Analysis of distributional Riemann curvature tensor in any dimension

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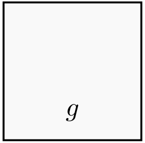
Riemannian manifolds and Regge metric

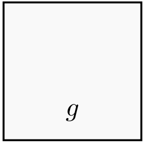
Riemannian manifolds

Riemannian manifold (Ω, g), $\Omega \subset \mathbb{R}^N$, g metric tensor



Riemannian manifolds

Riemannian manifold (Ω, g), $\Omega \subset \mathbb{R}^N$, g metric tensor



Levi-Civita connection ∇

$\nabla_X g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$

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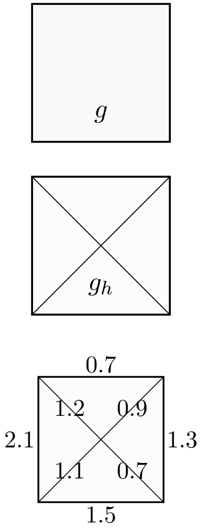
- Approximation of g on a triangulation
- Compute lengths and angles
- Regge calculus and finite elements



Riemannian manifolds

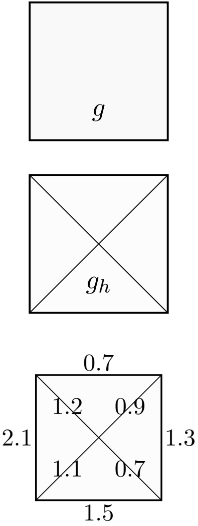
Riemannian manifold (Ω, g), $\Omega \subset \mathbb{R}^N$, g metric tensor

Levi-Civita connection ∇

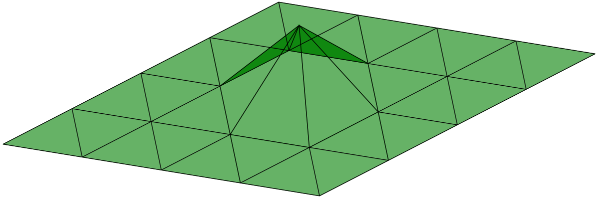
$\nabla_X g(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$

- Approximation of g on a triangulation
- Compute lengths and angles
- Regge calculus and finite elements
- How to compute curvature? Convergence?

$$\|\mathcal{R}(g_h) - \mathcal{R}(g)\|_? \leq \mathcal{O}(h^?)$$

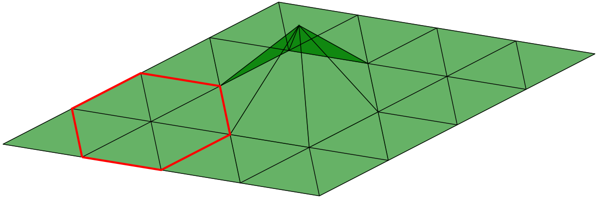


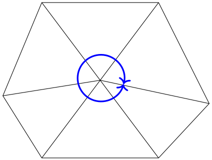
Regge calculus and Regge metrics



REGGE: General relativity without coordinates, *Il Nuovo Cimento* (1955-1965), 19 (1961).

Regge calculus and Regge metrics





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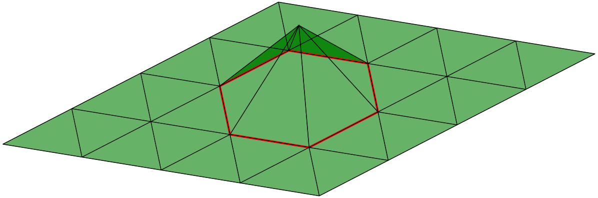

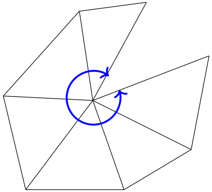

- angle defect

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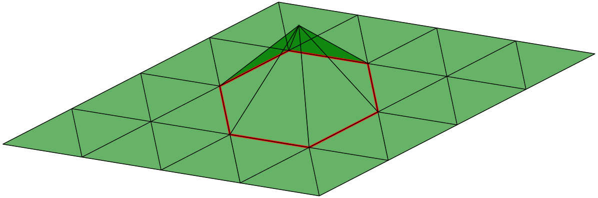
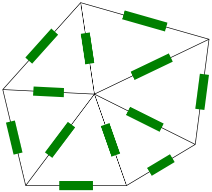
Regge calculus and Regge metrics



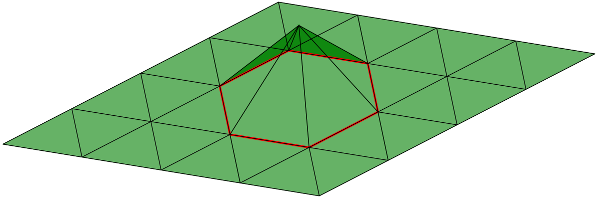


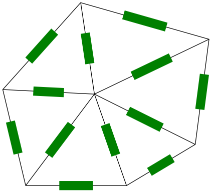
- metric tensor

REGGE: General relativity without coordinates, *Il Nuovo Cimento* (1955-1965), 19 (1961).

SORKIN: Time-evolution problem in Regge calculus, *Phys. Rev. D 12* (1975).

Regge calculus and Regge metrics





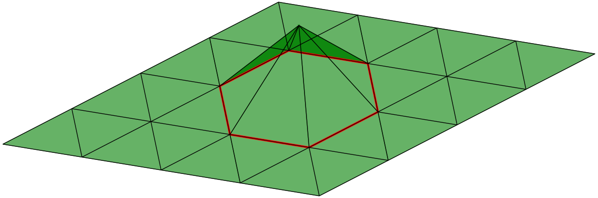
- metric tensor (tangential-tangential continuous)

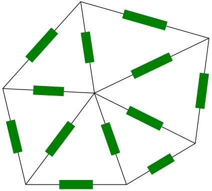
$$\mathrm{Reg}_h^0 = \{\varepsilon \in \mathcal{P}^0(\mathscr{T}, \mathbb{R}^{d\times d}_{\mathrm{sym}}) \mid [\![t^\top \varepsilon\, t]\!]_E = 0 \text{ for all edges } E\}$$

$$H(\mathrm{curl}\,\mathrm{curl}) = \{\varepsilon \in L^2(\Omega, \mathbb{R}^{d\times d}_{\mathrm{sym}}) \mid \mathrm{curl}^\top \mathrm{curl}(\varepsilon) \in H^{-1}(\Omega, \mathbb{R}^{(2d-3)\times(2d-3)})\}$$

CHRISTIANSEN: On the linearization of Regge calculus, *Numerische Mathematik 119, 4 (2011).*

Regge calculus and Regge metrics





- metric tensor (tangential-tangential continuous)

$$\mathrm{Reg}_h^k = \{\varepsilon \in \mathcal{P}^k(\mathscr{T}, \mathbb{R}^{d\times d}_{\mathrm{sym}}) \,|\, [\![t^\top \varepsilon\, t]\!]_E = 0 \text{ for all edges } E\}$$
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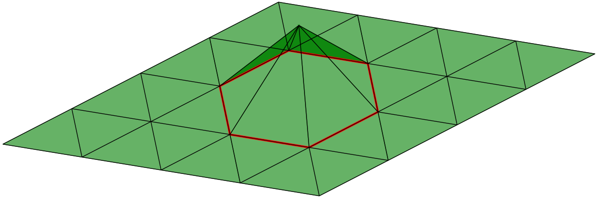
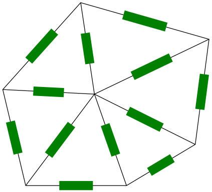
LI: Regge Finite Elements with Applications in Solid Mechanics and Relativity, *PhD thesis, University of Minnesota (2018).*





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Christiansen: On the linearization of Regge calculus, *Numerische Mathematik 119, 4 (2011).*

N.: Mixed Finite Element Methods for Nonlinear Continuum Mechanics and Shells, *PhD thesis, TU Wien (2021).*

Regge calculus and Regge metrics

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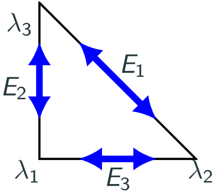
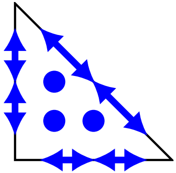
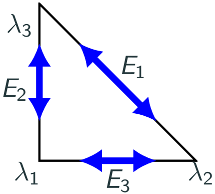


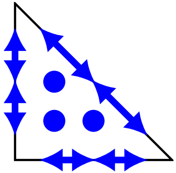


$$\varphi_{E_i} = \nabla\lambda_j \odot \nabla\lambda_k, \qquad t_j^\top \varphi_{E_i} t_j = c_i \delta_{ij}, \qquad \varphi_{T_i} = \lambda_i \nabla\lambda_j \odot \nabla\lambda_k$$

Regge calculus and Regge metrics

$$\mathrm{Reg}_h^k = \{\varepsilon \in \mathcal{P}^k(\mathscr{T}, \mathbb{R}^{d\times d}_{\mathrm{sym}}) \,|\, [\![t^\top \varepsilon\, t]\!]_E = 0 \text{ for all edges } E\}$$





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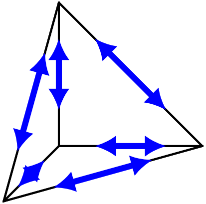
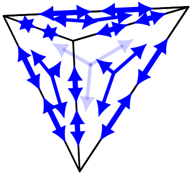




Definition distributional Riemann curvature tensor

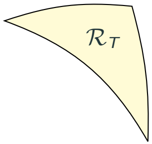
Riemann curvature tensor:

$$\mathcal{R}(X, Y, Z, W) = g(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, W)$$

$$\mathcal{R}_{ijkl} = \partial_i \Gamma_{jkl} - \partial_j \Gamma_{ikl} + \Gamma^p_{ik} \Gamma_{jpl} - \Gamma^p_{jk} \Gamma_{ipl}$$

Christoffel symbols: $\nabla_{\partial_j} \partial_k = \Gamma^l_{jk} \partial_l, \quad \{\partial_i\}_{i=1}^N$ coordinate frame

$$\Gamma^k_{ij}(g) = g^{kl} \frac{1}{2} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}) = g^{kl} \Gamma_{ijl}$$



Motivation Riemann curvature tensor I

Riemann curvature tensor:

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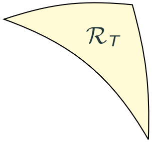
Contribution: Element-wise curvature $\mathcal{R}_T := \mathcal{R}(g_h)|_T$ for $T \in \mathscr{T}$

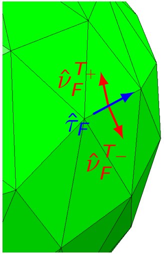


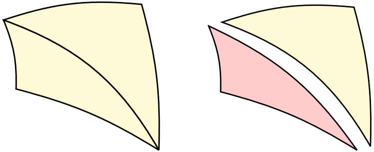
Second fundamental form: F hyper-surface with g-normal vector $\hat{\nu}$

$$\mathbb{I}_{\hat{\nu}}(X,Y) = -g(\nabla_X\hat{\nu},Y) = g(\hat{\nu},\nabla_X Y), \qquad X,Y\in\mathfrak{X}(F)$$

$$(\mathbb{I}_{\hat{\nu}})_{ij} = ({\delta_i}^l - \hat{\nu}_i\hat{\nu}^l)\,\Gamma_{lpk}\hat{\nu}^k\,({\delta^p}_j - \hat{\nu}^p\hat{\nu}_j), \qquad \hat{\nu}^i = \frac{1}{\|g^{-1}\nu\|}g^{ij}\nu_j$$

Metric g_h only tangential-tangential continuous $\quad\Rightarrow\quad \hat{\nu}_F^{T_+} \neq -\hat{\nu}_F^{T_-}, \quad F = T_+\cap T_-$





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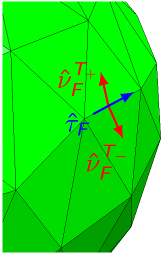
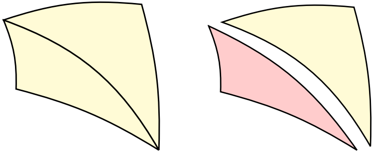
Contribution: Jump of second fundamental form $[\![\mathrm{I\!I}]\!]_F = \mathrm{I\!I}_{\hat{\nu}_F^{T_+}} + \mathrm{I\!I}_{\hat{\nu}_F^{T_-}}$ for $F \in \mathring{\mathscr{F}}$





Second fundamental form: F hyper-surface with g-normal vector $\hat{\nu}$

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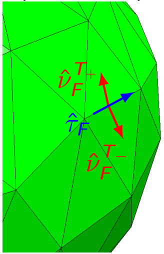
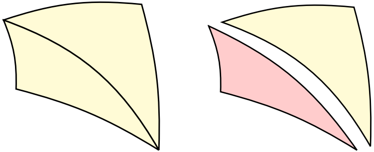
Contribution: Jump of second fundamental form $[\![\mathbb{I}]\!]_F = \mathbb{I}_{\hat{\nu}_F^{T_+}} + \mathbb{I}_{\hat{\nu}_F^{T_-}}$ for $F \in \mathring{\mathscr{F}}$

Motivation: Radial curvature equation

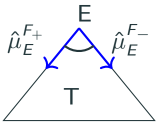
$$\mathcal{R}(X,\hat{\nu},\hat{\nu},Y) = (\nabla_{\hat{\nu}}\mathbb{I})(X,Y) - \mathbb{III}(X,Y), \quad X, Y \in \mathfrak{X}(F), \qquad \mathbb{III}(X,Y) = \langle \nabla_X\hat{\nu}, \nabla_Y\hat{\nu}\rangle$$



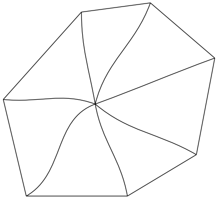


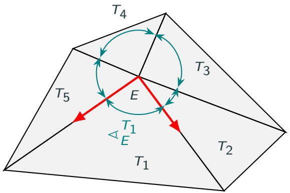
Angle defect:
At co-dimension 2 simplex E (Vertex in 2D, edge in 3D): 2-dimensional g-orthogonal plane

$$\Theta_E = 2\pi - \sum_{T \supset E} \arccos(g|_T(\hat{\mu}_E^{F_+}, \hat{\mu}_E^{F_-}))$$



Like classical angle defect for 2D manifolds





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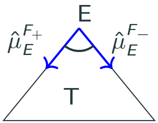
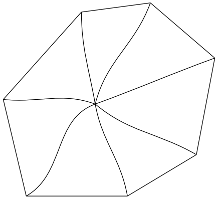
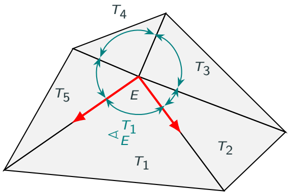


Like classical angle defect for 2D manifolds

Contribution: Θ_E for $E \in \mathring{\mathscr{E}}$





Distributional (densitized) Riemann curvature tensor

Test space:

$$\mathcal{A}(\mathscr{T}) = \{A \in T^4_0(\mathscr{T}) \,|\, A(X,Y,Z,W) = -A(Y,X,Z,W) = -A(X,Y,W,Z) = A(Z,W,X,Y),$$
$$A(\cdot,\hat{\nu},\hat{\nu},\cdot) \text{ is single-valued for all } F \in \mathring{\mathscr{F}},\; A(\hat{\mu},\hat{\nu},\hat{\nu},\hat{\mu}) \text{ is single-valued for all } E \in \mathring{\mathscr{E}}\}$$

$$\mathring{\mathcal{A}}(\mathscr{T}) = \{A \in \mathcal{A}(\mathscr{T}) \,:\, A(\cdot,\hat{\nu},\hat{\nu},\cdot) \text{ vanishes on all } F \in \mathscr{F}_\partial\}$$

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Distributional densitized Riemann curvature tensor

$$\widetilde{\mathcal{R}\omega}(A) = \sum_{T\in\mathscr{T}}\int_T \langle \mathcal{R}_T, A\rangle\,\omega_T + 4\sum_{F\in\mathring{\mathscr{F}}}\int_F \langle [\![\mathrm{II}]\!], A_{\cdot\hat\nu\hat\nu\cdot}\rangle\,\omega_F + 4\sum_{E\in\mathring{\mathscr{E}}}\int_E \Theta_E\, A_{\hat\mu\hat\nu\hat\nu\hat\mu}\,\omega_E, \quad A\in\mathring{\mathcal{A}}(\mathscr{T})$$

Gopalakrishnan, N., Schöberl, Wardetzky: Analysis of distributional Riemann curvature tensor in any dimension, *arXiv:2311.01603*.

Specialization to distributional Gauss curvature

Gauss curvature

$$K = \frac{\mathcal{R}(X,Y,Y,X)}{\|X\|_g\|Y\|_g - g(X,Y)^2} = \frac{\mathcal{R}_{1221}}{\det g}$$

Geodesic curvature

$$\kappa_{\hat{\nu}} = g(\hat{\nu}, \nabla_{\hat{\tau}}\hat{\tau}) = \mathbb{I}_{\hat{\nu}}(\hat{\tau},\hat{\tau})$$

Define test function $A(X,Y,Z,W) = -v\,\omega(X,Y)\omega(Z,W)$, $\quad v \in \mathring{\mathcal{V}} = \{u \in C^0(\Omega) \,|\, u|_{\partial\Omega} = 0\}$

Gauss curvature

$$K = \frac{\mathcal{R}(X,Y,Y,X)}{\|X\|_g\|Y\|_g - g(X,Y)^2} = \frac{\mathcal{R}_{1221}}{\det g}$$

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Define test function $A(X,Y,Z,W) = -v\,\omega(X,Y)\omega(Z,W)$, $\quad v \in \mathring{\mathcal{V}} = \{u \in C^0(\Omega) \,|\, u|_{\partial\Omega} = 0\}$

Distributional densitized Gauss curvature

$$\widetilde{K\omega}(v) = \frac{1}{4}\widetilde{\mathcal{R}\omega}(A) = \sum_{T\in\mathscr{T}}\int_T K_T\, v\,\omega_T + \sum_{F\in\mathring{\mathscr{F}}}\int_F [\![\kappa]\!]\, v\,\omega_F + \sum_{E\in\mathring{\mathscr{E}}}\Theta_E\, v(E).$$

Berchenko-Kogan, Gawlik: Finite element approximation of the Levi-Civita connection and its curvature in two dimensions, *Found Comput Math* (2022).

Gopalakrishnan, N., Schöberl, Wardetzky: Analysis of curvature approximations via covariant curl and incompatibility for Regge metrics, *SMAI J. Comput. Math.* (2023).

Specialization to distributional scalar curvature

Scalar curvature

$$S = g^{il} g^{jk} \mathcal{R}_{ijkl}$$

Mean curvature

$$H = \mathrm{tr}(\mathrm{I\!I}) = g^{ij} \mathrm{I\!I}_{ij}$$

- Kulkarni-Nomizu product $\owedge : T^2_0(\Omega) \times T^2_0(\Omega) \to T^4_0(\Omega)$

$$(h \owedge k)(X, Y, Z, W) = h(X, W)k(Y, Z) + h(Y, Z)k(X, W) - h(X, Z)k(Y, W) - h(Y, W)k(X, Z)$$

- Define test function $A = v\, g \owedge g, \qquad v \in \mathring{\mathcal{V}}$

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Distributional densitized scalar curvature

$$\widetilde{S\omega}(v) = \frac{1}{4}\widetilde{\mathcal{R}\omega}(A) = \sum_{T\in\mathscr{T}}\int_T S_T\, v\, \omega_T + 2\sum_{F\in\mathring{\mathscr{F}}}\int_F [\![H]\!]\, v\, \omega_F + 2\sum_{E\in\mathring{\mathscr{E}}}\int_E \Theta_E\, v\, \omega_E$$

Gawlik, N.: Finite element approximation of scalar curvature in arbitrary dimension, *arXiv:2301.02159*.

Specialization to distributional Ricci curvature tensor

Ricci tensor: $\mathrm{Ric}_{ij} = g^{ab}\mathcal{R}_{iabj}$

$A = g \owedge U$, $U \in \{V \in \mathcal{S}(\mathscr{T}) : V$ is *tt*- and *nn*-continuous, $V|_F$ and $V(\hat{\nu},\hat{\nu})$ vanish $\forall F \in \mathscr{F}_\partial\}$,

$$(g \owedge U)(X,\hat{\nu},\hat{\nu},Y) = U|_F(X,Y) + g|_F(X,Y)U(\hat{\nu},\hat{\nu})$$
$$(g \owedge U)(\hat{\mu},\hat{\nu},\hat{\nu},\hat{\mu}) = U(\hat{\mu},\hat{\mu}) + U(\hat{\nu},\hat{\nu}) = \mathrm{tr}(U) - \mathrm{tr}(U|_E).$$

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Distributional densitized Ricci curvature tensor

$$\widetilde{\mathrm{Ric}\,\omega}(U) = \frac{1}{4}\widetilde{\mathcal{R}\omega}(A) = \sum_{T\in\mathscr{T}}\int_T \langle \mathrm{Ric}_T, U\rangle\,\omega_T + \sum_{F\in\mathring{\mathscr{F}}}\int_F \langle [\![\mathrm{I\!I}]\!], U|_F + U(\hat\nu,\hat\nu)g|_F\rangle\,\omega_F$$
$$+ \sum_{E\in\mathring{\mathscr{E}}}\int_E \Theta_E\,(U(\hat\nu,\hat\nu) + U(\hat\mu,\hat\mu))\,\omega_E$$

Gopalakrishnan, N., Schöberl, Wardetzky: Analysis of distributional Riemann curvature tensor in any dimension, *arXiv:2311.01603*.

Error analysis

Integral representation of error

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- **Goal**: Find integral representation of H^{-2}-error

  parametrization $\tilde{g}(t) = g + t(g_h - g)$

$$\widetilde{\mathcal{R}\omega}(A)(g_h) - \widetilde{\mathcal{R}\omega}(A)(g) = \int_0^1 \frac{d}{dt}\widetilde{\mathcal{R}\omega}(A)(\tilde{g}(t))\,dt$$

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- **Problem**: test function $A = A_g$ depends on metric tensor

$$\mathcal{A}(\mathscr{T}) = \{A \in T^4_0(\mathscr{T})\,|\,A(X,Y,Z,W) = -A(Y,X,Z,W) = -A(X,Y,W,Z) = A(Z,W,X,Y),$$
$$A(\cdot,\hat{\nu},\hat{\nu},\cdot) \text{ is single-valued for all } F \in \mathring{\mathscr{F}},\ A(\hat{\mu},\hat{\nu},\hat{\nu},\hat{\mu}) \text{ is single-valued for all } E \in \mathring{\mathscr{E}}\}$$

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- **Solution**: Uhlenbeck trick

  transform to g-independent test functions U with $A_g = \mathbb{A}_g(U)$

Uhlenbeck trick

$$\mathcal{U}(\mathscr{T}) = \{U \in \Gamma(\bigwedge^{N-2}(\mathscr{T}) \odot \bigwedge^{N-2}(\mathscr{T})) \;:\; U|_F(X_1, \ldots, X_{N-2}, Y_1, \ldots, Y_{N-2}) \text{ is single valued for all } X_1, \ldots, X_{N-2}, Y_1, \ldots, Y_{N-2} \in \mathfrak{X}(F), F \in \mathring{\mathscr{F}}, U|_E(X_1, \ldots, X_{N-2}, Y_1, \ldots, Y_{N-2}) \text{ is single valued for all } X_1, \ldots, X_{N-2}, Y_1, \ldots, Y_{N-2} \in \mathfrak{X}(E),\ E \in \mathring{\mathscr{E}}\}$$

$U \in \mathcal{U}(\mathscr{T})$ is metric independent

Uhlenbeck trick

$$\mathcal{U}(\mathscr{T}) = \{U \in \Gamma(\textstyle\bigwedge^{N-2}(\mathscr{T}) \odot \bigwedge^{N-2}(\mathscr{T})) \; : \; U|_F(X_1, \dots, X_{N-2}, Y_1, \dots, Y_{N-2}) \text{ is single}$$

$$\text{valued for all } X_1, \dots, X_{N-2}, Y_1, \dots, Y_{N-2} \in \mathfrak{X}(F), F \in \mathring{\mathscr{F}},$$

$$U|_E(X_1, \dots, X_{N-2}, Y_1, \dots, Y_{N-2}) \text{ is single valued for all}$$

$$X_1, \dots, X_{N-2}, Y_1, \dots, Y_{N-2} \in \mathfrak{X}(E),\; E \in \mathring{\mathscr{E}}\}$$

$U \in \mathcal{U}(\mathscr{T})$ is metric independent

$$\mathbb{A} : \mathcal{U}(\mathscr{T}) \to T^4_0(\mathscr{T}), \qquad \mathbb{A}(U)^{ijkl} = \hat{\varepsilon}^{\alpha_1 \dots \alpha_{N-2} ij} \hat{\varepsilon}^{\beta_1 \dots \beta_{N-2} kl} U_{\alpha_1 \dots \alpha_{N-2} \beta_1 \dots \beta_{N-2}}$$

$$\hat{\varepsilon}^{\alpha_1 \dots \alpha_{N-2} ij} = \frac{1}{\sqrt{\det g}} \varepsilon^{\alpha_1 \dots \alpha_{N-2} ij}, \qquad \mathbb{A} = \mathbb{A}_g$$

Uhlenbeck trick

$$\mathcal{U}(\mathscr{T}) = \{U \in \Gamma(\bigwedge^{N-2}(\mathscr{T}) \odot \bigwedge^{N-2}(\mathscr{T})) \,:\, U|_F(X_1, \dots, X_{N-2}, Y_1, \dots, Y_{N-2}) \text{ is single valued for all } X_1, \dots, X_{N-2}, Y_1, \dots, Y_{N-2} \in \mathfrak{X}(F), F \in \mathring{\mathscr{F}}, U|_E(X_1, \dots, X_{N-2}, Y_1, \dots, Y_{N-2}) \text{ is single valued for all } X_1, \dots, X_{N-2}, Y_1, \dots, Y_{N-2} \in \mathfrak{X}(E),\ E \in \mathring{\mathscr{E}}\}$$

$U \in \mathcal{U}(\mathscr{T})$ is metric independent

$$\mathbb{A} : \mathcal{U}(\mathscr{T}) \to T_0^4(\mathscr{T}), \qquad \mathbb{A}(U)^{ijkl} = \hat{\varepsilon}^{\alpha_1 \dots \alpha_{N-2} ij} \hat{\varepsilon}^{\beta_1 \dots \beta_{N-2} kl} U_{\alpha_1 \dots \alpha_{N-2} \beta_1 \dots \beta_{N-2}}$$

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Lemma

The mapping $\mathbb{A}_g$ is bijective and there holds

$$\mathcal{A}(\mathscr{T}) = \{\mathbb{A}_g(U) \,:\, U \in \mathcal{U}(\mathscr{T})\}.$$

Lemma

Let $\sigma := \dot{g}(t)$, $A \in \mathcal{A}(\mathscr{T})$, $(SA)(X,Y,Z,W) = A(X,Z,Y,W)$ swaps second with third argument. There holds

$$\dot{A}(X,Y,Z,W) = -\operatorname{tr}(\sigma)A(X,Y,Z,W) + A(\sigma(X,\cdot)^\sharp,Y,Z,W) + A(X,\sigma(Y,\cdot)^\sharp,Z,W) + A(X,Y,\sigma(Z,\cdot)^\sharp,W) + A(X,Y,Z,\sigma(W,\cdot)^\sharp),$$

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Lemma

Let $\sigma := \dot{g}(t)$, $A \in \mathcal{A}(\mathscr{T})$, $(SA)(X,Y,Z,W) = A(X,Z,Y,W)$ swaps second with third argument. There holds

$$
\begin{aligned}
\dot{A}(X,Y,Z,W) = &-\mathrm{tr}(\sigma)A(X,Y,Z,W) + A(\sigma(X,\cdot)^\sharp,Y,Z,W) + A(X,\sigma(Y,\cdot)^\sharp,Z,W)\\
&+ A(X,Y,\sigma(Z,\cdot)^\sharp,W) + A(X,Y,Z,\sigma(W,\cdot)^\sharp),
\end{aligned}
$$

$$
\frac{d}{dt}\big(\langle \mathcal{R}, A\rangle\,\omega_T\big)|_{t=0} = \big(2\langle \nabla^2\sigma, S(A)\rangle + \langle \mathcal{R}(\sigma(\cdot,\cdot)^\sharp,\cdot,\cdot,\cdot), A\rangle - \frac{1}{2}\mathrm{tr}(\sigma)\langle \mathcal{R}, A\rangle\big)\omega_T,
$$

$$
\frac{d}{dt}\big(\langle [\![\mathrm{I\!I}]\!], A_{\cdot\hat{\nu}\hat{\nu}\cdot}\rangle\,\omega_F\big)|_{t=0} = \frac{1}{2}\langle [\![(\sigma(\hat{\nu},\hat{\nu}) - \mathrm{tr}(\sigma|_F))\mathrm{I\!I} + 2(\nabla_F\sigma)(\hat{\nu},\cdot)|_F - (\nabla_{\hat{\nu}}\sigma)|_F]\!], A_{\cdot\hat{\nu}\hat{\nu}\cdot}\rangle\,\omega_F,
$$

$$
\frac{d}{dt}\big(\Theta_E A_{\hat{\mu}\hat{\nu}\hat{\nu}\hat{\mu}}\,\omega_E\big)|_{t=0} = -\frac{1}{2}\Big(\sum_{F\supset E}[\![\sigma(\hat{\nu},\hat{\mu})]\!]_F^E + \mathrm{tr}(\sigma|_E)\Theta_E\Big)A_{\hat{\mu}\hat{\nu}\hat{\nu}\hat{\mu}}\,\omega_E.
$$

Proposition

Let $\sigma := \dot{g}$ and $A \in \mathring{\mathcal{A}}(\mathscr{T})$ with corresponding $U = \mathbb{A}^{-1}(A) \in \mathring{\mathcal{U}}(\mathscr{T})$. Then there holds

$$\frac{d}{dt}(\widetilde{\mathbb{A}^{-1}\mathcal{R}\omega})(g)(U)|_{t=0} = a_h(g;\sigma,U) + b_h(g;\sigma,U),$$

$$\begin{aligned} a_h(g;\sigma,U) = &\sum_{T\in\mathscr{T}}\int_T \left(\langle\mathcal{R}(\sigma(\cdot,\cdot)^\sharp,\cdot,\cdot,\cdot),\mathbb{A}(U)\rangle - \frac{1}{2}\mathrm{tr}(\sigma)\langle\mathcal{R},\mathbb{A}(U)\rangle\right)\omega_T\\ &-2\sum_{F\in\mathring{\mathscr{F}}}\int_F \left(\mathrm{tr}(\sigma|_F)\langle[\![\mathrm{I\!I}]\!],\mathbb{A}(U)_{\cdot\hat{\nu}\hat{\nu}\cdot}\rangle - [\![\mathrm{I\!I}]\!]:\sigma|_F:\mathbb{A}(U)_{\cdot\hat{\nu}\hat{\nu}\cdot}\right)\omega_F\\ &-2\sum_{E\in\mathring{\mathscr{E}}}\int_E \mathrm{tr}(\sigma|_E)\Theta_E\mathbb{A}(U)_{\hat{\mu}\hat{\nu}\hat{\nu}\hat{\mu}}\,\omega_E \end{aligned}$$

$[\![\mathrm{I\!I}]\!]:\sigma|_F:\mathbb{A}(U)_{\cdot\hat{\nu}\hat{\nu}\cdot} = [\![\mathrm{I\!I}]\!]_{ij}(\sigma|_F)^{jk}(\mathbb{A}(U)_{\cdot\hat{\nu}\hat{\nu}\cdot})_k{}^i.$

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Let $\sigma := \dot{g}$ and $A \in \mathring{\mathcal{A}}(\mathscr{T})$ with corresponding $U = \mathbb{A}^{-1}(A) \in \mathring{\mathcal{U}}(\mathscr{T})$. Then there holds

$$\frac{d}{dt}(\widetilde{\mathbb{A}^{-1}\mathcal{R}\omega})(g)(U)|_{t=0} = a_h(g;\sigma,U) + b_h(g;\sigma,U),$$

$$\begin{aligned} b_h(g;\sigma,U) = {}& 2\sum_{T\in\mathscr{T}}\int_T \langle \nabla^2\sigma, S(\mathbb{A}(U))\rangle\,\omega_T \\ &+ 2\sum_{F\in\mathring{\mathscr{F}}}\int_F \langle [\![\sigma(\hat{\nu},\hat{\nu})\mathbb{I} + (\nabla_F\sigma)(\hat{\nu},\cdot)|_F + \nabla_F(\sigma(\hat{\nu},\cdot))|_F - (\nabla_{\hat{\nu}}\sigma)|_F]\!], \mathbb{A}(U)_{\cdot\hat{\nu}\hat{\nu}\cdot}\rangle\,\omega_F \\ &- 2\sum_{E\in\mathring{\mathscr{E}}}\int_E \sum_{F\supset E}[\![\sigma(\hat{\nu},\hat{\mu})]\!]_F^E\,\mathbb{A}(U)_{\hat{\mu}\hat{\nu}\hat{\nu}\hat{\mu}}\,\omega_E. \end{aligned}$$

$b_h(g;\sigma,U) = 2\,\widetilde{\nabla^2}\sigma(S\mathbb{A}(U))$ is the distributional covariant incompatibility operator

$$\mathrm{inc}(\sigma)^{ij} = \mathrm{curl}(\mathrm{curl}(\sigma)^\top)^{ij} = \varepsilon^{ikl}\varepsilon^{jmn}\partial_k\partial_m\sigma_{ln}$$

Integral representation

- **Goal**: Estimate $\|(\widetilde{\mathbb{A}^{-1}\mathcal{R}\omega})(g_h) - (\mathbb{A}^{-1}\mathcal{R}\,\omega)(g)\|_{H^{-2}}$
- **Integral representation**: $\tilde{g}(t) = g + t(g_h - g)$, $\sigma = \frac{d}{dt}\tilde{g}(t) = g_h - g$

$$((\widetilde{\mathbb{A}^{-1}\mathcal{R}\omega})(g_h) - (\mathbb{A}^{-1}\mathcal{R}\,\omega)(g))(U) = \int_0^1 a_h(\tilde{g}(t);\sigma,U) + b_h(\tilde{g}(t);\sigma,U)\,dt$$

- **Proof strategy idea**: Estimate integrand

$$|a_h(\tilde{g}(t);\sigma,U)| \lesssim \|\sigma\|_{L^2}\|U\|_{H^2} = \|g_h - g\|_{L^2}\|U\|_{H^2}$$
$$|b_h(\tilde{g}(t);\sigma,U)| \lesssim \|g_h - g\|_{L^2}\|U\|_{H^2}$$

Extract convergence rate: $\|g_h - g\| \lesssim h^{k+1}$

Lemma

Let $\sigma \in \mathrm{Reg}(\mathscr{T})$, $\Psi \in \mathcal{A}(\mathscr{T})$ a smooth test function with compact support, and g a smooth metric tensor. Then the distributional covariant incompatibility operator $\widetilde{\nabla^2\sigma}(S\Psi)$ is

$$\widetilde{\nabla^2\sigma}(S\Psi) = \sum_{T\in\mathscr{T}}\Big[\int_T \langle \nabla^2\sigma, S\Psi\rangle\,\omega_T + \int_{\partial T}\langle(\nabla_F\sigma)(\cdot,\hat{\nu}) + \nabla_F(\sigma(\hat{\nu},\cdot)) - \nabla_{\hat{\nu}}\sigma$$

$$+\sigma(\hat{\nu},\hat{\nu})\mathrm{I\!I}_{\hat{\nu}}, (S\Psi)_{\cdot\hat{\nu}\hat{\nu}\cdot}\rangle\,\omega_{\partial T}\Big] - \sum_{E\in\mathring{\mathscr{E}}}\sum_{F\supset E}\int_E [\![\sigma(\hat{\nu},\hat{\mu})]\!]_F^E (S\Psi)_{\hat{\mu}\hat{\nu}\hat{\nu}\hat{\mu}}\,\omega_E.$$

Distributional covariant incompatibility operator

Lemma

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$$+\,\sigma(\hat{\nu},\hat{\nu})\mathrm{I\!I}_{\hat{\nu}}, (S\Psi)_{\cdot\hat{\nu}\hat{\nu}\cdot}\rangle\,\omega_{\partial T}\Big] - \sum_{E\in\mathring{\mathscr{E}}}\sum_{F\supset E}\int_E [\![\sigma(\hat{\nu},\hat{\mu})]\!]_F^E (S\Psi)_{\hat{\mu}\hat{\nu}\hat{\nu}\hat{\mu}}\,\omega_E.$$

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Definition (incompatibility operator)

Let U such that $U = \mathbb{A}^{-1}(A)$ with $A \in \mathcal{A}(\Omega)$. For a symmetric matrix $\sigma \in T_0^2(\Omega)$ we define the covariant incompatibility operator $\mathrm{inc}\,\sigma$ by

$$\langle \mathrm{inc}\,\sigma, U\rangle = -\langle \nabla^2\sigma, S(A)\rangle, \qquad \text{for all } A \in \mathcal{A}(\Omega).$$

Adjoint of distributional covariant incompatibility operator

Motivation:

$$|b_h(\tilde{g}(t);\sigma,U)| = \left|2\,\widetilde{\nabla^2\sigma}\big((S\mathbb{A})(U)\big)\right| \qquad \lesssim \|\sigma\|_{L^2}\|U\|_{H^2}$$

Motivation:

$$|b_h(\tilde{g}(t);\sigma,U)| = \left|2\,\widetilde{\nabla^2}\sigma\big((S\mathbb{A})(U)\big)\right| = \left|2\left(\widetilde{\mathrm{divdiv}}\big((S\mathbb{A})(U)\big)\right)(\sigma)\right| \lesssim \|\sigma\|_{L^2}\|U\|_{H^2}$$

Lemma

Let $\sigma \in \mathrm{Reg}(\mathscr{T})$, $A \in \mathring{\mathcal{A}}(\mathscr{T})$, and g a Regge metric. There holds $\widetilde{\nabla^2}\sigma(SA) = \widetilde{\mathrm{divdiv}}(SA)(\sigma)$ with

$$\begin{aligned}\widetilde{\mathrm{divdiv}}(SA)(\sigma) = \sum_{T\in\mathscr{T}} \Big[\int_T \langle\sigma,\mathrm{divdiv}(SA)\rangle\,\omega_T + \int_{\partial T} \big(\langle \sigma|_F, (\mathrm{div}(SA)+\mathrm{div}_F(SA))_{\hat\nu} + H\,(SA)_{\hat\nu\hat\nu}\rangle \\ - \sigma|_F : \mathrm{I\!I} : (SA)_{\hat\nu\hat\nu} - \langle \mathrm{I\!I}\otimes\sigma|_F, SA\rangle\big)\,\omega_{\partial T}\Big] - \sum_{E\in\mathring{\mathscr{E}}}\sum_{F\supset E}\int_E \langle \sigma|_E, [\![(SA)_{\hat\nu\hat\mu}]\!]_F^E\rangle\,\omega_E.\end{aligned}$$

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Proposition

Let $\tilde{g}(t) = g + (g_h - g)t$, $\sigma = g_h - g$, and $U \in H^2_0(\Omega, \mathcal{U})$. There holds for all $t \in [0,1]$

$$\left|a_h(\tilde{g}(t); \sigma, U)\right| \lesssim \left(1 + \max_{T\in\mathscr{T}_h} h_T^{-1} \|g_h - g\|_{W^{1,\infty}(T)} + \max_{T\in\mathscr{T}_h} h_T^{-2} \|g_h - g\|_{L^\infty(T)}\right) |||g_h - g|||_2 \|U\|_{H^2}.$$

Assume that $g_h = \mathcal{I}_h^k g$ is an optimal-order interpolant. Then for an integer $k \geq 1$

$$\left|a_h(\tilde{g}(t); \sigma, U)\right| \lesssim \left(\sum_{T\in\mathscr{T}_h} h_T^{p(k+1)} |g|^p_{W^{k+1,p}(T)}\right)^{1/p} \|U\|_{H^2} \approx h^{k+1} |g|_{W^{k+1,p}} \|U\|_{H^2}.$$

$$|||\sigma|||_2^2 = \|\sigma\|^2_{L^2} + h^2 \|\sigma\|^2_{H^1_h} + h^4 \|\sigma\|^2_{H^2_h}, \qquad \|\sigma\|^2_{H^1_h} = \sum_{T\in\mathscr{T}} \|\sigma\|^2_{H^1(T)}$$

Proposition

Let $\tilde{g}(t) = g + (g_h - g)t$, $\sigma = g_h - g$, and $U \in H^2_0(\Omega, \mathcal{U})$. There holds for all $t \in [0,1]$ for dimension $N \geq 3$

$$\left|b_h(\tilde{g}(t); \sigma, U)\right| \lesssim \left(1 + \max_{T \in \mathscr{T}_h} h_T^{-2} \|g_h - g\|_{L^\infty(T)} + \max_{T \in \mathscr{T}_h} h_T^{-1} \|g_h - g\|_{W^{1,\infty}(T)}\right) |||g_h - g|||_2 \|U\|_{H^2}$$

and for $N = 2$

$$\left|b_h(\tilde{g}(t); \sigma, U)\right| \lesssim \left(1 + \max_{T \in \mathscr{T}_h} h_T^{-1} \|g_h - g\|_{L^\infty(T)} + \|g_h - g\|_{W_h^{1,\infty}}\right) |||g_h - g|||_2 \|U\|_{H^2}.$$

Assume that $g_h = \mathcal{I}_h^k g$ is an optimal-order interpolant. Then for an integer $k \geq 1$ for $N \geq 3$ and $k \geq 0$ for $N = 2$

$$\left|b_h(\tilde{g}(t); \sigma, U)\right| \lesssim \left(\sum_{T \in \mathscr{T}_h} h_T^{p(k+1)} |g|^p_{W^{k+1,p}(T)}\right)^{1/p} \|U\|_{H^2} \approx h^{k+1} |g|_{W^{k+1,p}} \|U\|_{H^2}.$$

Theorem (Gopalakrishnan, N., Schöberl, Wardetzky 2023)

Assume $\{g_h\}_{h>0}$ is a family of Regge metrics on a shape regular family of triangulations $\{\mathscr{T}_h\}_{h>0}$ with $\lim_{h\to 0}\|g_h - g\|_{L^\infty} = 0$ and $\sup_{h>0}\max_{T\in\mathscr{T}_h}\|g_h\|_{W^{2,\infty}(T)} < \infty$. Then there exists $h_0 > 0$ such that for all $h \leq h_0$ in the two-dimensional case $N = 2$

$$\|(\widetilde{\mathbb{A}^{-1}\mathcal{R}\omega})(g_h) - (\mathbb{A}^{-1}\mathcal{R}\,\omega)(g)\|_{H^{-2}} \lesssim \Big(1 + \max_{T\in\mathscr{T}_h}\big(h_T^{-1}\|g - g_h\|_{L^\infty(T)}\big) + \|g - g_h\|_{W_h^{1,\infty}}\Big)\,|||g_h - g|||_2$$

and for higher dimensions $N \geq 3$

$$\begin{aligned}&\|(\widetilde{\mathbb{A}^{-1}\mathcal{R}\omega})(g_h) - (\mathbb{A}^{-1}\mathcal{R}\,\omega)(g)\|_{H^{-2}}\\ &\quad\lesssim \Big(1 + \max_{T\in\mathscr{T}_h}\big(h_T^{-2}\|g - g_h\|_{L^\infty(T)}\big) + \max_{T\in\mathscr{T}_h}\big(h_T^{-1}\|g - g_h\|_{W^{1,\infty}(T)}\big)\Big)\,|||g_h - g|||_2\,.\end{aligned}$$

$a_h(g;\sigma,U) = 0$ for $N = 2$

Theorem (Gopalakrishnan, N., Schöberl, Wardetzky 2023)

Let k be an integer with $k \geq 0$ for $N = 2$ and $k \geq 1$ for $N \geq 3$. Assume that $g_h = \mathcal{I}_h^k g \in \mathrm{Reg}_h^k$ is a family of optimal order interpolants on a shape regular family of triangulations $\{\mathscr{T}_h\}_{h>0}$ with $\sup_{h>0} \max_{T \in \mathscr{T}_h} \|g_h\|_{W^{2,\infty}(T)} < \infty$. Then there exists $h_0 > 0$ such that for all $h \leq h_0$ and $p \in [2, \infty]$ satisfying $p > \frac{m}{k+1}$

$$\|(\widetilde{\mathbb{A}^{-1}\mathcal{R}\omega})(g_h) - (\mathbb{A}^{-1}\mathcal{R}\,\omega)(g)\|_{H^{-2}} \lesssim \Big(\sum_{T \in \mathscr{T}_h} h_T^{p(k+1)} |g|^p_{W^{k+1,p}(T)} \Big)^{1/p} \approx h^{k+1} |g|_{W^{k+1,p}}$$

where m is the codimension index of $\mathcal{I}_h^k$.

$a_h(g; \sigma, U) = 0$ for $N = 2$

Specialization to 2D

Lemma

For $N = 2$ the distributional densitized Riemann curvature tensor simplifies to the distributional Gauss curvature

$$\widetilde{K\omega}(u) = \sum_{T\in\mathscr{T}} \int_T K_T\, u\, \omega_T + \sum_{F\in\mathring{\mathscr{F}}} \int_F [\![\kappa]\!]_F\, u\, \omega_F + \sum_{E\in\mathring{\mathscr{E}}} \Theta_E\, u(E), \qquad u \in \mathring{\mathcal{V}},$$

and there holds $\mathcal{U}(\mathscr{T}) = \mathring{\mathcal{V}}$ and

$$a_h(g;\sigma,u) = 0,$$

$$b_h(g;\sigma,u) = -2\sum_{T\in\mathscr{T}} \int_T \operatorname{inc}\sigma\, u\, \omega_T + 2\sum_{F\in\mathring{\mathscr{F}}} \int_F [\![\operatorname{curl}(\sigma)(\hat{\tau}) + \nabla_{\hat{\tau}}(\sigma(\hat{\nu},\hat{\tau}))]\!]_F\, u\, \omega_F - 2\sum_{E\in\mathring{\mathscr{E}}} \sum_{F\supset E} [\![\sigma(\hat{\nu},\hat{\mu})]\!]_F^E\, u(E).$$

Gopalakrishnan, N., Schöberl, Wardetzky: Analysis of curvature approximations via covariant curl and incompatibility for Regge metrics, *SMAI J. Comput. Math.* (2023).

Specialization to 3D

Curvature operator

$$\mathcal{Q}(X \wedge Y, W \wedge Z) := \mathcal{R}(X, Y, Z, W), \qquad X \wedge Y \in \bigwedge^2(\Omega)$$

$$\tilde{\mathcal{Q}}(\cdot,\cdot) := \mathcal{Q}(\star\cdot,\star\cdot) \in T^0_2(\Omega), \qquad \mathcal{U}(\mathscr{T}) = \mathrm{Reg}(\mathscr{T})$$

Lemma

$$\widetilde{\tilde{\mathcal{Q}}\omega}(U) = \sum_{T\in\mathscr{T}}\int_T \langle \tilde{\mathcal{Q}}_T, U\rangle\,\omega_T + \sum_{F\in\mathring{\mathscr{F}}}\int_F \langle [\![\mathrm{I\!I}]\!], (\hat{\nu}\otimes\hat{\nu})\times U\rangle\,\omega_F + \sum_{E\in\mathring{\mathscr{E}}}\int_E \Theta_E\, U(\hat{\tau},\hat{\tau})\,\omega_E$$

$$a_h(g;\sigma,U) = -2\sum_{T\in\mathscr{T}}\int_T \tilde{\mathcal{Q}} : \sigma : U\,\omega_T - 2\sum_{E\in\mathring{\mathscr{E}}}\int_E \Theta_E\,\sigma(\hat{\tau},\hat{\tau})\,U(\hat{\tau},\hat{\tau})\,\omega_E$$
$$-2\sum_{F\in\mathring{\mathscr{F}}}\int_F \big(\mathrm{tr}(\sigma|_F)\langle[\![\mathrm{I\!I}]\!],(\hat{\nu}\otimes\hat{\nu})\times U\rangle - [\![\mathrm{I\!I}]\!] : \sigma|_F : ((\hat{\nu}\otimes\hat{\nu})\times U)\big)\omega_F$$

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Curvature operator

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$$b_h(g;\sigma,U) = -2\sum_{T\in\mathscr{T}} \int_T \langle \mathrm{inc}\,\sigma, U\rangle\, \omega_T - 2\sum_{E\in\mathring{\mathscr{E}}} \int_E \sum_{F\supset E} [\![\sigma(\hat{\nu},\hat{\mu})]\!]^E_F U(\hat{\tau},\hat{\tau})\, \omega_E$$
$$+ 2\sum_{F\in\mathring{\mathscr{F}}} \int_F \langle [\![(\sigma(\hat{\nu},\hat{\nu})\mathrm{I\!I} + \nabla_F(\sigma(\hat{\nu},\cdot))) \times (\nu\otimes\nu) + Q(\mathrm{curl}\,\sigma)^\top \times \hat{\nu}]\!], U|_F\rangle\, \omega_F$$

Numerical examples

3D curvature

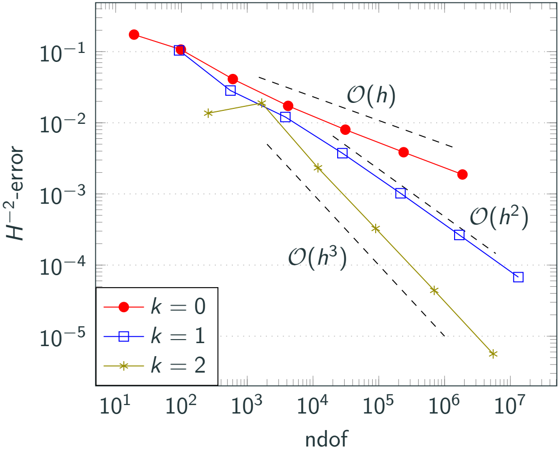
$$\Omega = (-1, 1)^3$$
$$\Phi(x, y, z) = (x, y, z, f(x, y, z)), \qquad f(x, y, z) = \tfrac{1}{2}(x^2 + y^2 + z^2) - \tfrac{1}{12}(x^4 + y^4 + z^4)$$
$$g = \nabla\Phi^\top \nabla\Phi$$

$$\tilde{\mathcal{Q}}_{xx} = \frac{9(z^2-1)(y^2-1)}{\det(g)\big(q(x)+q(y)+q(z)+9\big)},$$
$$\tilde{\mathcal{Q}}_{yy} = \frac{9(z^2-1)(x^2-1)}{\det(g)\big(q(x)+q(y)+q(z)+9\big)},$$
$$\tilde{\mathcal{Q}}_{zz} = \frac{9(x^2-1)(y^2-1)}{\det(g)\big(q(x)+q(y)+q(z)+9\big)},$$
$$\tilde{\mathcal{Q}}_{xy} = \tilde{\mathcal{Q}}_{xz} = \tilde{\mathcal{Q}}_{yz} = 0,$$

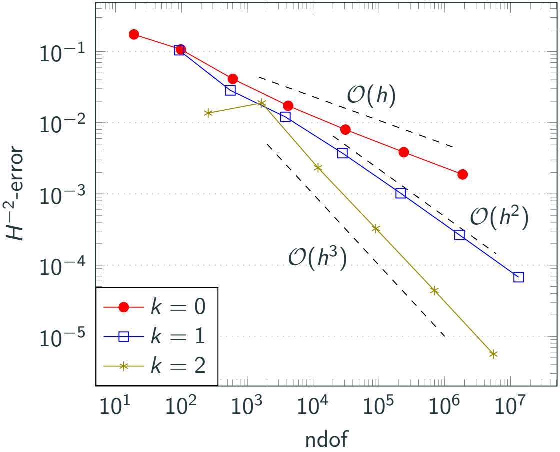
$$q(x) = x^2(x^2-3)^2$$

Perturb mesh with uniform random noise to avoid possible super-convergence!

- Confirms theory for $k > 1$
- For $k = 0$ linear convergence is observed?!



- Confirms theory for $k > 1$
- For $k = 0$ linear convergence is observed?!
- Test only parts where theory indicates no convergence

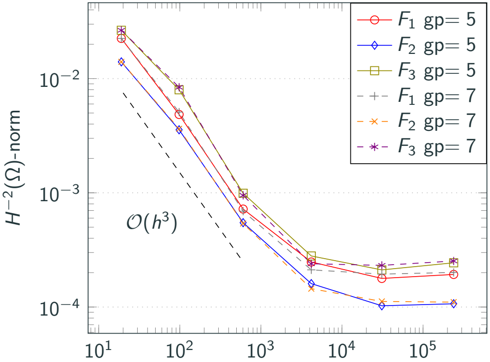


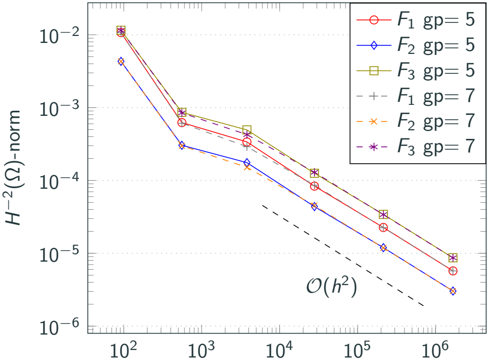
3D curvature

$$F_1 : U \mapsto \frac{1}{2}\int_0^1 \sum_{E\in\mathring{\mathscr{E}}} \int_E \sigma_{\hat\tau_{\tilde g(t)}\hat\tau_{\tilde g(t)}} \Theta_E(\tilde g(t)) U_{\hat\tau_{\tilde g(t)}\hat\tau_{\tilde g(t)}}\, \omega_E(\tilde g(t))\, dt$$

$$F_2 : U \mapsto -\frac{1}{2}\int_0^1 \sum_{E\in\mathring{\mathscr{E}}} \sum_{F\supset E} \int_E \sigma_{\hat\tau_{\tilde g(t)}\hat\tau_{\tilde g(t)}} [\![U_{\hat\nu_{\tilde g(t)}\hat\mu_{\tilde g(t)}}]\!]_F^E\, \omega_E(\tilde g(t))\, dt$$

$$F_3 = F_1 + F_2$$





Summary & Outlook

- Definition of densitized distributional Riemann curvature tensor
- Analysis in the H^{-2}-norm via integral representation and Uhlenbeck trick
- Includes Gauss, scalar, and Ricci curvature tensor

Summary & Outlook

- Definition of densitized distributional Riemann curvature tensor
- Analysis in the H^{-2}-norm via integral representation and Uhlenbeck trick
- Includes Gauss, scalar, and Ricci curvature tensor

- Define appropriate FE to compute L^2-representative and analyze in stronger norms
- Investigate PDEs involving curvature fields, e.g. numerical relativity

Literature

- Li: Regge Finite Elements with Applications in Solid Mechanics and Relativity, *PhD thesis, University of Minnesota* (2018).
- Berchenko-Kogan, Gawlik: Finite element approximation of the Levi-Civita connection and its curvature in two dimensions, *Found Comput Math* (2022).
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- Gopalakrishnan, N., Schöberl, Wardetzky: Analysis of distributional Riemann curvature tensor in any dimension, *arXiv:2311.01603*.

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Thank You for Your Attention!