- Process occurs as relativistic charged particles leave the region of the event horizon of a newly formed black hole created during supernova explosion. The beam of particles moving at relativistic speeds are focused for a few tens of seconds by the B-field of the exploding hypernova. The fusion explosion of the hypernova drives the energetics of the process.

Matter Interaction: When a GR passes through matter, the probability for absorption is proportional to the thickness of the layer, the density of the material, & the absorption cross section of the material.

- As it passes through matter, GR ionizes via 3 processes: (1) photoelectric effect (2) Compton Scat (3) Pair-production

(1) The case in which a Gamma photon interacts w/ & transfers its energy to an atomic electron, causing the ejection of that electron from the atom.

- KE of the resulting photon is equal to the E of the incident gamma photon minus the energy binding the electron to the atom.
- The Photoelec effect is the dominant E transfer mechanism for XR & GR w/ $E < 50$ KeV, but is less important at higher energies

(2) An incident GP loses enough energy to an atomic electron to cause its ejection, w/ the remainder of the original photon E emitted as a new, lower E GP whose emission direction differs from the incident GP.

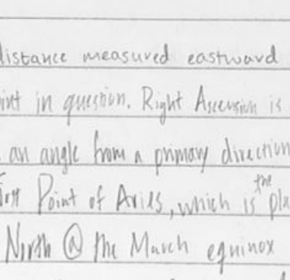
- The probability of Compton Scattering decreases w/ increasing photon energy.
- Principal absorption mechanism for GR in the intermediate energy range 100KeV to 10 MeV.
- Relatively independent of the atomic number of the absorbing material (why lead are only modestly better shields than less dense materials)

(3) - Become possible w/ GR exceeding 1.02 MeV. Important absorption mechanism at energies over 5MeV

- By interaction w/ the E-field of a nucleus, the energy of the incident photon is converted into the mass of an electron-positron pair.
- Any gamma energy in excess of the equivalent rest mass of the two particles appears as the KE of the pair & in the recoil of the emitting nucleus. At the end of the positron's range, it combines w/ a free electron, & the two annihilate, & the entire mass of these two is then converted into 2 GP of at least 0.51 MeV energy each

Right Ascention: is the angular distance measured eastward along the celestial equator from the vernal equinox to the hour circle of the point in question. Right Ascension is the celestial equivalent of terrestrial longitude. Both RA & the longitude measure an angle from a primary direction (a zero point) on an equator. RA is measured from the vernal equinox or the First Point of Aries, which is the place on the celestial sphere where the Sun crosses the equator from South to North @ the March equinox & is currently located in the constellation Pisces. RA is measured continuously in a full circle from that equinox towards the east

✓ On right ascension



$(x,y,z) \rightarrow (r,\theta,\phi)$

Cart → Spherical

$r = \sqrt{x^2+y^2+z^2}$

$\theta = \arctan \frac{y}{x}$

$\phi = \arctan \frac{\sqrt{x^2+y^2}}{z}$

Gamma-Rays

E above 100 KeV, almost always less than 10 MeV (1×10^{7})
f above 10 exahertz (or $> 10^{19}$ Hz)
λ less than 10 picometers (10^{-11} m)

- Are penetrating electromagnetic radiation of a kind arising from the radioactive decay of atomic nuclei.
- The decay of an atomic nucleus from a high energy state to a lower energy state, a process called gamma decay
- Defined by their energy, ranging to over 10 TeV (1×10^{13} eV) an energy too large to be a product of radioactive decay.
- Gamma-Ray bursts, of energy higher than can be produced by radioactive decay.
- These bursts of gamma rays are thought to be due to the collapse of stars called hypernovae
- Gamma Rays are produced by a number of astronomical processes in which very high-energy electrons are produced.
- Such electrons produce secondary GR by mechanisms of bremsstrahlung, inverse Compton Scattering, & sychrotron radiation
- Large fraction of such astronomical GR are screened by Earth's atmosphere & must be detected by spacecraft.
- In addition to nuclear emissions, they're often produced by sub-atomic particle & particle photon interactions.
- Those include electron-positron annihilation, neutral pion decay bremsstrahlung, inverse C.S. & synchrotron radiation.

Cosmic rays:

- high energy GR include the GR background produced when cosmic ray (high speed electron or proton) collide w/ ordinary matter, producing pair-production GR at 511 KeV (5×10^{5} eV)

Bremsstrahlung: produced at energies of tens of MeV or more when cosmic ray electrons interact w/ nuclei of sufficiently high atomic number

(common or higher) - The GR sky is dominated by the more common & longer-term production of GR that emanate from pulsars within the Milky Way
- Pulsars are neutron stars w/ relatively long-lived B-fields that produce focused beams of relativistic speed charged particles, which emit GR (bremsstrahlung) when those strike gas or dust in there medium, & are decellerated

Quasars - More powerful GR from distant quasars & active galaxies are thought to have a GR production source similar to particle accelerator
- High energy electrons produced by quasars, & subjected to inverse C.S., synchrotron radiation, or bremsstrahlung, are the likely source of the GR from those objects
- A supermassive blackhole at the center of such galaxies provide the power source that destroys stars & focuses the resulting charged particles into beams that emerge from their rotational poles.
- When these beams interact w/ gas, dust, or lower energy photons they produce GRs & XRs.

GR Bursts:

- The most intense source of GR. They're the "long duration bursts" (long here means a few tens of seconds).
- Short GR bursts (not produced by supernovae) are produced during the collision of neutron stars or a neutron star & a black hole
(These Short GR bursts last 2 seconds or fewer, & are far lower energy than the long GR bursts
- LDGRB produce a total energy of about 10^{44} Joules (6.24×10^{62} eV) within 20-40 secons (about 50% energy are GR)
- Compton Scattering & synchrotron radiation are the mechanisms of production of GR

X-Rays:

- Most X-rays have a wavelength ranging from 0.01 - 10 nanometers ($1\,nm \rightarrow 1\times10^{-9}\,m$)
- frequency in range of 30 PHz → 30 exahertz (3×10^{16} Hz - 3×10^{19} Hz)
- Energy ranges from 100eV → 100 KeV
- Hard (above 5-10 KeV, below 0.2-0.1 nm in wavelength) & soft (below that energy) x rays
- Emitted by electron.
- Has energy to ionize atoms & disrupt molecular bonds.
- Interacts with matter in three ways: (1) Photoabsorption (2) Compton Scattering (3) Rayleigh Scattering

(The strength of these interactions depend on energy & elemental composition of the material)

(1) A photoabsorbed photon transfers all of its energy to the electron w/ which it interacts, thus ionizing the atom to which the electron was bound & producing a photoelectron that likely ionizes more atoms in its path.

(2) Compton scattering is an inelastic scattering of the X-Ray photon by an outer shell electron.

- Part of the energy of photon is transferred to scattered electron, thereby ionizing the atom & increasing the wavelength
- Probability for different scattering angles are described by the Klein-Nishina formula
- Transferred E can be obtained from the scattering angle from the conservation of energy & momentum

(3) The dominant elastic scattering mechanism in the X-ray regime.

- Inelastic forward scattering gives rise to the refractive index, which for X-rays is only slightly below.
- Whenever charged particles (electrons or ions) of sufficient energy hits a material, X-rays are produced.
- X-rays can also be produced by fast protons or other positive ions

Celestial Coordinate System:

Systems

Horizontal: - Based on position of the observer on Earth

- Position of Celestial Object varies w/ time.
- Based on the position of Stars relative to an observer's ideal horizon.

Equatorial:

- Centered at Earth's center, fixed relative to the celestial poles & vernal equinox
- Based on location of stars relative to Earth's equator if it were projected out to an infinite dis.
- Normal Coordinate system, (follows the movement of the sky during the night).

Ecliptic:

- The fundamental plane is the plane of Earth's orbit.
- 2 principal variants: geocentric ecliptic coordinates & heliocentric ecliptic coordinates.
- Geocentric useful for computing the apparent motions of the Sun, Moon, & other solar system bodies.
- Heliocentric useful for computing the positions of solar bodies as well as defining their orbital elements.

Galactic System:

- Uses the approximate plane of our galaxy as its fundamental plane. Solar system still the center of the coordinate system.
- Zero point is defined as the direction towards the galactic center.
- Galactic latitude resembles the elevation above the galactic plane & galactic longitude determines direction relative to the center of the galaxy.

Supergalactic: Corresponds to a fundamental plane that contains a higher than average number of local galaxies in the sky as seen from Earth.

Chlorofile!

$f = 540$ THz (540×10^{12} Hz) $\longrightarrow$ $f = 526\text{-}606$ THz
$\lambda = 495\text{-}570$ nm

Photon energy of Green Light: 2.17 - 2.5 eV.

An amount of light compatible w/ everyday experience is the energy of 1 mole of photons
its energy can be computed by multiplying the photon energy by the Avogadro Constant ($N_A \approx 6.022 \times 10^{23}$ mol^{-1})
The result is that green light of wavelength 555nm has an energy of 216 KJ/mol,
a typical energy of everyday life.

Look up AA battery

Planck Constant:

- Related to the quantization of light & matter,
- Energy multiplied by time, or momentum multiplied by distance, or angular momentum,
- Expressed in Joule-seconds (J·s or N·m·s or $kg \cdot m^2 \cdot s^{-1}$).
- Planck-einstein relation connects the particular photon energy E w/its associated wave frequency f: $E = hf$
- Conceived as the proportionality constant between the minimal increment of energy E, of a hypothetical electrically charged oscillator in a cavity that contained black-body radiation, & the frequency f, of its associated electromagnetic wave

Inverse Compton Scattering:

- Charged particles (usually electrons) impart energy to low-energy photons boosting them to higher energy photons
- Such impacts of photons on relativistic charged particle beams is another possible mechanism of GR production

Aaamouth

Why is glass transparent?

Photons pass through glass because they're not absorbed. This is because there is nothing which "absorbs" light in visual frequencies in glass. UV photons are absorbed by glass, so glass is not transparent for them. Exactly the same happens w/ X-rays for which our body is nearly transparent whilst a metal plate absorbs it.
Any photon has a certain frequency - which for visible light is related to the color of light, whilst for lower or upper frequencies in the EM spectrum it is simply a measure of the energy transported by the photon. A material's absorbtion spectrum (which frequencies are absorbed & how much) depends on the structure of the material at atomic scale. Absorption may be from atoms which absorb photons (remember - electrons go to upper energetic states by absorbing photons), from molecules, or from lattices. There are important differences in these absorption possibilities.

1. Atoms absorb well-defined discrete frequencies. Usually single atoms absorb only a few frequencies - it depends on the energy spectrum of its electrons. Regarding atomic absorption, the graph of absorption (plotted as a function of frequency of light) contains well-defined peaks for frequencies when absorption occurs, & no absorption at all between them.
2. molecules absorb discrete frequencies but there are many more absorption lines b/c even a simple molecule has many more energetic levels than any atom. So molecules absorb much more light.
3. Crystalline ~~[illegible]~~ lattices may absorb not only discrete frequencies, but also continuous bands of frequencies, mainly b/c of discrepancies in the crystalline structure.

As glass is non-crystalline, overcooled fluid, consisting of molecules, its absorption occurs in the 1^{st} & 2^{nd} ways, but because of the matter it is composed of, it absorbs outside or visible spectrum.

- Mass of proton = $0.938\ GeV/c^2$. All hadrons are of order $1\ GeV/c^2$
Electron Volt, is the amount of energy gained (or lost) by the charge of a single electron moving across an electric potential difference of one volt.

- A particle w/ charge q has an energy $E = qV$ after passing through the potential V; if q is quoted in integer units of the elementary charge & the terminal bias in volts, one gets an energy in eV

$$-\frac{\hbar^2}{2m}\frac{\partial^2 u}{\partial r^2} + \left[-\frac{e^2}{4\pi\varepsilon_0 r} + \frac{\hbar^2}{2m}\frac{L(L+1)}{r^2}\right]u = Eu \quad u = rR$$

Ridburg Constant

hydrogen radius (Bohr rad).

Links Scale: 0.529 Å

We want bound states, $E<0$. Define $\alpha = \sqrt{\frac{-2ME}{\hbar^2}}$.

Now get equation into dimensionless form

Put $\rho = \alpha r$ & $\rho_0 = Me^2/2\pi\varepsilon_0\hbar^2\alpha$ $\quad E = \frac{-\hbar^2}{2M}\alpha^2 = -\frac{\hbar^2}{2M}\left(Me^2/2\pi\varepsilon_0\hbar^2\rho_0\right)^2$

$\Rightarrow \frac{d^2u}{d\rho^2} = \left(1 - \frac{\rho_0}{\rho} + \frac{L(L+1)}{\rho^2}\right)u$. Analyze asymptotics: Large $\rho \to u \sim e^{-\rho}$

Small $\rho \to u \sim \rho^{L+1}$

try substitution: $u = \rho^{L+1}e^{-\rho}\nu(\rho) \Rightarrow \rho\frac{d^2u}{d\rho^2} + 2(L+1-\rho)\frac{d\nu}{d\rho} + [\rho_0 - 2(L+1)]\nu = 0$

Then try series expansion for $\nu(\rho)$ & requires series to terminate.

$\rightarrow \rho_0 = 2n$ where $n \geq L+1$ is integer. Solutions $\nu_{nL}(\rho) = L^{2L+1}_{n-L-1}(2\rho)$

Associated Laguerre polynomials give oscillations & nodes $U(r)$: $L_0^0 = 1$, $L_1^0 = -x+1$ $\quad L_2^0 = x^2 - 4x + 2$

$\rightarrow R(r) = \frac{U_{nL}(r)}{r} = \frac{(\alpha r)^{L+1}}{r}e^{-\alpha r}\nu_{nL}(\alpha r)$

$E_n = -\frac{\hbar^2}{2M}\left(\frac{Me^2}{2\pi\varepsilon_0\hbar^2\rho_0}\right) = \frac{-e^4}{32\pi^2\varepsilon_0\hbar^2 n^2} = \frac{R_y}{n^2}$ $\quad R_y = 13.6$ eV

as we found long ago.

Important

Hydrogen eigenfunctions $\varphi = RY^m_L$ $\quad \boxed{\varphi_{nLm} = A_{nLm}e^{-r/na}\left(\frac{2r}{na}\right)L^{2L+1}_{n-L-1}\left(\frac{2r}{na}\right)Y^m_L(\theta,\phi)}$ already normalized.

$A_{nLm} = \sqrt{\left(\frac{2}{na}\right)^3\frac{(n-L-1)!}{2n[(n+L)!]^3}}$ $\quad$ orthonormality: $\int \varphi^*_{nLm}\varphi_{n'L'm'}\,d^3r = \delta_{nn'}\delta_{LL'}\delta_{mm'}$

$d^3r = r^2\sin\theta\, dr\, d\theta\, d\phi$

degeneracy: E_n is independent of L & m. (# of wavefunctions w/ the same energy).

$|m| \leq L < n$ $\quad$ count $\sum_{L=0}^{n-1}\sum_{m=-L}^{+L} 1 = \boxed{n^2}$ - Degeneracy of hydrogen.

$\hbar = 1.05457148 \times 10^{-34}\ kg\,m^2/s$

3 unexplained Experiments: Blackbody Spectrum, Photoelectric effect, Bright line Spectra:

Black Body Spectrum: $\boxed{\text{HOT}}\ T$

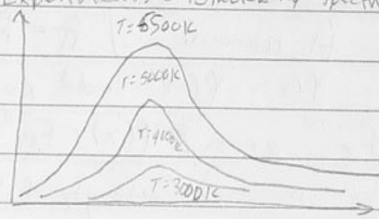
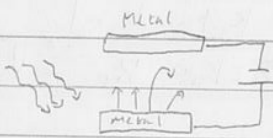


Photo Electric Effect:



Classical E&M:

Intensity ↑ ⟶ $|E|$↑ ⟹ energy ↑ Ustop ↑

freq ↑ ⟶ $|E|$ same ⟹ energy & Ustop same

Reality:

U_{stop} Same

U_{stop} ↑

Bright line Spectra: Bright lines on spectra of element would be indicated as peaks/packets (energy comes quantized)

When does QM apply: When angular momentum $L \sim \hbar$, when uncertainty $\Delta p \Delta x \sim \hbar$, when uncertainties $\Delta E \Delta t \sim \hbar$, when action $S \sim \hbar$

Ex: electron in hydrogen atom: energy $10eV \sim \frac{p^2}{2m} \Rightarrow p \sim 1.4 \times 10^{-24}\ kg \frac{m}{s}$. Size of atom: $0.1nm \sim 10^{-16} m = \Delta x \Rightarrow \Delta p \Delta x \sim 1.7 \times 10^{-34}\ kg\,m^2/s$ $\approx \hbar!$ $\hbar$

QM: Single particles, Lasers, Semiconductors, Low temperatures (< 100 K)

The Wavefunction: $\Psi(x,y,z,t)$, complex function, represents the state of the system, gives probability

Operators: Connect Ψ to observables, hats $\hat{x}$ & $\hat{p}$, acts on Ψ $\hat{x}\Psi$ & $\hat{p}\Psi$, $\hat{p} = -\hbar \frac{d}{dx}$

S.E. $i\hbar \frac{\partial \Psi}{\partial t} = \hat{H}\Psi$, $\hat{H}$ = "energy operator", Hamiltonian $= (\hat{KE} + \hat{V})\Psi = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V(x)\Psi$

Operators (3) $\xrightarrow{\text{Used in}}$ S.E. (4)

Operators $\xrightarrow{\text{act on}}$ Wavefunction (2)

Operators ⟶ observables

S.E. $\xrightarrow{\text{solution to}}$ Wavefunction (2)

Wavefunction ⟶ observables

Wavefunction ⟶ probabilities (1)

(5) examples QM

(6) Formalism

(7) Applications

Complex Numbers: $i^2 = -1$, $i = \sqrt{-1}$, $z = \underset{\text{real}}{x} + i\underset{\text{imag}}{y}$, $\mathrm{Re}\{x+iy\} = x$, $\mathrm{Im}\{x+iy\} = y$ $x \,\&\, y \in \mathbb{R}$

$i = \sqrt{-1}$

$e^{iy} = \cos y + i \sin y$. Polar Form: $\underset{\text{rectangular}}{x+iy} = \sqrt{x^2+y^2}\,(\cos\theta + i\sin\theta) = Re^{i\theta}$ (Taylor Series!)

$\theta = \arctan\left(\frac{y}{x}\right)$

Operations (Rectangular Form): $(a+ib)+(c+id) = (a+c) + i(b+d)$ - addition/subtraction

$(a+ib)(c+id) = (ac-bd) + i(ad+bc)$ - Multiplication/distribute

$\frac{a+ib}{c+id} = \frac{a+ib}{c+id}\,\frac{c-id}{c-id} = \frac{(ac+bd)+i(bc-ad)}{c^2+d^2}$ - Division (like rationalizing the denominator)

(Polar Form): $qe^{i\theta} + re^{i\phi} = \ldots$ complicated - Addition

$(qe^{i\theta})(re^{i\phi}) = (qr)e^{i(\theta+\phi)}$ - Multiplication (multiply radii, add angles)

$\frac{qe^{i\theta}}{re^{i\phi}} = \frac{q}{r}e^{i(\theta-\phi)}$ - Division (divid radii, subtract angles)

$|x+iy|^2 =$

Magnitudes:

Complex conjugates: $(x+iy)^* = x - iy$

Squared magnitudes (always real, positive):

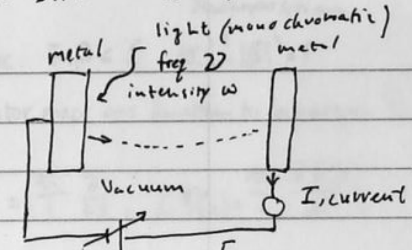
$|Z|^2 = ZZ^* = \mathrm{Re}\{z\}^2 + \mathrm{Im}\{z\}^2$

1.2 Why Quantum Mechanics? Continued... Energy comes in packets. $E \sim h\cdot\lambda$, $\lambda = \frac{h}{p}$

Can explain Black Body Radiation. Atoms radiate distinct spectral lines.
If electron acts as wave w/ $\lambda = \frac{h}{p}$ & forms standing waves in its orbit, $n\lambda = 2\pi r \rightarrow L = rp = n\hbar$

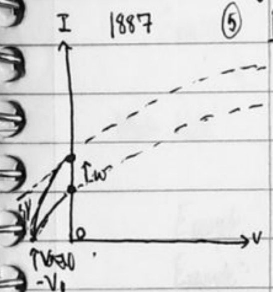
10.3

(Closed Circuit)!



Bohr-Sommerfeld:

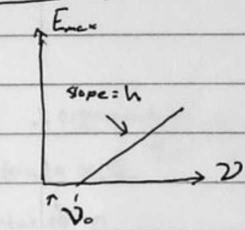
(Demo 10.5)



1887 ⑤ Photoelectric Effect: Hertz. Measured correctly in $\boxed{1914}$ Millikan in the U.S.

a) In dark (at $\omega = 0$), no current. $I \propto \omega$.

Number of emitted electrons is proportional to power or intensity of the source light. d)



b) For $V < -V_0$, $I = 0$ for any $\omega \Rightarrow$ max Kinetic energy of emitted electrons $E_{max} = eV_0$

no electrons for $\nu < \nu_0$

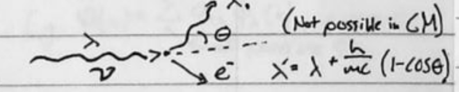
c) E_{max} is independent of ω!!!

$h\nu$ $\rightarrow E_{max}$ $\boxed{h\nu = \phi + E_{max}}$

Einstein ⇒ light comes in packets, explains everything!

ϕ - work function - extracting from solid. $\boxed{h\nu_0 = \phi}$

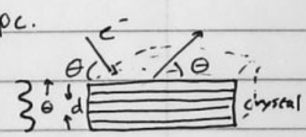
ϕ (work function) depends on material.



$\lambda' = \lambda + \frac{h}{mc}(1 - \cos\theta)$

1923 U.S. 6) Compton Scattering: of x-rays from solids. Found that x-rays are red-shifted (frequency is lowered).

Can be derived by conservation of Energy & Momentum assuming light comes in packets. $E = h\nu$, $p = \frac{h}{\lambda}$, $E = pc$.



1923 U.S. & U.K. 7) Electron Diffraction: Davisson-Germer Experiment. Found strong scattering when $\boxed{\sin\theta = \frac{n\lambda}{2d}}$ (n integer), $\lambda = \frac{h}{p}$.

d = spacing of lattice planes.

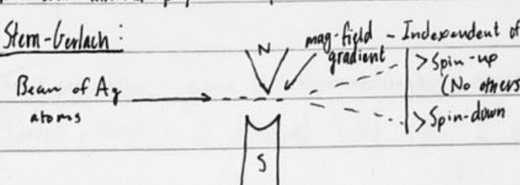
1924 de Broglie: all fundamental particles have $\boxed{\lambda = \frac{h}{p}}$ de Broglie relation

8) Randomness: Each diffracted particle is detected at a random position (on screen). Events are probabilistic. Some θ are more probable than others.

This randomness appears fundamental to nature.

9) There exist "internal" properties of particles, that take distinct values. "spin", charge, lepton number, etc.

1922 Stern-Gerlach:



Postulates of Quantum Mechanics: Not universal $\Psi(x,t)$ - particle moving in 1 dimension.

1. The state of a quantum system is specified by wave function Ψ. Ψ takes complex values. Func. of space, time, other variables (e.g. spin)

Binomial Distribution: $P(X=x) = \frac{n!}{x!(n-x)!}\pi^x(1-\pi)^{(n-x)}$, $\sigma=\sqrt{n\pi(1-\pi)}$ (Bernoulli)

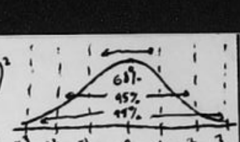
$E(x), \mu_x = n\pi$, $V(x), \sigma_x^2 = n\pi(1-\pi)$

Poisson: $p(x) = \frac{e^{-\lambda}\lambda^x}{x!}$, $E(x)=V(x)=\lambda$, $\lambda = n\pi$

$\bar{x} = \sum_{i=1}^{n} \frac{x_i}{n}$, Dis: $\sum x p(x)$, Continuous: $E(x)$: $\int_{-\infty}^{\infty} x f(x)dx = E(x)$

Types of Data: - Continuous: Height (in), weight (lb), corrosion (mpy)
- Categorical: : Nominal (race, gender, student major)
: : Ordinal (pain scale, good/better/best)

Gaussian (C): $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ $\mu=0$, $\sigma=1$



If X is norm. dis. w/ param. μ & σ then $Z = \frac{X-\mu}{\sigma}$. The area under $f(x)$ between a & b is equiv. to the area under $f(z)$ between $\frac{a-\mu}{\sigma}$ & $\frac{b-\mu}{\sigma}$

Sample Variance: $s^2 = \sum_{i=1}^{n} \frac{(x_i-\bar{x})^2}{n-1} = \sum_{i=1}^{n} \frac{x_i^2 - n\bar{x}^2}{n-1}$, Samp. SD: $s = \sqrt{s^2} = \sigma$,

For discrete: $V(X) = \sum (X-E(X))^2 p(X)$
" Continuous: $V(X) = \int_{-\infty}^{\infty} (X-E(X))^2 f(X)$

Median: c s.t. $\int_{-\infty}^{c} f(x)dx = 0.5$ or $\int_{c}^{\infty} f(x)dx = 0.5$

To find the q^{th} quantile, or the $(100q)^{th}$ percentile, find c:
c s.t. $\int_{-\infty}^{c} f(x) = q$, or $\int_{c}^{\infty} f(x)dx = 1-q$

The sample correlation between 2 var. X & Y: $r = \frac{1}{n-1}\frac{\sum_{i=1}^{n}(X_i-\bar{X})(Y_i-\bar{Y})}{S_x S_y}$, S_x = SD of X, S_y = SD of Y
r measures strength of linear relation between x & y

$Y_i = \alpha + \beta X_i + e_i$, Y: Dependent, X = indep/predictor, α = intercept, β = Slope.

Linear Regression, Notation: Sample mean x: $\sum_{i=1}^{n} \frac{x_i}{n} = \bar{x}$

$\bar{y} = \sum_{i=1}^{n} \frac{y_i}{n}$, $S_y = \sqrt{\frac{\sum_{i=1}^{n}(y_i-\bar{y})^2}{(n-1)}}$, $S_x = \sqrt{\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{(n-1)}}$

$S_{yy} = \sum_{i=1}^{n}(y_i-\bar{y})^2$, $S_{xx} = \sum_{i=1}^{n}(x_i-\bar{x})^2$,
$\overline{xy} = \sum_{i=1}^{n} x_i y_i / n$, $\overline{x^2} = \sum_{i=1}^{n} x_i^2/n$, $S_{xy} = \sum_{i=1}^{n}(y_i-\bar{y})(x_i-\bar{x})$
$S_y = \sqrt{S_{yy}/(n-1)}$, $S_x = \sqrt{S_{xx}/(n-1)}$, $r = S_{xy}/\sqrt{S_{xx}S_{yy}}$,

Prediction: $\hat{Y}_i = (\hat{\alpha} + \hat{\beta}X_i)$. $R^2 = \frac{SS_{EXP}}{SST} = \frac{(SST-SSE)}{SST}$

$SS_{TOTAL} = SS_{UNEXPLAINED} + SS_{EXPLAINED}$

$\sum_{i=1}^{n}(y_i-\bar{y})^2 = \sum_{i=1}^{n}(y_i-\hat{y})^2 + \sum_{i=1}^{n}(\hat{y}_i-\bar{y})^2$

Correlation is invariant to shifting & Scaling

$SS_{EXP.} = SSR, SS_{regression}, SST_r, SS\beta$
$SS_{UNEXP.} = SSE, SS_{resid.}$ $SS_{TOTAL} = SS_{TOT}$

Uniform: $\mu = \frac{a+b}{2}$, $Var = \frac{(b-a)^2}{12}$, Exponential $\mu = \frac{1}{\lambda}$, $Var = \frac{1}{\lambda^2}$

• To get the ordinary least squares (OLS) fit, minimize $\sum_{i=1}^{n} e_i^2$, the sum of squared errors (SSE)

• Recall: $e_i = y_i - \hat{y}_i$, so $SSE = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n}(y_i - \hat{\alpha} - x_i\hat{\beta})^2$.

- Minimize SSE by differentiating it w.r.t $\hat{\alpha}$ & $\hat{\beta}$. Set equations to 0, Solve for $\hat{\alpha}$ & $\hat{\beta}$

$\frac{\partial SSE}{\partial\beta} = \sum_{i=1}^{n}\frac{\partial(y_i-\hat{\alpha}-x_i\hat{\beta})^2}{\partial\beta} = 2\sum(y_i-\hat{\alpha}-x_i\hat{\beta})(-x_i) = -2(\sum y_i x_i - \hat{\alpha}\sum x_i - \hat{\beta}\sum x_i^2)$
$= -2(n\overline{xy} - n\hat{\alpha}\bar{x} - n\hat{\beta}\overline{x^2})$, where $\overline{xy} = \sum x_i y_i/n$, $\overline{x^2} = \sum x_i^2/n$

$\frac{\partial SSE}{\partial\alpha} = \sum_{i=1}^{n}\frac{\partial(y_i-\hat{\alpha}-x_i\hat{\beta})^2}{\partial\alpha} = 2\sum(y_i-\hat{\alpha}-x_i\hat{\beta})(-1) = -2n(\bar{y}-\hat{\alpha}-\hat{\beta}\bar{x})$

- Set derivative eq. to 0 (Normal eq.), and solve for $\hat{\alpha}$ and $\hat{\beta}$.
1) $-2n(\overline{xy} - \hat{\alpha}\bar{x} - \hat{\beta}\overline{x^2}) = 0$. So $\hat{\beta} = \frac{(\overline{xy}-\bar{x}\bar{y})}{(\overline{x^2}-\bar{x}^2)}$ & $\hat{\alpha} = \bar{y} - \hat{\beta}\bar{x}$
2) $-2n(\bar{y} - \hat{\alpha} - \hat{\beta}\bar{x}) = 0$.

Lin. Reg. & Analysis of Variance: $\sum_{i=1}^{n}(y_i-\bar{y})^2 = \sum_{i=1}^{n}(y_i-\hat{y}_i)^2 + \sum_{i=1}^{n}(\hat{y}_i-\bar{y})^2$,
$(y_i-\bar{y}) = (y_i-\hat{y}_i) + (\hat{y}_i-\bar{y})$,
Square both sides & sum over all: $\sum_{i=1}^{n}(y_i-\bar{y})^2 = \sum_{i=1}^{n}(y_i-\hat{y}_i)^2 + \sum_{i=1}^{n}(\hat{y}_i-\bar{y})^2 + \sum_{i=1}^{n}2(\hat{y}_i-\bar{y})(y_i-\hat{y}_i)$

Simple Lin. Reg. gives $\hat{\alpha} = \bar{y} - \hat{b}\bar{x}$. What follows depends on the following:
$\sum_{i=1}^{n}2(\hat{y}_i-\bar{y})(y_i-\hat{y}_i) = \sum 2((\bar{y}-\hat{b}\bar{x}+\hat{b}x_i)-\bar{y})(y_i-\hat{y}_i) = \sum 2((\bar{y}+\hat{b}(x_i-\bar{x}))-\bar{y})(y_i-\hat{y}_i)$
$= \sum 2(\hat{b}(x_i-\bar{x}))(y_i-\hat{y}_i) = \sum 2\hat{b}(x_i-\bar{x})(y_i-(\bar{y}+\hat{b}(x_i-\bar{x}))) = \sum 2\hat{b}((y_i-\bar{y})(x_i-\bar{x})-\hat{b}(x_i-\bar{x})^2)$

$\hat{b} = \frac{\sum(x_i-\bar{x})(y_i-\bar{y})}{\sum(x_i-\bar{x})^2} \Rightarrow \sum 2(\hat{y}_i-\bar{y})(y_i-\hat{y}_i) = \sum 2\hat{b}((y_i-\bar{y})(x_i-\bar{x})-\hat{b}(x_i-\bar{x})^2)$
$= 2\hat{b}\left(\sum(y_i-\bar{y})(x_i-\bar{x}) - \hat{b}\sum(x_i-\bar{x})^2\right) = 2\hat{b}\sum((y_i-\bar{y})(x_i-\bar{x})-(y_i-\bar{y})(x_i-\bar{x}))$
$= 2\hat{b}\cdot 0 = 0$

For Continuous x: $f(x)$ = density function; Must satisfy: $f(x)\geq 0$, $\int_{-\infty}^{\infty} f(x)dx = 1$, $\int_a^b f(x)dx$ = prop. of x in between a & b

Discrete X: $p(x)$ = mass function; Must satisfy: $p(x)\geq 0$, $\sum_{-\infty}^{\infty} p(x) = 1$, $\sum_a^b p(x)$ = prop of x vals in between a & b