem above are piecewise linear, so we need to reformulate into a LP by introducing addition variables, $v$ and $t$.

$$\begin{aligned}
\min_{\beta,t,v} \quad & \frac{1}{m}\sum_{i=1}^{m} t_i + \lambda \sum_{j=1}^{n} v_j \\
\text{s.t.} \quad & t_i \geq x_i^T\beta - y_i \ \forall i = 1,\dots,m \\
& t_i \geq -(x_i^T\beta - y_i) \ \forall i = 1,\dots,m \\
& v_j \geq \beta_j \ \forall j = 1,\dots,n \\
& v_j \geq -\beta_j \ \forall j = 1,\dots,n
\end{aligned}$$

[7]:

```julia
using JuMP
using HiGHS
using Random, Distributions
using LinearAlgebra

#Make sure notebook always produces the same results
Random.seed!(0)

n=10;
m=500;
```

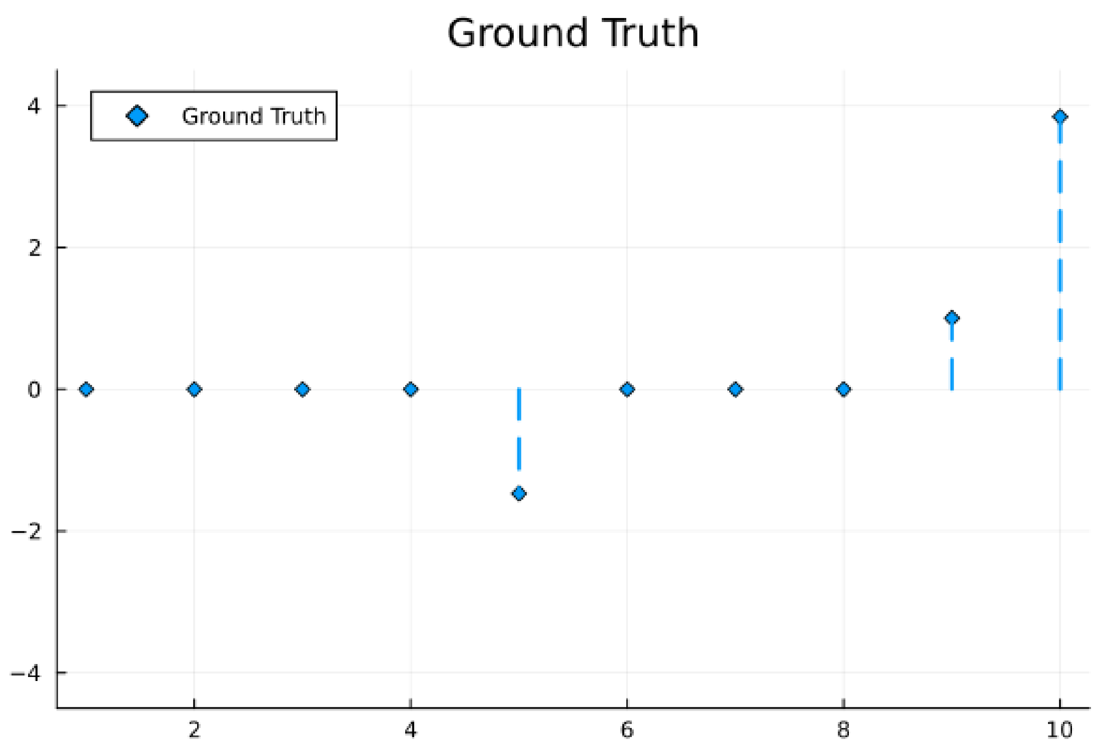
```
numsamples = m;

#Well known symmetric, heavy-tailed distribution, see, e.g. https://en.
 ↪wikipedia.org/wiki/Cauchy_distribution for details.
d = Cauchy()

#Generate a random vector using uniformly at random noise,
#just going to be used to select some indices which will
#be non-zero in our "ground truth" vector
v = rand(Uniform(0,1), n)
#Find the 3 indices with the largest values in v
nonzeros = partialsortperm(v, 1:3)

#Create a vector of 0's
groundtruth = zeros(n)
#For the indices that will be non-zero, give them a random value
for i=1:length(nonzeros)
    groundtruth[nonzeros[i]] = rand(Normal(0,5))
end

#Create a random matrix using the matrix normal distribution
X = rand(MatrixNormal(zeros(m,n), Diagonal(ones(m,m)), Diagonal(ones(n,n))))

#Set the RHS vector be to be A*groundtruth + noise
#(coming from the distribution d) defined above
y = X*groundtruth + 10*rand(d, m)

#Run this with no regularization, regularization parameter of .02,
#regularization parameter of .05 and regularization parameter of .1
lambdas = [0, .02, .05, .1]

 out = zeros(n,length(lambdas))

i = 1
#Run this loop once per regularization parameter
#Store the result in a column of xout
for   in lambdas
    sparse = Model(HiGHS.Optimizer)
    @variable(sparse,  [1:n])
    #Add epigraph variables
    @variable(sparse, t[1:m] >= 0)
    @variable(sparse, v[1:n] >= 0)

    #Add epigraph constraints for the absolute value of x
    @constraint(sparse, epigraph1y[i in 1:m], t[i] >= sum(X[i,j]* [j] for j in␣
 ↪1:n)-y[i])
```

```
    @constraint(sparse, epigraph2y[i in 1:m], t[i] >= -(sum(X[i,j]* [j] for j␣
↪in 1:n)-y[i]))

    @constraint(sparse, epigraph1t[i in 1:n], v[i] >=  [i])
    @constraint(sparse, epigraph2t[i in 1:n], v[i] >= - [i])

    #Objective is least squares +  *\|x\|_1 (which is handled by the epigraph␣
↪constraints)
    @objective(sparse, Min, (1/m)*sum(t[i] for i in 1:m) +  *sum(v[i] for i in␣
↪1:n))

    optimize!(sparse);
     out[:,i] = value.( )
    i = i+1
end
```

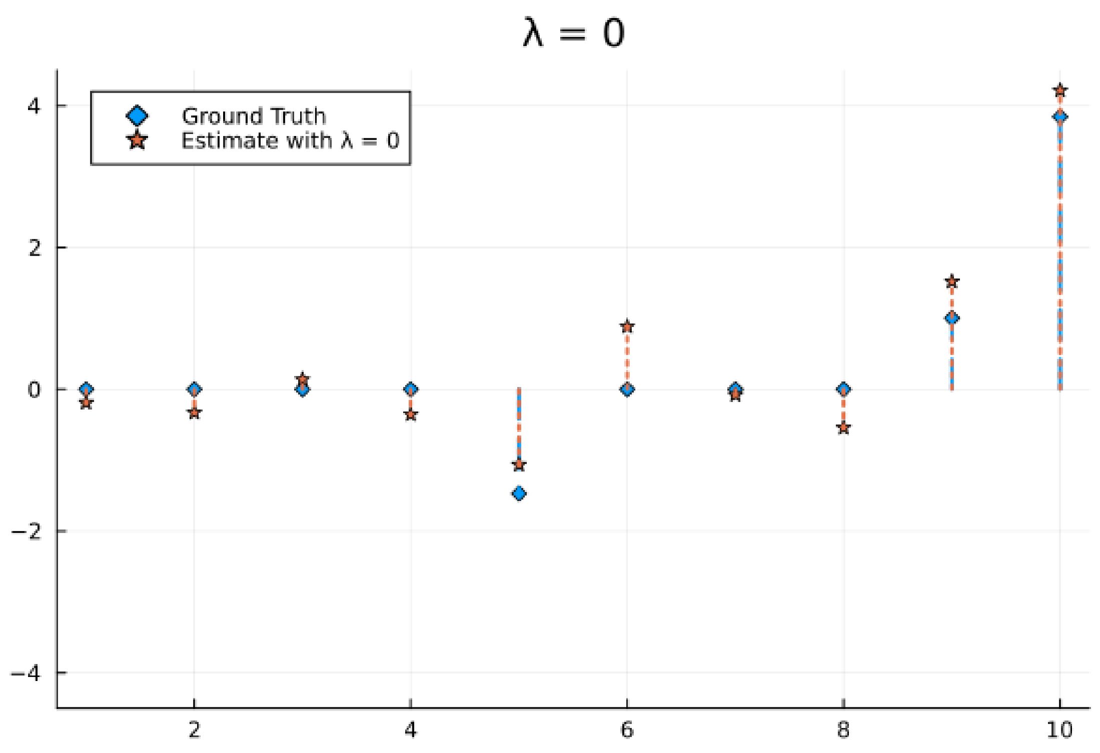
```
Running HiGHS 1.6.0: Copyright (c) 2023 HiGHS under MIT licence terms
Presolving model
1000 rows, 510 cols, 11000 nonzeros
1000 rows, 510 cols, 11000 nonzeros
Presolve : Reductions: rows 1000(-20); columns 510(-10); elements 11000(-40)
Solving the presolved LP
Using EKK dual simplex solver - serial
  Iteration        Objective     Infeasibilities num(sum)
          0     0.0000000000e+00 Pr: 500(3800.52); Du: 0(2.8813e-10) 0s
        613     5.7646846423e+01 Pr: 0(0) 0s
Solving the original LP from the solution after postsolve
Model   status      : Optimal
Simplex   iterations: 613
Objective value     :  5.7646846423e+01
HiGHS run time      :          0.02
Running HiGHS 1.6.0: Copyright (c) 2023 HiGHS under MIT licence terms
Presolving model
1020 rows, 520 cols, 11040 nonzeros
1020 rows, 520 cols, 11040 nonzeros
Presolve : Reductions: rows 1020(-0); columns 520(-0); elements 11040(-0) - Not
reduced
Problem not reduced by presolve: solving the LP
Using EKK dual simplex solver - serial
  Iteration        Objective     Infeasibilities num(sum)
          0     0.0000000000e+00 Pr: 500(3800.52); Du: 0(4.12477e-10) 0s
        646     5.7805473792e+01 Pr: 0(0) 0s
Model   status      : Optimal
Simplex   iterations: 646
Objective value     :  5.7805473792e+01
HiGHS run time      :          0.02
Running HiGHS 1.6.0: Copyright (c) 2023 HiGHS under MIT licence terms
```

```
Presolving model
1020 rows, 520 cols, 11040 nonzeros
1020 rows, 520 cols, 11040 nonzeros
Presolve : Reductions: rows 1020(-0); columns 520(-0); elements 11040(-0) - Not
reduced
Problem not reduced by presolve: solving the LP
Using EKK dual simplex solver - serial
  Iteration        Objective     Infeasibilities num(sum)
          0     0.0000000000e+00 Pr: 500(3800.52); Du: 0(4.12543e-10) 0s
        648     5.7974234027e+01 Pr: 0(0) 0s
Model   status      : Optimal
Simplex   iterations: 648
Objective value     :  5.7974234027e+01
HiGHS run time      :          0.02
Running HiGHS 1.6.0: Copyright (c) 2023 HiGHS under MIT licence terms
Presolving model
1020 rows, 520 cols, 11040 nonzeros
1020 rows, 520 cols, 11040 nonzeros
Presolve : Reductions: rows 1020(-0); columns 520(-0); elements 11040(-0) - Not
reduced
Problem not reduced by presolve: solving the LP
Using EKK dual simplex solver - serial
  Iteration        Objective     Infeasibilities num(sum)
          0     0.0000000000e+00 Pr: 500(3800.52); Du: 0(4.1299e-10) 0s
        660     5.8169351541e+01 Pr: 0(0) 0s
Model   status      : Optimal
Simplex   iterations: 660
Objective value     :  5.8169351541e+01
HiGHS run time      :          0.02
```

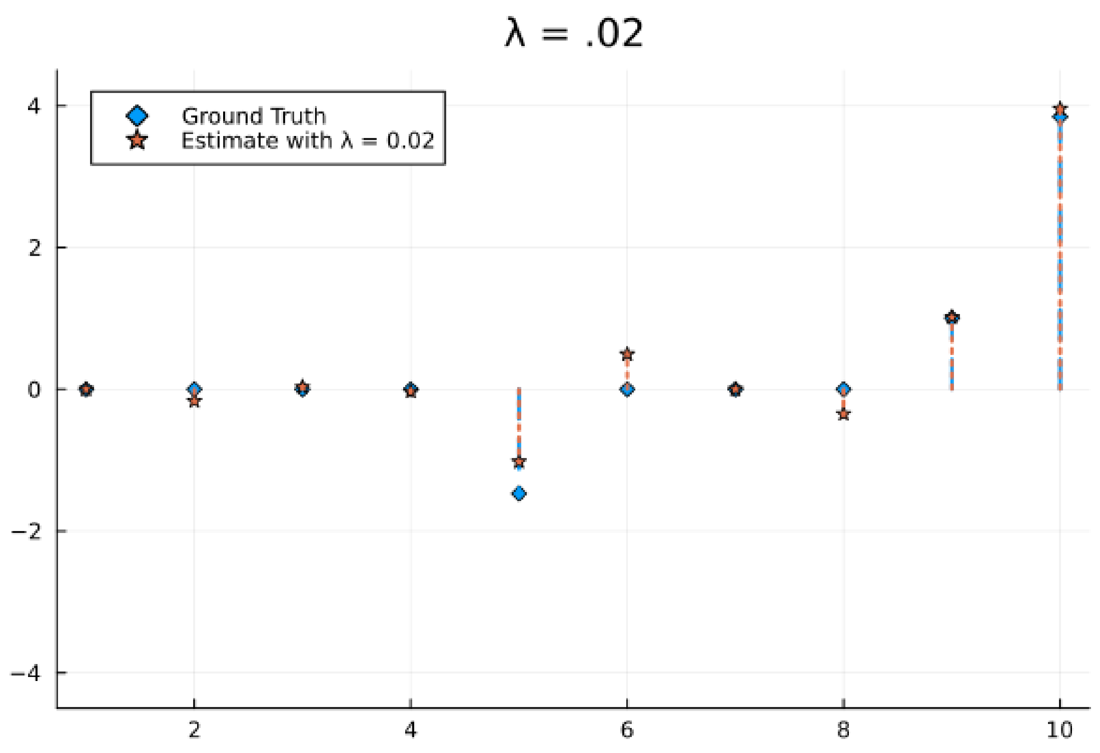
[13]:
```
xaxis = 1:n

using Plots

#Plotting the ground truth vector
plot(xaxis, groundtruth, seriestype = :sticks, markershape = :diamond,␣
  ↪linewidth = 2, markerwidth = 5, linestyle = :dash, title = "Ground Truth",␣
  ↪ylimits=(-4.5,4.5), label="Ground Truth")
```



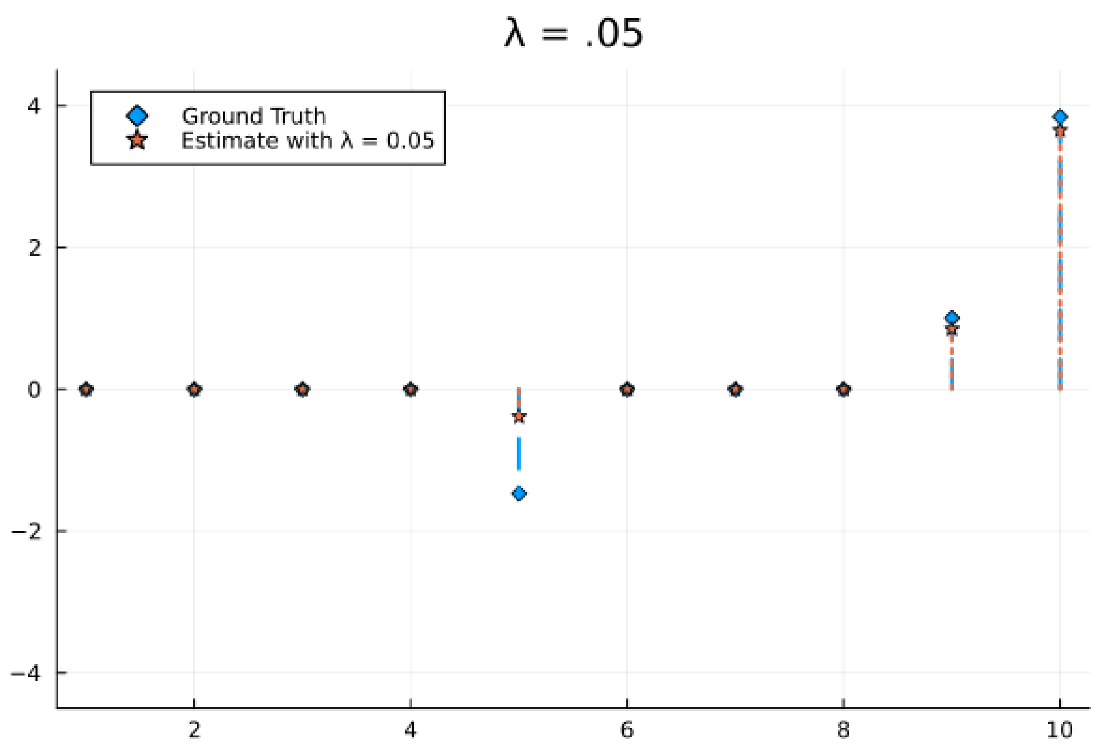
[9]:
```
xaxis = 1:n

#Plot the results for different values of
#Blue diamonds are the "ground truth", red stars are the solution for that␣
 ↪lambda parameter
plot1 = plot(xaxis, groundtruth, seriestype = :sticks, markershape = :diamond,␣
 ↪linewidth = 2, markerwidth = 5, linestyle = :dash, title = "  = 0",␣
 ↪ylimits=(-4.5,4.5), label="Ground Truth")
plot1 = plot!(xaxis,  out[:,1], seriestype = :sticks, markershape = :star5,␣
 ↪linewidth = 2, markerwidth = 5, linestyle = :dot, label="Estimate with   = 0")
```



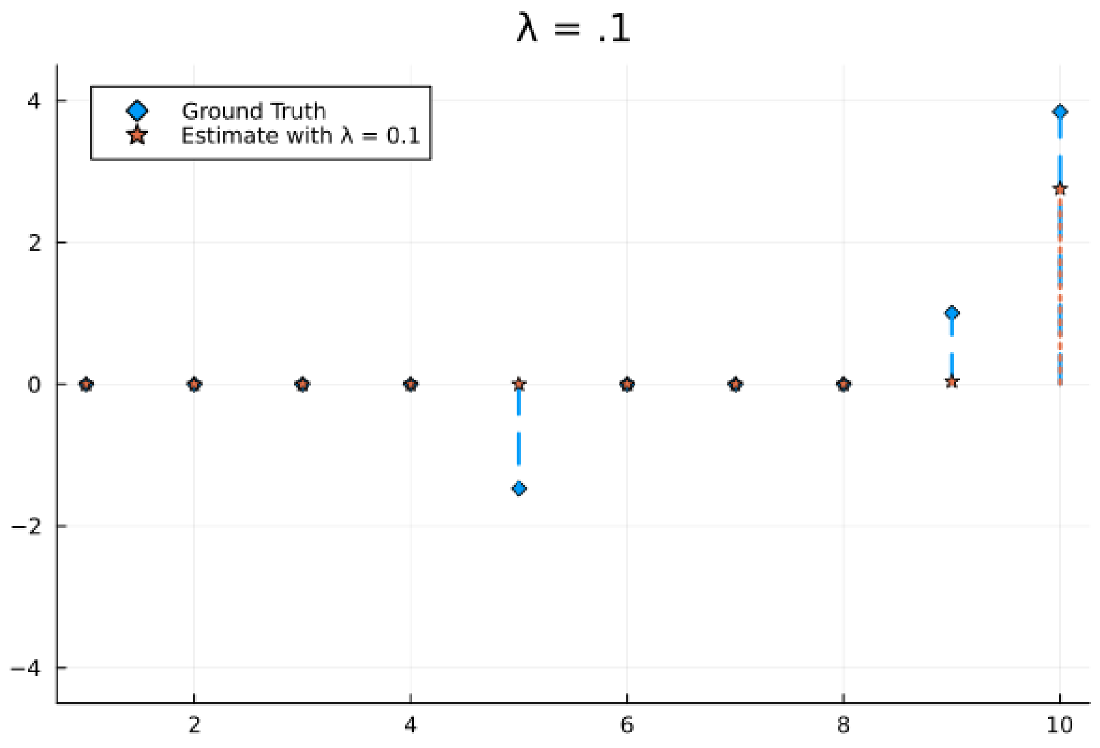
```
[10]: plot2 = plot(xaxis, groundtruth, seriestype = :sticks,markershape = :diamond,␣
        ↪linewidth = 2, markerwidth = 5, linestyle = :dash, title = "  = .02",␣
        ↪ylimits=(-4.5,4.5), label="Ground Truth")
      plot2 = plot!(xaxis,  out[:,2], seriestype = :sticks, markershape = :star5,␣
        ↪linewidth = 2, markerwidth = 5, linestyle = :dot, label="Estimate with   = 0.
        ↪02")
```



[11]:
```
plot3 = plot(xaxis, groundtruth, seriestype = :sticks,markershape = :diamond,␣
  ↪linewidth = 2, markerwidth = 5, linestyle = :dash, title = "  = .05",␣
  ↪ylimits=(-4.5,4.5), label="Ground Truth")
plot3 = plot!(xaxis,  out[:,3], seriestype = :sticks, markershape = :star5,␣
  ↪linewidth = 2, markerwidth = 5, linestyle = :dot, label="Estimate with   = 0.
  ↪05")
```



[12]:
```
plot3 = plot(xaxis, groundtruth, seriestype = :sticks,markershape = :diamond,␣
↪linewidth = 2, markerwidth = 5, linestyle = :dash, title = "  = .1",␣
↪ylimits=(-4.5,4.5), label="Ground Truth")
plot3 = plot!(xaxis,  out[:,4], seriestype = :sticks, markershape = :star5,␣
↪linewidth = 2, markerwidth = 5, linestyle = :dot, label="Estimate with   = 0.
↪1")
```


λ = .1
Ground Truth
Estimate with λ = 0.1
4
2
0
−2
−4
2
4
6
8
10