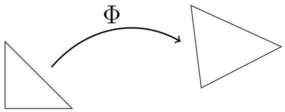
Surface PDEs, Plates and Shells

Michael Neunteufel, Joachim Schöberl

2nd NGSolve User Meeting, Göttingen, July 4-6, 2018

PDEs on Surfaces

PDEs on surfaces



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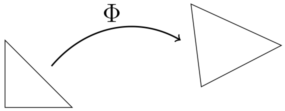

- $F = \nabla_{\hat{x}} \Phi \in \mathbb{R}^{2\times 2}$
- $J = \det(F)$

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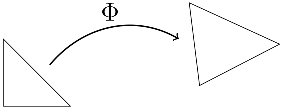

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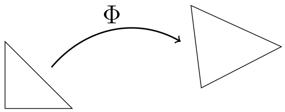

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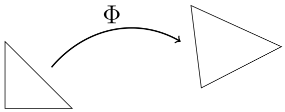

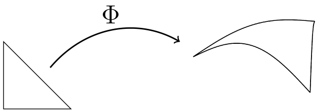

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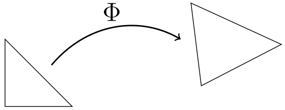

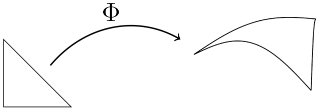

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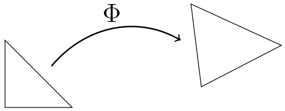

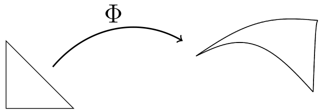

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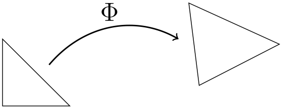

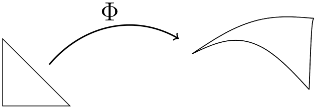

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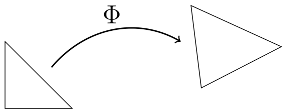

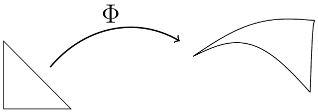

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$$\int_T f(x) g(x)\, dx = \int_{\hat{T}} f(\Phi(\hat{x})) g(\Phi(\hat{x}))\; J \; d\hat{x}$$

$$\int_T \nabla_\Gamma f(x) \nabla_\Gamma g(x)\, dx = \int_{\hat{T}} ({F^{\dagger}}^T \nabla_{\hat{x}} f)(\Phi(\hat{x}))({F^{\dagger}}^T \nabla_{\hat{x}} g)(\Phi(\hat{x}))\, J\, d\hat{x}$$

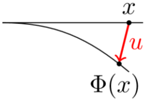
Plates and Shells

Continuum mechanics

Deformation $\Phi : \Omega \to \mathbb{R}^3$

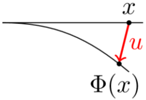
Continuum mechanics

Deformation	$\Phi : \Omega \to \mathbb{R}^3$
Displacement	$u := \Phi - id$



Continuum mechanics

Deformation	$\Phi : \Omega \to \mathbb{R}^3$
Displacement	$u := \Phi - id$
Deformation gradient	$F := \nabla\Phi$



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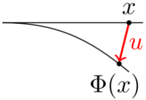

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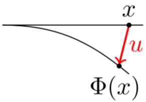
Cauchy-Green strain tensor	$C := F^T F$



$$\frac{||\Phi(x + \Delta x) - \Phi(x)||^2}{||\Delta x||^2} = \frac{\Delta x^T F^T F \Delta x}{||\Delta x||^2} + \mathcal{O}(||\Delta x||)$$

Continuum mechanics

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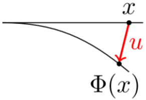
Green strain tensor	$E := \frac{1}{2}(C - I)$



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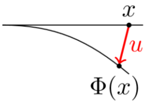
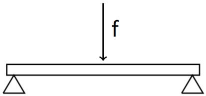
Linearized strain tensor	$\epsilon(u) := \frac{1}{2}(\nabla u^T + \nabla u)$



Hellan-Herrmann-Johnson method

- Discretization method for 4th order elliptic problems **[Comodi, 1989]**

$$\mathrm{div}(\,\mathrm{div}(\nabla^2 u)) = f$$



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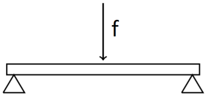

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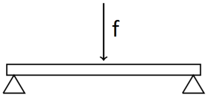

Variational problem

Find $u \in H_0^2(\Omega)$ s.t.

$$\int_\Omega \nabla^2 u : \nabla^2 v \, dx = \int_\Omega f \cdot v \, dx \qquad \forall v \in H_0^2(\Omega).$$

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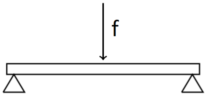


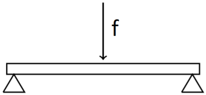
Energy minimization problem

Find $u \in H_0^2(\Omega)$ s.t.

$$\mathcal{W}(u) = \frac{1}{2}\int_\Omega \nabla^2 u : \nabla^2 u \, dx - \int_\Omega f \cdot u \, dx \to \min!$$

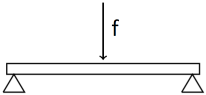
Hellan-Herrmann-Johnson method

$$\operatorname{div}(\operatorname{div}(\nabla^2 u)) = f$$



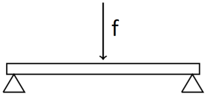
Hellan-Herrmann-Johnson method

$$\begin{aligned} \sigma &= \nabla^2 u, \\ \mathrm{div}(\,\mathrm{div}(\sigma)) &= f, \end{aligned}$$



Hellan-Herrmann-Johnson method

$$\begin{aligned} \sigma &= \nabla^2 u, &&\Rightarrow u \in H^1(\Omega) \\ \operatorname{div}(\operatorname{div}(\sigma)) &= f, &&\Rightarrow \sigma \in H(\operatorname{divdiv}, \Omega) \end{aligned}$$



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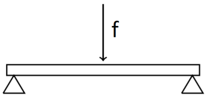

Hellan-Herrmann-Johnson

Find $u \in H^1(\Omega)$ and $\sigma \in H(\operatorname{divdiv}, \Omega)$ s.t.

$$\begin{aligned} -a(\sigma, \tau) + b(u, \tau) &= 0 && \forall \tau \in H(\operatorname{divdiv}, \Omega) \\ b(v, \tau) &= -\int_\Omega f \cdot v \, dx && \forall v \in H^1(\Omega), \end{aligned}$$

$$a(\sigma, \tau) := \int_\Omega \sigma : \tau \, dx$$
$$b(u, \tau) := \sum_{T \in \mathcal{T}} \int_T \nabla u \cdot \operatorname{div}(\tau) \, dx - \int_{\partial T} \nabla_t u \cdot \tau_n \, ds$$

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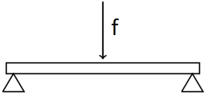

Hellan-Herrmann-Johnson (Saddle point)

Find $u \in H^1(\Omega)$ and $\sigma \in H(\operatorname{divdiv}, \Omega)$ s.t. the saddle point problem

$$\mathcal{W}(u,\sigma) = -\frac{1}{2}a(\sigma,\sigma) + b(u,\sigma) - \int_\Omega f \cdot u \, dx$$

is solved.

$$a(\sigma,\tau) := \int_\Omega \sigma : \tau \, dx$$
$$b(u,\tau) := \sum_{T\in\mathcal{T}} \int_T \nabla u \cdot \operatorname{div}(\tau) \, dx - \int_{\partial T} \nabla_t u \cdot \tau_n \, ds$$

The space H(divdiv)

$$M_h^k := \{\sigma \in [\Pi^k(\mathcal{T}_h)]^{d\times d}_{sym} \mid n^T \sigma n \text{ is continuous over elements}\}$$

A. Pechstein and J. Schöberl: The TDNNS method for Reissner-Mindlin plates, *J. Numer. Math. (2017) 137, pp. 713-740.*

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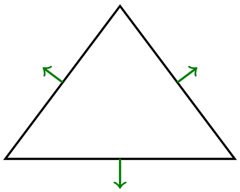

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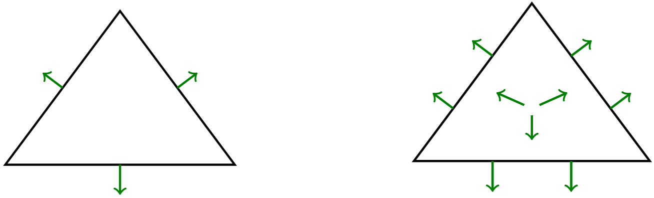

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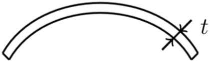
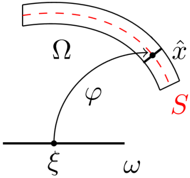
Thin-walled structures

- Model of reduced dimensions

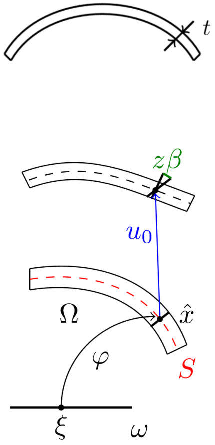
Thin-walled structures





- Model of reduced dimensions
- $\Omega = \left\{ \varphi(\xi) + z\hat{n}(\xi) : \xi \in \omega, z \in \left[-\frac{t}{2}, \frac{t}{2}\right] \right\}$

Thin-walled structures



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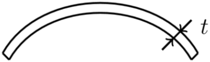
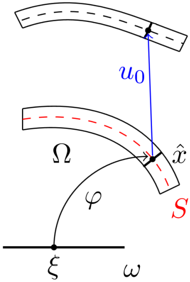
- $u(\hat{x} + z\hat{n}(\xi)) = u_0(\hat{x}) + z\,(n + \beta)\,(\hat{x})$

Thin-walled structures





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- $u(\hat{x} + z\hat{n}(\xi)) = u_0(\hat{x}) + z n(\hat{x})$

Shell energy (Kirchhoff-Love)

$$\mathcal{W}(\mathbf{u}) = \|E_{\tau\tau}(\mathbf{u})\|^2 + \frac{t^2}{2}\|\hat{\kappa}(\mathbf{u}) - \kappa_R\|^2$$

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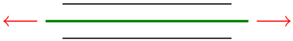

- Membrane energy

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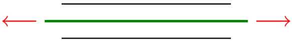
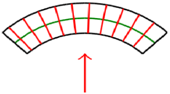
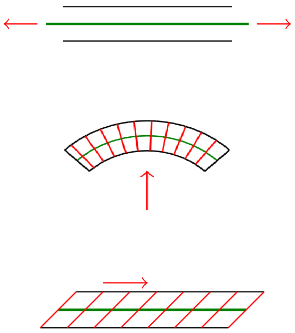




- Membrane energy
- Bending energy

Shell energy (Kirchhoff-Love)

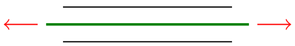
$$\mathcal{W}(\mathbf{u}) = \|E_{\tau\tau}(\mathbf{u})\|^2 + \frac{t^2}{2}\|\hat{\kappa}(\mathbf{u}) - \kappa_R\|^2$$

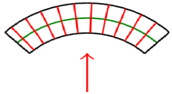


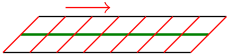
- Membrane energy
- Bending energy
- Shearing energy

Shell energy (Kirchhoff-Love)

$$\mathcal{W}(\mathbf{u},\sigma) = \|E_{\tau\tau}(\mathbf{u})\|^2 - \frac{1}{2t^2}a(\sigma,\sigma) + b(\mathbf{u},\sigma)$$







- Membrane energy
- Bending energy
- Shearing energy

Shell energy (Reissner-Mindlin)

$$\mathcal{W}(\mathbf{u}, \sigma) = \|E_{\tau\tau}(\mathbf{u})\|^2 - \frac{1}{2t^2} a(\sigma, \sigma) + b(\mathbf{u}, \sigma)$$

$$a(\sigma, \tau) := \int_S \sigma : \tau \, dx$$

$$b(\mathbf{u}, \tau) := \sum_{T \in \mathcal{T}} \int_T (n^T \nabla \mathbf{u}) \cdot \operatorname{div}(\tau) \, dx - \int_{\partial T} (n^T \nabla \mathbf{u})_t \cdot \tau_n \, ds$$

Shell energy (Reissner-Mindlin)

$$\mathcal{W}(\mathbf{u},\sigma,\beta) = \|E_{\tau\tau}(\mathbf{u})\|^2 - \frac{1}{2t^2}a(\sigma,\sigma) + b(\beta,\sigma) + c(\mathbf{u},\beta)$$

$$a(\sigma,\tau) := \int_S \sigma : \tau \, dx$$

$$b(\beta,\tau) := \sum_{T\in\mathcal{T}} \int_T \beta \cdot \operatorname{div}(\tau)\, dx - \int_{\partial T} \beta_t \cdot \tau_n \, ds$$

Shell energy (Reissner-Mindlin)

$$\mathcal{W}(\mathbf{u},\sigma,\beta) = \|E_{\tau\tau}(\mathbf{u})\|^2 - \frac{1}{2t^2}a(\sigma,\sigma) + b(\beta,\sigma) + c(\mathbf{u},\beta)$$

$$a(\sigma,\tau) := \int_S \sigma : \tau \, dx$$

$$b(\beta,\tau) := \sum_{T\in\mathcal{T}} \int_T \beta \cdot \operatorname{div}(\tau)\, dx - \int_{\partial T} \beta_t \cdot \tau_n \, ds$$

$$c(\mathbf{u},\beta) := \int_S (n^T\nabla\mathbf{u} - \beta)\cdot(n^T\nabla\mathbf{u} - \beta)\, dx = \|n^T\nabla\mathbf{u} - \beta\|^2_{L^2(S)}$$

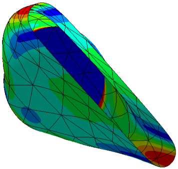
Shell energy (Reissner-Mindlin)

$$\mathcal{W}(\mathbf{u},\sigma,\beta) = \|E_{\tau\tau}(\mathbf{u})\|^2 - \frac{1}{2t^2}a(\sigma,\sigma) + b(\beta,\sigma) + c(\mathbf{u},\beta)$$

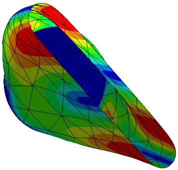
$$a(\sigma,\tau) := \int_S \sigma : \tau \, dx$$

$$b(\beta,\tau) := \sum_{T\in\mathcal{T}} \int_T \beta \cdot \operatorname{div}(\tau)\, dx - \int_{\partial T} \beta_t \cdot \tau_n \, ds$$

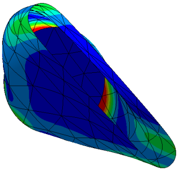
$$c(\mathbf{u},\beta) := \int_S (n^T\nabla\mathbf{u} - \beta)\cdot(n^T\nabla\mathbf{u} - \beta)\, dx = \|n^T\nabla\mathbf{u} - \beta\|^2_{L^2(S)}$$



Moments

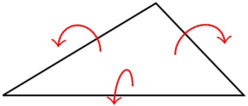


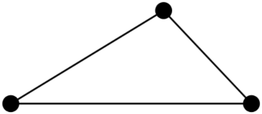
Rotation

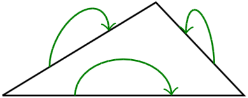


Shear stress

Finite element spaces

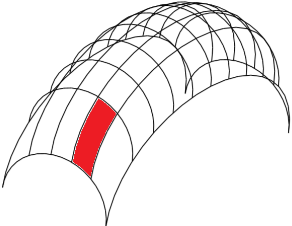






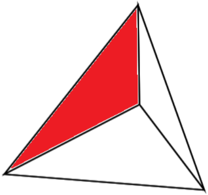
Finite Element Space M_{Surf}:

- 2D elements of M_h as face-elements



Finite Element Space V_h and B_h:

- Traces of 3D elements



Nonlinear Shells

Example