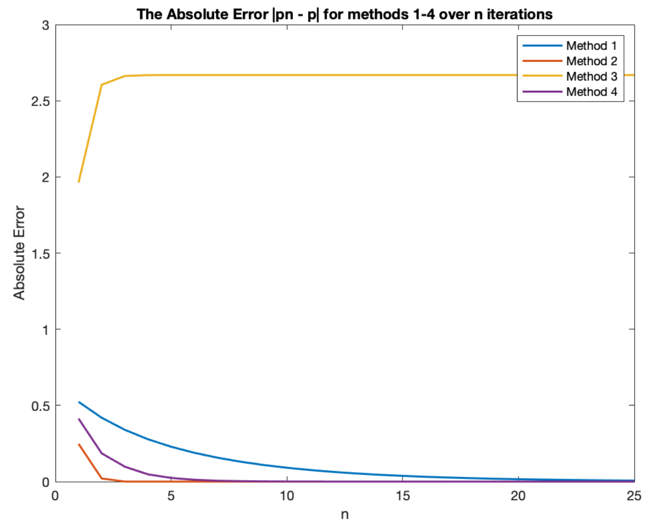
Part A

Methodology: I used a set number of iterations for all aspects of this project, with $n = 25$.

1. Method 2
2. Method 4
3. Method 1
4. Method 3



Note that method 3 converges second fastest, but is set up as a root finding problem, so it converges to 0, not to the desired solution.

Part B

Methods 1, 2, and 4 are set up as fixed-point problems: they search for where $g(x) = x$. The fewer iterations required for their absolute errors to equal 0, the faster the convergence. From the graph of the absolute errors: method 2 converges fastest, then method 4, then method 1. Method 3 is a root-finding problem of the form $f(x) = g(x) - x = 0$ where $g(x) = x$. For large enough n we have $p_n = 0$, so $|p_n - p| = |0 - p| = p$. Thus it diverges from our desired solution, but the sequence produced converges to 0, while the absolute error at n converges to $19^{1/3}$ for large enough n.

Part C

My sincerest thanks to Garrett during workshop hours for assistance with the following process. Let $|p_n - p| = e_n$, then $\frac{e_{n+1}}{e_n^\alpha} \approx \lambda \Rightarrow e_{n+1} \approx \lambda e_n^\alpha$ for large n. Now consider:

$$e_{n+2} \approx \lambda e_{n+1}^\alpha$$

$$\Rightarrow \frac{e_{n+2}}{e_{n+1}} \approx \frac{\lambda e_{n+1}^\alpha}{\lambda e_n^\alpha} = \frac{e_{n+1}^\alpha}{e_n^\alpha}$$

$$\Rightarrow \log \frac{e_n+2}{e_n+1} \approx \log \frac{e_{n+1}^\alpha}{e_n^\alpha} = \alpha(\log e_{n+1} - \log e_n) = \alpha \log \frac{e_{n+1}}{e_n}$$

$$\Rightarrow \alpha \approx \frac{log(\frac{e_{n+2}}{e_{n+1}})}{log(\frac{e_{n+1}}{e_n})}$$

By first solving for α, I then solved for $\frac{e_{n+1}}{e_n^\alpha} \approx \lambda$, for each method. Note that I used $n = 25$ iterations throughout.

- Method 1: For $n = 25$, $\alpha \approx 0.9994$, $\lambda \approx 0.8391$. So, this sequence converges to p of order 1 with asymptotic error constant $\lambda \approx 0.8391 < 1$ and the sequence is said to be linearly convergent.
- Method 2: For $n = 25$ this sequences converged to p, so $\frac{|p_{n+1}-p|}{|p_n-p|^\alpha} = \frac{0}{0}$, which is undefined.
- Method 3: For $n = 25$, $\alpha \approx \frac{log(\frac{e_{n+2}}{e_{n+1}})}{log(\frac{e_{n+1}}{e_n})} = \frac{0}{0}$, which is undefined.
- Method 4: For $n = 25$, $\alpha \approx 1$, $\lambda \approx 0.5$. So this sequence converges to p of order 1 with asymptotic error constant $\lambda \approx 0.5 < 1$ and the sequence is said to be linearly convergent.