ABSTRACT

COVARIANCE MATRIX FORECASTING USING COPULA-BASED SIMULATION

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This paper introduces a covariance matrix forecasting method based on copula-GARCH simulated returns. The accuracy of this method is compared against the traditional forecasting methods of Dynamic Conditional Covariance (DCC) and rolling historical. The results found the Clayton vine copula dependency structure performed best and ultimately led to superior forecasts over traditional methods. This is shown to be due to an increased accuracy in modeling the correlation structure of the assets. To ensure robustness of the results, multiple datasets and loss functions are used in the analysis.

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COVARIANCE MATRIX FORECASTING USING COPULA-BASED SIMULATION

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0.1 Introduction

Equity portfolio returns, while often assumed to be multivariate normal, have shown to have complex joint distributions. Fortunately the use of copulas, specifically vine copulas, has shown to be an effective tool for creating accurate joint distributions (Jondeau & Rockinger, 2006). Copulas can be combined with GARCH models as a method to create simulated future returns. These simulated future returns have been applied to determining portfolio value at risk (Aas & Berg, 2009; Nikoloulopoulos et al., 2012; Özgür et al., 2021) and optimal portfolio allocations (Yew Low et al., 2016; Sahamkhadam et al., 2022; Sahamkhadam & Stephan, 2023).

This paper extends the previous research of copula-GARCH simulated returns by using them for the purpose of creating a covariance matrix forecast. The covariance matrix is used as an input to many commonly used mean-variance based portfolio allocation strategies. When a vector of asset weights is applied to the covariance matrix, it easily summarizes portfolio risk by representing portfolio variance in quadratic form. Improving the accuracy of the covariance matrix forecast will lead to a more accurate efficient frontier and asset allocation weights being determined.

The goal of this paper is to compare the copula-GARCH covariance matrix forecasts against other traditionally used methods to find out how well it performs. The benchmark forecasting methods are Dynamic Conditional Correlation (DCC-) GARCH and rolling historical. DCC-GARCH was chosen as a benchmark because it is known to perform well and is regarded as the best traditional method to use when forecasting the covariance matrix. The rolling historical method was chosen as a benchmark because it is the easiest to calculate and is therefore the most frequently used method.

The Clayton copula and Clayton vine copula (CVC) dependency structures are used in this paper because they are asymmetric copulas that are able to measure lower tail dependence. Equity portfolio returns have dependency structures that are often asymmetric. Asymmetric dependence is described by Patton to be when 'stock returns appear to be more highly correlated during market downturns than during market upturns' (Patton, 2004). This has been shown to hold true in many papers (Erb et al., 1994; Longin & Solnik, 2001; Ang & Chen, 2002; Okimoto, 2008; Alcock & Hatherley, 2009). When a min-CVaR portfolio allocation strategy is used, the Clayton copula has shown able to reduce downside loss exposure more than than the multivariate normal distribution (Hatherley & Alcock, 2007). This was then extended to show the use of a canonical Clayton vine copula can outperform a standard Clayton copula when the number of dimensions are increased (Low et al., 2013).

The results of this paper show the Clayton vine copula is able to outperform the alternative copula methods that were tested. When the Clayton vine copula dependency structure is used, the copula-GARCH forecasting method was able to outperform the traditionally used benchmark methods. The driving force of the superior performance is the copulas' ability to create a superior correlation structure. When an application using portfolio allocation was used, the results held up.

0.2 Copula Background Information

Copulas are functions that model the dependency structure between random variables. Sklar's theorem below shows that when the marginal distributions ($F_i(x_i)$) of the random variables (x_i) are known, the data can be transformed to the uniform [0,1] scale and be used with a copula (C) to create a joint distribution function ($F(x_1, ..., x_K)$) (Sklar, 1959).

Sklar's Theorem:

$$F(x_1, ..., x_K) = C(F_1(x_1), ..., F_K(x_K)) \tag{1}$$

Copulas can be bivariate (paired copulas), multivariate, or vine copulas. Paired copulas model the dependency structure of two variables, multivariate copulas model the dependency structure of greater than two variables, and vine copulas create a multivariate copula using only paired copulas that are linked together via conditioning. Vine copulas are often able to offer improvement over traditional multivariate copulas due to their increased flexibility. This is because for each paired dependency between the variables, the best suited bivariate copula can be selected and the optimal parameters can be individually estimated. Additionally, the structure of how the variables are linked together can be modified. The example below shows how the bivariate copulas can be linked together to form a multivariate distribution function with a canonical vine (c-vine) structure. In the example, $F_i(x_i)$ is the marginal distribution of the random variable (x_i). $F_{i|j}(x_i|x_j)$ is the marginal distribution of random variable x_i conditional on random variable x_j. C_{ij} is the bivariate copula for the dependency between random variables x_i and x_j. $C_{ij|l}$ is the bivariate copula of random variables x_i and x_j when x_i and x_j are both conditional on x_l. $F(x_i, x_j, x_l)$ is the resulting joint distribution function created by the vine copula. A deeper review of vine copulas can be found in Czado et al., 2022.

$$\begin{aligned} F(x_1, x_2, x_3) =& C_{13;2}(F_{1|2}(x_1|x_2), F_{3|2}(x_3|x_2); x_2) \\ & \times C_{23}(F_2(x_2), F_3(x_3)) \\ & \times C_{12}(F_1(x_1), F_2(x_2)) \end{aligned} \tag{2}$$

Once the joint distribution is known, it can then be sampled. When M samples are taken from a multivariate copula of K dimensions, the result would be an $M \times K$ data frame. The use of a large sample size will give a good approximation of the joint distribution function in a tangible form.

0.3 Data

The time frame of the data is between July 1963 and December 2022 for a total of 714 months. To help ensure robustness of results, multiple data sets are used. This paper uses the CRSP portfolio data sets of Beta, Standard Deviation (SD), and Market Capitalization (Mkt Cap). Each data set contains the returns of various portfolios, which are then considered to be asset returns and are used to forecast the covariance matrix. Each CRSP dataset will represent a portfolio of 10 assets.

These datasets are based on the returns of real firms. The Industry and Mkt Cap data sets are created by combining all of the stocks on the NYSE, AMEX, and NASDAQ. The Beta and SD is based on NYSE and AMEX returns only, NASDAQ was not included. The Beta, SD, and Mkt Cap data sets create portfolios by sorting firms based on their correlation to the market, standard deviation, or market capitalization. The firms are sorted into 10 decile portfolios with decile 1 containing the lowest ranked 10% of firms and decile 10 containing the highest ranked 10% of firms. The returns of the 10 decile portfolios based on the sorting methods used become the datasets used in this paper. The returns of these datasets are then considered to be the assets used to create the covariance matrix.

For the CRSP data, the daily data was sourced and then aggregated to be on a monthly scale. The CRSP 30 day monthly T-bill rate is subtracted and the excess returns are then multiplied by 100.

0.4 Method

A rolling window on the monthly scale is the basis for all of the results. The rolling window size is 240 months (20 years), so the first forecast is for the month of July 1983. There are a total of 474 forecasts. The forecasts are made based on the monthly data and are used to forecast the one month ahead covariance matrix. The analysis was performed in R with the packages of Rugarch, Rmgarch, Rvinecopulib, and Copula.

The forecasting methods used in this paper are rolling historical (HISTORICAL), DCC-GARCH (DCC), and a copula-GARCH method. For simplicity, all data sets use the same GARCH specification assumptions. All GARCH models are assumed to have stationary means, follow a GARCH(1,1) process, and have innovations from a standard normal distribution. When copula methods are used, the marginal distributions are always assumed to be normal distributions regardless of the copula specification. The normal marginal distribution assumption is used for applying the probability and quantile data transformations necessary for fitting and sampling from a copula. The typical copula data transformations are made using the fitdist, pdist, and qdist functions in the Rugarch package. The appendix tables show alternative results based on the student t distribution. When the t distribution is used the results hold up.

Since the true monthly covariance matrix is not observable, the results of this paper are based on using its proxy (Σ_t). Creating a proxy will allow for distance measures to be used to measure the forecast error and analyze the performance of each method. Σ_t is the realized covariance matrix computed using daily data. The proxy variance values (σ_t) within Σ_t are found using equation 3.

$$\sigma_t = \sum_{j=1}^{nd_t} (r_j - \bar{r}_t)^2 \tag{3}$$

Where nd_t is the number of days during month t, r_j is the daily return on day j, $\bar{r}_t$ is the mean of daily returns in month t.

0.4.1 Forecasting Methods

0.4.1.1 Rolling Historical (HISTORICAL)

The rolling historical method uses the historical returns within the rolling window to create a covariance matrix. That covariance matrix then becomes the forecast for next month. For this paper, the use of a hat over a variable represents it is a forecasted value. This helps to differentiate a forecast from its realized proxy. The use of t+1 does not always indicate forecast, it simply indicates it is a different time period from t. In the R code, the covariance matrix was found using the built in cov() function. This function can be equivalently expressed using equation 4.

$$\hat{\Sigma}_{t+1}^{HISTORICAL} = (N-1)^{-1} Y_t' Y_t - \{N(N-1)\}^{-1} Y_t' \mathbb{1}\mathbb{1}' Y_t \tag{4}$$

Where:

$$a_{it} = r_{it} - \mu_{it} \qquad \forall\, i \in 1, ..., K, \qquad \forall\, t \in 1, ..., N$$

$$Y_t = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,K} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,K} \\ \vdots & \vdots & \vdots & \vdots \\ a_{N,1} & a_{N,2} & \cdots & a_{N,K} \end{pmatrix}$$

There are K total assets in the portfolio, with i being an individual asset within K. The subscript t indicates the month and N indicates the total number of months in the rolling window. Each assets monthly returns, r_{it}, can be decomposed into the assets historical rolling window mean, μ_{it}, and its error term a_{it}. These error terms can be found and used to create matrix Y_t. Y_t is an $N \times K$ matrix and $\mathbb{1}$ is an $N \times K$ matrix filled with values of 1. The variance values within $\hat{\Sigma}_{t+1}^{HISTORICAL}$ can be found using equation 5.

$$\hat{\sigma}_{t+1}^{HISTORICAL} = \frac{\sum_{j=1}^{N}(r_j - \mu_t)^2}{N-1} \tag{5}$$

0.4.1.2 DCC-GARCH (DCC)

The GARCH model was introduced for univariate time series data as a way to adjust for heteroskedasticity. It is able to predict next periods variance using weighted past residual observations and past realized variances (Bollerslev, 1986). Equation 6 is the equation for the GARCH(1,1) model that is used in this paper. In the equation the error term can be rewritten as $a_{it} = \sqrt{\sigma_{it}}\epsilon_{it}$ with $\epsilon \sim N(0,1)$. The scalar values of α_0, α_1, and β_1 are found using maximum likelihood estimation.

$$\hat{\sigma}_{t+1}^{GARCH} = \alpha_0 + \alpha_1 a_t^2 + \beta_1 \sigma_t \tag{6}$$

Additionally GJR-GARCH variations are used in the results. GJR was developed to incorporate asymmetric effects into GARCH models (Glosten et al., 1993)(Ghalanos, n.d.). The formula for GJR-GARCH is shown in equation 7.

$$\hat{\sigma}_{t+1}^{GJRGARCH} = \alpha_0 + \alpha_1 a_t^2 + \gamma_t I_t a_t^2 + \beta_1 \sigma_t \tag{7}$$

Where I_t is an indicator function which equals 1 when $a_t < 0$ and 0 otherwise. γ_t is the leverage term which is used to incorporate the asymmetrically effects of shocks.

DCC-GARCH is an extension of GARCH to the multivariate case, it works well compared to other multivariate GARCH models due to its comparable simplicity and ability to have a time varying correlation matrix, R_t (Engle & Sheppard, 2001). As seen below, DCC is based on using univariate GARCH models to first estimate the variances and obtain the GARCH error terms. These values are then used to estimate the time varying correlation matrix and obtain the covariance matrix forecast using equation 8.

$$\hat{\Sigma}_{t+1}^{DCC} = D_{t+1} R_{t+1}^{DCC} D_{t+1} \tag{8}$$

Where:

$$D_{t+1} = diag(\sqrt{\hat{\sigma}_{1,t+1}^{GARCH}}, ..., \sqrt{\hat{\sigma}_{K,t+1}^{GARCH}})$$
$$R_{t+1} = Q_{t+1}^{*-1} Q_{t+1} Q_{t+1}^{*-1}$$
$$Q_{t+1} = (1 - \theta_1 - \theta_2)\overline{Q} + \theta_1 e_t e_t' + \theta_2 Q_t$$

The notation θ_1 and θ_2 are weights obtained by maximum likelihood estimation, $A_t = (a_{1,t}, ..., a_{K,t})'$ are the univariate GARCH error terms, $e_t = D_t^{-1} A_t$ are the standardized GARCH error terms, $\overline{Q} = E[e_t e_t']$ is the unconditional covariance matrix, $Q_{t+1}^{*} = (diag(Q_{t+1}))^{\frac{1}{2}}$.

0.4.1.3 Copula-GARCH

The copula-GARCH method combines the use of GARCH models with the use of copula.

Step 1. Apply univariate GARCH to each asset to obtain $\hat{\mu}_{i,t+1}$, $\hat{\sigma}_{i,t+1}$, and the vector of standardized fitted residuals, $Z_{i,t}$. $Z_{i,t}$ is found using equation 9. $\hat{\mu}_{i,t+1}$ is obtained from an ARMA(0,0) process.

$$Z_{i,t} = \frac{r_{i,t} - \hat{\mu}_{i,t}}{\sqrt{\hat{\sigma}_{i,t}}} \qquad \forall\, i \in 1, ..., K, \qquad \forall\, t \in 1, ..., N \tag{9}$$

Step 2. For each asset the individual vectors of standardized fitted residuals are then transformed to be on the Uniform [0,1] scale using the assumed standard Normal marginal distributions. The vectors are combined together to create an $N \times K$ matrix of residuals which is used to fit the copula parameters.

Step 3. The copula is then sampled 10,000 times to obtain a $10,000 \times K$ matrix of copula sampled standardized residuals. The inverse CDF is then used to transorm the residuals out of Uniform [0,1] form. Equation 10 is applied to convert each assets copula sampled standardized residuals ($\hat{Z}_{i,t+1}$) into return form ($\hat{r}_{it+1}$). Where $\hat{\mu}_{i,t+1}$ and $\hat{\sigma}_{i,t+1}$ are the scalar values obtained in step 1 and $\hat{Z}_{i,t+1}$ is a $10,000 \times 1$ vector taken from the $10,000 \times K$ matrix of copula sampled standardized residuals. This results in $\hat{r}_{it+1}$ being a $10,000 \times 1$ vector of future returns. When applied for each asset, this results in $\hat{r}_{t+1}$ a $10,000 \times K$ matrix of future returns.

$$\hat{r}_{it+1} = \hat{\mu}_{i,t+1} + \sqrt{\hat{\sigma}_{i,t+1}}\hat{Z}_{i,t+1} \qquad \forall\, i \in 1, ..., K \tag{10}$$

Step 4. These forecasted joint returns ($\hat{r}_{t+1}$) are then used to calculate the correlation matrix (R_{t+1}).

Step 5.

$$\hat{\Sigma}_{t+1}^{Copula-GARCH} = D_{t+1} R_{t+1}^{Copula-GARCH} D_{t+1} \tag{11}$$

Where:

$$D_{t+1} = diag(\sqrt{\hat{\sigma}^{GARCH}_{1,t+1}}, ..., \sqrt{\hat{\sigma}^{GARCH}_{K,t+1}})$$

The copula margins are always assumed to be normal, but a few different dependency structures are considered. This is because changing the copula will change the joint distribution that is created. Multiple copulas are evaluated to reduce mis-specification and ensure the most accurate joint distribution is created to model the data. The copulas used are the multivariate Gaussian, multivariate Student T, multivariate Clayton, and the c-vine Clayton copula. When the multivariate Gaussian copula is used with normal marginal distributions it becomes equivalent to a multivariate normal distribution. Once the best performing copula methods become known, the lower performing versions were dropped out of the analysis. The survival BB7 vine copula was also looked into but showed to be outperformed by the CVC dependency structure.

0.4.2 Evaluation

The evaluation of the forecast methods is based on using a proxy for the true covariance matrix and measuring the error of the forecast during each time period compared to the proxy. The error is measured by taking the absolute error of each variance and covariance value in the forecasted matrix and summing the absolute error to obtain the absolute forecast error (AFE). The mean of these error loss values are then taken and presented in the results tables. The method with the lowest mean absolute forecasting error (MAFE) is the best performing method. In equation , T is the total number of out of sample forecasts made. In this paper T will equal 474.

$$\mathrm{AFE}_t = \sum_{i=1}^{K}\sum_{j=1}^{K} |\sigma_{ij,t} - \hat{\sigma}_{ij,t}| \tag{12}$$

$$\mathrm{MAFE} = \frac{1}{T}\sum_{t=1}^{T} \mathrm{AFE}_t \tag{13}$$

In addition to looking at the MAFE level, statistical significance is also established using the Model Confidence Set (MCS) test (Hansen et al., 2011). While the MAFE is the mean of a series of losses, the MCS test is performed on the series of losses directly. The MCS test was performed in the Rugarch package. The MCS test compares the performance of the forecast methods to create a subset of methods that are considered superior. The MCS test works by starting with all methods being included in the superior set then testing for equal predictive ability (EPA) of all forecast methods. If this test fails, the prediction ability is not equal and it eliminates the worst performing method. This is an iterative process of removing the worst performing model then retesting for EPA until all remaining methods have equal predictive ability. At the end of testing, the models that remain in the superior set have the best forecasts, and the models that were eliminated were inferior. The method is described in (Bernardi & Catania, 2018; Bernardi & Catania, 2016).

0.4.3 Application

An additional check on the results is an application towards portfolio theory using equal weighting portfolio allocations as shown in equation 14. Under this allocation method, weights are determined solely by the number of total assets in the portfolio and does not require any additional information. Applying portfolio weighting will convert information contained in the covariance matrix into a singular value of portfolio standard deviation.

Analyzing the portfolio standard deviation values will show the main results hold when different loss functions are used.

The squared tracking error (STE) formula below shows that the squared error is found between each forecast methods standard deviation ($\sqrt{\hat{\sigma}_{p,t}}$) and the actual proxy's standard deviation ($\sqrt{\sigma_{p,t}}$). The mean squared tracking error (MSTE) is found by averaging the STE values. Additionally, an alternative loss function based on the absolute error values is provided. Equation 19 shows how the absolute tracking error (ATE) is determined and equation 20 shows how the values are averaged to become the mean absolute tracking error (MATE).

$$w_i = \frac{1}{K} \qquad \forall i \in 1, ..., K \tag{14}$$

$$\sigma_{p,t} = w_t' \Sigma_t w_t \tag{15}$$

$$\hat{\sigma}_{p,t} = w_t' \hat{\Sigma}_t w_t \tag{16}$$

$$\text{STE}_t = (\sqrt{\sigma_{p,t}} - \sqrt{\hat{\sigma}_{p,t}})^2 \tag{17}$$

$$\text{MSTE} = \frac{1}{T} \sum_{t=1}^{T} \text{STE}_t \tag{18}$$

$$\text{ATE}_t = |\sqrt{\sigma_{p,t}} - \sqrt{\hat{\sigma}_{p,t}}| \tag{19}$$

$$\text{MATE} = \frac{1}{T}\sum_{t=1}^{T} \text{ATE}_t \tag{20}$$

0.5 Results

0.5.1 Evaluate Matrix Forecasts

The results for the full matrix analysis is shown on Table 1. The table shows the results for the previously described methods of HISTORICAL, DCC-GARCH (using standard GARCH and GJR GARCH variations), and copula-GARCH (with variations for both the copula used and the GARCH method used).

The standard GARCH and the GJR GARCH based methods are not comparable to each other; however, both can be compared to the Historical method. In all tables the comparable methods are grouped together (separated by a hash line).

The CG-GN-CVC-N notation in the table can be interpreted as the method that represents the copula-GARCH blended method using normal innovations for each GARCH model with the residuals being modeled with a Clayton vine copula with normal margins. Alternatively, MVN indicates the multivariate normal distribution which is equivalent to the Gaussian copula with use of normal margins. SC-N indicates the standard multivariate version of the Clayton copula used with normal margins

The results of the table show the copula-GARCH method using the Clayton vine copula dependency structure to be best performing. This holds for both the standard GARCH and GJR GARCH model variations. The CVC dependency structure outperforms both other copula structures and the benchmark methods. These methods were considered overall best performing based on having the lowest error values and by being included in the superior set of the model confidence set test. The second best performing method was DCC. The historical method is always the worst performing method. These results hold regardless of the data set used.

Table 1: MAFE Full Matrix

	Method	Beta	SD	Mkt Cap
1A.	HISTORICAL	2765.56	2654.57	2721.89
	DCC NORMAL	2245.52	2206.18	2359.99
	CG-GN-MVN	2240.46	2215.96	2358.92
	CG-GN-SC-N	2237.12	2264.83	2388.75
	CG-GN-CVC-N	2122.29*	2011.19*	2179.90*
1B.	HISTORICAL	2765.56	2654.57	2721.89
	DCC GJR NORMAL	2139.89	2083.61	2356.00
	CG-GJR-GN-MVN	2144.95	2110.95	2355.51
	CG-GJR-GN-SC-N	2138.51	2150.68	2386.59
	CG-GJR-GN-CVC-N	2036.14*	1904.58*	2169.43*

This table shows the forecasting accuracy of the full matrix using various data sets. Error is computed using Eq. 13. The * indicates the method is in the superior set of models obtained by a MCS test with 95% confidence. There are two MCS tests performed. The MCS test is performed using only comparable methods. Comparable methods are grouped into sections 1A for standard GARCH based methods and 1B for GJR-GARCH based methods.

To gain a better understanding of the results, the full matrix is segmented and the variances and covariance values are analyzed separately on Tables 2 and 3 respectively. The results show that the variances are the same values for the DCC and copula-GARCH methods. This is expected as both of these methods are based on the same univariate GARCH generated standard deviation values. In other words, Eq. 8 and Eq. 11 both use the exact same D_{t+1} matrix and since the correlation matrix it is multiplying with will always have diagonal values equal to 1, this will result in equivalent variance values of the resulting covariance matrix. The covariance results show the previous overall best performing CVC methods remained the best performing methods.

Both the DCC and copula-GARCH methods are based on using the same GARCH forecasted values. The copula-GARCH method requires an additional $\hat{\mu}_{i,t+1}$ forecast, but the accuracy of its forecasted value does not have an impact on the forecast of a covariance matrix since the linear correlation formula is based on the covariance formula which works by first subtracting out each assets mean as shown in Eq. 5. That leaves the correlation structure as the main difference between the two methods. The covariance values of the CVC methods are more accurate than the covariance values of the DCC method because the correlation structure of the copula method is more accurate.

Table 2: MAFE Variances

	Method	**Beta**	**SD**	**Mkt Cap**
2A.	HISTORICAL	310.03	321.56	310.47
	DCC NORMAL	256.73	279.43	272.14
	CG-GN-MVN	256.73	279.44	272.14
	CG-GN-SC-N	256.73	279.44	272.14
	CG-GN-CVC-N	256.73	279.44	272.14
2B.	HISTORICAL	310.03	321.56	310.47
	DCC GJR NORMAL	243.11	272.97	273.25
	CG-GJR-GN-MVN	243.11	273.06	273.25
	CG-GJR-GN-SC-N	243.11	273.06	273.25
	CG-GJR-GN-CVC-N	243.11	273.06	273.25

This table analyzes the variance values of the 10 dimensional matrix forecasts. Error is computed using Eq. 13 when $i = j$. Comparable methods are grouped into sections 2A for standard GARCH based methods and 2B for GJR-GARCH based methods.

Table 3: MAFE Covariances

	Method	Beta	SD	Mkt Cap
3A.	HISTORICAL	1227.77	1166.50	1205.71
	DCC NORMAL	994.39	963.38	1043.92
	CG-GN-MVN	991.86	968.26	1043.39
	CG-GN-SC-N	990.19	992.69	1058.30
	CG-GN-CVC-N	932.78*	865.87*	953.88*
3B.	HISTORICAL	1227.77	1166.50	1205.71
	DCC GJR NORMAL	948.39	905.32	1041.38
	CG-GJR-GN-MVN	950.92	918.94	1041.13
	CG-GJR-GN-SC-N	947.70	938.81	1056.67
	CG-GJR-GN-CVC-N	896.52*	815.76*	948.09*

This table analyzes the covariance values of the 10 dimensional matrix forecasts. Error is computed using Eq. 13 when $i < j$. The * indicates the method is included in the superior set of models obtained by a MCS test with 95% confidence. There are two MCS tests performed. The MCS test is performed using only comparable methods. Comparable methods are grouped into sections 3A for standard GARCH based methods and 3B for GJR-GARCH based methods.

0.5.2 Application Results

Table 4 shows when the alternative loss function of MSTE (Eq. 18) is used, the CVC copula-GARCH methods continue to be able to outperform their comparable benchmark forecasting methods of DCC and rolling historical. In the appendix, table 5 shows the results when the MATE (Eq. 20) loss function is used. The results held for both loss functions.

Table 4: Equal Weighting Application Results MSTE

	Method	Beta	SD	Mkt Cap
4A.	HISTORICAL	12.33	11.93	11.10
	DCC NORMAL	8.59	8.54	8.84
	CG-GN-CVC-N	8.13	7.83	8.07
4B.	HISTORICAL	12.33	11.93	11.10
	DCC GJR NORMAL	8.10	7.86	8.71
	CG-GJR-GN-CVC-N	7.65	7.13	7.87

This table shows the MSTE (Eq. 18) when equal portfolio weighting is applied to obtain portfolio standard deviation values. Comparable methods are grouped into sections 4A for standard GARCH based methods and 4B for GJR-GARCH based methods.

0.6 Conclusion

The main goal of this paper was to analyze the copula simulated returns method of creating a covariance matrix forecast by comparing its accuracy against traditionally used benchmark methods that could alternatively be used to forecast the matrix.

The results of this paper show that copulas add predictive power towards creating covariance matrix forecasts. For both the standard GARCH and GJR-GARCH forms of copula methods tested, the CVC dependency structure were shown to work well. The copula-GARCH method was able to out perform both DCC and rolling historical to become the best performing method tested. By studying both the variance and covariance values, it helped to understand why the copula-GARCH method performed so well. The results show it performs well because it has the most accurate covariance values due to the superior correlation structure of the copula model.

To show robustness of results, multiple data sets and multiple loss functions were tested, and the results held up. There was however a trade off to using multiple data sets, which was more assumptions had to be made to ensure the methods and results were comparable to each other. These assumptions were to use a simple GARCH(1,1) model without any use of an ARMA model and to use normally distributed marginal distributions for the copula model. While this paper made simplifying assumptions, eliminating those assumptions could potentially increase the predictive ability of the copula-GARCH method. Perhaps the results obtained by this paper could be improved further by using a more general regular vine copula structure, fitting an additional ARMA model, using non normal margins, or by using a more sophisticated form of GARCH model.

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0.7 Appendix

Table 5: Equal Weighting Application Results MATE

	Method	**Beta**	**SD**	**Mkt Cap**
5A.	HISTORICAL	2.73	2.68	2.81
	DCC NORMAL	2.20	2.21	2.41
	CG-GN-CVC-N	2.08	2.01	2.25
5B.	HISTORICAL	2.73	2.68	2.81
	DCC GJR NORMAL	2.12	2.10	2.42
	CG-GJR-GN-CVC-N	1.99	1.90	2.25

This table shows the MATE (Eq. 20) when equal portfolio weighting is applied to obtain portfolio standard deviation values. Comparable methods are grouped into sections 5A for standard GARCH based methods and 5B for GJR-GARCH based methods.

Table 6: MAFE Full Matrix, Student T

	Method	**Beta**	**SD**	**Mkt-Cap**
6A.	HISTORICAL	2765.56	2654.57	2721.89
	DCC STUDENT T	2175.55	2122.25	2243.24
	CG-GT-MVT	2162.12	2115.44	2241.39
	CG-GT-SC-T	2110.39	2114.61	2197.10
	CG-GT-CVC-T	1723.76*	1682.09*	1719.66*
6B.	HISTORICAL	2765.56	2654.57	2721.89
	DCC GJR STUDENT T	2186.82	2083.02	2355.11
	CG-GJR-GT-MVT	2180.52	2093.40	2350.41
	CG-GJR-GT-SC-T	2125.64	2085.61	2304.68
	CG-GJR-GT-CVC-T	1782.17*	1729.67*	1806.84*

This table shows the forecasting accuracy of the full matrix on various data sets. Error is computed using Eq. 13. The * indicates the method is in the superior set of models obtained by a MCS test with 95% confidence. There are two MCS tests performed. The MCS test is performed using only comparable methods. Comparable methods are grouped into sections 6A for standard GARCH based methods and 6B for GJR-GARCH based methods.

Table 7: MAFE Variances, Student T

	Method	Beta	SD	Mkt-Cap
7A.	HISTORICAL	310.03	321.56	310.47
	DCC STUDENT T	248.90	271.91	261.02
	CG-GT-MVT	248.90	271.89	261.02
	CG-GT-SC-T	248.90	271.89	261.02
	CG-GT-CVC-T	248.90	271.89	261.02
7B.	HISTORICAL	310.03	321.56	310.47
	DCC GJR STUDENT T	247.48	265.54	275.05
	CG-GJR-GT-MVT	247.48	265.57	275.05
	CG-GJR-GT-SC-T	247.48	265.57	275.05
	CG-GJR-GT-CVC-T	247.48	265.57	275.05

This table analyzes the variance values of the 10 dimensional matrix forecasts. Error is computed using Eq. 13 when $i = j$. Comparable methods are grouped into sections 7A for standard GARCH based methods and 7B for GJR-GARCH based methods.

Table 8: MAFE Covariances, Student T

	Method	Beta	SD	Mkt-Cap
8A.	HISTORICAL	1227.77	1166.50	1205.71
	DCC STUDENT T	963.33	925.17	991.11
	CG-GT-MVT	956.61	921.77	990.18
	CG-GT-SC-T	930.74	921.36	968.04
	CG-GT-CVC-T	737.43*	705.10*	729.32*
8B.	HISTORICAL	1227.77	1166.50	1205.71
	DCC GJR STUDENT T	969.67	908.74	1040.03
	CG-GJR-GT-MVT	966.52	913.92	1037.68
	CG-GJR-GT-SC-T	939.08	910.02	1014.81
	CG-GJR-GT-CVC-T	767.34*	732.05*	765.89*

This table analyzes the covariance values of the 10 dimensional matrix forecasts. Error is computed using Eq. 13 when $i < j$. The * indicates the method is included in the superior set of models obtained by a MCS test with 95% confidence. There are two MCS tests performed. The MCS test is performed using only comparable methods. Comparable methods are grouped into sections 8A for standard GARCH based methods and 8B for GJR-GARCH based methods.

Table 9: Equal Weighting Application Results MSTE, Student T

	Method	Beta	SD	Mkt-Cap
9A.	HISTORICAL	12.33	11.93	11.10
	DCC STUDENT T	8.17	8.03	8.04
	CG-GT-CVC-T	6.35	6.41	5.61
9B.	HISTORICAL	12.33	11.93	11.10
	DCC GJR STUDENT T	8.33	7.85	8.70
	CG-GJR-GT-CVC-T	6.56	6.49	6.00

This table shows the MSTE (Eq. 18) when equal portfolio weighting is applied to obtain portfolio standard deviation values. Comparable methods are grouped into sections 9A for standard GARCH based methods and 9B for GJR-GARCH based methods.

Table 10: Equal Weighting Application Results MATE, Student T

	Method	**Beta**	**SD**	**Mkt-Cap**
10A.	HISTORICAL	2.73	2.68	2.81
	DCC STUDENT T	2.12	2.11	2.28
	CG-GT-CVC-T	1.50	1.53	1.70
10B.	HISTORICAL	2.73	2.68	2.81
	DCC GJR STUDENT T	2.10	2.04	2.35
	CG-GJR-GT-CVC-T	1.56	1.55	1.76

This table shows the MATE (Eq. 20) when equal portfolio weighting is applied to obtain portfolio standard deviation values. Comparable methods are grouped into sections 10A for standard GARCH based methods and 10B for GJR-GARCH based methods.