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# Motion Variables Leading to Efficient Equations of Motion

## Abstract

*In order to formulate equations that govern the motion of a mechanical system, the analyst must choose time-dependent variables that characterize the configuration and motion of the system. Generally, the choice of variables made by the analyst has a profound effect on the efficiency of the resulting equations—"efficiency" here referring to relative simplicity, ease of manipulation for purposes of designing automatic control systems, and minimal consumption of computer time during numerical solution. In this article, guidelines are set forth for the selection of motion variables that lead to exceptionally efficient dynamical differential equations for systems belonging to a large class frequently encountered in robotics.*

## 1. Introduction

Figure 1 is a schematic representation of a robotic manipulator supported by a spacecraft. If the resultant of the set of external forces acting on this system is equal to zero, the system can be treated as one possessing nine degrees of freedom, so that its motions are governed by 18 first-order differential equations, nine of which are called *kinematical*, while the rest, derived from a principle of dynamics, are termed *dynamical*. However, before one can write either set of equations, one must choose variables for the characterization of the motion of the system, and this choice can be made in infinitely many ways. To show that the choice of motion variables has a profound effect on the complexity of the resulting equations, we consider two choices, the first a customary one, the second apparently new, and report some of the resulting equations in Tables 1 and 2. (The two sets of motion variables will be described in detail later. The symbols $s_i$, $c_i$, and $t_i$ stand for $\sin q_i$, $\cos q_i$, and $\tan q_i$, respectively, while $s_{ij}$ and $c_{ij}$ are abbreviations for $\sin(q_i + q_j)$ and $\cos(q_i + q_j)$, where $q_i$ ($i = 1, \ldots, 9$) are generalized coordinates.) Table 1 shows the nine kinematical differential equations associated with the customary and the new motion variables. The shortest of the nine dynamical differential equations corresponding to each set of of motion variables appears in Table 2. As can be seen at a glance, some of the customary kinematical equations in Table 1 are a bit simpler than their new counterparts; but the first dynamical equation in Table 2 is vastly more complex than the second. Indeed, the first dynamical equation has not been displayed in full, because less than 11% of it fits on one page. As for the rest of the dynamical differential equations governing the new variables, three of these are of the same length as the new equation displayed in Table 2, and the remaining five new equations, while substantially longer, are significantly simpler than the customary equations they replace. For example, an equation less than one line long replaces one that is more than 2.2 *million* lines long after simplification.

The purpose of the sequel is to address the following questions: Exactly what are the new variables? How does one use them? What are the resulting benefits? To answer these questions, the notion of "generalized speeds" is examined briefly, guidelines are stated for the selection of generalized speeds, the use of the guidelines is illustrated with three relatively simple examples, and the spacecraft-supported robotic manipulator depicted in Figure 1 is examined once more. These considerations support the conclusion that use of the new motion variables yield very substantial benefits.

## 2. Generalized Speeds

As is well known, *generalized coordinates* are quantities that characterize the instantaneous *configuration* of a system, for which reason one might call them *configuration variables*. Possibly less well known is the fact that it is the choice of *motion variables*, or *generalized speeds*, defined as linear combinations of time derivatives of generalized coordinates (Kane and Levinson 1985), that determines the efficiency of dynamical equations of motion. Let us examine this proposition in the light of a simple example.

Figure 2 shows a rocket A moving in an X-Y plane fixed in a Newtonian reference frame N. Generalized

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**Table 1. Kinematical Differential Equations**

**Customary**

$$
\begin{aligned}
\dot{q}_1 &= (c_3u_1 - s_3u_2)/c_2 \\
\dot{q}_2 &= s_3u_1 + c_3u_2 \\
\dot{q}_3 &= u_3 + t_2(s_3u_2 - c_3u_1) \\
\dot{q}_4 &= u_4 \\
\dot{q}_5 &= u_5 \\
\dot{q}_6 &= u_6 \\
\dot{q}_7 &= (c_9u_7 - s_9u_8)/c_8 \\
\dot{q}_8 &= s_9u_7 + c_9u_8 \\
\dot{q}_9 &= u_9 + t_8(s_9u_8 - c_9u_7)
\end{aligned}
$$

**New**

$$
\begin{aligned}
\dot{q}_1 &= (c_3u_1 - s_3u_2)/c_2 \\
\dot{q}_2 &= s_3u_1 + c_3u_2 \\
\dot{q}_3 &= u_3 + t_2(s_3u_2 - c_3u_1) \\
\dot{q}_4 &= u_4 - u_2 \\
\dot{q}_5 &= u_5 - s_4u_1 - c_4u_3 \\
\dot{q}_6 &= u_6 - u_5 \\
\dot{q}_7 &= (c_9u_7 - s_9u_8)/c_8 + c_7t_8u_6 + (s_4c_{56} - s_7t_8s_{56})u_3 + c_4(s_7t_8s_{56} - c_{56})u_1 - (s_{56} + s_7t_8c_{56})u_4 \\
\dot{q}_8 &= s_9u_7 + c_9u_8 + c_4c_7s_{56}u_1 - s_7u_6 - c_7c_{56}u_4 - s_4c_7s_{56}u_3 \\
\dot{q}_9 &= u_9 + t_8(s_9u_8 - c_9u_7) + (s_7c_{56}u_4 + s_4s_7s_{56}u_3 - c_7u_6 - s_7c_4s_{56}u_1)/c_8
\end{aligned}
$$

**Table 2. Dynamical Differential Equations**

**Customary**

$\dot{u}_2 = (I_1^B - I_3^B)(s_4u_1 + c_4u_3)(s_4u_3 - c_4u_1) + c_5(I_1^C(u_5 + s_4u_1 + c_4u_3)(s_4c_5u_3 - s_5u_2 - s_5u_4 - c_4c_5u_1) - I_3^C(u_5 + s_4u_1 + c_4u_3)(s_4c_5u_3 - s_5u_2 - s_5u_4 - c_4c_5u_1) - I_2^C(s_5u_4(s_4u_1 + c_4u_3) - u_5(s_5(u_2 + u_4) - c_5(s_4u_3 - c_4u_1)))) + c_{56}(I_1^D(u_5 + u_6 + s_4u_1 + c_4u_3)(s_4c_{56}u_3 - s_{56}u_2 - s_{56}u_4 - c_4c_{56}u_1) - I_3^D(u_5 + u_6 + s_4u_1 + c_4u_3)(s_4c_{56}u_3 - s_{56}u_2 - s_{56}u_4 - c_4c_{56}u_1) - I_2^D(s_6(c_5u_4(s_4u_1 + c_4u_3) - u_5(c_5(u_2 + u_4) + s_5(s_4u_3 - c_4u_1))) + c_6(s_5u_4(s_4u_1 + c_4u_3) - u_5(s_5(u_2 + u_4) - c_5(s_4u_3 - c_4u_1))) - u_6(s_6(c_5u_2 + c_5u_4 + s_4s_5u_3 - s_5c_4u_1) - c_6(s_4c_5u_3 - s_5u_2 - s_5u_4 - c_4c_5u_1)))) + (s_9c_8s_{56} - c_{56}(c_7c_9 - s_7s_8s_9))(I_1^E(u_9 + c_7c_8u_5 + c_7c_8u_6 + (s_8s_{56} - s_7c_8c_{56})u_2 + (s_8s_{56} - s_7c_8c_{56})u_4 + (s_4c_7c_8 + s_8c_4c_{56} + s_7c_4c_8s_{56})u_1 + (c_4c_7c_8 - s_4s_8c_{56} - s_4s_7c_8s_{56})u_3)(u_7 + (s_7s_9 - s_8c_7c_9)u_5 + (s_7s_9 - s_8c_7c_9)u_6 + (c_8c_9s_{56} + c_{56}(s_9c_7 + s_7s_8c_9))u_2 + (c_8c_9s_{56} + c_{56}(s_9c_7 + s_7s_8c_9))u_4 + (c_4c_8c_9c_{56} + s_4(s_7s_9 - s_8c_7c_9) - c_4s_{56}(s_9c_7 + s_7s_8c_9))u_1 - (s_4c_8c_9c_{56} - c_4(s_7s_9 - s_8c_7c_9) - s_4s_{56}(s_9c_7 + s_7s_8c_9))u_3) - I_3^E(u_9 + c_7c_8u_5 + c_7c_8u_6 + (s_8s_{56} - s_7c_8c_{56})u_2 + (s_8s_{56} - s_7c_8c_{56})u_4 + (s_4c_7c_8 + s_8c_4c_{56} + s_7c_4c_8s_{56})u_1 + (c_4c_7c_8 - s_4s_8c_{56} - s_4s_7c_8s_{56})u_3)(u_7 + (s_7s_9 - s_8c_7c_9)u_5 + (s_7s_9 - s_8c_7c_9)u_6 + (c_8c_9s_{56} + c_{56}(s_9c_7 + s_7s_8c_9))u_2 + (c_8c_9s_{56} + c_{56}(s_9c_7 + s_7s_8c_9))u_4 + (c_4c_8c_9c_{56} + s_4(s_7s_9 - s_8c_7c_9) - c_4s_{56}(s_9c_7 + s_7s_8c_9))u_1 - (s_4c_8c_9c_{56} - c_4(s_7s_9 - s_8c_7c_9) - s_4s_{56}(s_9c_7 + s_7s_8c_9))u_3) - I_2^E((s_7c_9 + s_8s_9c_7)u_4(s_4u_3 - c_4u_1) - u_7(c_7c_8(u_5 + u_6 + s_4u_1 + c_4u_3) - s_8(s_4c_{56}u_3 - s_{56}u_2 - s_{56}u_4 - c_4c_{56}u_1) - s_7c_8(c_{56}u_2 + c_{56}u_4 + s_4s_{56}u_3 - c_4s_{56}u_1)) - u_9(c_8c_9(s_4c_{56}u_3 - s_{56}u_2 - s_{56}u_4 - c_4c_{56}u_1) - (s_7s_9 - s_8c_7c_9)(u_5 + u_6 + s_4u_1 + c_4u_3) - (s_9c_7 + s_7s_8c_9)(c_{56}u_2 + c_{56}u_4 + s_4s_{56}u_3 - c_4s_{56}u_1)) - s_9c_8(c_6(c_5u_4(s_4u_1 + c_4u_3) - u_5(c_5(u_2 + u_4) + s_5(s_4u_3 - c_4u_1))) - s_6(s_5u_4(s_4u_1 + c_4u_3) - u_5(s_5(u_2 + u_4) - c_5(s_4u_3 - c_4u_1))) - u_6(c_6(c_5u_2 + c_5u_4 + s_4s_5u_3 - s_5c_4u_1) + s_6(s_4c_5u_3 - s_5u_2 - s_5u_4 - c_4c_5u_1))) - (c_7c_9 - s_7s_8s_9)(s_6(c_5u_4(s_4u_1 + c_4u_3) - u_5(c_5(u_2 + u_4) + s_5(s_4u_3 - c_4u_1))) + c_6(s_5u_4(s_4u_1 + c_4u_3) - u_5(s_5(u_2 + u_4) - c_5(s_4u_3 - c_4u_1))) - u_6(s_6(c_5u_2 + c_5u_4 + s_4s_5u_3 - s_5c_4u_1) - c_6(s_4c_5u_3 - s_5u_2 - s_5u_4 - c_4c_5u_1))))) + T_2^A - \quad \ldots$

**New**

$$\dot{u}_2 = [\, T_2^A - T^{A/B} - (I_1^A - I_3^A)\, u_1u_3 \,] \,/\, I_2^A$$

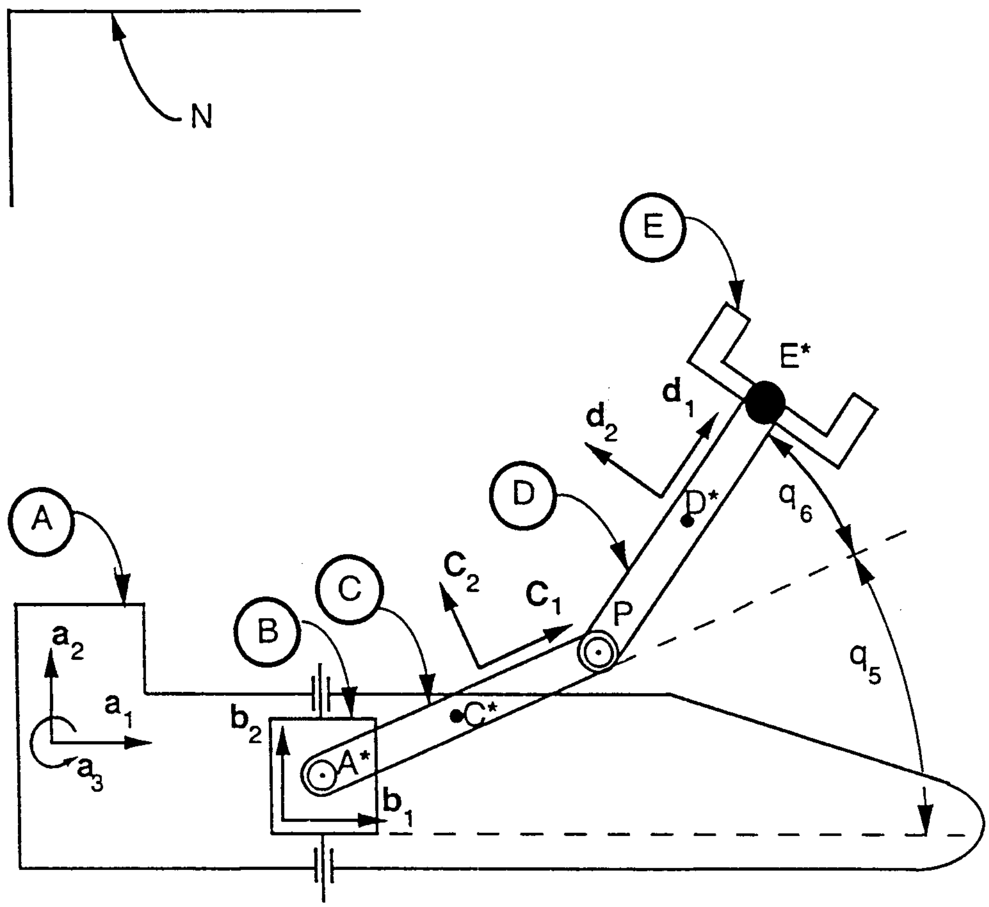


*Fig. 1. Robot manipulator supported by a spacecraft.*

coordinates $q_1$, $q_2$, and $q_3$ are defined, respectively, as the distance from the origin O of the X-Y plane to $A^*$, the mass center of A; the angle between the X-axis and line $OA^*$; and the angle between line $OA^*$ and the longitudinal axis of A. Here, as always, the simplest way to introduce the generalized speeds $u_1$, $u_2$, and $u_3$ is to define each as the first time derivative of a generalized coordinate (i.e., to let $u_i = \dot{q}_i (i = 1, 2, 3)$), in which event the associated kinematical differential equations are

$$\dot{q}_1 = u_1 \tag{1}$$

$$\dot{q}_2 = u_2 \tag{2}$$

$$\dot{q}_3 = u_3 \tag{3}$$

However, this is not the only possible choice of generalized speeds. For instance, after introducing $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ as mutually perpendicular unit vectors fixed in A, with $\mathbf{a}_1$ parallel to the long axis of A, $\mathbf{a}_3$ normal to the X-Y plane, and $\mathbf{a}_2 = \mathbf{a}_3 \times \mathbf{a}_1$, as shown in Figure 2, one can define $u_i$ as the $\mathbf{a}_i (i = 1, 2)$ measure numbers of the velocity of $A^*$ in N, and $u_3$ as the $\mathbf{a}_3$ measure number of the angular velocity of A in N, in which event the associated kinematical differential equations are

$$\dot{q}_1 = u_1 \cos q_3 - u_2 \sin q_3 \tag{4}$$

$$\dot{q}_2 = (u_1 \sin q_3 + u_2 \cos q_3)/q_1 \tag{5}$$

$$\dot{q}_3 = -(u_1 \sin q_3 + u_2 \cos q_3)/q_1 + u_3 \tag{6}$$

Now, what is the difference between the dynamical differential equations corresponding to these two sets of generalized speeds? To answer this question, we let M denote the mass of A, and I the moment of inertia of A about the line passing through $A^*$ and parallel to $\mathbf{a}_3$, and we replace the set of all contact and distance forces acting on A with a couple of torque $T\mathbf{a}_3$, together with a force $F_1\mathbf{a}_1 + F_2\mathbf{a}_2$ applied at $A^*$. Under these circumstances, the dynamical differential equations corresponding to equations (1)–(3) can be written

$$\dot{u}_1 = (F_1 \cos q_3 - F_2 \sin q_3)/M + q_1 u_2^2 \tag{7}$$

$$\dot{u}_2 = (F_1 \sin q_3 + F_2 \cos q_3)/(M q_1) - 2u_1 u_2/q_1 \tag{8}$$

$$\dot{u}_3 = -(F_1 \sin q_3 + F_2 \cos q_3)/(M q_1) + 2u_1 u_2/q_1 + T/I \tag{9}$$

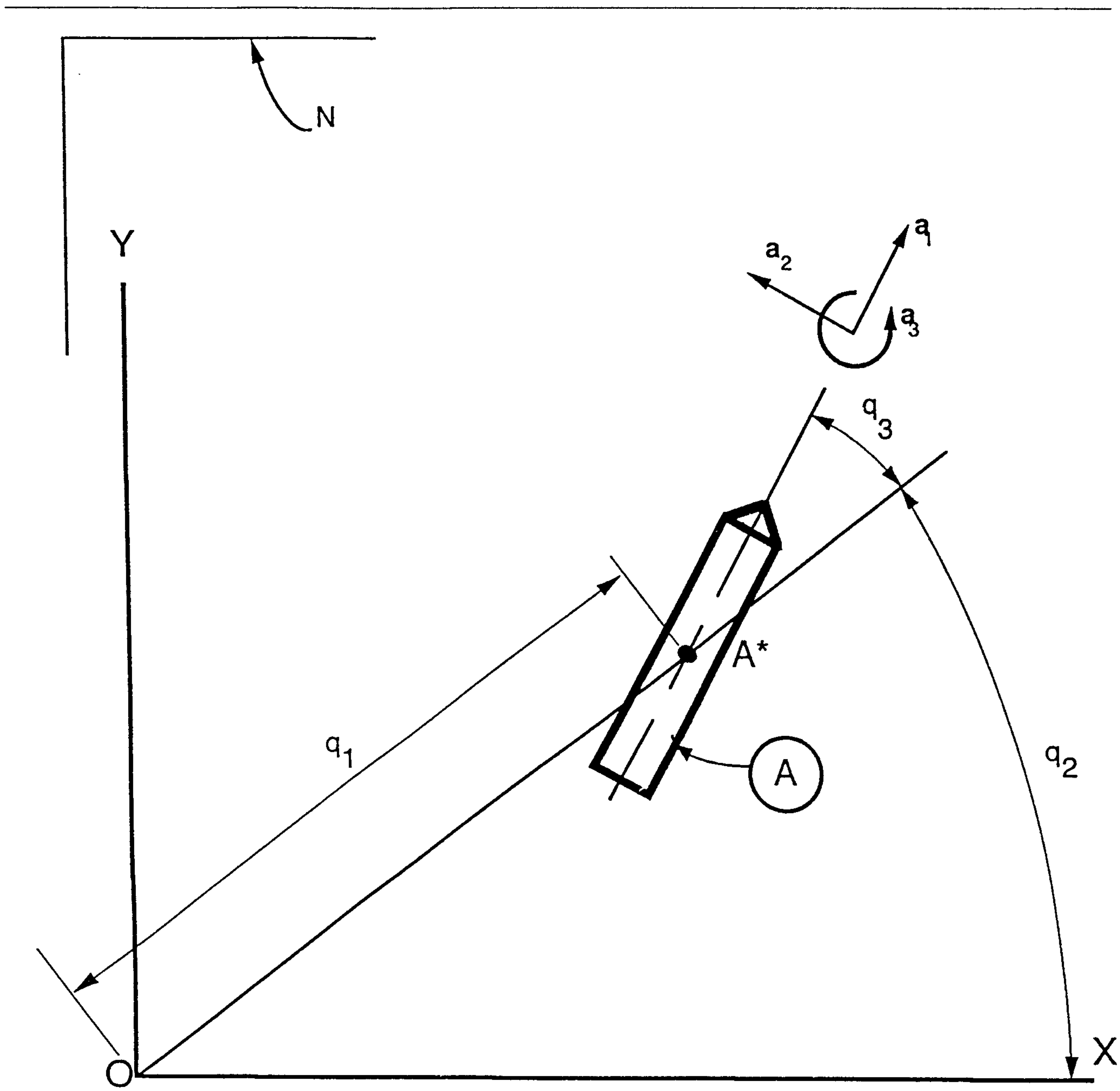


*Fig. 2. Rocket moving in a vertical plane.*

while those associated with equations (4)–(6) are

$$\dot{u}_1 = F_1/M + u_2 u_3 \tag{10}$$

$$\dot{u}_2 = F_2/M - u_1 u_3 \tag{11}$$

$$\dot{u}_3 = T/I \tag{12}$$

Clearly, the choice of generalized speeds has a significant effect on the form of both the kinematical and the dynamical equations. The first set of kinematical differential equations is linear in $q_i$ $(i = 1, 2, 3)$, whereas the second set is nonlinear in $q_i$ $(i = 1, 2, 3)$. This hardly matters, for neither set can be solved independently of the associated dynamical equations. However, there are two noteworthy differences between the two sets of dynamical differential equations: the equations of the first set are longer than those of the second, and the first set involves the generalized coordinates, whereas the second set does not. The point intended to be made by this example is that the choice of generalized speeds has a profound effect on the form of equations of motion. The point of the sequel is to show how to use this choice for maximum benefit.

## 3. Guidelines for Choosing Generalized Speeds

The rationale underlying three guidelines to be set forth below is that it is advantageous to choose generalized speeds that lead to the simplest possible expressions for the inertial angular velocities of rigid bodies of a system under consideration, for these angular velocities figure prominently in the formation of expressions for other important quantities, such as velocities, accelerations, angular accelerations, inertia torques, partial angular velocities, partial velocities, and kinetic energy. Accordingly, each guideline is accompanied by an expression for the inertial angular velocity of a rigid body.

For a system S of rigid bodies moving in a Newtonian reference frame N, let A and B be generic bodies of S, and use the symbols ${}^A\boldsymbol{\omega}^B$, ${}^N\boldsymbol{\omega}^B$, and ${}^N\boldsymbol{\omega}^A$ to denote the angular velocity of B in A, the angular velocity of B in N, and the angular velocity of A in N, respectively. Consider the following three types of connection of A to B, noting that these are the ones most frequently encountered when changes in the relative orientation of A and B are meant to occur.

**Type 1.** A and B are connected in such a way that ${}^A\boldsymbol{\omega}^B$ can be written as $\sigma_1\mathbf{i}$, where $\sigma_1$ is a time-dependent scalar and $\mathbf{i}$ is a unit vector fixed in both A and B. Under these circumstances, at most one generalized speed (e.g., $u_x$) is needed to characterize the time rate of change of the orientation of B in A. (No generalized speed is needed if $\sigma_1$ is a specified function of time.) When it is needed, define the generalized speed as

$$u_x = {}^N\boldsymbol{\omega}^B \cdot \mathbf{i}. \tag{13}$$

When $u_x$ is defined as in equation (13), the angular velocity of B in N can be written

$${}^N\boldsymbol{\omega}^B = {}^N\boldsymbol{\omega}^A + (u_x - {}^N\boldsymbol{\omega}^A \cdot \mathbf{i})\mathbf{i}. \tag{14}$$

This can be seen as follows. By hypothesis,

$${}^A\boldsymbol{\omega}^B = \sigma_1\mathbf{i}. \tag{15}$$

In accordance with the angular velocity addition theorem (Kane and Levinson 1985),

$${}^N\boldsymbol{\omega}^B = {}^N\boldsymbol{\omega}^A + {}^A\boldsymbol{\omega}^B. \tag{16}$$

From equations (15) and (16),

$${}^N\boldsymbol{\omega}^B = {}^N\boldsymbol{\omega}^A + \sigma_1\mathbf{i}. \tag{17}$$

Substituting from equation (17) into equation (13), one obtains

$$u_x = {}^N\boldsymbol{\omega}^A \cdot \mathbf{i} + \sigma_1, \tag{18}$$

which, solved for $\sigma_1$, yields

$$\sigma_1 = u_x - {}^N\boldsymbol{\omega}^A \cdot \mathbf{i}. \tag{19}$$

Substitution from equation (19) into equation (17) leads directly to equation (14).

**Type 2.** A and B are connected in such a way that ${}^A\boldsymbol{\omega}^B$ can be expressed as $\sigma_1\mathbf{i} + \sigma_2\mathbf{j}$, where $\sigma_1$ and $\sigma_2$ are time-dependent scalars, while $\mathbf{i}$ and $\mathbf{j}$ are unit vectors permanently fixed in A and B, respectively. Under these circumstances, at most two generalized speeds (e.g., $u_x$ and $u_y$) are needed to characterize the time rate of change of the orientation of B in A. (One or both of $\sigma_1$ and $\sigma_2$ may be a specified function of time, in which case fewer than two generalized speeds are needed.) If two generalized speeds are needed, define them as

$$u_x = ({}^N\boldsymbol{\omega}^B - \sigma_2\mathbf{j}) \cdot \mathbf{i}, \tag{20}$$

$$u_y = {}^N\boldsymbol{\omega}^B \cdot \mathbf{j}. \tag{21}$$

The angular velocity of B in N then can be written

$${}^N\boldsymbol{\omega}^B = {}^N\boldsymbol{\omega}^A + (u_x - {}^N\boldsymbol{\omega}^A \cdot \mathbf{i})\mathbf{i} + [u_y - {}^N\boldsymbol{\omega}^A \cdot \mathbf{j} - (u_x - {}^N\boldsymbol{\omega}^A \cdot \mathbf{i})\mathbf{i} \cdot \mathbf{j}]\mathbf{j}. \tag{22}$$

To prove this, we begin by noting that equation (16) applies. Given the hypothesis at hand, one thus has

$${}^N\boldsymbol{\omega}^B = {}^N\boldsymbol{\omega}^A + \sigma_1\mathbf{i} + \sigma_2\mathbf{j}. \tag{23}$$

Substituting from equation (23) into equation (20), and solving for $\sigma_1$, one finds that

$$\sigma_1 = u_x - {}^N\boldsymbol{\omega}^A \cdot \mathbf{i}. \tag{24}$$

Similarly, substituting from equation (23) into equation (21), and solving for $\sigma_2$ yields, with the aid of equation (24),

$$\sigma_2 = [u_y - {}^N\boldsymbol{\omega}^A \cdot \mathbf{j} - (u_x - {}^N\boldsymbol{\omega}^A \cdot \mathbf{i})\mathbf{i} \cdot \mathbf{j}], \tag{25}$$

whereupon substitution from equations (24) and (25) into equation (23) completes the proof.

**Type 3.** A and B are connected in such a way that ${}^A\boldsymbol{\omega}^B$ can be expressed as $\sigma_1\mathbf{i} + \sigma_2\mathbf{j} + \sigma_3\mathbf{k}$, where $\sigma_1$, $\sigma_2$, and $\sigma_3$ are time-dependent scalars, while $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are nonparallel unit vectors. Under these circumstances, at most three generalized speeds (e.g., $u_x$, $u_y$, and $u_z$) are needed to characterize the time rate of change of the orientation of B in A. Define these as

$$u_x = {}^N\boldsymbol{\omega}^B \cdot \mathbf{c}_1, \tag{26}$$

$$u_y = {}^N\boldsymbol{\omega}^B \cdot \mathbf{c}_2, \tag{27}$$

$$u_z = {}^N\boldsymbol{\omega}^B \cdot \mathbf{c}_3, \tag{28}$$

where $\mathbf{c}_i (i = 1, 2, 3)$ are nonparallel, noncoplanar unit vectors. The choice of $\mathbf{c}_i (i = 1, 2, 3)$ can have a significant effect on the relative complexity of dynamical differential equations. One choice that works well is to let each of $\mathbf{c}_i$ $(i = 1, 2, 3)$ be parallel to a central principal axis of inertia of B. Another possibility, one that may be efficacious when A and B possess a permanently common point, is to let $\mathbf{c}_1$ be parallel to the line that joins the common point to the mass center of B, and to make $\mathbf{c}_2$ and $\mathbf{c}_3$ orthogonal to each other and to $\mathbf{c}_1$. In any event, as long as $\mathbf{c}_i$ $(i = 1, 2, 3)$ are mutually perpendicular, it follows directly from equations (26)–(28) that the angular velocity of B in N can be written

$${}^N\boldsymbol{\omega}^B = u_x\mathbf{c}_1 + u_y\mathbf{c}_2 + u_z\mathbf{c}_3. \tag{29}$$

The three types of connection of A to B considered so far all permit relative rotation of A and B. When relative translation of A and B also is possible, there frequently exist additional generalized speeds that lead to simplified equations of motion. Specifically, if a point P of B is constrained to move on A in a direction parallel to a unit vector $\mathbf{i}$ fixed in A, and one generalized speed (e.g., $u_x$) suffices to characterize the time rate of change of the distance between P and a point fixed in A, then it may be beneficial to define $u_x$ as ${}^N\mathbf{v}^P \cdot \mathbf{i}$, where ${}^N\mathbf{v}^P$ denotes the velocity of P in N. Alternatively, one can define $u_x$ as ${}^A\mathbf{v}^P \cdot \mathbf{i}$, where ${}^A\mathbf{v}^P$ is the velocity of P in A. Which of these choices leads to the simpler equations of motion depends on the system under consideration.

A word of caution: motion constraints, which lead to interdependencies among the generalized speeds, may affect the dynamical equations so as to diminish, although only slightly, the benefits provided by the choice of generalized speeds advocated above.

## 4. Illustrative Examples

**Example 1.** Figure 3 shows a system formed by two rigid bodies A and B connected to each other at a point P by means of a revolute joint whose axis is parallel to a unit vector $\mathbf{a}_1$ that is fixed in both A and B; $\mathbf{a}_2$ and $\mathbf{a}_3$ are fixed in A, and perpendicular to each other as well as to $\mathbf{a}_1$. The orientation of A in a Newtonian reference frame N is presumed to be changing in such a way that ${}^N\boldsymbol{\omega}^A$ can be expressed as $\omega_1\mathbf{a}_1 + \omega_2\mathbf{a}_2 + \omega_3\mathbf{a}_3$. To deal with the motion of this system in N, let $u_1$ be a generalized speed defined in accordance with equation (13) as $u_1 = {}^N\boldsymbol{\omega}^B \cdot \mathbf{a}_1$. From equation (14), it then follows that the angular velocity of B in N is given by

$${}^N\boldsymbol{\omega}^B = u_1\mathbf{a}_1 + \omega_2\mathbf{a}_2 + \omega_3\mathbf{a}_3. \tag{30}$$

This equation is simpler than the one with which it must be replaced if $u_1$ is taken to be ${}^A\boldsymbol{\omega}^B \cdot \mathbf{a}_1$, which is the customary choice.

**Example 2.** In Figure 4, A and B designate rigid bodies connected to each other with a Hooke joint formed by mounting in bearings fixed in A and B a rigid cross-piece C consisting of two arms that form a right angle. Mutually perpendicular unit vectors $\mathbf{a}_i$ $(i = 1, 2, 3)$ are fixed in A, with $\mathbf{a}_1$ parallel to the arm of C that is mounted in A, and $\mathbf{b}_2$ is a unit vector parallel to the arm of C mounted in B. The orientation of A in N is presumed to be changing in such a way that ${}^N\boldsymbol{\omega}^A$ can be expressed as $\omega_1\mathbf{a}_1 + \omega_2\mathbf{a}_2 + \omega_3\mathbf{a}_3$. Because of the way C connects A to B, there exist scalars $\sigma_1$ and $\sigma_2$ such that the angular velocities of C in A and B in C can be expressed as ${}^A\boldsymbol{\omega}^C = \sigma_1\mathbf{a}_1$ and ${}^C\boldsymbol{\omega}^B = \sigma_2\mathbf{b}_2$, respectively; and the angular velocity of B in A therefore can be written ${}^A\boldsymbol{\omega}^B = \sigma_1\mathbf{a}_1 + \sigma_2\mathbf{b}_2$. Hence, after defining $u_1$ as ${}^N\boldsymbol{\omega}^C \cdot \mathbf{a}_1$ and $u_2$ as ${}^N\boldsymbol{\omega}^B \cdot \mathbf{b}_2$, which is compatible with the definitions proposed in equations (20) and (21), one obtains with the aid of equation (22) the following simple expression for ${}^N\boldsymbol{\omega}^B$ in terms of $u_1$ and $u_2$:

$${}^N\boldsymbol{\omega}^B = {}^N\boldsymbol{\omega}^A + (u_1 - \omega_1)\mathbf{a}_1 + (u_2 - \omega_2)\mathbf{b}_2. \tag{31}$$

**Example 3.** Figure 5 depicts two rigid bodies A and B connected by means of a ball-and-socket joint. Two sets of mutually perpendicular unit vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, and $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$, are fixed in A and B, respectively, and the orientation of A in N is presumed to be changing in such a way that the angular velocity of A in N is given by $\omega_1\mathbf{a}_1 + \omega_2\mathbf{a}_2 + \omega_3\mathbf{a}_3$. The connection between A and B being of type 3, we set $u_i = {}^N\boldsymbol{\omega}^B \cdot \mathbf{b}_i$ $(i = 1, 2, 3)$ and note that, in conformity with equation (29), the angular velocity of B in N is given by

$${}^N\boldsymbol{\omega}^B = u_1\mathbf{b}_1 + u_2\mathbf{b}_2 + u_3\mathbf{b}_3. \tag{32}$$

**Example 4.** Figure 1, as mentioned earlier, is a schematic representation of a system S formed by a spacecraft A that supports a manipulator. The manipulator consists of a rotatable shoulder B, an upper arm C, a forearm D, and a hand E. Dextral sets of orthonormal unit vectors $\mathbf{a}_i$, $\mathbf{b}_i$, $\mathbf{c}_i$, $\mathbf{d}_i$, $\mathbf{e}_i$, and $\mathbf{n}_i$ $(i = 1, 2, 3)$ are fixed in A, B, C, D, E, and a Newtonian reference frame N, respectively, with $\mathbf{a}_i$ $(i = 1, 2, 3)$ parallel to central principal axes of inertia of A, and similarly for $\mathbf{e}_i$ $(i = 1, 2, 3)$

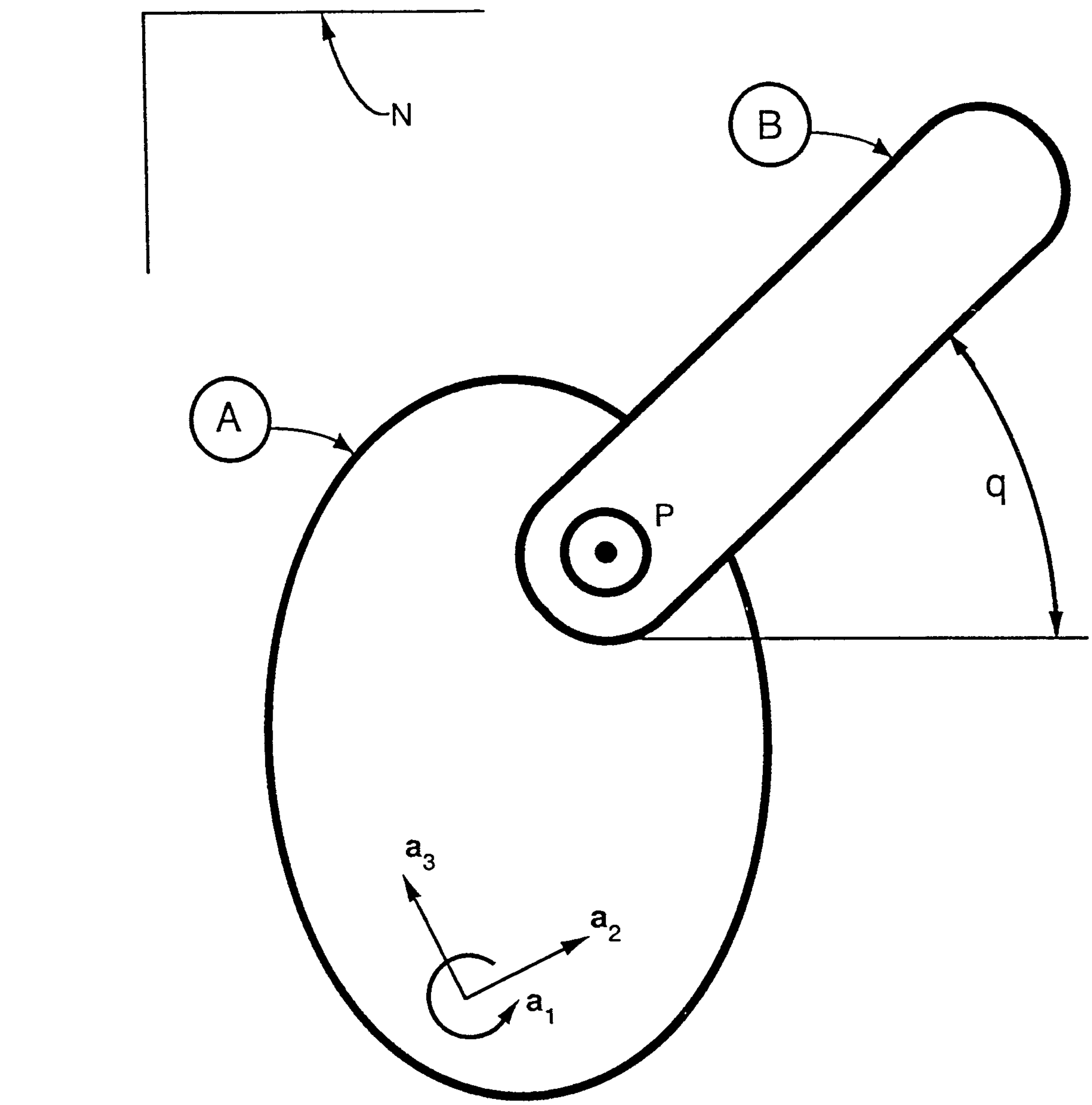


*Fig. 3. Rigid bodies connected by a revolute joint.*

and E. B is connected to A at $A^*$, the mass center of A, by means of bearings mounted in A so as to permit rotation of B relative to A about an axis parallel to $\mathbf{b}_2$ (or, equivalently, $\mathbf{a}_2$); C is connected to B at $A^*$ with a revolute joint whose axis is parallel to $\mathbf{c}_3$ $(= \mathbf{b}_3)$; D is connected to C at point P with a revolute joint whose axis is parallel to $\mathbf{d}_3$ $(= \mathbf{c}_3)$; and E is connected to D at $E^*$, the mass center of E, with a ball-and-socket joint. $C^*$, the mass center of C, is at the midpoint between $A^*$ and P, and $D^*$, the mass center of D, is at the midpoint between P and $E^*$.

The masses of A, ..., E are $M_A, \ldots, M_E$, and A has a moment of inertia $I_i^A$ about a line passing through $A^*$ and parallel to $\mathbf{a}_i$ $(i = 1,2,3)$. Similarly, $I_i^B, \ldots, I_i^E$ $(i = 1,2,3)$ denote central principal moments of inertia of B, ..., E.

For purposes of attitude control, a couple of torque $\mathbf{T}^A$ is exerted on A by thrusters (not shown in Fig. 1). Associated scalars $T_i^A$ $(i = 1,2,3)$ are defined as $T_i^A = \mathbf{T}^A \cdot \mathbf{a}_i$ $(i = 1,2,3)$. To drive B, ..., E, motors (not shown in Fig. 1) transmit forces from A to B, B to C, C to D, and D to E. The set of such forces exerted by A on B is equivalent to a couple of torque $\mathbf{T}^{A/B}$ together with a force $\mathbf{F}^{A/B}$ applied at a point on the axis of relative rotation of A and B. The only way $\mathbf{T}^{A/B}$ and $\mathbf{F}^{A/B}$ manifest themselves in the equations of motion is through the appearance of $T^{A/B}$, defined as $T^{A/B} = \mathbf{T}^{A/B} \cdot \mathbf{a}_2$. Similarly, $T^{B/C}$, $T^{C/D}$, and $T_i^E$ $(i = 1,2,3)$, defined as $T^{B/C} = \mathbf{T}^{B/C} \cdot \mathbf{b}_3$,

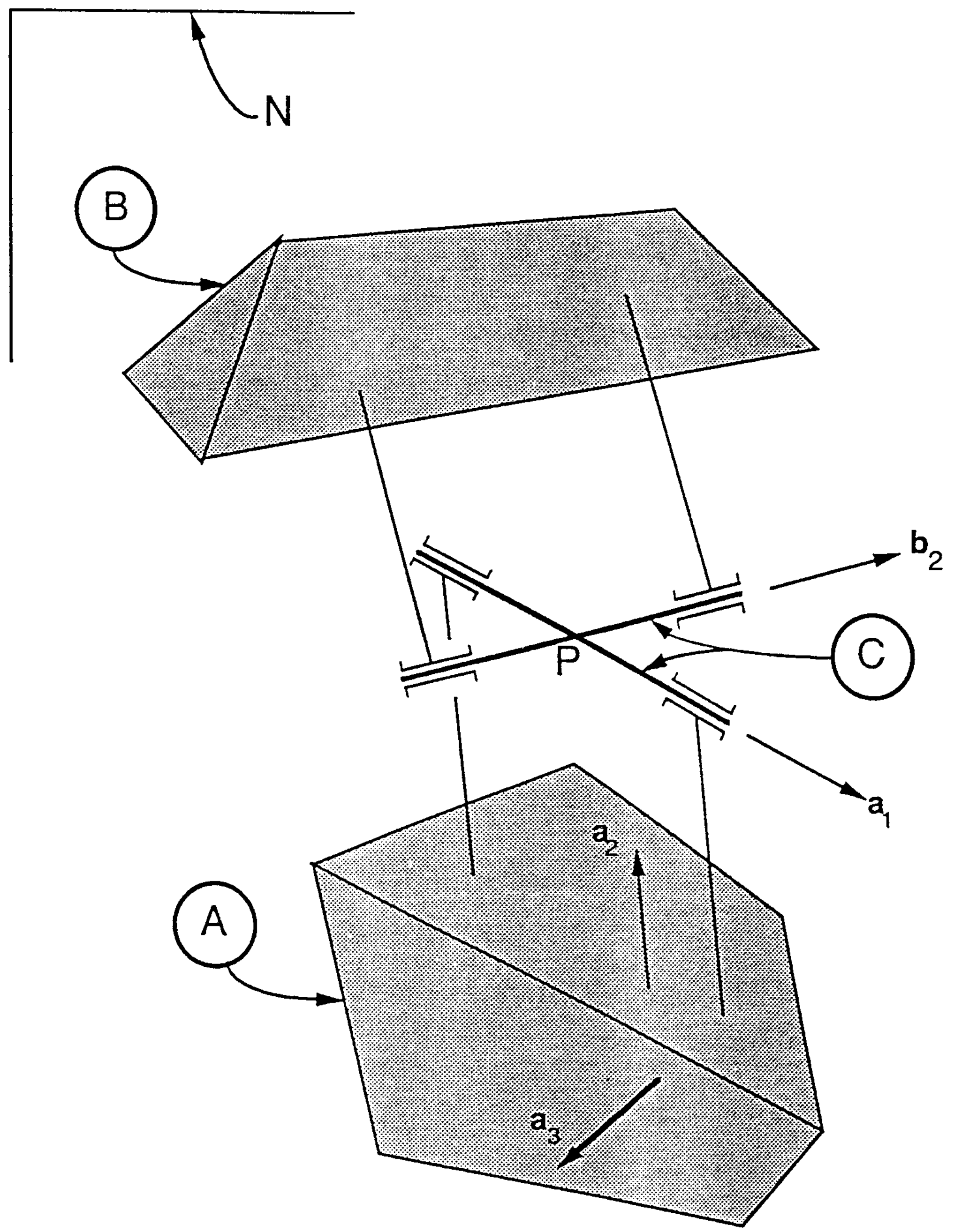


*Fig. 4. Rigid bodies connected by a Hooke's joint.*

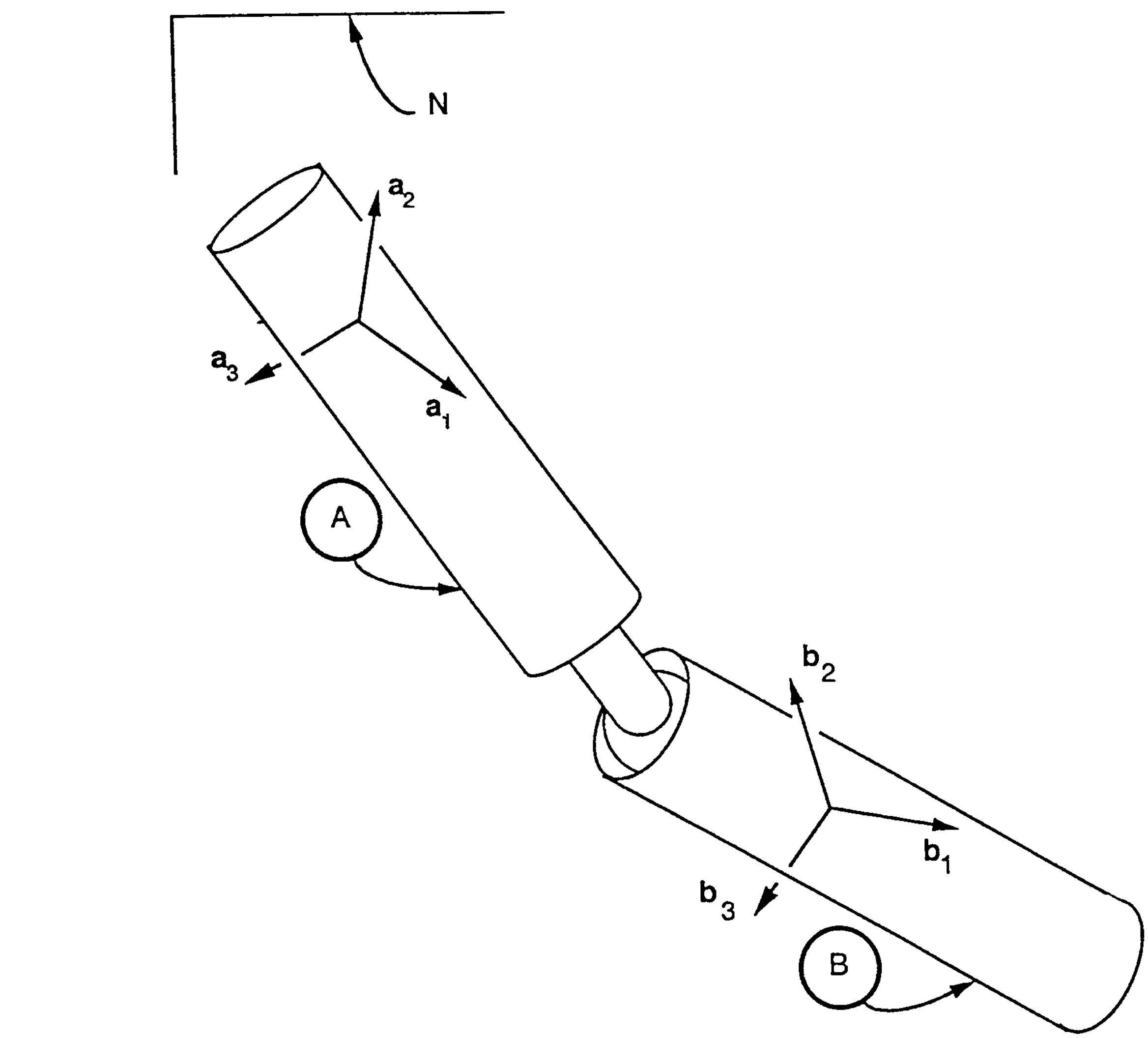


*Fig. 5. Rigid bodies connected by a ball-and-socket joint.*

$T^{C/D} = \mathbf{T}^{C/D} \cdot \mathbf{c}_3$, and $T_i^E = \mathbf{T}^{D/E} \cdot \mathbf{e}_i$ $(i = 1, 2, 3)$, enter the equations of motion.

Generalized coordinates $q_1, \ldots, q_9$ are introduced as follows. To bring A into a general orientation in N, $\mathbf{a}_i$ is aligned with $\mathbf{n}_i$ $(i = 1, 2, 3)$, and A is then subjected to successive right-hand rotations in N of amounts $q_1$, $q_2$, and $q_3$ about $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$, respectively. The orientations of B in A, C in B, and D in C are characterized, respectively, by the angle $q_4$ between $\mathbf{a}_3$ and $\mathbf{b}_3$, the angle $q_5$ between $\mathbf{b}_1$ and $\mathbf{c}_1$, and the angle $q_6$ between $\mathbf{c}_1$ and $\mathbf{d}_1$. Finally, to orient E in D, $\mathbf{e}_i$ is aligned with $\mathbf{d}_i$ $(i = 1, 2, 3)$, and E is then subjected to successive right-hand rotations of amounts $q_7$, $q_8$, and $q_9$ about $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$, respectively.

For the introduction of generalized speeds $u_1, \ldots, u_9$, we consider three choices. The first, termed a "simple" choice, is to let

$$u_i = \dot{q}_i \qquad (i = 1, \ldots, 9). \tag{33}$$

This choice is made by default when one formulates equations of motion via Lagrange's method and leads to extremely complex dynamical equations. A second possibility is to let

$$u_i = {}^N\boldsymbol{\omega}^A \cdot \mathbf{a}_i \qquad (i = 1, 2, 3), \tag{34}$$

$$u_i = \dot{q}_i \qquad (i = 4, 5, 6), \tag{35}$$

$$u_i = {}^D\boldsymbol{\omega}^E \cdot \mathbf{e}_{i-6} \qquad (i = 7, 8, 9). \tag{36}$$

This is the choice that many experienced dynamicists have been making throughout the last three decades (see references 1–13), for which reason we call it "customary."

**Table 3. Operations Count**

| Operation | Simple | Customary | New |
|---|---|---|---|
| Addition | 430 | 272 | 120 |
| Subtraction | 919 | 680 | 278 |
| Multiplication | 2310 | 1568 | 741 |
| Division | 45 | 48 | 21 |
| Sine | 10 | 8 | 8 |
| Cosine | 10 | 9 | 9 |
| Tangent | 0 | 2 | 2 |
| Negation | 59 | 44 | 27 |
| Assignment | 669 | 564 | 254 |
| Total | 4452 | 3195 | 1460 |

Finally, noting that the connections between A and B, B and C, and C and D all are of type 1, while that between D and E is of type 3, one is led by the guidelines set forth above to let

$$u_i = {}^N\boldsymbol{\omega}^A \cdot \mathbf{a}_i \qquad (i = 1, 2, 3) \tag{37}$$

$$u_4 = {}^N\boldsymbol{\omega}^B \cdot \mathbf{b}_2 \tag{38}$$

$$u_5 = {}^N\boldsymbol{\omega}^C \cdot \mathbf{c}_3 \tag{39}$$

$$u_6 = {}^N\boldsymbol{\omega}^D \cdot \mathbf{d}_3 \tag{40}$$

$$u_i = {}^N\boldsymbol{\omega}^E \cdot \mathbf{e}_{i-6} \quad (i = 7, 8, 9). \tag{41}$$

The sets of kinematical differential equations corresponding to the customary (equations (34)–(36)) and the new (equations (37)–(41)) choices of generalized speeds are reported in Table 1. Each of the first three equations in the customary set is identical to the corresponding equation in the new set, but each of the remaining equations in the novel set is somewhat more complicated than its counterpart in the customary set. This is the price one must pay for working with the new generalized speeds rather than with the familiar or simple ones. As will become apparent presently, it is a very small price indeed.

Frequently, the purpose of formulating equations of motion is to produce motion simulations by numerical integration of dynamical and kinematical differential equations. In this context, it is disadvantageous to write differential equations in the fully expanded form employed in Table 1, for this engenders repetitious calculations. The correct course of action is to introduce intermediate variables for the specific purpose of reducing the number of calculations required at each integration step. Such variables are introduced automatically when the symbol manipulation program AUTOLEV (Schaechter and Levinson 1988) is employed to formulate the equations of motion. Thereafter, the efficiency of the equations of motion can be measured by counting the number of operations required to evaluate the time derivatives of the generalized coordinates and generalized speeds. Table 3 reports the results of the operations counts for the evaluation of $\dot{q}_i$ and $\dot{u}_i$ $(i = 1, \ldots, 9)$ for the simple (equation (33)), customary (equations (34)–(36)) and new (equations (37)–(41)) generalized speeds. The superiority of the new generalized speeds to the customary and simple ones is obvious in the light of these results. A related, corroborative finding is that 10,000 evaluations of $\dot{q}_i$ and $\dot{u}_i$ $(i = 1, \ldots, 9)$ require 154.83 seconds when the simple generalized speeds are used, 112.48 seconds when the customary generalized speeds are used, but only 54.21 seconds when the new ones are employed (calculations performed on an HP 386 VECTRA 25 megahertz computer). As may be expected, the computation time required for numerical simulation is substantially reduced when the new generalized speeds are utilized. For example, with a fifth-order Kutta-Merson integrator, it takes 212.39 seconds to generate a 300-second time history for $\dot{q}_i$ and $\dot{u}_i$ $(i = 1, 2, \ldots, 9)$ with the simple generalized speeds, 125.39 seconds with the customary generalized speeds, but only 69.04 seconds with the new generalized speeds.

As a note of interest, it turns out that $q_i$ $(i = 1, 2, 3)$ do not appear in any of the dynamical or kinematical differential equations for $\dot{q}_i$ $(i = 4, \ldots, 9)$ or $\dot{u}_i$ $(i = 1, \ldots, 9)$ when the customary or new generalized speeds are used. One consequence of this fact is that the analytical scheme used to describe the orientation of A in N, i.e., Euler angles, Euler parameters, Rodrigues parameters, etc., may be chosen after the dynamical equations of motion have been generated, a luxury not afforded by the simple choice of generalized speeds.

Since the results just reported apply to a particular system, it is natural to wonder about the extent to which choosing generalized speeds in accordance with the guidelines in this article is likely to lead to significant reductions in the complexity of equations of motion for systems other than the one considered here. To determine this, we have examined many other systems and have found in each case that use of the new generalized speeds is highly advantageous. For example, Rosenthal and Sherman's (1986) equations of motion for the Stanford Arm necessitate 966 arithmetic operations at each integration step, whereas equations governing generalized speeds chosen in accordance with our guidelines require only 689 such operations. As is self-evident, incorporation of this new methodology in multibody general-purpose programs is computationally advantageous and of great importance in situations requiring real-time numerical solutions.

## 5. Conclusion

In this article, guidelines were presented for selecting motion variables that lead to efficient equations of motion. Each guideline was accompanied by an illustrative example. Several comparative studies were performed to quantify the differences between the efficiency of equations of motion generated with the motion variables chosen in accordance with the guidelines, and variables chosen in a more conventional fashion.

The new motion variables have been used successfully in a recent analysis of tethered spacecraft (Thornburg 1995) and have been found to be highly computationally advantageous in the Order-N formulation (Banerjee 1994). However, it is not known what advantages they may have in connection with other applications, such as control system design by the computed torque method, or what benefits may be derived from using them in conjunction with other schemes that improve computational efficiency of equations of motion (Singer and Seering 1993).

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