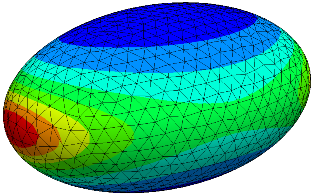
Distributional curvature approximations with applications to shells

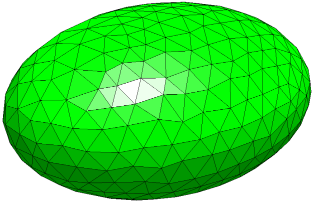
Michael Neunteufel (TU Wien)
Jay Gopalakrishnan (Portland State University)
Joachim Schöberl (TU Wien)
Max Wardetzky (University of Göttingen)

ENUMATH, Lisbon, Sep 4th, 2023

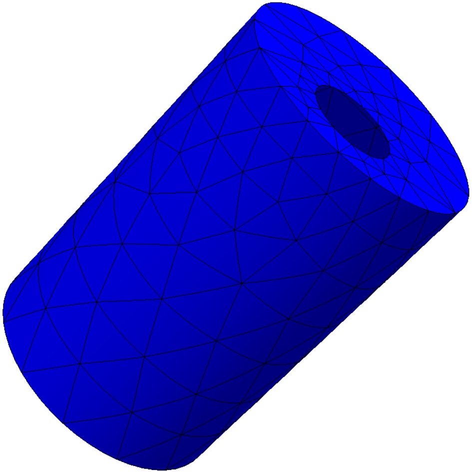
Motivation

Approximate extrinsic/intrinsic curvature of non-smooth surfaces



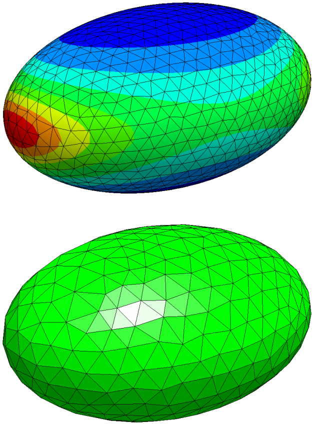


Application to shells

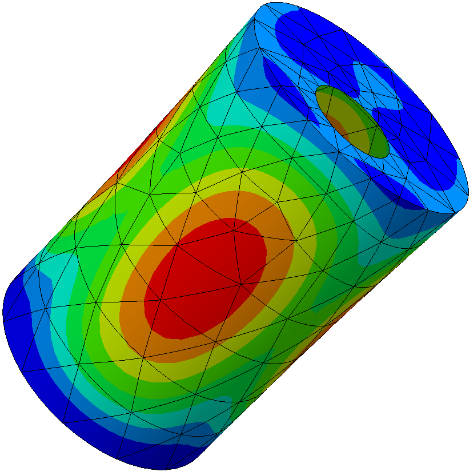


Motivation

Approximate extrinsic/intrinsic curvature of non-smooth surfaces



Application to shells

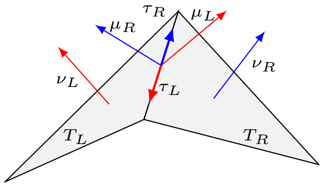


Contents

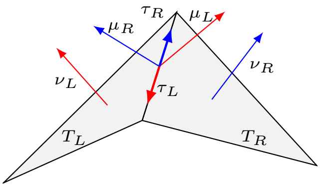
Distributional extrinsic and intrinsic curvature

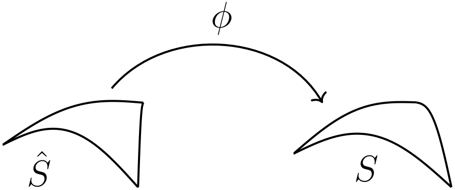
Differential geometry

- Normal vector ν
  Tangent vector τ
  Element normal vector $\mu = \nu \times \tau$

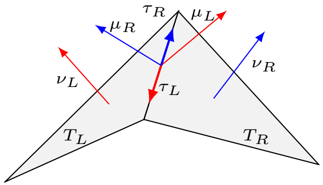


- Normal vector $\hat{\nu}$
  Tangent vector $\hat{\tau}$
  Element normal vector $\hat{\mu} = \hat{\nu} \times \hat{\tau}$

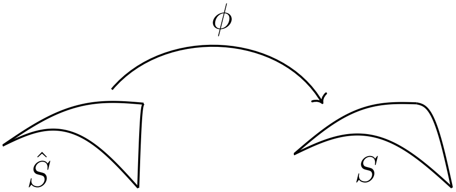




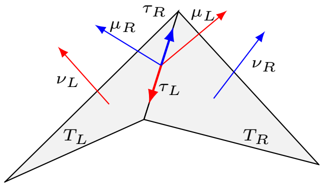
- Normal vector $\hat{\nu}$
  Tangent vector $\hat{\tau}$
  Element normal vector $\hat{\mu} = \hat{\nu} \times \hat{\tau}$



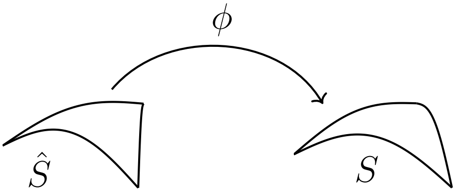
- $\boldsymbol{F} = \nabla_{\hat{\tau}}\phi$, $J = \sqrt{\det(\boldsymbol{F}^\top \boldsymbol{F})}$



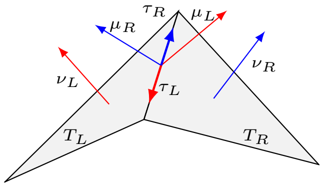
- Normal vector $\hat{\nu}$
  Tangent vector $\hat{\tau}$
  Element normal vector $\hat{\mu} = \hat{\nu} \times \hat{\tau}$



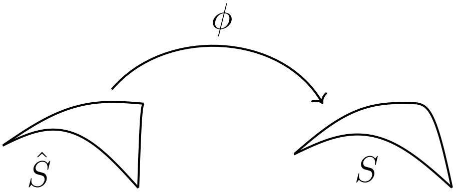
- $\boldsymbol{F} = \nabla_{\hat{\tau}}\phi$, $J = \|\operatorname{cof}(\boldsymbol{F})\|_F$



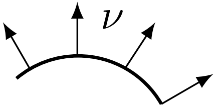
- Normal vector $\hat{\nu}$
  Tangent vector $\hat{\tau}$
  Element normal vector $\hat{\mu} = \hat{\nu} \times \hat{\tau}$



- $\boldsymbol{F} = \nabla_{\hat{\tau}} \phi$, $J = \|\operatorname{cof}(\boldsymbol{F})\|_F$
- $\nu \circ \phi = \frac{1}{J} \operatorname{cof}(\boldsymbol{F}) \hat{\nu}$
  $\tau \circ \phi = \frac{1}{J_B} \boldsymbol{F} \hat{\tau}$
  $\mu \circ \phi = \nu \circ \phi \times \tau \circ \phi$

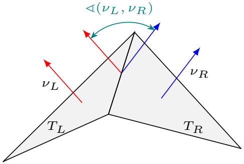


- Change of normal vector measures curvature $\nabla\nu$



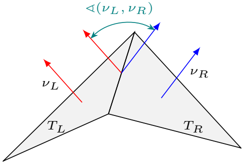
Distributional extrinsic curvature

- Change of normal vector measures curvature $\nabla\nu$
- How to define $\nabla\nu$ for discrete surface?



GRINSPUN, GINGOLD, REISMAN AND ZORIN: Computing discrete shape operators on general meshes, *Computer Graphics Forum 25*, 3 (2006), pp. 547–556.

Distributional extrinsic curvature



- Change of normal vector measures curvature $\nabla\nu$
- How to define $\nabla\nu$ for discrete surface?

  - **Distributional Weingarten tensor**

$$\langle \nabla\nu, \boldsymbol{\sigma}\rangle_{\mathscr{T}} = \sum_{T\in\mathscr{T}_h}\int_T \nabla\nu|_T : \boldsymbol{\sigma}\, dx + \sum_{E\in\mathcal{E}_h}\int_E \sphericalangle(\nu_L, \nu_R)\boldsymbol{\sigma}_{\mu\mu}\, ds$$

  - Measure jump of normal vector
  - Test function $\boldsymbol{\sigma}$ symmetric, normal-normal continuous $\Rightarrow$ Hellan–Herrmann–Johnson finite elements

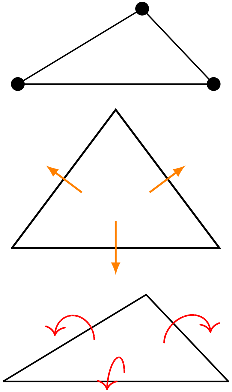
N., Schöberl, Sturm, Numerical shape optimization of Canham-Helfrich-Evans bending energy, *J. Comput. Phys.* (2023).

Hellan–Herrmann–Johnson finite elements

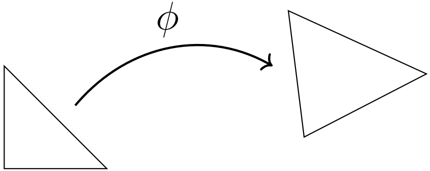
$$H^1(\Omega) = \{u \in L^2(\Omega) \,|\, \nabla u \in [L^2(\Omega)]^d\}$$
$$\mathcal{L}_h^k(\mathscr{T}_h) = \mathcal{P}^k(\mathscr{T}_h) \cap C(\Omega)$$

$$H(\mathrm{div}, \Omega) = \{\sigma \in [L^2(\Omega)]^d \,|\, \mathrm{div}\sigma \in L^2(\Omega)\}$$
$$BDM^k = \{\sigma \in [\mathcal{P}^k(\mathscr{T}_h)]^d \,|\, [\![\sigma_n]\!]_F = 0\}$$

$$H(\mathrm{divdiv}, \Omega) = \{\sigma \in [L^2(\Omega)]^{d\times d}_{\mathrm{sym}} \,|\, \mathrm{div}\mathbf{div}\sigma \in H^{-1}(\Omega)\}$$
$$M_h^k(\mathscr{T}_h) = \{\sigma \in [\mathcal{P}^k(\mathscr{T}_h)]^{d\times d}_{\mathrm{sym}} \,|\, [\![n^T \sigma n]\!]_F = 0\}$$



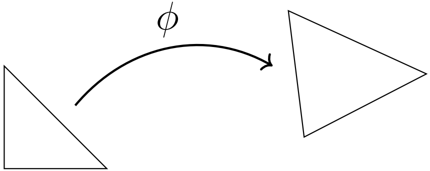
A. Pechstein and J. Schöberl: The TDNNS method for Reissner-Mindlin plates, *J. Numer. Math. (2017) 137, pp. 713-740.*


ϕ

Mapping to the surface

- Piola transformation (H(div)-conforming RT/BDM elements)

$$v \circ \phi = \frac{1}{J}\boldsymbol{F}\hat{v}, \quad \hat{v} \in BDM/RT(\hat{T}) \qquad\qquad \boldsymbol{F} = \nabla_{\hat{x}}\phi, J = \det(\boldsymbol{F})$$

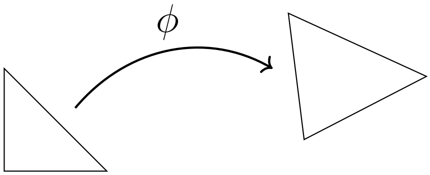


- Piola transformation (H(div)-conforming RT/BDM elements)

$$v \circ \phi = \frac{1}{J}\boldsymbol{F}\hat{v}, \quad \hat{v} \in BDM/RT(\hat{T}) \qquad\qquad \boldsymbol{F} = \nabla_{\hat{x}}\phi, J = \det(\boldsymbol{F})$$

- Double-Piola transformation (Hellan–Herrmann–Johnson elements)

$$\boldsymbol{\sigma} \circ \phi = \frac{1}{J^2}\boldsymbol{F}\hat{\boldsymbol{\sigma}}\boldsymbol{F}^T, \quad \hat{\boldsymbol{\sigma}} \in M_h^k(\hat{T})$$



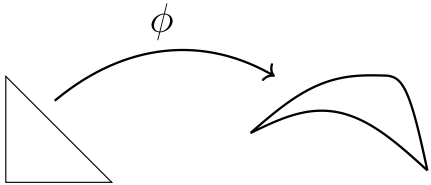
Mapping to the surface

- Piola transformation (H(div)-conforming RT/BDM elements)

$$v \circ \phi = \frac{1}{J}\boldsymbol{F}\hat{v}, \quad \hat{v} \in BDM/RT(\hat{T}) \qquad \boldsymbol{F} = \nabla_{\hat{x}}\phi, J = \sqrt{\det(\boldsymbol{F}^T\boldsymbol{F})} = \|\operatorname{cof}(\boldsymbol{F})\|_F$$

- Double-Piola transformation (Hellan–Herrmann–Johnson elements)

$$\boldsymbol{\sigma} \circ \phi = \frac{1}{J^2}\boldsymbol{F}\hat{\boldsymbol{\sigma}}\boldsymbol{F}^T, \quad \hat{\boldsymbol{\sigma}} \in M_h^k(\hat{T})$$



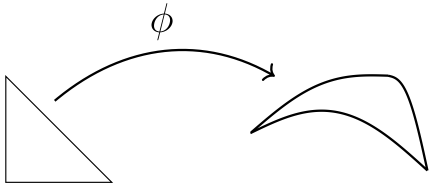
Mapping to the surface

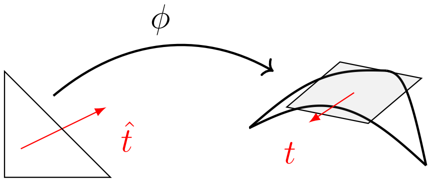
- Piola transformation (H(div)-conforming RT/BDM elements)

$$v \circ \phi = \frac{1}{J}\boldsymbol{F}\hat{v}, \quad \hat{v} \in BDM/RT(\hat{T}) \qquad \boldsymbol{F} = \nabla_{\hat{x}}\phi, J = \sqrt{\det(\boldsymbol{F}^T\boldsymbol{F})} = \|\operatorname{cof}(\boldsymbol{F})\|_F$$

- Double-Piola transformation (Hellan–Herrmann–Johnson elements)

$$\boldsymbol{\sigma} \circ \phi = \frac{1}{J^2}\boldsymbol{F}\hat{\boldsymbol{\sigma}}\boldsymbol{F}^T, \quad \hat{\boldsymbol{\sigma}} \in M_h^k(\hat{T})$$





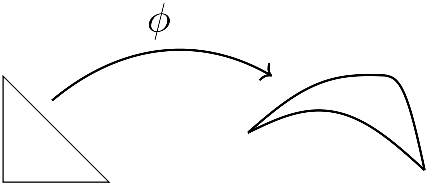
Mapping to the surface

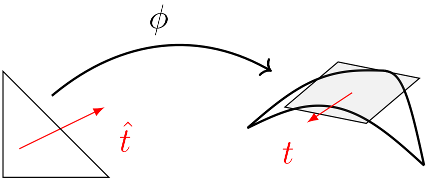
- Piola transformation (H(div)-conforming RT/BDM elements)

$$v \circ \phi = \frac{1}{J} \boldsymbol{F} \hat{v}, \quad \hat{v} \in BDM/RT(\hat{T}) \qquad \boldsymbol{F} = \nabla_{\hat{x}} \phi, J = \sqrt{\det(\boldsymbol{F}^T \boldsymbol{F})} = \| \operatorname{cof}(\boldsymbol{F}) \|_F$$

- Double-Piola transformation (Hellan–Herrmann–Johnson elements)

$$\boldsymbol{\sigma} \circ \phi = \frac{1}{J^2} \boldsymbol{F} \hat{\boldsymbol{\sigma}} \boldsymbol{F}^T, \quad \hat{\boldsymbol{\sigma}} \in M_h^k(\hat{T})$$





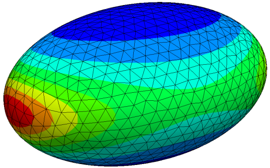
- v and $\boldsymbol{\sigma}$ are tangential vector/matrix fields on S! $\qquad v \cdot \nu = \boldsymbol{\sigma} \nu = \nu^T \boldsymbol{\sigma} = 0$

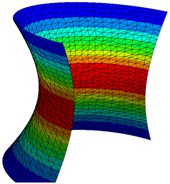
Lifted distributional curvature

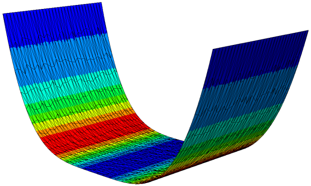
Lifting of distributional Weingarten tensor

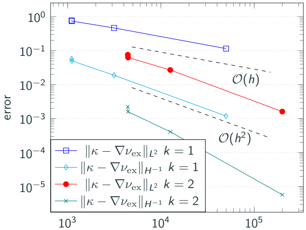
Find $\boldsymbol{\kappa} \in M_h^{k-1}$ for $\mathscr{T}_h$ curving order k s.t. for all $\boldsymbol{\sigma} \in M_h^{k-1}$

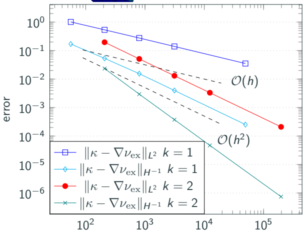
$$\int_{\mathscr{T}_h} \boldsymbol{\kappa} : \boldsymbol{\sigma} \, dx = \langle \nabla \nu, \boldsymbol{\sigma} \rangle_{\mathscr{T}}.$$

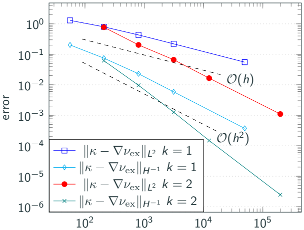






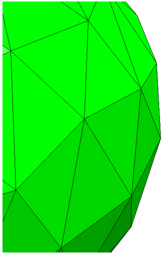






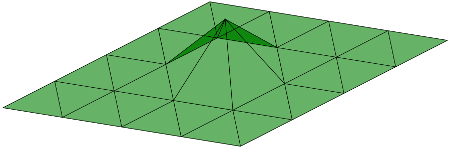
Distributional intrinsic curvature

Gauss Theorema Egregium: Gauss curvature depends on metric, $K = \kappa_1 \kappa_2$



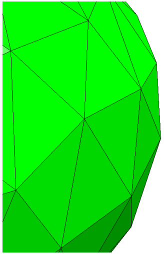
angle defect (DDG, Regge calculus)

metric $g = \nabla\Phi^\top \nabla\Phi$



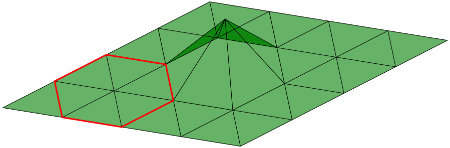
REGGE: General relativity without coordinates, *Il Nuovo Cimento* (1955-1965), 19 (1961).

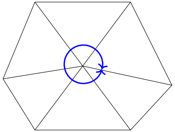
Gauss Theorema Egregium: Gauss curvature depends on metric, $K = \kappa_1 \kappa_2$



angle defect (DDG, Regge calculus)

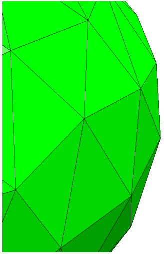
metric $g = \nabla\Phi^\top \nabla\Phi$





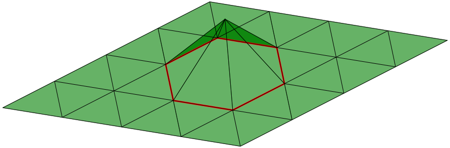
REGGE: General relativity without coordinates, *Il Nuovo Cimento* (1955-1965), 19 (1961).

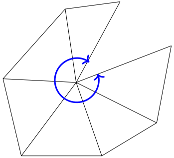
Gauss Theorema Egregium: Gauss curvature depends on metric, $K = \kappa_1\kappa_2$



angle defect (DDG, Regge calculus)

metric $g = \nabla\Phi^\top\nabla\Phi$

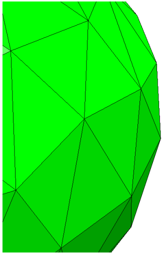




REGGE: General relativity without coordinates, *Il Nuovo Cimento* (1955-1965), 19 (1961).

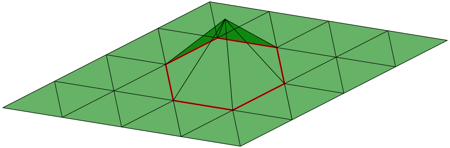
Distributional intrinsic curvature

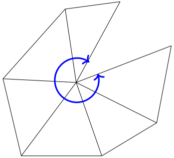
Gauss Theorema Egregium: Gauss curvature depends on metric, $K = \kappa_1\kappa_2$

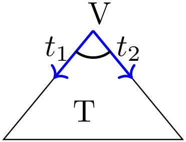


angle defect (DDG, Regge calculus)

metric $g = \nabla\Phi^\top\nabla\Phi \qquad \omega = \sqrt{\det g}$





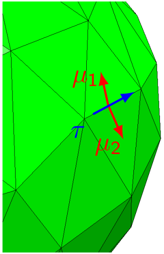


Let $g \in \mathrm{Reg}_h^0(\mathscr{T}_h)$ and $\varphi \in V_h^1$

$$\langle (K\omega)(g), \varphi\rangle = \sum_{V\in\mathscr{V}_h} K_V(\varphi, g), \quad K_V(\varphi,g) = \big(2\pi - \sum_{T:V\subset T} \sphericalangle_V^T(g)\big)\varphi(V)$$

Distributional intrinsic curvature

Gauss Theorema Egregium: Gauss curvature depends on metric, $K = \kappa_1\kappa_2$



angle defect (DDG, Regge calculus)

metric $g = \nabla\Phi^\top\nabla\Phi$ $\qquad \Gamma_{ij}^k(g) = \frac{1}{2}g^{kl}\left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right) = g^{kl}\Gamma_{ijl}(g)$

$$\kappa_g = g(\nabla_\tau\tau, \mu) = \frac{\sqrt{\det g}}{g_{\hat\tau\hat\tau}^{3/2}}\left(\partial_{\hat\tau}\hat\tau\cdot\hat\mu + \Gamma_{\hat\tau\hat\tau}^{\hat\mu}\right)$$

$$K(g) = \frac{1}{\det g}\left(\partial_1\Gamma_{212} - \partial_2\Gamma_{112} + \Gamma_{11}^p\Gamma_{22p} - \Gamma_{22}^p\Gamma_{11p}\right)$$

Let $g \in \mathrm{Reg}_h^k(\mathscr{T}_h)$ and $\varphi \in V_h^{k+1}$

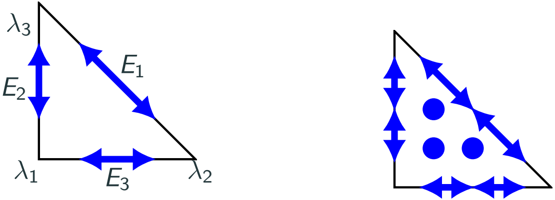
$$\langle (K\omega)(g), \varphi\rangle = \sum_{V\in\mathscr{V}_h} K_V(\varphi, g) + \int_{\mathscr{T}_h} K(g)\,\varphi\,\omega_T + \sum_{E\in\mathscr{E}_h}\int_E [\![\kappa(g)]\!]\varphi\,\omega_E$$

Berchenko-Kogan, Gawlik: Finite element approximation of the Levi-Civita connection and its curvature in two dimensions, *Found Comput Math* (2022).

TECHNISCHE UNIVERSITÄT WIEN

$$H(\operatorname{curl}\operatorname{curl}) := \{\sigma \in [L^2(\Omega)]^{2\times 2}_{sym} \mid \operatorname{curl}\operatorname{curl}\sigma \in H^{-1}(\Omega)\}$$

$$\mathrm{Reg}_h^k := \{\varepsilon \in [\mathcal{P}^k(\mathscr{T}_h)]^{d\times d}_{\mathrm{sym}} \mid [\![t^\top \varepsilon\, t]\!]_E = 0 \text{ for all edges } E\}$$



$$\varphi_{E_i} = \nabla\lambda_j \odot \nabla\lambda_k, \qquad t_j^\top \varphi_{E_i} t_j = c_i\delta_{ij}, \qquad \varphi_{T_i} = \lambda_i \nabla\lambda_j \odot \nabla\lambda_k$$

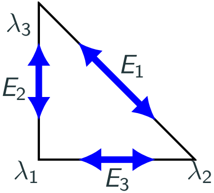
CHRISTIANSEN: On the linearization of Regge calculus, *Numerische Mathematik 119, 4 (2011).*

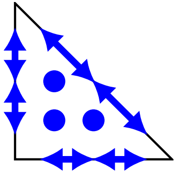
LI: Regge Finite Elements with Applications in Solid Mechanics and Relativity, *PhD thesis, University of Minnesota (2018).*

N.: Mixed Finite Element Methods For Nonlinear Continuum Mechanics And Shells, *PhD thesis, TU Wien (2021).*

$$H(\mathrm{curl}\,\mathrm{curl}) := \{\sigma \in [L^2(\Omega)]^{2\times 2}_{sym} \mid \mathrm{curl}\,\mathrm{curl}\,\sigma \in H^{-1}(\Omega)\}$$

$$\mathrm{Reg}_h^k := \{\varepsilon \in [\mathcal{P}^k(\mathscr{T}_h)]^{d\times d}_{\mathrm{sym}} \mid [\![t^\top \varepsilon\, t]\!]_E = 0 \text{ for all edges } E\}$$





$$\varphi_{E_i} = \nabla\lambda_j \odot \nabla\lambda_k, \qquad t_j^\top \varphi_{E_i} t_j = c_i \delta_{ij}, \qquad \varphi_{T_i} = \lambda_i \nabla\lambda_j \odot \nabla\lambda_k$$

$$\mathcal{R}_h^k : C^0(\Omega) \to \mathrm{Reg}_h^k$$ canonical interpolant

$$\int_E (g - \mathcal{R}_h^k g)_{tt}\, q\, dl = 0 \text{ for all } q \in \mathcal{P}^k(E)$$

$$\int_T (g - \mathcal{R}_h^k g) : Q\, da = 0 \text{ for all } Q \in \mathcal{P}^{k-1}(T, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$$

TECHNISCHE UNIVERSITÄT WIEN

Lifting of distributional Gauss curvature

For $g \in \mathrm{Reg}_h^k$ find $K_h \in \mathcal{L}_h^{k+1}(\mathscr{T}_h)$ such that for all $\varphi \in \mathcal{L}_h^{k+1}(\mathscr{T}_h)$

$$\int_\Omega K_h \varphi\, \omega = \langle (K\omega)(g), \varphi \rangle.$$

Theorem

Let $g_h \in \mathrm{Reg}_h^k$ optimal-order interpolant, $-1 \leq l \leq k-2$

$$\|K_h - K\|_{H_h^l} \leq C\, h^{-l+k-1}(|g|_{W^{k+1,\infty}} + |K|_{H^k})$$

GAWLIK: High-Order Approximation of Gaussian Curvature with Regge Finite Elements, *SIAM J. Numer. Anal.* (2020).

TECHNISCHE UNIVERSITÄT WIEN

Lifting of distributional Gauss curvature

For $g \in \mathrm{Reg}_h^k$ find $K_h \in \mathcal{L}_h^{k+1}(\mathscr{T}_h)$ such that for all $\varphi \in \mathcal{L}_h^{k+1}(\mathscr{T}_h)$

$$\int_\Omega K_h \varphi\, \omega = \langle (K\omega)(g), \varphi \rangle.$$

Theorem

Let $g_h \in \mathrm{Reg}_h^k$ optimal-order interpolant, $-1 \leq l \leq k-2$

$$\|K_h - K\|_{H^{-1}} \leq C\, h^k (|g|_{W^{k+1,\infty}} + |K|_{H^k})$$

GAWLIK: High-Order Approximation of Gaussian Curvature with Regge Finite Elements, *SIAM J. Numer. Anal.* (2020).

TECHNISCHE UNIVERSITÄT WIEN

Lifting of distributional Gauss curvature

For $g \in \mathrm{Reg}_h^k$ find $K_h \in \mathcal{L}_h^{k+1}(\mathscr{T}_h)$ such that for all $\varphi \in \mathcal{L}_h^{k+1}(\mathscr{T}_h)$

$$\int_\Omega K_h \varphi\, \omega = \langle (K\omega)(g), \varphi \rangle.$$

Theorem (Gopalakrishnan, N., Schöberl, Wardetzky 2022)

Let $g_h = \mathcal{R}_h^k g \in \mathrm{Reg}_h^k$, $-1 \leq l \leq k-1$

$$\|K_h - K\|_{H_h^l} \leq C\, h^{-l+k}(|g|_{W^{k+1,\infty}} + |K|_{H^k})$$

Gopalakrishnan, N., Schöberl, Wardetzky: Analysis of curvature approximations via covariant curl and incompatibility for Regge metrics, *SMAI J. Comput. Math.* (accepted).

Lifting of distributional Gauss curvature

For $g \in \mathrm{Reg}_h^k$ find $K_h \in \mathcal{L}_h^{k+1}(\mathscr{T}_h)$ such that for all $\varphi \in \mathcal{L}_h^{k+1}(\mathscr{T}_h)$

$$\int_\Omega K_h \varphi\, \omega = \langle (K\omega)(g), \varphi \rangle.$$

Theorem (Gopalakrishnan, N., Schöberl, Wardetzky 2022)

Let $g_h = \mathcal{R}_h^k g \in \mathrm{Reg}_h^k$, $-1 \leq l \leq k-1$

$$\|K_h - K\|_{H^{-1}} \leq C\, h^{k+1}(|g|_{W^{k+1,\infty}} + |K|_{H^k})$$

Gopalakrishnan, N., Schöberl, Wardetzky: Analysis of curvature approximations via covariant curl and incompatibility for Regge metrics, *SMAI J. Comput. Math.* (accepted).

Idea: Integral representation orthogonality properties of interpolant

$$\frac{d}{dt}((K\omega)(g+t\sigma))|_{t=0} = -\frac{1}{2}\,\mathrm{curl}_g\,\mathrm{curl}_g\,\sigma\,\omega_g \qquad \mathrm{inc}_g := \mathrm{curl}_g\,\mathrm{curl}_g$$

Integral representation $(G(t) = g_h + t(g-g_h),\ \sigma = G'(t) = g - g_h)$

$$\langle (K\omega)(g), u_h\rangle - \langle (K\omega)(g_h), u_h\rangle = -\frac{1}{2}\int_0^1 \langle \mathrm{inc}_{G(t)}\,\sigma, u_h\rangle\,dt$$

$$\langle \mathrm{inc}_g\,\sigma, u\rangle = \sum_{T\in\mathscr{T}_h}\int_T \mathrm{inc}_g\,\sigma\,u\,\omega_T - \int_{\partial T} u\,g(\mathrm{curl}_g\,\sigma - \mathrm{grad}_g\,\sigma(n_g,t_g),t_g)\,\omega_{\partial T} - \sum_{V\in\mathscr{V}_T}[\![\sigma(n_g,t_g)]\!]_V^T u(V)$$

Gawlik: High-Order Approximation of Gaussian Curvature with Regge Finite Elements, *SIAM J. Numer. Anal.* (2020).

Idea: Integral representation orthogonality properties of interpolant

$$\frac{d}{dt}((K\omega)(g+t\sigma))|_{t=0} = -\frac{1}{2}\,\mathrm{curl}_g\,\mathrm{curl}_g\,\sigma\,\omega_g \qquad \mathrm{inc}_g := \mathrm{curl}_g\,\mathrm{curl}_g$$

Integral representation $(G(t) = g_h + t(g-g_h),\ \sigma = G'(t) = g - g_h)$

$$\langle (K\omega)(g), u_h\rangle - \langle (K\omega)(g_h), u_h\rangle = -\frac{1}{2}\int_0^1 \langle \mathrm{inc}_{G(t)}\,\sigma, u_h\rangle\,dt$$

$$\langle \mathrm{inc}_g\,\sigma, u\rangle = \sum_{T\in\mathscr{T}_h}\int_T \mathrm{inc}_g\,\sigma\,u\,\omega_T - \int_{\partial T} u\,g(\mathrm{curl}_g\,\sigma - \mathrm{grad}_g\,\sigma(n_g,t_g),t_g)\,\omega_{\partial T} - \sum_{V\in\mathscr{V}_T}[\![\sigma(n_g,t_g)]\!]_V^T u(V)$$

Orthogonality property of Regge interpolant

$$\int_E(\sigma-\mathcal{R}_h^k\sigma)_{tt}\,q\,dl = 0 \text{ for all } q\in\mathcal{P}^k(E), \qquad \int_T(\sigma-\mathcal{R}_h^k\sigma):q\,da = 0 \text{ for all } q\in\mathcal{P}^{k-1}(T,\mathbb{R}^{2\times 2})$$

$$\langle \mathrm{inc}(\sigma-\mathcal{R}_h^k\sigma), u_h\rangle = 0 \text{ for all } u_h\in\mathcal{L}_h^{k+1}(\mathscr{T}_h)$$

Gopalakrishnan, N., Schöberl, Wardetzky: Analysis of curvature approximations via covariant curl and incompatibility for Regge metrics, *SMAI J. Comput. Math.* (accepted).

Idea: Integral representation orthogonality properties of interpolant

$$\frac{d}{dt}((K\omega)(g+t\sigma))|_{t=0} = -\frac{1}{2}\operatorname{curl}_g \operatorname{curl}_g \sigma\,\omega_g \qquad \operatorname{inc}_g := \operatorname{curl}_g \operatorname{curl}_g$$

Integral representation $(G(t) = g_h + t(g - g_h),\ \sigma = G'(t) = g - g_h)$

$$\langle (K\omega)(g), u_h\rangle - \langle (K\omega)(g_h), u_h\rangle = -\frac{1}{2}\int_0^1 \langle \operatorname{inc}_{G(t)}\sigma, u_h\rangle\, dt$$

$$\langle \operatorname{inc}_g \sigma, u\rangle = \sum_{T\in\mathscr{T}_h}\int_T \operatorname{inc}_g \sigma\, u\,\omega_T - \int_{\partial T} u\, g(\operatorname{curl}_g \sigma - \operatorname{grad}_g \sigma(n_g, t_g), t_g)\,\omega_{\partial T} - \sum_{V\in\mathscr{V}_T} [\![\sigma(n_g, t_g)]\!]_V^T u(V)$$

Orthogonality property of Regge interpolant

$$\int_E (\sigma - \mathcal{R}_h^k\sigma)_{tt}\, q\, dl = 0 \text{ for all } q\in\mathcal{P}^k(E), \qquad \int_T (\sigma - \mathcal{R}_h^k\sigma) : q\, da = 0 \text{ for all } q\in\mathcal{P}^{k-1}(T,\mathbb{R}^{2\times 2})$$

$$\langle \operatorname{inc}(\sigma - \mathcal{R}_h^k\sigma), u_h\rangle = 0 \text{ for all } u_h\in\mathcal{L}_h^{k+1}(\mathscr{T}_h)$$

$$|\langle \operatorname{inc}_g(\sigma - \mathcal{R}_h^k\sigma), u_h\rangle| \leq C \qquad \|\sigma - \mathcal{R}_h^k\sigma\|\,\|u_h\|_{H^1} \text{ for all } u_h\in\mathcal{L}_h^{k+1}(\mathscr{T}_h)$$

Gopalakrishnan, N., Schöberl, Wardetzky: Analysis of curvature approximations via covariant curl and incompatibility for Regge metrics, *SMAI J. Comput. Math.* (accepted).

Idea: Integral representation orthogonality properties of interpolant

$$\frac{d}{dt}((K\omega)(g+t\sigma))|_{t=0} = -\frac{1}{2}\,\mathrm{curl}_g\,\mathrm{curl}_g\,\sigma\,\omega_g \qquad \mathrm{inc}_g := \mathrm{curl}_g\,\mathrm{curl}_g$$

Integral representation $(G(t) = g_h + t(g-g_h),\ \sigma = G'(t) = g - g_h)$

$$\langle (K\omega)(g), u_h\rangle - \langle (K\omega)(g_h), u_h\rangle = -\frac{1}{2}\int_0^1 \langle \mathrm{inc}_{G(t)}\,\sigma, u_h\rangle\,dt$$

$$\langle \mathrm{inc}_g\,\sigma, u\rangle = \sum_{T\in\mathscr{T}_h}\int_T \mathrm{inc}_g\,\sigma\,u\,\omega_T - \int_{\partial T} u\,g(\mathrm{curl}_g\,\sigma - \mathrm{grad}_g\,\sigma(n_g,t_g),t_g)\,\omega_{\partial T} - \sum_{V\in\mathscr{V}_T}[\![\sigma(n_g,t_g)]\!]_V^T u(V)$$

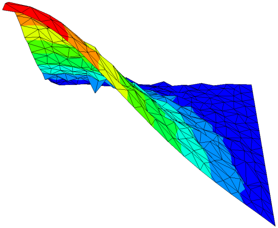
Orthogonality property of Regge interpolant

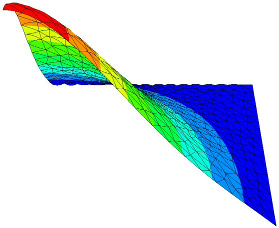
$$\int_E (\sigma - \mathcal{R}_h^k\sigma)_{tt}\,q\,dl = 0 \text{ for all } q\in\mathcal{P}^k(E), \qquad \int_T (\sigma - \mathcal{R}_h^k\sigma) : q\,da = 0 \text{ for all } q\in\mathcal{P}^{k-1}(T,\mathbb{R}^{2\times 2})$$

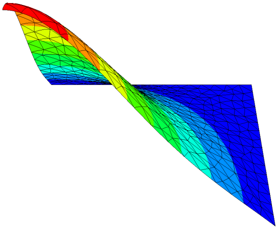
$$\langle \mathrm{inc}(\sigma - \mathcal{R}_h^k\sigma), u_h\rangle = 0 \text{ for all } u_h \in \mathcal{L}_h^{k+1}(\mathscr{T}_h)$$

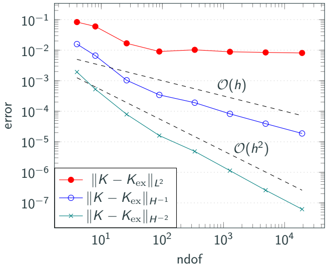
$$|\langle \mathrm{inc}_g(\sigma - \mathcal{I}_k\sigma), u_h\rangle| \le C\,h^{-1}\|\sigma - \mathcal{R}_h^k\sigma\|\|u_h\|_{H^1} \text{ for all } u_h\in\mathcal{L}_h^{k+1}(\mathscr{T}_h)$$

Gopalakrishnan, N., Schöberl, Wardetzky: Analysis of curvature approximations via covariant curl and incompatibility for Regge metrics, *SMAI J. Comput. Math.* (accepted).

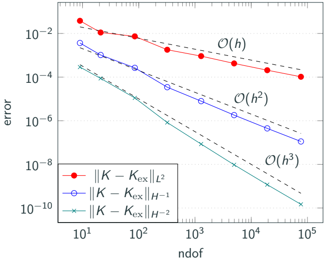




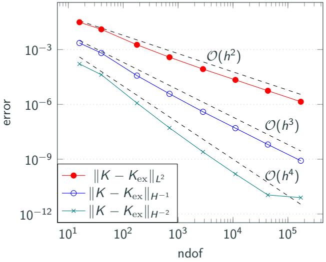




10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
error
10^{1}
10^{2}
10^{3}
10^{4}
ndof
$\mathcal{O}(h)$
$\mathcal{O}(h^2)$
$\|K - K_{\mathrm{ex}}\|_{L^2}$
$\|K - K_{\mathrm{ex}}\|_{H^{-1}}$
$\|K - K_{\mathrm{ex}}\|_{H^{-2}}$



10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
error
10^{1}
10^{2}
10^{3}
10^{4}
10^{5}
ndof
$\mathcal{O}(h)$
$\mathcal{O}(h^2)$
$\mathcal{O}(h^3)$
$\|K - K_{\mathrm{ex}}\|_{L^2}$
$\|K - K_{\mathrm{ex}}\|_{H^{-1}}$
$\|K - K_{\mathrm{ex}}\|_{H^{-2}}$



10^{-3}
10^{-6}
10^{-9}
10^{-12}
error
10^{1}
10^{2}
10^{3}
10^{4}
10^{5}
ndof
$\mathcal{O}(h^2)$
$\mathcal{O}(h^3)$
$\mathcal{O}(h^4)$
$\|K - K_{\mathrm{ex}}\|_{L^2}$
$\|K - K_{\mathrm{ex}}\|_{H^{-1}}$
$\|K - K_{\mathrm{ex}}\|_{H^{-2}}$

Nonlinear shells

Koiter shell

$$\mathcal{W}(u) = \frac{t}{2}\|\boldsymbol{E}(u)\|_{\mathcal{M}}^2 + \frac{t^3}{24}\|\boldsymbol{F}^T\nabla(\nu \circ \phi) - \nabla\hat{\nu}\|_{\mathcal{M}}^2$$

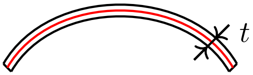
u . . . displacement of mid-surface

t . . . thickness

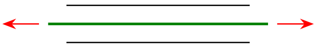
$\mathcal{M}$ . . . material tensor

$\boldsymbol{F} = \nabla u + \boldsymbol{P} = \nabla\phi, \qquad \boldsymbol{P} = \boldsymbol{I} - \hat{\nu} \otimes \hat{\nu}$

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^\top \boldsymbol{F} - \boldsymbol{P}) = \frac{1}{2}(\nabla u^\top \nabla u + \nabla u^\top \boldsymbol{P} + \boldsymbol{P}\nabla u)$$



$$\mathcal{W}(u) = \frac{t}{2}\|\boldsymbol{E}(u)\|^2_{\mathcal{M}} + \frac{t^3}{24}\|\boldsymbol{F}^T\nabla(\nu\circ\phi) - \nabla\hat{\nu}\|^2_{\mathcal{M}}$$



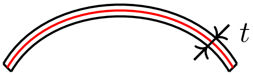
u ... displacement of mid-surface

t ... thickness

$\mathcal{M}$ ... material tensor

$$\boldsymbol{F} = \nabla u + \boldsymbol{P} = \nabla\phi, \qquad \boldsymbol{P} = \boldsymbol{I} - \hat{\nu}\otimes\hat{\nu}$$

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^\top\boldsymbol{F} - \boldsymbol{P}) = \frac{1}{2}(\nabla u^\top\nabla u + \nabla u^\top\boldsymbol{P} + \boldsymbol{P}\nabla u)$$



$$\mathcal{W}(u) = \frac{t}{2}\|\boldsymbol{E}(u)\|^2_{\mathcal{M}} + \frac{t^3}{24}\|\boldsymbol{F}^T\nabla(\nu\circ\phi) - \nabla\hat{\nu}\|^2_{\mathcal{M}}$$

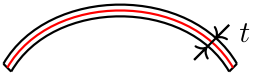
$u\ldots$ displacement of mid-surface

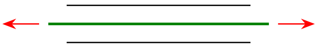
$t\ldots$ thickness

$\mathcal{M}\ldots$ material tensor

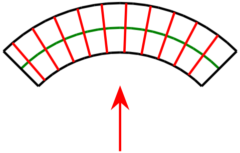
$$\boldsymbol{F} = \nabla u + \boldsymbol{P} = \nabla\phi, \qquad \boldsymbol{P} = \boldsymbol{I} - \hat{\nu}\otimes\hat{\nu}$$

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^\top\boldsymbol{F} - \boldsymbol{P}) = \frac{1}{2}(\nabla u^\top\nabla u + \nabla u^\top\boldsymbol{P} + \boldsymbol{P}\nabla u)$$





membrane energy



bending energy

Distributional curvature for Koiter shell

- Lifted curvature difference $\boldsymbol{\kappa}^{\text{diff}}$ via three-field formulation

$$\mathcal{L}(u,\boldsymbol{\kappa}^{\text{diff}},\boldsymbol{\sigma}) = \frac{t}{2}\|\boldsymbol{E}(u)\|_{\mathcal{M}}^2 + \frac{t^3}{12}\|\boldsymbol{\kappa}^{\text{diff}}\|_{\mathcal{M}}^2 - \langle f,u\rangle$$
$$+\sum_{T\in\mathcal{T}_h}\int_T \left(\boldsymbol{\kappa}^{\text{diff}} - (\boldsymbol{F}^T\nabla(\nu\circ\phi) - \nabla\hat{\nu})\right):\boldsymbol{\sigma}\,dx$$
$$+\sum_{E\in\mathcal{E}_h}\int_E (\sphericalangle(\nu_L,\nu_R) - \sphericalangle(\hat{\nu}_L,\hat{\nu}_R))\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\,ds$$

- Lagrange parameter $\boldsymbol{\sigma}\in M_h^k$ moment tensor
- Eliminate $\boldsymbol{\kappa}^{\text{diff}} \to$ two-field formulation in $(u,\boldsymbol{\sigma})$

N., Schöberl: The Hellan–Herrmann–Johnson and TDNNS method for linear and nonlinear shells, *arXiv:2304.13806*.

Formulation: Koiter shell

Shell problem

Find $u \in [\mathcal{L}_h^{k+1}(\mathscr{T}_h)]^3$ and $\boldsymbol{\sigma} \in M_h^{k-1}$ for $(\boldsymbol{H}_\nu := \sum_i (\nabla^2 u_i)\nu_i)$

$$
\begin{aligned}
\mathcal{L}(u,\boldsymbol{\sigma}) =& \frac{t}{2}\|\boldsymbol{E}(u)\|^2_{\mathcal{M}} - \frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\mathcal{M}^{-1}} - \langle f, u\rangle \\
&+ \sum_{T\in\mathcal{T}_h} \int_T \boldsymbol{\sigma} : (\boldsymbol{H}_\nu + (1-\hat{\nu}\cdot\nu)\nabla\hat{\nu})\,dx \\
&+ \sum_{E\in\mathcal{E}_h} \int_E (\sphericalangle(\nu_L,\nu_R) - \sphericalangle(\hat{\nu}_L,\hat{\nu}_R))\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\,ds
\end{aligned}
$$

Use hybridization to eliminate $\boldsymbol{\sigma}$ $\rightarrow$ recover minimization problem

N., Schöberl: The Hellan–Herrmann–Johnson method for nonlinear shells, *Comput. Struct. 225 (2019).*

Membrane locking

Membrane locking

$$\mathcal{W}(u) = t\, E_{\text{mem}}(u) + t^3\, E_{\text{bend}}(u) - f \cdot u, \qquad f = t^3 \tilde{f}$$

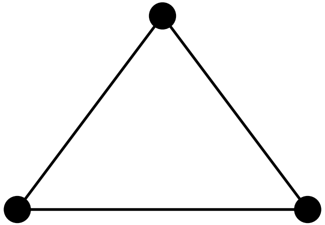
Membrane locking

$$\mathcal{W}(u) = t^{-2} E_{\text{mem}}(u) + E_{\text{bend}}(u) - \tilde{f} \cdot u, \qquad f = t^3 \tilde{f}$$

Enforces $E_{\text{mem}}(u) = 0$ in the limit $t \to 0$

$$\mathcal{W}(u) = t^{-2} E_{\text{mem}}(u) + E_{\text{bend}}(u) - \tilde{f} \cdot u, \qquad f = t^3 \tilde{f}$$

Enforces $E_{\text{mem}}(u) = 0$ in the limit $t \to 0$

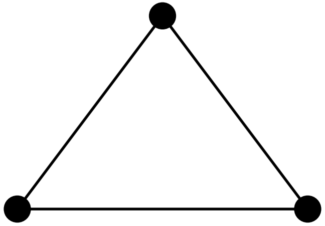


$$\mathcal{L}_h(\mathscr{T}_h) = \mathcal{P}(\mathscr{T}_h) \cap C(\Omega) \subset H^1(\Omega)$$

$$\mathcal{W}(u) = t^{-2}E_{\text{mem}}(u) + E_{\text{bend}}(u) - \tilde{f}\cdot u, \qquad f = t^3\tilde{f}$$

Enforces $E_{\text{mem}}(u) = 0$ in the limit $t \to 0$

$$E_{\text{mem}}(u) = 0 \quad \not\Rightarrow \quad E_{\text{mem}}(u_h) = 0$$



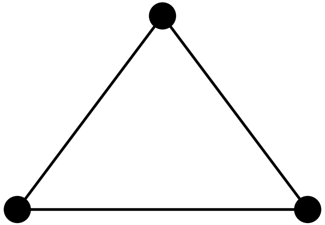
$$\mathcal{L}_h(\mathscr{T}_h) = \mathcal{P}(\mathscr{T}_h) \cap C(\Omega) \subset H^1(\Omega)$$

Membrane locking

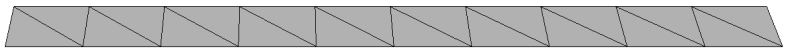
$$\mathcal{W}(u) = t^{-2} E_{\text{mem}}(u) + E_{\text{bend}}(u) - \tilde{f} \cdot u, \qquad f = t^3 \tilde{f}$$

Enforces $E_{\text{mem}}(u) = 0$ in the limit $t \to 0$

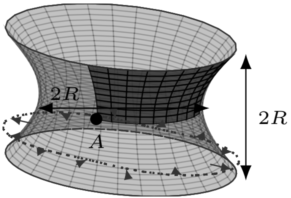
$$E_{\text{mem}}(u) = 0 \quad \not\Rightarrow \quad E_{\text{mem}}(u_h) = 0$$

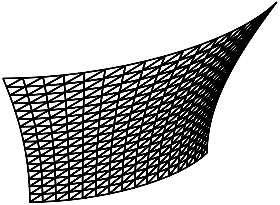


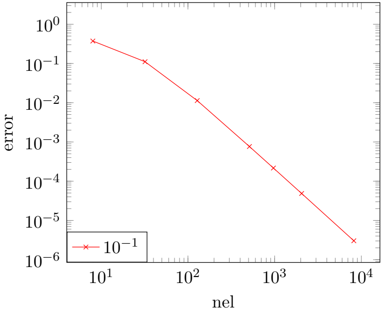
$$\mathcal{L}_h(\mathscr{T}_h) = \mathcal{P}(\mathscr{T}_h) \cap C(\Omega) \subset H^1(\Omega)$$



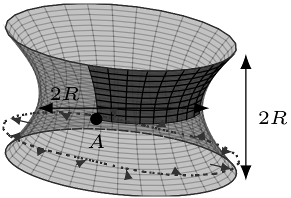
Hyperboloid with free ends

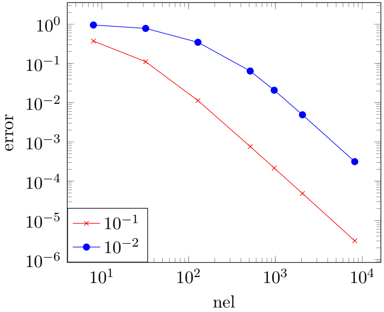


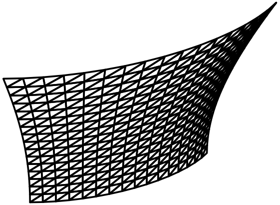




Hyperboloid with free ends

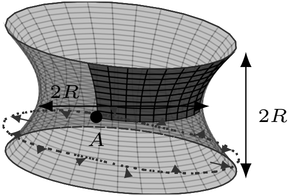


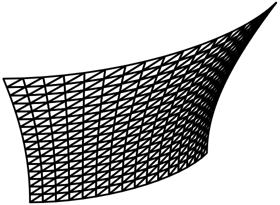


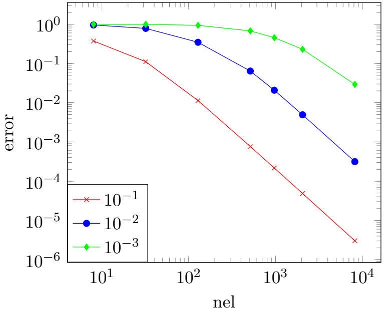


- Pre-asymptotic regime

Hyperboloid with free ends

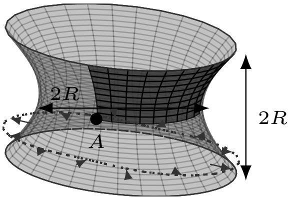


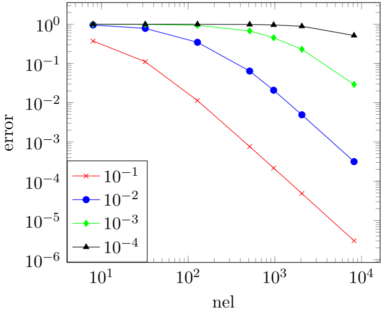


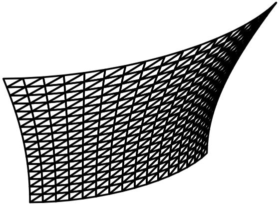


- Pre-asymptotic regime

Hyperboloid with free ends







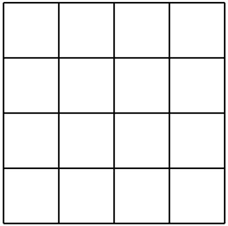
- Pre-asymptotic regime

Membrane locking

$$\frac{1}{t^2}\| \quad \boldsymbol{E}(u_h)\|_{\mathbb{M}}^2$$

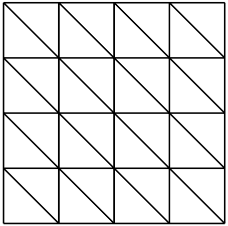
Membrane locking

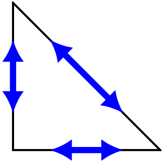
$$\frac{1}{t^2}\|\Pi^k_{L^2}\boldsymbol{E}(u_h)\|^2_{\mathbb{M}}$$



- Reduced integration for quadrilateral meshes

$$\frac{1}{t^2}\|\mathcal{I}_{\mathcal{R}}^{k}\boldsymbol{E}(u_h)\|_{\mathbb{M}}^2$$

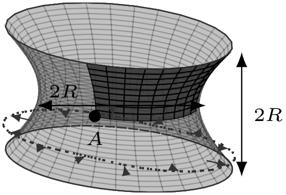


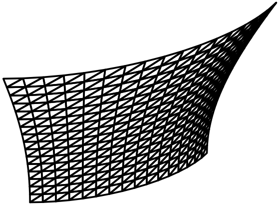


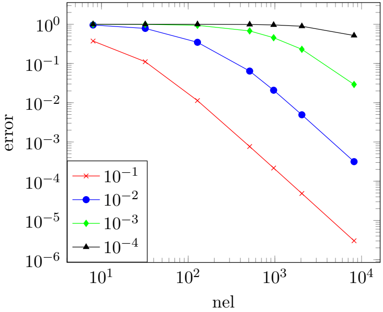
- Reduced integration for quadrilateral meshes
- Regge interpolant for triangles
- Connection to MITC shell elements

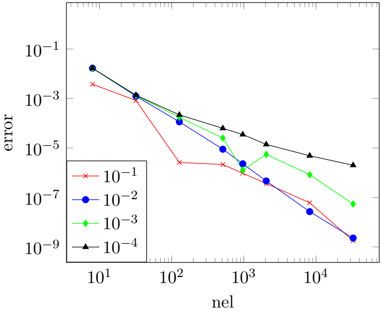
N., Schöberl: Avoiding membrane locking with Regge interpolation, *Comput. Methods Appl. Mech. Engrg 373 (2021).*

Hyperboloid with free ends



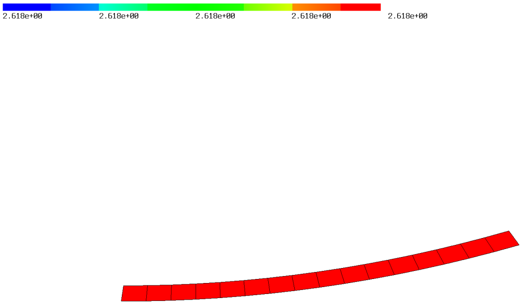


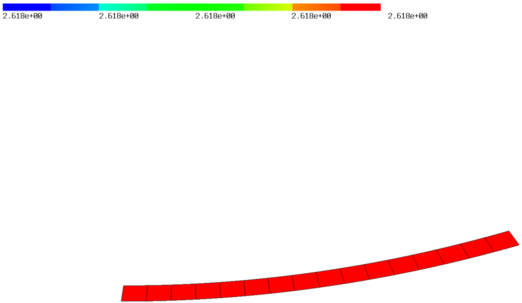




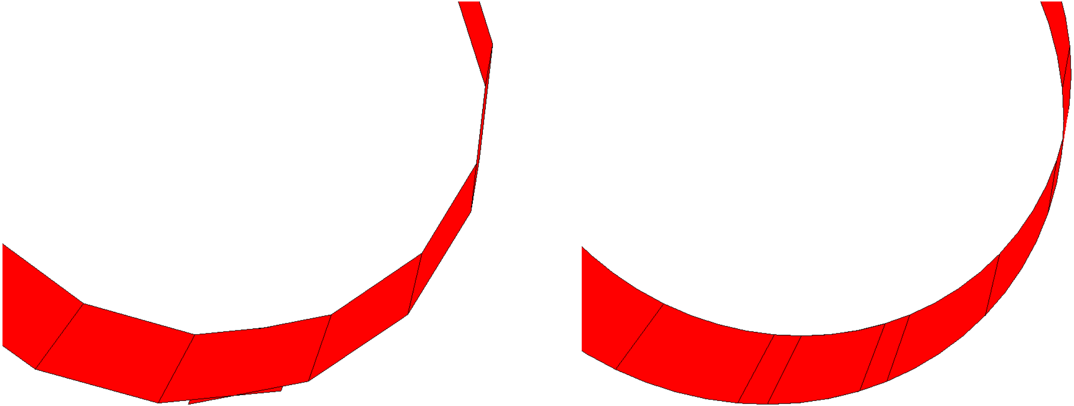
Numerical examples

Cantilever subjected to end moment

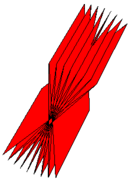


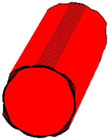


Cantilever subjected to end moment

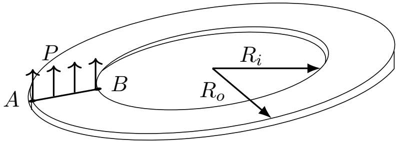


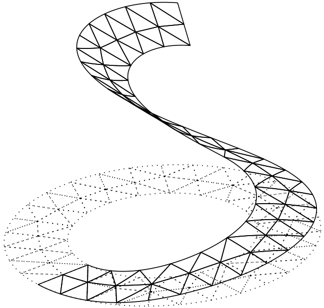
Cantilever subjected to end moment

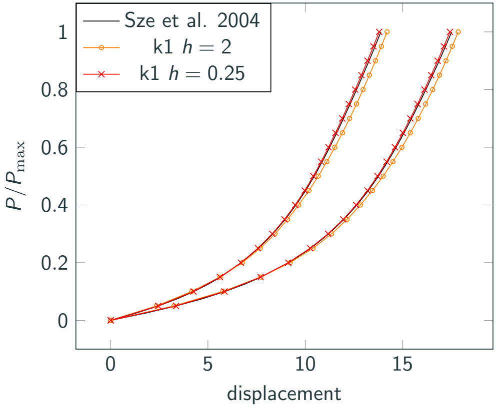




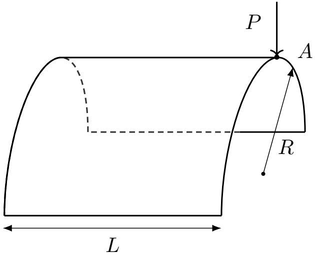
Slit annular plate

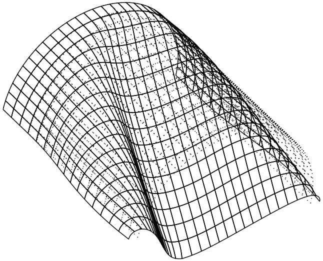


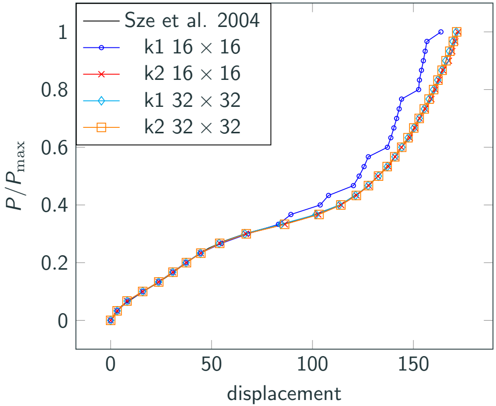




Pinched cylinder







Summary & outlook

- Distributional extrinsic/intrinsic curvature
- Application to nonlinear shells
- Hellan–Herrmann–Johnson and Regge finite elements for stress and strain/metric fields

Summary & outlook

- Distributional extrinsic/intrinsic curvature
- Application to nonlinear shells
- Hellan–Herrmann–Johnson and Regge finite elements for stress and strain/metric fields
- Naghdi shells
- Coupling for 3D elasticity
- Distributional curvature higher dimension $\rightarrow$ general relativity

- Distributional extrinsic/intrinsic curvature
- Application to nonlinear shells
- Hellan–Herrmann–Johnson and Regge finite elements for stress and strain/metric fields
- Naghdi shells
- Coupling for 3D elasticity
- Distributional curvature higher dimension $\rightarrow$ general relativity

Thank You for Your attention!

Literature

- N., Schöberl: The Hellan–Herrmann–Johnson and TDNNS method for linear and nonlinear shells, *arXiv:2304.13806*.
- N., Schöberl: The Hellan–Herrmann–Johnson method for nonlinear shells, *Comput. Struct. 225 (2019)*.
- N., Schöberl: Avoiding membrane locking with Regge interpolation, *Comput. Methods Appl. Mech. Engrg 373 (2021)*.
- N.: Mixed Finite Element Methods For Nonlinear Continuum Mechanics And Shells, *PhD thesis, TU Wien (2021)*.
- Gopalakrishnan, N., Schöberl, Wardetzky: Analysis of curvature approximations via covariant curl and incompatibility for Regge metrics, *SMAI J. Comput. Math.* (accepted), *arXiv:2206.09343*.
- N., Schöberl, Sturm, Numerical shape optimization of Canham-Helfrich-Evans bending energy, *J. Comput. Phys.* (2023).
- Gawlik, N.: Finite element approximation of scalar curvature in arbitrary dimension, *arXiv:2301.02159*.