Kaplansky's Direct Finiteness Conjecture and Sofic Groups

Lucas Levrouw Michaël Maex Tomas Reunbrouck
Supervisor: Gabor Szabo

November 29, 2019

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1 Introduction

The Lay of the Land

This report discusses the conjecture of Kaplansky which states that any group ring $K[G]$ with K a field is directly finite. That is, the left-invertibility of elements implies their right-invertibility. For general groups and fields, this conjecture remains unanswered. It has been resolved, however, for specific cases, such as the case in which the group G is sofic (the definition of soficity follows in section 3). Some cases require the assumption of another conjecture. For example, when we talk about finite fields K we have a positive result for Kaplansky's conjecture for surjunctive groups (definition in section 6). The class of surjunctive groups is very large and possibly contains all countable groups. Therefore, if we were to have that all countable groups are surjunctive, this positively resolves Kaplansky's conjecture for finite fields and countable groups. The assumption that all countable groups are surjunctive is known as the conjecture of Gottschalk and will be discussed in detail.

Historical Context

We can trace the origins of the concepts we shall discuss back to the research by John von Neumann throughout the 20th century. First, he introduced the notion of amenable groups in the context of the Banach-Tarski Paradox. It was meant as a generalisation of commutative groups on the one hand and various versions of finite groups on the other. Secondly, he lay the foundations of the study of so-called 'cellular automata'. What a cellular automaton is, can be explained the following way. Take a set A (finite or infinite), of which we call the elements 'states', and a 'universe' of 'cells', being a countable group G. Now A^G (the set of maps $G \to A$) is the set of configurations. The automaton, then, is a map from A^G to itself, satisfying the property that the state of a specific cell in the image is determined by the initial states of the cells in a bounded neighbourhood of this cell [Neu66] .

These two concepts may, at first glance, seem completely unrelated. The connection becomes clear, however, when considering the question of surjectivity of these maps on the configuration space. Surjectivity in this context is, intuitively, the property that any configuration can be reached as a result from some other configuration. A configuration that can not be produced because it is not in the image of the (non-surjective) automaton is called a 'Garden of Eden'. Importantly, such a Garden of Eden always contains a so-called 'orphan' which is a finite subset which cannot be produced (a finite Garden of Eden, as it were). The Garden of Eden Theorem states that a configuration is a Garden of Eden if and only if it contains such an orphan. From this theorem it follows that the injectivity of the automaton implies its surjectivity. However, the validity of this theorem has only been proven for amenable groups. Now, groups G for which it is the case that injectivity of an automaton using any finite alphabet implies surjectivity of that automaton are the afore mentioned surjunctive groups. This, then, brought research eventually to sofic groups, as this is the largest known class of groups of which all members are known to be surjunctive [CC10].

Goal of this Report

We shall mainly focus on the following two concepts: soficity and surjunctivity. First we shall discuss what it means for a group to be sofic. It is a very broad concept. In fact, there exist no known examples of non-sofic groups [Wei00]. This will be illustrated by proving soficity for a few large classes of groups. Furthermore, we shall discuss its connections to other important concepts and group properties such as amenability. Secondly, this report will cover surjunctivity which is related to soficity but will be used to discuss the connection between the conjectures of Kaplansky and Gottschalk. Specifically, it is known that all sofic groups are surjunctive as well. Finally, we will discuss Kaplansky's conjecture in the context of finite fields. More precisely we will give the details of a proof that the group ring a surjunctive group and finite field is directly finite. Figure 1 shows how how the different concepts in this report relate to each other.

$$\begin{tikzcd}
G \text{ is Residually Finite} \arrow[dr, Rightarrow] & & & \\
& G \text{ is Sofic} \arrow[r, Rightarrow] & G \text{ is Surjunctive} \arrow[r, Rightarrow] & \mathbb{F}_{p^k}[G] \text{ is Directly finite} \\
G \text{ is Amenable} \arrow[ur, Rightarrow] & & &
\end{tikzcd}$$

Figure 1: A dependency graph of the relevant theorems/properties.

2 Group Rings

The setting of Kaplansky's direct finiteness conjecture is about so-called group rings. These are defined as follows

Definition 2.1. *Let $\mathbb{K}$ be a field and G be a group. The* ***group ring*** *$\mathbb{K}[G]$ is the set of formal linear combinations*

$$\sum_{g\in G} \lambda_g g,$$

where $\lambda_g \in \mathbb{K}$ for all $g \in G$ and only finitely many λ_g are nonzero. The operations $+$ and $\cdot$ are defined by

$$\sum_{g\in G} \lambda_g g + \sum_{g\in G} \mu_g g = \sum_{g\in G} (\lambda_g + \mu_g) g$$

$$\left(\sum_{g\in G} \lambda_g g\right) \cdot \left(\sum_{g\in G} \mu_g g\right) = \sum_{g\in G} \left(\sum_{h\in G} \lambda_h \mu_{h^{-1}g}\right) g.$$

Proposition 2.2. *Let $\mathbb{K}$ be a field and G a group. Then $\mathbb{K}[G]$ is a unital ring.*

The proof of this theorem is no more than checking the group axioms one by one. For completeness you can find it in section 10.1.

Remark 2.3. A group ring in general is not commutative. In fact $\mathbb{K}[G]$ is commutative if and only if G is a commutative group. So we will have to be careful when dealing with these rings.

We can now state Kaplansky's conjecture by the example of the introduction in [ES03].

Conjecture 2.4 (Kaplansky)**.** *Let $\mathbb{K}$ be a field and G a group. Then $\mathbb{K}[G]$ is directly finite i.e. for all $a, b \in \mathbb{K}[G]$, the equation $ab = 1$ implies $ba = 1$.*

To put it shortly, Kaplansky's conjecture states that for all elements in the group ring, left-invertible implies right-invertible. If $\mathbb{K}$ is a finite field and G is a finite group, it can quite easily be proven that $\mathbb{K}[G]$ is directly finite.

Theorem 2.5. *Let $\mathbb{K}$ be a field and G be a finite group. Then $\mathbb{K}[G]$ is directly finite.*

Proof. Let $a, b \in \mathbb{K}[G]$ and suppose $ab = 1$. Now consider the map

$$\phi : \mathbb{K}[G] \to \mathbb{K}[G] : x \mapsto xa$$

This map is injective. To see this, let $x, y \in \mathbb{K}[G]$ such that $xa = ya$. Multiply on the right with b and use the associativity to conclude $x = y$. But since $\mathbb{K}[G]$ is a finite set, the map must also be surjective. Therefore there exists an $x \in \mathbb{K}[G]$ such that $xa = 1$. Furthermore, $x = xab = b$. So we have $b = a$. □

Note how the argument hinges on the fact that the map ϕ is guaranteed to be surjective by the finiteness of $\mathbb{K}$ and G. In what follows we will define several "finiteness" properties of groups and try to generalise the proof above under weaker assumptions.

3 Sofic Groups

3.1 Definition and Motivation

In this section we will introduce the concept of a *sofic group*. A sofic group is a group that is "well approximated" by a finite permutation group. Recall that every finite group is isomorphic to the subgroup of a permutation group.

To define this, we will need a way to measure a distance between permutations. It is quite natural to define the distance between two permutation as the fraction of elements on which they differ [CL15, subsection 2.1].

Definition 3.1. *The* ***Hamming distance*** *on* $\mathrm{Sym}(n)$ *is defined by*

$$d_{\mathrm{Sym}(n)}(\sigma, \tau) = \frac{1}{n} \left|\{k \in \{1, 2, \ldots, n\} \mid \sigma(k) \neq \tau(k)\}\right|,$$

for every $\sigma, \tau \in \mathrm{Sym}(n)$.

It is easy to show that $d_{\mathrm{Sym}(n)}$ is a metric. We might call a group well-approximated by finite permutation groups if there exists a sequence of "approximate morphisms" $(\Phi_i : G \to \mathrm{Sym}(d_i))_{i\in\mathbb{N}}$ which mimic the group structure of G better and better in $\mathrm{Sym}(d_i)$ as i tends to infinity. For that, we first of all want that $d_{\mathrm{Sym}(d_i)}(\Phi_i(g) \circ \Phi_i(h), \Phi_i(gh))$ converges to zero, for every g and h in G. Additionaly, we want that in the limit these maps have a sort of injectivity property. Therefore we will require all elements of G except the identity element to converge out of the kernel of these maps. We add the following condition [1]

$$\lim_{i\to\infty} d_{\mathrm{Sym}(d_i)}(\mathrm{id}, \Phi(g)) = \begin{cases} 0 & \text{if } g = e \\ 1 & \text{otherwise,} \end{cases}$$

where e denotes the identity element of G and id is the identity permutation.

We can now state the definiton of a sofic group based on [KL10, section 1].

[1] One might ask why we do not require that every $g \neq 0$ converges to some fixed strictly positive number $r(g)$ instead. In fact one can show that if that is the case, one can construct another sequence of maps $\Phi_i' : G \to \mathrm{Sym}\, d_i'$ such that $d_{\mathrm{Sym}(d_i')}(\mathrm{id}, \Phi_i'(g))$ converges to 1 for every g ([CL15], definition 2.2.2)

Definition 3.2. *Let G be a countable group. We say that G is **sofic** if there exists a sequence of natural numbers $(d_i)_{i\in\mathbb{N}}$ with associated maps $\Phi_i : G \to \mathrm{Sym}(d_i)$ such that for all $g, h \in G$:*

(i) $\lim_{i\to\infty} d_{\mathrm{Sym}(d_i)}(\Phi_i(g)\circ\Phi_i(h), \Phi_i(gh)) = 0$

(ii) $\lim_{i\to\infty} d_{\mathrm{Sym}(d_i)}(\mathrm{id}, \Phi(g_i)) = \begin{cases} 0 & \text{if } g = e \\ 1 & \text{otherwise,} \end{cases}$

We will call such a family of maps $(\Phi_i : G \to \mathrm{Sym}(d_i))_{i\in\mathbb{N}}$ a *sofic approximation sequence* for G. It is unkown if every group is sofic.

We can also reformulate the two conditions in a form more suitable for calculations.

Lemma 3.3. *Let G be a countable group. Then G is sofic if and only if there exists a sequence of natural numbers $(d_i)_{i\in\mathbb{N}}$ with associated maps $\Phi_i : G \to \mathrm{Sym}(d_i)$ such that for all $g, h \in G$:*

(i) $\lim_{i\to\infty} \frac{1}{d_i} \left|\{k \in \{1,\dots,d_i\} \mid (\Phi_i(g)\circ\Phi_i(h))(k) = \Phi_i(gh)(k)\}\right| = 1$

(ii) $\lim_{i\to\infty} \frac{1}{d_i} \left|\{k \in \{1,\dots,d_i\} \mid \Phi_i(g)(k) = k\}\right| = \begin{cases} 1 & \text{if } g = e \\ 0 & \text{otherwise,} \end{cases}$

3.2 Examples of Sofic Groups

The Sofic groups are a very large class of groups. In fact there are no known examples of non-sofic groups . Here we will list some groups that we know are sofic.

reference for no known non-sofic

Example 3.4. Consider a finite group G. Look at the map $\phi : G \to \mathrm{Sym}(G) : g \mapsto \sigma_g$, where σ_g is defined by $\sigma_g(h) = gh$. Clearly, ϕ is a homomorphism. Now we can choose $d_i = |G|$ and $\Phi_i = \phi$ for all $i \in \mathbb{N}$. Here we have identified $\mathrm{Sym}(G)$ with $\mathrm{Sym}(d_i)$. The first condition is satisfied since ϕ is a morphism. The second condition follows from the fact that $\sigma_g(h) = gh = h$ if and only if g is the identity. Hence every finite group is sofic.

Example 3.5. Consider the group $\mathbb{Z}$. Take the sequence of maps $\Phi_i : G \to \mathrm{Sym}(i) : g \mapsto \Phi_i(g)$ where $\Phi_i(g)$ is defined as the map such that for any $k \in \{1,\dots,i\} : \Phi_i(g)(k) = (k+g) \mod i$. The first condition of soficity is satisfied as for any $i \in \mathbb{N}, g, h \in \mathbb{Z}, k \in \{1,\dots,i\}$ it can be said that

$$\begin{aligned}(\Phi_i(g)\circ\Phi_i(h))(k) &= (k+h+g) \mod i \\ &= (k+g+h) \mod i \\ &= \Phi_i(g+h)(k)\end{aligned}$$

Now for the second condition. It is clear that for any $i \in \mathbb{N}, k \in \{1,\dots,i\} : \Phi_i(0)(k) = k$. Now take any $g \neq 0$ and consider $\Phi_i(g)$. For any $k \in \{1,\dots,i\}$ it is perfectly possible that $\Phi_i(g)(k) = k$. However this is not the case as soon as $i > |g|$. In that case, we have that $|(k+g)-k| < i$, from which it follows that $(k+g) \neq k \mod i$. Thus we have that $(k+g) \neq k \mod i$. Therefore, $\Phi_i(g)(k) \neq k$, proving the second condition.

Example 3.6. The direct product of two sofic groups is again sofic. Consider two sofic groups G and H. Let $(\phi_i : G \to \mathrm{Sym}(d_i))_{i\in\mathbb{N}}$ and $(\psi_i : G \to \mathrm{Sym}(e_i))_{i\in\mathbb{N}}$ be sofic approximation sequences for G resp. H. Now consider $\Phi_i : G \times H \to \mathrm{Sym}(d_i + e_i)$ for every $i \in \mathbb{N}$, where

$\Phi_i(g,h) \in \mathrm{Sym}(d_i + e_i)$ is defined by

$$\Phi_i(g,h)(k) = \begin{cases} \phi_i(g)(k) & \text{if } 1 \leq k \leq d_i \\ \psi_i(h)(k) + d_i & \text{if } d_i + 1 \leq k \leq e_i \end{cases}.$$

It is not hard to show that $(\Phi_i)_{i\in\mathbb{N}}$ is a sofic approximation sequence for $G \times H$. Hence finite direct products conserve the soficity property.

Write out proof

In the next sections we will introduce two classes of groups that are sofic, *amenable groups* and *residually finite groups*. Since it is often easier to check these properties than to work with the definition of soficity, this will allow us to give some more examples. Some notable examples are

- countable abelian groups (proposition 4.8)
- countable solvable groups (proposition 4.12)
- finitely generated free groups (example 5.4)

4 Amenability

4.1 Definition and Examples

Amenable groups are a large class of groups. Many equivalent definitions exist. An extensive study of amenable groups goes beyond the scope of this project. Hence we will not give multiple definitions and prove equivalence nor will we provide proofs for every property of amenable groups.

For our purposes we will only define amenability for countable groups and take the Følner criterion as definition. In [KL16, subsection 4.1] an alternative definition is given, and it is proven that this is equivalent to the Følner criterion, as formulated below.

Definition 4.1. *Let G be a countable group. We say that G satisfies the* ***Følner criterion*** *if there exists a sequence of finite subsets (not necessarily subgroups) $F_1, F_2, \dots \subset G$ such that for all $g \in G$*

$$\lim_{i\to\infty} \frac{|gF_i \triangle F_i|}{|F_i|} = 0,$$

where $\triangle$ denotes the symmetric difference.

Figure 2 gives a visual intuition for this notion. If a countable group statisfies the Følner condition, we will call the group *amenable.*[2]

reference for the footnote

By the following lemma, we can not only exploit this for individual elements, but also for finite subsets.

Moet dit lemma naar achter?

Lemma 4.2. *Let F_n be a Følner sequence. Then for every finite subset $B \subset G$ it holds that*

$$\lim_{n\to\infty} \frac{|BF_n \triangle F_n|}{|F_n|} = 0.$$

[2] One can also define amenability also for uncountable groups. In that case however, it is defined differently and the Følner criterion is not equivalent to the amenability.

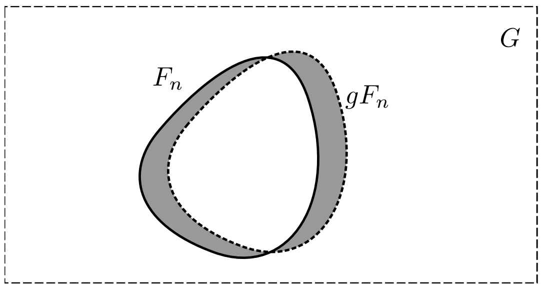

Figure 2: An illustration of the Følner criterion. The limit means that the gray part will become very small compared to F_n itself as n gets large.

Proof. Suppose that $B = \{b_1, b_2, \ldots, b_r\}$. Have a look at the following calculation:

$$\begin{aligned}
\lim_{n\to\infty} \frac{|BF_n \triangle F_n|}{|F_n|} &= \lim_{n\to\infty} 2\frac{|BF_n \setminus F_n|}{|F_n|} \\
&= \lim_{n\to\infty} 2\frac{\left|\bigcup_{i=1}^{k}(b_iF_n \setminus F_n)\right|}{|F_n|} \\
&\leq \lim_{n\to\infty} 2\frac{\sum_{i-1}^{k} |b_iF_n \setminus F_n|}{|F_n|} \\
&= \lim_{n\to\infty} \sum_{i=1}^{k} \frac{|b_iF_n \triangle F_n|}{|F_n|} = 0.
\end{aligned}$$

.

□

Example 4.3. Every finite group trivially satisfies the Følner condition. Indeed, we can take $F_i = G$ for every $i \in \mathbb{N}$. Note that $gG\triangle G = G\triangle G = \emptyset$. Hence $|gF_i\triangle F_i| = 0$ for all $i \in \mathbb{N}$.

Example 4.4. The group of integers $\mathbb{Z}$ is also amenable. Let $F_i = \{-i, -i+1, \ldots, i-1, i\}$ for every $i \in \mathbb{N}$ and let k be an arbitrary integer. Then

$$\frac{|(k+F_i)\triangle F_i|}{|F_i|} = \frac{|\{k-i, k-i+1, \ldots, k+i-1, k+i\}\triangle\{-i, -i+1, \ldots, i-1, i\}|}{2i+1} \leq \frac{2|k|}{2i+1},$$

and this converges to zero as $i \to \infty$.

Example 4.5. Counterexamples are at least as insightful as examples. The free group $F_n = \langle x_1, x_2, \ldots, x_n\rangle, n \geq 2$ is not amenable. For those not familiar with the free group, a brief introduction is given in section 10.2. To prove this we will have to show that there does not exist a Følner sequence. We will prove that for every non-empty finite subset $H \subset \bar{n}$ it holds that

$$\left|\bigcup_{i=1}^{n} x_iH \cup x_i^{-1}H\right| \geq (2n-1)\,|H|\,.$$

We can prove this using induction on the size of H.

$|H| = 1$**:** Then $H = \{g\}$. Hence the left hand side set is $\{x_1 g, x_1^{-1}, x_2 g, x_2^{-1} g, \ldots, x_n g, x_n^{-1} g\}$, which has cardinality $2n$.

$|H| > 1$**:** Let $h \in H$ be an element that has maximum number of letters in reduced form. Consider the set $H' = H \setminus \{h\}$. By the induction hypothesis it holds that

$$\left| \bigcup_{i=1}^{n} x_i H' \cup x_i^{-1} H' \right| \geq (2n-1) \left| H' \right|.$$

For at least $2n-1$ elements of $\{x_1, x_1^{-1}, x_2, x_2^{-1}, \ldots, x_n, x_n^{-1}\}$ the words $x_i h$ or $x_i^{-1} h$ have more letters in reduced form than h. Let $\Delta = \{h_1, h_2, \ldots, h_n\}$ be that set. As multiplying by one generator changes the number of letters by ± 1, we see that elements of Δ are not of the form $x_i h'$ of $x_i^{-1} h'$ for some generator x_i and $h' \in H$. Hence

$$\left| \bigcup_{i=1}^{n} x_i H \cup x_i^{-1} H \right| \geq \left| \bigcup_{i=1}^{n} x_i H' \cup x_i^{-1} H' \right| + 2n - 1.$$

So

$$\left| \bigcup_{i=1}^{n} x_i H \cup x_i^{-1} H \right| \geq (2n-1) \left| H' \right| + (2n-1) = (2n-1) \left| H \right|.$$

This clearly contradicts the condition in lemma 4.2. We conclude that the free group is not amenable.

4.2 Properties of Amenable Groups

To prove that each group that satisfies the Følner criterion is sofic, we first need the following set-theoretic lemma.

Lemma 4.6. *Let A and B be finite sets such that $|A| = |B|$. Given $A' \subset A, B' \subset B$ and a bijection $f' : A' \to B'$ there exists a bijection $f : A \to B$ such that $f|_{A'} = f'$.*

Proof. Since $|A| = |B|$ and $|A|' = |B|'$

$$|A \setminus A'| = |A| - |A|' = |B| - |B|' = |B \setminus B'|.$$

Therefore there exists a bijection $g : A \setminus A' \to B \setminus B'$. Define $f : A \to B$ by $f(x) = f'(x)$ for all $x \in A'$ and $f(x) = g(x)$ for all $x \in A \setminus A'$. Then f is a bijection and $f|_{A'} = f'$. □

We now state the proof.

Szabo: Is this your theorem? If you present something like this which is well-known, either provide a reference or at least point out that it is known.

Theorem 4.7. *Every countable amenable group G is sofic.*

Proof. Let $F_1, F_2, F_3, \ldots$ be a sequence such that that the Følner criterion is satisfied. Choose for every $i \in \mathbb{N} : d_i = |F_i|$. We will implicitly identify $\mathrm{Sym}(F_i)$ with $\mathrm{Sym}(d_i)$. Now choose maps $\Phi_i : F_i \to \mathrm{Sym}(F_i)$ such that $\Phi(g)(h) = gh$ whenever $gh \in F_i$, which is possible due to lemma 4.6.

Fix $g, h \in G$. By the Følner criterion we know that

$$\begin{aligned}
\lim_{i\to\infty} \frac{\left|g^{-1}F_i \triangle F_i\right|}{|F_i|} &= 0 \\
&= \lim_{i\to\infty} \frac{\left|g^{-1}F_i \setminus F_i\right| + \left|F_i \setminus g^{-1}F_i\right|}{d_i} \\
&= \lim_{i\to\infty} \frac{2\left|F_i \setminus g^{-1}F_i\right|}{d_i} \\
&= \lim_{i\to\infty} \frac{2\left|\{x \in F_i \mid gx \notin F_i\}\right|}{d_i}.
\end{aligned}$$

This implies that

$$\lim_{i\to\infty} \frac{|\{x \in F_i \mid gx \in F_i\}|}{d_i} = 1, \tag{1}$$

or equivalently

$$\lim_{i\to\infty} \frac{|\{x \in F_i \mid gx \notin F_i\}|}{d_i} = 0 \tag{2}$$

As $\Phi(g)(x) = gx$ whenever $gx \in F_i$, we can see as well that

$$\lim_{i\to\infty} \frac{|\{x \in F_i \mid \Phi(g)(x) \neq gx\}|}{d_i} = 0.$$

Which proves the second condition for soficity.

To prove the first condition, we can apply lemma 4.2, which gives that

$$\lim_{i\to\infty} \frac{|\{g, h, gh\}F_i \triangle F_i|}{d_i} = 0.$$

Which means that

$$\lim_{i\to\infty} \frac{|\{x \in F_i \mid gx, hx, ghx \in F_i\}|}{d_i} = 1.$$

As $\Phi(g)(x) = gx$ whenever $gx \in F_i$, we can see as well that

$$\lim_{i\to\infty} \frac{|\{x \in F_i \mid (\Phi(g) \circ \Phi(h))(x) = \Phi(gh)(x) \in F_i\}|}{d_i} = 1.$$

This proves the first condition for soficity and concludes the proof. $\square$

Proposition 4.8. *Every countable abelian group is amenable.*

Proof. Let $G, +$ be a countable abelian group. Since G is countable, we can write $G = \{e, g_1, g_2, g_3, \dots\}$. Consider the sequence of sets

$$F_n = \left\{ \sum_{i=1}^{n} a_i g_i \mid -n \leq a_i \leq n \right\}.$$

Fix $g \in G$. Take any $n > j$. Let

$$H_n = \left\{ \sum_{\substack{i=1 \\ g_i \neq g}}^{n} a_i g_i \mid -n \leq a_i \leq n \right\}.$$

Notice that $\bigcup_{k=-n}^{n}(kg+H_n) = F_n$. Define the finite sequence of sets $(B_m^{(n)})_m$ for $m \in \{-n,\dots,n+1\}$ such that $B_m^{(n)} = (mg+H_n) \setminus \bigcup_{k=-n}^{m-1}(kg+H_n)$. It should be clear that the elements of $(B_m^{(n)})_m$ are pairwise disjoint and that

$$\bigcup_{k=-n}^{n} B_k^{(n)} = F_n.$$

It follows that $\sum_{k=-n}^{n} \left|B_k^{(n)}\right| = |F_n|$. We know that

$$\begin{aligned}
|(g+F_n)\triangle F_n| &= 2\,|(g+F_n)\setminus F_n| \\
&= 2\left|\bigcup_{k=-n}^{n}((k+1)g+H)\setminus \bigcup_{k=-n}^{n}(kg+H_n)\right| \\
&= 2\left|((n+1)g+H_n)\setminus \bigcup_{k=-n}^{n}(kg+H_n)\right| \\
&= 2\left|B_{n+1}^{(n)}\right|
\end{aligned}$$

So to show that the F_n adheres to the Følner criterion, it is sufficient to show that $\frac{\left|B_{n+1}^{(n)}\right|}{|F_n|} \to 0$ as $n \to \infty$.

We claim that $\left(\left|B_m^{(n)}\right|\right)_m$ is a non-increasing sequence. The following calculation shows this.

$$\begin{aligned}
\left|B_m^{(n)}\right| &= \left|(mg+H_n)\setminus \bigcup_{k=-n}^{m-1} kg+H_n\right| \\
&= \left|((m-1)g+H_n)\setminus \bigcup_{k=-n}^{m-1} (k-1)g+H_n\right| \\
&\le \left|((m-1)g+H_n)\setminus \bigcup_{k=-n}^{m-2} (k-1)g+H_n\right| = \left|B_{m-1}^{(n)}\right|
\end{aligned}$$

So $|F_n| = \sum_{i=-n}^{n} \left|B_i^{(n)}\right| \le (2n+1)\left|B_{n+1}^{(n)}\right|$, which means that

$$\frac{\left|B_{n+1}^{(n)}\right|}{|F_n|} \le \frac{1}{2n+1}.$$

So we find that $\lim_{n\to\infty} \frac{\left|B_{n+1}^{(n)}\right|}{|F_n|} = 0$ which concludes the proof. □

check Szabo's remarks!

As an immediate corollary we get the following well-known result.

Corollary 4.9. *Every countable abelian group is sofic.*

The following lemma is a known fact about about amenable groups, that we will use without proof. The statement is a slight reformulation of [KL16, prop. 4.2.(ii)-(iii)], where a proof can be found as well.

Lemma 4.10. *Let A, B, C be groups and let $f : A \to B$ be an injective morphism and $g : B \to C$ a surjective morphism such that*

$$A \xhookrightarrow{f} B \xrightarrow{g} \!\!\!\!\rightarrow C$$

is an exact sequence i.e. $\operatorname{im} f = \ker g$*. Then A and B are amenable if and only if C is amenable.*

Corollary 4.11. *Every quotient group of an amenable group is amenable. Every normal subgroup of an amenable group is amenable.*

Proof. Let G be a amenable group and let H be a quotient group such that $G = \frac{H}{N}$. Consider the following sequence.

$$N \xhookrightarrow{\iota} G \xrightarrow{\pi} \!\!\!\!\rightarrow H$$

It is clear that this sequence is exact. Applying the previous lemma shows that N and H are amenable. □

Proposition 4.12. *Every countable solvable group is amenable.*

Proof. Let G be a countable solvable group. Then there exists a finite sequence of subgroups such that

$$\{e\} = G_0 \lhd G_1 \lhd G_2 \lhd \cdots \lhd G_n = G,$$

and for every $i, 0 \leq i < n$, the quotient $\frac{G_{i+1}}{G_i}$ is commutative. It is clear that G_0 is amenable. We will now use finite induction to show that G is amenable as well. Take any $i, 0 \leq i < n$ and assume that G_i is amenable. Then

$$G_i \xhookrightarrow{Id} G_{i+1} \xrightarrow{\pi} \!\!\!\!\rightarrow \frac{G_{i+1}}{G_i}$$

is an exact sequence. Because $\frac{G_{i+1}}{G_i}$ is commutative we know by proposition 4.8 that it is amenable. We assumed that G_i is amenable as well. By lemma 4.10 we find that G_{i+1} is amenable as well. This finishes the induction and thus we can conclude that $G_n = G$ is amenable. □

Theorem 4.13. *Let G be a countable group, for which there exists a set of increasing amenable subgroups $G_1 \subset G_2 \subset G_3 \subset \ldots$ such that $G = \bigcup_{i=1}^{\infty} G_i$. Then G is amenable.*

Proof. Let $G = \{g_1, g_2, g_3, \ldots\}$. For every G_i we choose a Følner sequence $(F_{i,n})_n$. We will use these Følner sequences to construct a Følner sequence for G, $\mathcal{F}_n$. Let us fix an n, and consider the set $H = G_n \cap \{g_1, \ldots, g_n\}$. As H is a finite subset of G_n we can find a Følner set $F_{n,N}$ such that for all $h \in H$:

1. $h \in F_{n,N}$
2. $\frac{|hF_{n,N} \triangle F_{n,N}|}{|hF_{n,N}|} < \frac{1}{n}$

We define $\mathcal{F}_m = F_{n,N}$. As we can do this for every n we can construct $\mathcal{F}_n$ in this way.

Now I claim that $\mathcal{F}_n$ is a Følner sequence of G. To verify this, take N such that $N > i$ and $g_i \in G_N$. For all $m > N$ we see that

$$\frac{|g_i \mathcal{F}_m \triangle \mathcal{F}_m|}{|\mathcal{F}_m|} \leq \frac{1}{m}.$$

Hence we find that

$$\lim_{n\to\infty} \frac{|g_i \mathcal{F}_n \triangle \mathcal{F}_n|}{|\mathcal{F}_n|} = 0.$$

We conclude that $\mathcal{F}_n$ is a Følner sequence and that G is amenable. $\square$

Lemma 4.14. *Every finitely generated abelian group is amenable.*

Proof. Let $G = \langle a_1, a_2, \ldots, a_k \rangle$ be a finitely generated group. Then

$$\phi : \mathbb{Z}^k \to G : (n_1, \ldots, n_k) \mapsto a_1^{n_1} a_2^{n_2} \ldots a_k^{n_k}$$

is a surjective groups morphism. Hence $G = \frac{Z^k}{\ker \phi}$. By corollary 4.11 we see that G is amenable. $\square$

Remark 4.15. Lemma 4.14 and theorem 4.13 give an alternative proof of the fact that every countable group is amenable as every countable group $G = \{g_1, g_2, g_3, \ldots\}$ can be written as the union of $G_n = \langle g_1, g_2, g_3, \ldots, g_n \rangle$.

4.3 About Følner Sequences

merge this section with the previous one

In this section we list some more properties obout the Følner criterion and Følner sequences. These will be used in subsection 7.1.

Lemma 4.16. *If a group G satisfies the Følner criterion, then there also exists a Følner sequence for the right action of G on itself. I.e. there exists a sequence F_n of finite subsets of G such that*

1. *for every $g \in G$ there exists an N such that for all $n > N$ it holds that $g \in F_n$*
2. *For every $g \in G$ the folowing is true:*

$$\lim_{n\to\infty} \frac{|F_n g \triangle F_n|}{|F_n|} = 0.$$

Proof. Let F_n be a Følner sequence for the left action. It is easy to check that F_n^{-1} will be a Følner sequence for the right action. $\square$

Remark 4.17. A result analogous to lemma 4.2 holds for a right Følner sequence.

Lemma 4.18. *Let F_n, E_n be two (right) Følner sequences for G. Then $F_n \cup E_n$ is a (right) Følner sequence for G.*

Proof. We will prove it for ordinary (left) Følner sequences as the proof is completely analogous for right Følner sequences. The first property is obivously satisfied. Let us have a look at the limit.

$$\begin{aligned}
\lim_{n\to\infty} \frac{|g(F_n \cup E_n) \triangle (F_n \cup E_n)|}{|F_n \cup G_n|} &= 2 \lim_{n\to\infty} \frac{|(gF_n \cup gE_n) \setminus (F_n \cup E_n)|}{|F_n \cup E_n|} \\
&= 2 \lim_{n\to\infty} \frac{|F_n \setminus (gF_n \cup gE_n) \cup E_n \setminus (gE_n \cup gF_n)|}{|F_n \cup E_n|} \\
&\leq 2 \lim_{n\to\infty} \frac{|gF_n \setminus F_n| + |gE_n \setminus E_n|}{|F_n \cup E_n|} \\
&\leq 2 \lim_{n\to\infty} \frac{|gF_n \setminus F_n|}{|F_n|} + \frac{|gE_n \setminus E_n|}{|E_n|} \\
&= \lim_{n\to\infty} \frac{|gF_n \triangle F_n|}{|F_n|} + \frac{|gE_n \triangle E_n|}{|E_n|} = 0.
\end{aligned}$$

Hence we see that the claim is valid. □

We find the following corollary.

Corollary 4.19. *Let $B \subset G$ be a finite subset and F_n be a (left or right) Følner sequence of G. Then are BF_n and F_nB again Følner sequences.*

5 Residually Finite Groups

5.1 Definition and examples

Another property that implies soficity is the following.

Definition 5.1 ([Hal59, ch. 1 p. 16])**.** *We say that a group G is* ***residually finite*** *if for every $g \in G \setminus \{e\}$ there exists a finite group H and a homomorphisms $f : G \to H$ such that $f(g) \neq e$.*

Example 5.2. Finite groups G are residually finite. Let $g \in G \setminus \{e\}$ and consider $\mathrm{id} : G \to G : g \mapsto g$. Obviously $\mathrm{id}(g) \neq e$.

Example 5.3. The group of integers $\mathbb{Z}$ is residually finite. Let $k \in =z$. Consider the map $f_k : \mathbb{Z} \to \mathbb{Z}/(k+1)\mathbb{Z} : m \mapsto [m]_{k+1}$. Clearly, f_k is a homomorphism and $f_k(k) = [k]_{k+1} \neq 0$.

Example 5.4. The free group generated by n elements is residually finite. Consider $F_k = \langle x_1, x_2, x_3, \ldots, x_k \rangle$. Let Reference

$$a = x_{i_r}^{\epsilon_r n_r} x_{i_{r-1}}^{\epsilon_{r-1} n_{r-1}} x_{i_{r-2}}^{\epsilon_{r-2} n_{r-2}} x_{i_{r-3}}^{\epsilon_{r-3} n_{r-3}} \ldots x_{i_1}^{\epsilon_1 n_1},$$

where for every i it holds that $n_i \in \mathbb{Z}_{>0}$ and $\epsilon_i \in \{-1, 1\}$, be any non-trivial word in reduced form. Our goal is to find a morphism to a finite group that does not map a to 1. Define the sequence $(N_i)_i$ as $N_i = \sum_{j=1}^{i} n_j$. We will construct a map from F_k to S_{N_r+1}, which is the symmetric group on $\{0, 1, 2, \ldots, N_r\}$. By the universal property of free groups it to define a map it is sufficient to choose for every m the image of x_m. We will define it as follows

Szabo heeft hier problemen met de notatie, ik heb geen flauw idee waarom

$$x_m \mapsto \prod_{\substack{j \in \{1,\ldots,r\} \\ m = i_j}} (N_{j-1} N_{j-1} + 1 \ldots N_j)^{\epsilon_i},$$

where the brackets denote cycle notation.

Using this morphism we can let F_k act on the set $\{0, 1, \ldots, N_r\}$. We will deterimne the position of 0 after the action of a by keeping track of the position of 0 after every subsequent action of $x_{i_m}^{\epsilon_m n_m}$.

We claim that after acting with $x_{i_m}^{\epsilon_m n_m} x_{i_m}^{\epsilon_m n_m} \ldots x_{i_1}^{\epsilon_1 n_1}$ the location of 0 is at N_m. We will prove this claim by induction. If $m = 0$ we essentially act with the trivial element. So 0 is at position 0. Suppose that $m \neq 0$ to zero and that the action of $x_{i_m}^{\epsilon_m n_m} x_{i_m}^{\epsilon_m n_m} \ldots x_{i_1}^{\epsilon_1 n_1}$ maps 0 to N_m. We can act with $x_{i_{m+1}}^{\epsilon_{m+1} n_{m+1}}$. If we look at the cycles of $x_{i_{m+1}}$ we can see that N_m (the position of 0) is in the cycle $(N_m \ldots N_{m+1})^{\epsilon_{m+1}}$. As zero is in this cycle can solely focus on what happens within this cycle when dealing with powers of $x_{i_{m+1}}$. Taking the $\epsilon_{m+1} n_{m+1}$-th power means that this becomes the cycle $(N_m \ldots N_m + n_{m+1})^{n_{m+1}} = (N_{m+1} N_{m+1} - 1 \ldots N_m + 1 N_m)$. So we see that 0 now gets mapped to N_m, which proves our claim.

As the action of a maps 0 to $N_r \neq 0$, we see that the image of a in S_n cannot be trivial. We conclude that F_k is residually finite.

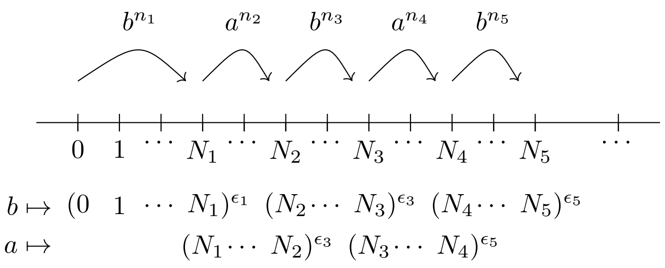

Figure 3: Illustration of the action in the case of $F_2 = \langle a, b \rangle$

5.2 Properties of Residually Finite Groups

It is a well-known result that residually finite groups are sofic. reference

Theorem 5.5. *Every countable residually finite group is sofic.*

Proof. Let G be a countable residually finite group. As G is countable we can order the elements non-identity elements $g_1, g_2, g_3, \ldots$ By the first property of residually finite there exist finite groups $G_1, G_2, G_3, \ldots$ and functions $f_1, f_2, f_3, \ldots$ such that $f_i : G \to G_i$ is a homomorphism. Consider the direct products $H_n = \prod_{i=1}^{n} G_i$ with the induced homomorphisms $h_i : G \to H_i : g \mapsto (f_1(g), f_2(g), \ldots, f_i(g))$. Now we will look at the image of G under these morphisms $\Gamma_i = h_i(G)$.

Notice that Γ_i is always finite. We will implicitly identify $\mathrm{Sym}(\Gamma_i)$ with $\mathrm{Sym}(|\Gamma_i|)$. We'll choose $\forall\ i \in \mathbb{N} : d_i = |\Gamma_i|$ and $\phi_i : G \to \mathrm{Sym}(\Gamma_i) : g \mapsto (a \mapsto ga)$. We claim that these maps satisfy the conditions for G to be sofic.

For every $k \in \Gamma_i, g, h \in G$ we can see that $(\phi(g) \circ \phi(h))(k) = ghk = \phi(gh)$. So $\{k \in \Gamma_i \mid (\phi(g) \circ \phi(h))(k) = \phi(gh)\} = \Gamma_i$. Then we can see that for all $g, h \in G$:

$$\lim_{i \to \infty} \frac{1}{d_i} |\{k \in \Gamma_i | (\phi(g) \circ \phi(h))(k) = \phi(gh)\}| = 1.$$

For every $g \in G, g \neq 1$ there is an $N \in \mathbb{N}$ such that for every $i > N$, $h_i(G) \neq e$. So it is clear that

$$\lim_{i\to\infty} \frac{1}{d_i} |\{k \in \Gamma_i \mid \phi(g) = gk \neq k\}| = 1.$$

For $g = e$ we find that

$$\lim_{i\to\infty} \frac{1}{d_i} |\{k \in \Gamma_i \mid \phi(g) = k \neq k\}| = 0,$$

which proves the second condition. □

6 Surjunctivity

First we define a basic concept in group theory.

Definition 6.1. *Let G be a group and X be a set. Then $\alpha : G \times X \to X : (g, x) \mapsto \alpha_g(x)$ is called an* ***action*** *of G on X if*

(i) $\forall x \in X : \alpha_e(x) = x$

(ii) $\forall g, h \in G, x \in X : \alpha_{gh}(x) = \alpha_g(\alpha_h(x))$.

In this section we will encounter the set of all maps from G to a finite set A. This set will be denoted by A^G. We will consider this set as a topological space, the topology on A^G being the product topology, where A carries the discrete topology.

Definition 6.2. *Let G be a countable group and A a finite set. Then the* ***right shift*** *σ of G on A^G is defined by*

$$\sigma : G \times A^G \to A^G : (g, (x_h)_{h\in G}) \mapsto \sigma_g(\omega),$$

where σ_g is defined by $\sigma_g\left((x_h)_{h\in G}\right) = (x_{hg})_{h\in G}$.

One can see that the right shift σ is a G-action on A^G. We say that a map $\phi : A^G \to A^G$ *commutes with the right shift* if $\sigma_g \circ \phi = \phi \circ \sigma_g$ for every $g \in G$.

Definition 6.3. *Let G be a countable group. The group G is called* ***surjunctive*** *if for any finite set A and every injective and continuous $\phi : A^G \to A^G$ that commutes with the right shift σ of G on A^G is also surjective.* reference

Conjecture 6.4 (Gottschalk)**.** *Every countable group is surjunctive.*

It can be proven that every sofic group is surjunctive. However, the proof is not easy. For that reason we will prove it first for the special case of residually finite groups. We will need the following definition

Definition 6.5. *Let G be a countable group and A a finite set. Let H be a normal subgroup of finite index. Then the set of H-**periodic points** is the following subset of A^G.*

$$P_H = \{\omega \in A^G \mid \forall h \in H : \sigma_h\omega = \omega\}$$

Notice that this means that any $\omega \in P_H$ can be factored into the natural quotient map $i : G \to \frac{G}{H}$ and a function $f : \frac{G}{H} \to A$ such that $\omega = f \circ i$. As we require H to be normal and of finite index, we get that $\frac{G}{H}$ is a finite set. So f is a function from and to finite sets. There are only a finite number of such functions. Hence P_H is a finite set. A key lemma we will further need in our proof of theorem 6.7 is the following:

Lemma 6.6. *Let G be a countable group and A a finite set. Then the set of all periodic points (with respect to all normal subgroups of finite index)*

$$\bigcup_{\substack{H \triangleleft G \\ [G:H] < \infty}} P_H$$

is dense in A^G.

Proof. Let $f : G \to A \in A^G$. We will prove that there exists a sequence of periodic points that converges to f. As G is countable we can list its elements $G = \{1, g_1, g_2, g_3, \dots\}$. Let $F_n = \{1, g_1, g_2, \dots, g_n\}$ and fix n. As G is residually finite, for every $g \in F_n^{-1} \cdot F_n,\ g \neq 1$, we can find a map $\pi_g : G \to H_g$, where H_g is a finite group and $\pi_g(g) \neq 1$. Consider the map

$$\pi = \prod_{\substack{g \in F_n^{-1} \cdot F_n \\ g \neq 0}} \pi_g.$$

Observe that $\mathrm{im}(\pi)$ is finite, thus $\ker \pi$ is a normal subgroup of finite index of G. For $g_i, g_j \in F_n, i \neq j$ it is clear that $\pi(g_i) \neq \pi(g_j)$ as $\pi(g_j^{-1} g_i) \neq 1$. This means that $\pi|_{F_n} : F_n \to \pi(F_n)$ is a bijection. Define $\beta : \pi(F_n) \to A : h \mapsto \left(f \circ (\pi|_F)^{-1}\right)(h)$. Notice that for all $g_i \in F_n$: $(\beta \circ \pi)(g_i) = f(g_i)$. Take any extension $\alpha : \mathrm{im}(\pi) \to A$ of β. Notice that $\alpha \circ \pi \in A^G$. Let $h \in \ker \pi$ and consider the right h shift, σ_h. Take any $g \in G$. Then

$$\sigma_h(\alpha \circ \pi)(g) = (\alpha \circ \pi)(hg) = \alpha(\pi(h)\pi(g)) = \alpha(\pi(g)) = (\alpha \circ \pi)(g).$$

Thus $\sigma_g(\alpha \circ \pi) = (\alpha \circ \pi)$. So $\alpha \circ \pi$ is a $\ker \pi$ periodic point.

For every $F_n = \{1, g_1, g_2, \dots, g_n\}$ we can construct such a function $\alpha_n \circ \pi_n$ which is an $\ker \pi_n$ periodic point and that agrees with f on F_n. It is clear that $(\alpha_n \circ \pi_n)_{n \in \mathbb{N}}$ converges to f pointwise as a function. This is sufficient to say that $(\pi_n)_n$ converges to f in the product topology. □

Now we can quite easily prove the following result. We follow the proof of Weiss [Wei00] here.

Theorem 6.7. *Every residually finite group is surjunctive.*

Proof. Let G be a residually finite group and A a finite set. Let $\phi : A^G \to A^G$ be a continuous map which commutes with the right shift σ. Assume further that ϕ is injective. We need to prove that ϕ is also surjective. Since ϕ commutes with the shift, it follows readily that it maps H-periodic points to H-periodic points, for any normal subgroup H of finite index. Because P_H is finite (for H of finite index), and ϕ is injective, the restriction of ϕ to P_H is surjective, thus $P_H \subset \phi(A^G)$. Now since A^G is compact, continuous maps from A^G to itself map closed sets to closed sets. Therefore

$$\phi(A^G) = \overline{\phi(A^G)} \supset \overline{\bigcup_{\substack{\text{subgroup } H \\ [G:H] < \infty}} \phi(P_H)} = \overline{\bigcup_{\substack{\text{subgroup } H \\ [G:H] < \infty}} P_H} = A^G,$$

where we have used lemma 6.6. Thus we conclude ϕ is surjective. □

7 Surjunctivity of Amenable and Sofic Groups

Szabo:The phrasing here suggests that what you are about to do has something novel to it. Is that really the case? If so, point out explicitly what it is.

In subsection 7.1 we present a proof of the fact that amenable groups are surjunctive. In the literature, it is usual to do this using the concept of entropy (as in [KL10, section 4] or [CL15, subsection 2.13]), or using graph theory (as in [Wei00, section 3]). We will however, follow an alternative approach and prove this well-known result directly from our definitions. We were unable to find a proof in the literature that follows the same approach, but it is important to point out that our proof and the aforementioned proofs have many conceptual similarities.

Alas, we were unable to generalise these techniques to show that sofic groups are surjunctive. In subsection 7.2, we sketch how one can obtain the surjunctivity result for sofic groups using the concept of entropy.

7.1 Amenable Groups are Surjunctive

In the following section G will be a fixed amenable group, A a fixed finite set and $\phi : A^G \to A^G$ a fixed continuous injective function that commutes with the right shift. We will also denote a fixed element of A with 0.

Lemma 7.1. *There exists a finite subset $\Gamma \subset G$ such that the value of $\phi((x_g)_{g\in G})(1_G)$ is only dependent on the values of x_h where $h \in \Gamma$. I.e. if for some $(x_g)_{g\in G}, (y_g)_{g\in G} \in A^G$ it holds that $x_h = y_h$ for all $h \in \Gamma$, then $\phi((x_g)_{g\in G})(1_G) = \phi((y_g)_g)(1_G)$.*

Proof. Choose any $a \in A$. As ϕ is continuous we know that

$$U_a = \phi^{-1}\left(\prod_{g\in G}\begin{cases}\{a\} & g = 1_G\\ A & g \neq 1_G\end{cases}\right)$$

is open. Hence we can choose some basis opens $U_{i,a}$ for some index $i \in I$ such that

$$U_a = \bigcup_{i\in I} U_{i,a}.$$

Notice that $\bigcup_{a\in A} U_a = A^G$ is a disjoint union. Hence $\bigcup_{a\in A, i\in I} U_{i,a} = A^G$. As A^G is compact, we see that there must be a finite subset of indices K such that

$$A^G = \bigcup_{(i,a)\in K} U_{i,a}.$$

Every one of these basis opens is of the form

$$U_{i,a} = \prod_{g\in G}\begin{cases}V_{i,a,g} & g \in \Gamma_{i,a}\\ A & g \notin \Gamma_{i,a}\end{cases},$$

for some open sets $V_{i,a,g} \subset A$ and finite subsets $\Gamma_{i,a} \subset G$.

We now claim that $\Gamma = \bigcup_{(i,a)\in K} \Gamma_{i,a}$ is set we are looking for. Obviously Γ is finite. Further, suppose that $(x_g)_{g\in G} \in A^G$ and $(y_g)_{g\in G} \in A^G$ agree on Γ, meaning that for every $\gamma \in \Gamma$ it holds that $x_\gamma = y_\gamma$. We know that for some $(i, a) \in K$ the point $(x_g)_g \in U_{i,a}$. In particular we see that $x_\gamma = y_\gamma$ for every $\gamma \in \Gamma_{i,a}$. So $(y_g)_{g\in G} \in U_{i,a}$. Hence $(x_g)_{g\in G}, (y_g)_{g\in G}$ are elements of U_a. So $\phi((x_g)_{g\in G})(1_G) = a = \phi((y_g)_{g\in G})(1_G)$. We conclude that Γ is indeed the set we're looking for. $\square$

From this point on we will also fix Γ.

Corollary 7.2. *For any $k \in G$ the value of $\phi((x_g)_{g\in G})(k)$ is only dependent on the values of x_{kh} where $h \in \Gamma$.*

Proof. As ϕ commutes with the right shift σ_k we know that $\sigma_k\phi = \phi\sigma_k$. Applying the previous lemma immediately gives the result. $\square$

Corollary 7.3. *If $(x_g)_{g\in G}, (y_g)_{g\in G} \in A^G$ only differ on a set $H \subset G$, ie $\forall g \notin H : x_g = y_g$, then $\phi((x_g)_{g\in G})$ and $\phi((y_g)_{g\in G})$ will only differ on $H\Gamma^{-1}$.*

Proof. Take $g \notin H\Gamma^{-1}$. Then $\phi((x_h)_{h\in G})(g)$ and $\phi((x_h)_{h\in G})(g)$ will only depend on $\{x_{g\gamma} \mid \gamma \in \Gamma\}$ and $\{y_{g\gamma} \mid \gamma \in \Gamma\}$. But $g\Gamma \cap H = \emptyset$. Hence $x_{g\gamma} = y_{g\gamma}$ for all $\gamma \in \Gamma$. By the previous corollary we see that $\phi((x_h)_{h\in G})(g) = \phi((y_h)_{h\in G})(g)$. $\square$

Definition 7.4. *We say that $(x_g)_{g\in G} \in A^G$ has support H if $x_g = 0$ for all $g \notin H$.*

Let A_H be the set of all element of A^G with support H. We can implicitly identify A_H with A^H.

Lemma 7.5. *Let H be any finite subset of G. Consider the function*

$$\psi : A_H \to A^{H\Gamma^{-1}} : (x_g)_{g\in G} \mapsto (h \mapsto \phi((x_g)_{g\in G})(h))\,.$$

This function is an injection.

Proof. Suppose that $x = (x_g)_{g\in G}$ and $y = (y_g)_{g\in G} \in A^H$ have the same image under ψ. As x, y can only differ on H, their images under ϕ cannot differ in $G \setminus H\Gamma^{-1}$ (see corollary 7.3). Because they also have the same image under ψ, we know that $\phi(x)$ and $\phi(y)$ must also agree on $H\Gamma^{-1}$. Hence $\phi(x) = \phi(y)$. By the injectivity of ϕ we find that $x = y$. $\square$

At this point we are almost ready to prove that G is surjunctive, i.e. ϕ is surjective. In order to show this will show that for any finite subset $B \subset G$ and a map $f : B \to A$ we can find a $x \in A^G$ such that $\phi(x)|_B = f$. Then we can finish the proof by showing that they image of ϕ is dense and using compactness.

For now, lets fix the set B and the map $f : B \to A$. We will first introduce the notion of a copy of B. [3]

Definition 7.6. *A set $A \subset G$ is a copy of B if and only if there exists an $s \in G$ such that $A = B \cdot s$.*

Lemma 7.7. *Let F be any finite subset of G. There exists a subset $F' \subset F$ such that for any two $s, t \in F', s \neq t$ it holds that $B \cdot s$ and $B \cdot t$ are disjoint and $|F'| \geq \frac{|F|}{|B|^2}$*

In other words, the set BF contains at least $\frac{|F|}{|B^2|}$ disjoint copies of B.

Proof. We will prove this using strong induction. The case where $F = \emptyset$ is trivial.

[3] This is not standard notation. It is just a term we made up to aid in this specific proof.

Case $1 \leq |F| \leq |B|^2$: Let $f \in F$ be an arbitrary element of F. The set $\{f\}$ has 1 element which is atleast $\frac{|F|}{|B|^2} \leq 1$.

Case $|F| > |B|^2$: Take any $f \in F$. Suppose that for some $s \in G$, Bf and Bs are not disjoint. Hence there exists $a, b \in B$ such that $af = bs$. Thus $s = b^{-1}af$. So $s \in fB^{-1}B$. Notice that $\left|fB^{-1}B\right| \leq |B|^2$. This means that for any $f' \in F \setminus fB^{-1}B$, Bf and Bf' will be disjoint. Notice that $|F| > \left|F \setminus fB^{-1}B\right| \geq |F| - |B|^2$.

Applying the induction hypothesis shows that there exists a $F' \subset \left(F \setminus fB^{-1}B\right)$ such that for all $s, t \in F'$, Bs and Bf are disjoint, and that

$$|F'| \geq \frac{\left|F \setminus fB^{-1}B\right|}{|B|^2}.$$

On top of that for every $f' \in F \setminus fB^{-1}B$, Bf and Bf' will be disjoint. So for every $s, t \in \{f\} \cup F'\}$, Bs and Bt will be disjoint. Further

$$|\{f\} \cup F'| = 1 + |F'| \geq 1 + \frac{\left|F \setminus fB^{-1}B\right|}{|B|^2} \geq \frac{|F|}{|B|^2}.$$

We conclude that $\{f\} \cup F'$ is the set we are looking for.

□

We are finally setup to proof that all amenable groups are surjunctive.

Theorem 7.8. *Let G be a countable amenable group and A be finite set, where we will choose one element and denote it with 0. Let $\phi : A^G \to A^G$ be a injective, continuous map that commutes with the right shift. Then ϕ is surjective.*

Proof. Lets fix a finite subset $B \subset G$ and a map $f : B \to A$. Suppose that there does not exists a $x \in A^G$ such that $\phi(x)|_B = f$. Notice that this means that for any copy of B, take $B \cdot s$, the image of ϕ does not contains a function $\phi(x)$ such that when restrited to $B \cdot s$ it is $f' : B \cdot s \to A : x \mapsto f\left(xs^{-1}\right)$, as applying the right shift $\sigma_{s^{-1}}$ then gives $\phi(\sigma_{s^{-1}}x)|_B = f$.

Let F_n be a right Følner sequence of G. By corollary 4.19 we know that BF_n is a right Følner sequence as well.

Fix n for a moment. By lemma 7.5 the function

$$\psi : A^{BF_n} \to A^{BF_n\Gamma^{-1}} : x \mapsto (g \mapsto \phi(x)(g)).$$

is injective. Lemma 7.7 states that we can choose $\left\lceil \frac{|F_n\Gamma|}{|B|^2} \right\rceil$ disjoint copies of B in $BF_n\Gamma$. We know that for every such copy of B the number of possible values at the coordinates of this copy is atmost $|A|^{|B|} - 1$ by our assumption at the start of this proof. Hence there exists a set $\mathcal{A} \subset A^{BF_n\Gamma^{-1}}$ of cardinality

$$|\mathcal{A}| = |A|^{\left|BF_n\Gamma^{-1}\right| - |B|\left\lceil \frac{\left|F_n\Gamma^{-1}\right|}{|B|^2} \right\rceil} \cdot \left(|A|^B - 1\right)^{\left\lceil \frac{\left|F_n\Gamma^{-1}\right|}{|B|^2} \right\rceil}$$

such that $\operatorname{im}\psi \subset \mathcal{A}$. As ψ is injective it follows that $\left|A^{BF_n}\right| \leq |\mathcal{A}|$. Hence

$$|A|^{|BF_n|} \leq |A|^{\left|BF_n\Gamma^{-1}\right| - |B|\left\lceil \frac{\left|F_n\Gamma^{-1}\right|}{|B|^2} \right\rceil} \cdot \left(|A|^B - 1\right)^{\left\lceil \frac{\left|F_n\Gamma^{-1}\right|}{|B|^2} \right\rceil}.$$

By taking the logarithm of both sides we obtain

$$|BF_n|\ln(|A|) \leq \left|BF_n\Gamma^{-1}\right|\ln(|A|) - \left\lceil \frac{\left|F_n\Gamma^{-1}\right|}{|B|^2} \right\rceil \left(\ln(|A|^{|B|}) - \ln(|A|^{|B|}-1)\right).$$

By dividing both sides by $|BF_n|$ we see

$$\ln(|A|) \leq \frac{\left|BF_n\Gamma^{-1}\right|}{|BF_n|}\ln(|A|) - \frac{1}{|BF_n\Gamma^{-1}|} \left\lceil \frac{\left|F_n\Gamma^{-1}\right|}{|B|^2} \right\rceil \left(\ln(|A|^{|B|}) - \ln(|A|^{|B|}-1)\right).$$

Lemma 4.2 shows that $\lim_{n\to\infty} \frac{\left|BF_n\Gamma^{-1}\right|}{|BF_n|} = 1$. So we know as well that $\lim_{n\to\infty} \frac{1}{|BF_n\Gamma^{-1}|} \left\lceil \frac{\left|F_n\Gamma^{-1}\right|}{|B|^2} \right\rceil \geq \frac{1}{|B|^3}$. Therefore

$$\ln(|A|) \leq \ln(|A|) - \frac{1}{|B|^3}\left(\ln(|A|^{|B|}) - \ln(|A|^B - 1)\right),$$

which yields a contradiction. So our initial assumption is false.

This means that there exists a $x \in A^G$ such that $\phi(x)|_B = f$. As this is true for every B and $f : B \to A$ we see that the image of ϕ must be dense in A^G. As a map on compact spaces maps closed sets to closed sets we see that the image of ϕ must be A^G. We conclude that ϕ is surjective!

□

7.2 Sofic Groups, Entropy and Surjunctive Groups

To prove that sofic groups are always surjunctive we define sofic entropy. The definition we present is a particular case of the more general definition formulated in paragraph 2.13.6 in [CL15] and is equivalent to the one given in definition 4.2 of [KL10].

Definition 7.9. *A subshift $X \subset A^G$ with A a finite set and $G := \{g_0 = e, g_1, g_2, ...\}$ any countable group is a closed subset which is invariant under the shift.*

Suppose now that G is sofic and $\Sigma := (\Phi_i : G \to \mathrm{Sym}(d_i))_{i\in\mathbb{N}}$ is a sofic approximation sequence. Take a subshift X and a finite subset $F \subset G$. For any $\delta > 0$ and $n \in \mathbb{N}$ we define the function

Where does this definition come from?

$$H_n(\delta) := \{(a_i)_i \in A^n | \exists (x_i)_i \in X_n : \forall g \in F : d_{S_n}((a_{\Phi_n^{-1}(g)(i)})_i, (x_{gg_i})_i) < \delta\}$$

where by X_n we mean the sequences in X restricted to $\{e, g_1, g_2, ..., g_{n-1}\}$.

Now we define entropy as

$$h_\Sigma(X) := \inf_{\delta>0} \limsup_{n\to\infty} \frac{1}{n} \log |H_n(\delta)|$$

We shall use the following intermediate result, for which a simplified proof is provided. For the complete proof, we refer again to either [CL15] (paragraph 2.13.7) or [KL10] (Theorem 4.12).

Lemma 7.10. *For $X \subset A^G$ a subshift we have that*

$$h_\Sigma(X) = \log |A| \Leftrightarrow X = A^G$$

Proof. For the implication $X = A^G \Rightarrow h_\Sigma(X) = \log |A|$: assume that $X = A^G$. Now it is quite clear that for any $n \in \mathbb{N}$ and $\delta > 0$: $|H_n(\delta)| = |A|^n$. Indeed, for any $(a_i)_i \in A^n$ we can find a sequence $(x_i)_i \in X^n$ such that $\forall g \in F : d_{S_n}((a_{\Phi_n^{-1}(g)(i)})_i, (x_{gg_i})_i) < \delta$. Namely $(a_i)_i$ itself. Therefore $h_\Sigma(X) = \log |A|$.

For the implication $h_\Sigma(X) = \log |A| \Rightarrow X = A^G$: assume that $X \subsetneq A^G$. It is possible that a certain element of A is never a component in one of the elements of X. However, then it is immediately clear that $h_\Sigma(X) \leq \log(|A| - 1)$ and thus $h_\Sigma(X) < \log |A|$. Assume, therefore, that every element of A is a component in at least one of the elements of X. Now there must exist some finite $F \subset G$ such that $Y_F \subsetneq A^G$. It can then be shown that, using this F, we have that

$$h_\Sigma(X) \leq \log |A| - \frac{1}{|F|^2} \log \left(\frac{|A|^{|F|}}{|A|^{|F|} - 1} \right) < \log |A|$$

which proves the implication by contrapositive. $\square$

Now we can prove the surjunctivity of sofic groups.

Theorem 7.11. *Any sofic group G is surjunctive.*

Proof. Suppose that we have a map $\phi : A^G \to A^G$ which is continuous, injective and which commutes with the shifts σ_g. Now consider $X := \phi(A^G) \subset A^G$. We claim that this is a subshift. Indeed, because ϕ is continuous and A^G is compact, X is closed. Furthermore, X is invariant under any shift σ_g:

$$\begin{aligned} \sigma_g(X) &= (\sigma_g \circ \phi)(A^G) \\ &= (\phi \circ \sigma_g)(A^G) \\ &= \phi(A^G). \end{aligned}$$

The inverse $\phi^{-1} : X \to A^G$ is well defined, continuous (since A^G and therefore also X is Hausdorff) and commutes with the shifts. Therefore, ϕ^{-1} is determined by a finite restriction $\phi_0^{-1} : X_\Gamma \to A^\Gamma$ with Γ a finite subset of G. Indeed, since ϕ^{-1} is continuous, the value of any component, say $(\phi^{-1}((x_g)_{g\in G}))_e \in A$ is determined by the values $(x_g)_{g\in\Gamma}$ of a finite subset $\Gamma \subset G$, meaning

$$(\phi^{-1}((x_g)_{g\in G}))_e = (\phi_0^{-1}((x_g)_{g\in\Gamma}))_e$$

Now, take a sequence $(x_g)_{g\in G} \in X$ and its restriction $(x_g)_{g\in\Gamma} \in X_\Gamma$. Now for any $h \in G$ it is the case that

$$(\phi^{-1}((x_g)_{g\in G}))_h = (\sigma_h \circ \phi^{-1}((x_g)_{g\in G}))_e = (\phi^{-1} \circ \sigma_h((x_g)_{g\in G}))_e$$

This knowledge allows us to write

$$\begin{aligned} (\phi^{-1}((x_g)_{g\in G}))_h &= (\sigma_h \circ \phi^{-1}((x_g)_{g\in G}))_e \\ &= (\phi^{-1}((x_{gh})_{g\in G}))_e \\ &= (\phi_0^{-1}((x_{gh})_{g\in\Gamma}))_e \\ &= (\phi_0^{-1}((x_g)_{g\in\Gamma h}))_e \end{aligned}$$

Now fix any $\delta > 0$. Consider $H_n(\delta)$ for any $n \in \mathbb{N}$. Look at any sequence $(a_i)_i \in A^n$. Take then $(x_i)_i = \phi((a_i)_i) \in X$. As Γ is only finite it is possible to take n large enough such that $d_{S_n}((a_{\Phi_n^{-1}(g)(i)})_i, (x_{gg_i})_i)) < \delta$ for every $g \in \Gamma$. This is equivalent to saying that for $n \in \mathbb{N}$ large enough $(a_i)_i \in H_n(\delta)$ for every $(a_i)_i \in A^n$. Therefore, $\limsup_{n\to\infty} |H_n(\delta)| = |A|^n$ for any $\delta > 0$ and $h_\Sigma(X) = \log |A|$, which, by the previous lemma, means that $X = A^G$. Thus, ϕ is surjective.

$\square$

8 Kaplansky's Direct Finiteness Condition for Finite Fields

Let us recall the statement of Kaplansky's direct finiteness condition.

Definition 8.1. *Let K be a field and G be a group. The group ring $K[G]$ is called **directly finite** if every $a \in K[G]$, that is left invertible, is right invertible as well.*

Kaplansky conjectured that all group rings are directly finite and Weiss showed that this is true for all sofic groups [Wei00]. In the case where R is a finite field, Weiss did this by first proving the the surjunctivity of sofic groups. He finished the argument by proving that the group rings of finite fields and surjunctive groups are directly finite.

In this section we will give the details of a proof of that last step. i.e we will prove the following theorem.

Theorem 8.2. *Let $\mathbb{F}$ be a finite field and G be a surjunctive group. Then $\mathbb{F}[G]$ is directly finite.*

It looks like we will need to use the surjuctivity on $\mathbb{F}^G$. Hence we need some statement that relates $\mathbb{F}^G$ to $\mathbb{F}[G]$.

Lemma 8.3. *For a group G and field $\mathbb{K}$ we can consider $\mathbb{K}^G$ as a left $\mathbb{K}[G]$-module, with pointwise addition and multiplication defined as*

$$\cdot : (\mathbb{K}[G], \mathbb{K}^G) \longrightarrow \mathbb{K}^G$$
$$\left(\sum_{g\in G} a_g g, (x_g)_{g\in G}\right) \longmapsto \left(\sum_{h\in G} a_h x_{h^{-1}g}\right)_{g\in G}.$$

Proof. The proof is given in the appendix. $\square$

Remark 8.4. The way $\mathbb{F}^G$ is defined as a $\mathbb{F}[G]$-module is not completely abitrary. It should be easy to see that $\mathbb{F}^G$ can be represented as formal sums $\sum_{g\in G} x_g g$, just like $\mathbb{F}[G]$, but where we now allow infinite sums. The definition of the multiplication between $\mathbb{F}[G]$ and $\mathbb{F}^G$ agrees with the obvious multiplication with $\mathbb{F}^G$ represented as above.

We are now ready to give a proof of theorem 8.2.

Proof. Let $\mathbb{F}$ be a finite field and G be a surjunctive group. Let $a = \sum_{g\in G} a_g g \in \mathbb{F}[G]$ be a left invertible element with left inverse b, i.e. $ba = 1$. Consider the map $\phi : \mathbb{F}^G \to \mathbb{F}^G : x \mapsto a \cdot x$. Notice that this map

- is injective. Suppose that $\phi(x) = \phi(y)$. Then $a \cdot x = a \cdot y$. So $ba \cdot x = x$ and $ba \cdot y = 1 \cdot y = y$. Hence $x = y$.

- commutes with the bernoulli shift σ_k.

$$\sigma_k\phi(x) = \sigma_k\left(\sum_{h\in G} a_h x_{h^{-1}g}\right)_{g\in G} = \left(\sum_{h\in G} a_h x_{h^{-1}gk}\right)_{g\in G} = \left(\sum_{h\in G} a_h h\right)\cdot(x_{gk})_{g\in G} = \phi(\sigma_k(x))$$

- is continuous. One can easily check that the set $\{U_{h,t}\}$, where

$$U_{h,t} := \prod_{g\in G}\begin{cases}\mathbb{F} & h\neq g\\ \{t\} & h=g\end{cases}.$$

  is a subbasis of $\mathbb{F}^G$. Hence, to prove that ϕ is continuous it is sufficient to check that $\phi^{-1}(U_{h,t})$ is open for every $h\in G, t\in\mathbb{F}$. Notice that an $(x_g)_{g\in G}\in\mathbb{F}^G$ is in $\phi^{-1}(U_{h,t})$ if and only if $\sum_{k\in G} a_k x_{k^{-1}h} = t$. Remember that a_k is nonzero only for a finite number subset X, $k\in X\subset G$. So $\phi^{-1}(U_{h,t})$ is closed under addition with an element of

$$\prod_{g\in G}\begin{cases}\mathbb{F} & hg^{-1}\notin X\\ \{0\} & hg^{-1}\in X\end{cases}.$$

  Let $(x_g)\in\phi^{-1}(U_{g,t})$. We see that

$$(x_g)\in\prod_{g\in G}\begin{cases}\mathbb{F} & hg^{-1}\notin X\\ \{x_g\} & hg^{-1}\in X\end{cases}\subset\phi^{-1}(U_{h,t}).$$

  So $\phi^{-1}(U_{h,t})$ is open. Hence ϕ is continuous.

By the surjunctivity of G we find that ϕ is surjective as well. In particular this means that there exists a $y=(y_g)_{g\in G}$ such that $\phi(y)=(\delta_{g,1})_{g\in G}$, where δ is the kronecker delta. By multiplying with b on the left we find that

$$y = ba\cdot y = b\cdot(\delta_{g,1})_{g\in G} = \left(\sum_{h\in G} b_h\delta_{h^{-1}g,1}\right)_{g\in G} = \left(\sum_{h\in G} b_h\delta_{h,g}\right)_{g\in G} = (b_g)_{g\in G}.$$

This means that

$$a\cdot y = \left(\sum_{h\in G} a_h b_{h^{-1}g}\right)_{g\in G} = (\delta_{g,1})_{g\in G}.$$

In other words

$$\sum_{h\in G} a_h b_{h^{-1}g} = \delta_{g,1}.$$

But this means exactly that $ab=1$. So a is right invertible as well. □

9 Conclusion

To be written

10 Appendices

10.1 Group Rings are Rings

Proof. We need to prove associativity of +, commutativity of +, existence of zero element, existence of additive inverses, associativity for ·, left distributivity, right distributivity and existence of a unit element.

associativity of $+$

$$\begin{aligned}
\left(\sum_{g\in G}\lambda_g g+\sum_{g\in G}\mu_g g\right)+\sum_{g\in G}\nu_g g &= \sum_{g\in G}\left((\lambda_g g+\mu_g g)+\nu_g g\right)\\
&= \sum_{g\in G}\left(\lambda_g g+(\mu_g g+\nu_g g)\right)\\
&= \sum_{g\in G}\lambda_g g+\left(\sum_{g\in G}\mu_g g+\sum_{g\in G}\nu_g g\right)
\end{aligned}$$

commutativity of $+$

$$\begin{aligned}
\sum_{g\in G}\lambda_g g+\sum_{g\in G}\mu_g g &= \sum_{g\in G}(\lambda_g g+\mu_g g)\\
&= \sum_{g\in G}(\mu_g g+\lambda_g g)\\
&= \sum_{g\in G}\mu_g g+\sum_{g\in G}\lambda_g g
\end{aligned}$$

existence of 0

$$\begin{aligned}
\sum_{g\in G}\lambda_g g+\sum_{g\in G}0g &= \sum_{g\in G}(\lambda_g+0)g\\
&= \sum_{g\in G}\lambda_g g
\end{aligned}$$

existence of additive inverse

$$\begin{aligned}
\sum_{g\in G}\lambda_g g+\sum_{g\in G}(-\lambda_g g) &= \sum_{g\in G}(\lambda_g-\lambda_g)g\\
&= 0
\end{aligned}$$

associativity of $\cdot$

$$\begin{aligned}\left[\left(\sum_{g\in G}\lambda_g g\right)\cdot\left(\sum_{g\in G}\mu_g g\right)\right]\cdot\sum_{g\in G}\nu_g g &= \left(\sum_{g\in G}\left(\sum_{h\in G}\lambda_h\mu_{h^{-1}g}\right)g\right)\cdot\sum_{g\in G}\nu_g g\\ &= \sum_{g\in G}\sum_{x\in G}\left(\sum_{h\in G}\lambda_h\mu_{h^{-1}x}\right)\nu_{x^{-1}g}\, g\\ &= \sum_{g\in G}\sum_{y\in G}\sum_{h\in G}\lambda_h\mu_y\nu_{y^{-1}h^{-1}g}\, g\\ &= \sum_{g\in G}\sum_{h\in G}\lambda_h\left(\sum_{y\in G}\mu_y\nu_{y^{-1}(h^{-1}g)}\right) g\\ &= \left(\sum_{g\in G}\lambda_g g\right)\cdot\left[\left(\sum_{g\in G}\mu_g g\right)\cdot\sum_{g\in G}\nu_g g\right]\end{aligned}$$

left distributivity

$$\begin{aligned}\sum_{g\in G}\lambda_g g\cdot\left(\sum_{g\in G}\mu_g g+\sum_{g\in G}\nu_g g\right) &= \sum_{g\in G}\lambda_g g\cdot\sum_{g\in G}(\mu_g+\nu_g)g\\ &= \sum_{g\in G}\sum_{h\in G}\lambda_h(\mu_{h^{-1}g}+\nu_{h^{-1}g})g\\ &= \sum_{g\in G}\left(\sum_{h\in G}\lambda_h\mu_{h^{-1}g}+\lambda_h\nu_{h^{-1}g}\right)g\\ &= \sum_{g\in G}\lambda_g g\cdot\sum_{g\in G}\mu_g g+\sum_{g\in G}\lambda_g g\cdot\sum_{g\in G}\nu_g g\end{aligned}$$

right distributivity

$$\begin{aligned}\left(\sum_{g\in G}\lambda_g+\sum_{g\in G}\mu_g g\right)\cdot\sum_{g\in G}\nu_g g &= \left(\sum_{g\in G}(\lambda_g g+\mu_g)g\right)\cdot\sum_{g\in G}\nu_g g\\ &= \sum_{g\in G}\sum_{h\in G}(\lambda_h+\mu_h)\nu_{h^{-1}g}g\\ &= \sum_{g\in G}\sum_{h\in G}\lambda_h\nu_{h^{-1}g}g+\mu_h\nu_{h^{-1}g}g\\ &= \sum_{g\in G}\lambda_g g\cdot\sum_{g\in G}\nu_g g+\sum_{g\in G}\mu_g g\cdot\sum_{g\in G}\nu_g\end{aligned}$$

existence of 1

$$\begin{aligned}\sum_{g\in G}\delta_{g,e}g\cdot\sum_{g\in G}\lambda_g g &= \sum_{g\in G}\left(\sum_{h\in H}\delta_{h,e}\lambda_{h^{-1}g}\right)\\ &= \sum_{g\in G}\lambda_{e^{-1}g}g=\sum_{g\in G}\lambda_g g.\end{aligned}$$

$\square$

10.2 A Brief Introduction to Free Groups

This section is a brief introduction to the free groups, as many of our readers may not be familiar with this family of groups. This is in no way meant to be a rigorous construction of the group and its properties, rather it is an informal overview of its construction and its most important property. The details can be found in many algebra text books such as [Art91, section 7.9] or [Rot15, section 5.5].

The construction of a free group starts with the notion of an *alphabet*, which is a set of *letters*/symbols. We usually denote the alphabet with S. In the case where the alphabet only contains two letters we usually denote these with a, b. In other cases we usually index the letters x_i for some index $i \in I$, such that $S = \{x_i \mid i \in I\}$.

Using the letters of the alphabet $S = \{x_i \mid i \in I\}$ we can make *words*. Words are finite sequences of letters and inverses of letters. At this point inverses of letters are just more symbols. We denote $W(S)$ for the words made with letters in S. For example if $S = \{a, b\}$ the following are words a, b, a^{-1}, ab^{-1}, $aa^{-1}b$, $aabba^{-1}$, 1. Notice that there is also the empty word containing no letters (denoted by 1). We define the *reduction of a word*, $\operatorname{red} w$, as the operation that repeatedly scraps any pairs of letters adjacent to its inverse, $x_i x_i^{-1}$ and $x_i^{-1} x_i$, until there are no more such pairs. For example $\operatorname{red}(a^{-1}ab) = b$, $\operatorname{red}(aa^{-1}) = 1$ $\operatorname{red}(aba^{-1}b^{-1}) = aba^{-1}b^{-1}$, $\operatorname{red}(a^{-1}bb^{-1}a^{-1}) = 1$, $\operatorname{red}(x_1x_2x_3) = x_1x_2x_3$.

We can define a *multiplication* on words by concatenating the sequences and reducing the result. For example $(ab) \cdot (ba) = abba$, $(ab) \cdot (b^{-1}a) = \operatorname{red}(abb^{-1}a) = aa$. Using this multiplication it turns out that we can define the *free group generated by S* as

$$F_S = \{\text{reduced words in } W(S)\}, \cdot.$$

We also write $F_S = \langle S \rangle$, or in the case $S = \{x_1, x_2, \ldots, x_n\}$, we write $F_S = \langle x_1, x_2, \ldots, x_n \rangle$. If S is finite with $n = |S|$, we also write F_n as for two alphabets of the same cardinality S, S' the groups F_S and $F_{S'}$ are isomorphic.

In some sense, the free group, F_S is the largest group that can be generated by the elements of S. This is formalized in the following *universal property of the free group*, that characterizes the free group.

Property 10.1 ([Rot15, ch. 5, p. 298]). *If S is a subset of a group F, then F is the free group generated by S if and only if for every G and function $f : S \to G$ there exists a unique morphism $\phi : F \to G$ such that $\phi(s) = f(s)$ for all $s \in S$. I.e. such that ϕ makes the following diagram commute:*

$$\begin{tikzcd} F \arrow[dr, "\phi"] & \\ S \arrow[u, hook] \arrow[r, "f"] & G \end{tikzcd}$$

10.3 $\mathbb{K}^G$ is a $\mathbb{K}[G]$-module

We check the module axioms one by one. In the following calculations $r = \sum_{g \in G} r_g g$, $s = \sum_{g \in G} s_g g$ are arbitrary elements of $\mathbb{K}[G]$ and $x = (x_g)_{g \in G}, y = (y_g)_{g \in G}$ are arbitrary elements of $\mathbb{K}^G$.

left distributivity

$$
\begin{aligned}
r \cdot (x + y) &= \left(\sum_{h \in G} r_h (x_{h^{-1}g} + y_{h^{-1}g}) \right)_{g \in G} \\
&= \left(\sum_{h \in G} r_h x_{h^{-1}g} \right)_{g \in G} + \left(\sum_{h \in G} r_h y_{h^{-1}g} \right)_{g \in G} \\
&= r \cdot x + r \cdot y.
\end{aligned}
$$

right distributivity

$$
\begin{aligned}
(r + s) \cdot x &= \left(\sum_{h \in G} (r_h + s_h) x_{h^{-1}g} \right)_{g \in G} \\
&= \left(\sum_{h \in G} r_h x_{h^{-1}g} \right)_{g \in G} + \left(\sum_{h \in G} s_h x_{h^{-1}g} \right)_{g \in G} \\
&= r \cdot x + s \cdot y.
\end{aligned}
$$

associativity

$$
\begin{aligned}
(rs) \cdot x &= \left(\sum_{g_1 \in G} \sum_{g_2 \in G} r_{g_1} s_{g_2} g_1 g_2 \right) \cdot (x_g)_{g \in G} \\
&= \left(\sum_{h \in G} \sum_{k \in G} r_k s_{k^{-1}h} h \right) \cdot (x_g)_{g \in G} \\
&= \left(\sum_{h \in G} \sum_{k \in G} r_k s_{k^{-1}h} x_{h^{-1}g} \right)_{g \in G} \\
&= \left(\sum_{h \in G} \sum_{k \in G} r_k s_{k^{-1}h} x_{(k^{-1}h)^{-1}(k^{-1}g)} \right)_{g \in G} \\
&= \left(\sum_{h \in G} r_h h \right) \cdot \left(\sum_{k \in G} s_k x_{k^{-1}g} \right)_{g \in G} \\
&= r(s \cdot x).
\end{aligned}
$$

unity

$$
\begin{aligned}
1 \cdot x &= (x_{1^{-1}g})_{g \in G} \\
&= x
\end{aligned}
$$

.

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