Generalizing Riemann curvature to Regge metrics

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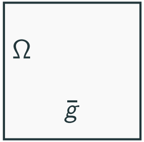
Riemannian manifolds

- N-dimensional Riemannian manifold $(\Omega, \bar{g})$ with $\bar{g}$ smooth metric tensor (spd bilinear form)



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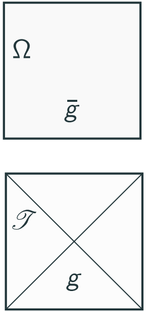
  with $\bar{g}$ smooth metric tensor (spd bilinear form)

- g piecewise smooth metric on triangulation $\mathscr{T}$ approximating $\bar{g}$



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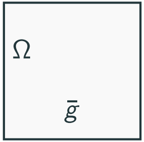
- Can we approximate Riemann curvature of $(\Omega, \bar{g})$ with $(\mathscr{T}, g)$?
  Can we generalize curvatures to piecewise smooth metrics g?



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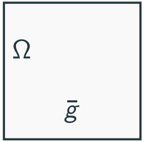

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- Can we approximate Riemann curvature of $(\Omega, \bar{g})$ with $(\mathscr{T}, g)$?
  Can we generalize curvatures to piecewise smooth metrics g?

Yes, if g is in the Regge space!

- $\mathfrak{X}(\mathscr{T})$, $\Lambda^k(\mathscr{T})$, $\mathcal{T}_l^k(\mathscr{T})$ piecewise smooth vector, k-forms, and (k,l)-tensor fields
- $g \in \mathcal{S}(\mathscr{T}) = \{\sigma \in \mathcal{T}^2(\mathscr{T}) : \sigma(X,Y) = \sigma(Y,X) \text{ for all } X, Y \in \mathfrak{X}(\mathscr{T})\}$
- Elementwise Riemann curvature tensor $\mathcal{R} \in \mathcal{T}^4(\mathscr{T})$ by $(X, Y, Z, W \in \mathfrak{X}(\mathscr{T}))$

$$\mathcal{R}(X,Y,Z,W) = g(\mathcal{R}_{X,Y}Z, W) = g(\nabla_X\nabla_Y - \nabla_Y\nabla_X Z - \nabla_{[X,Y]Z,W})$$

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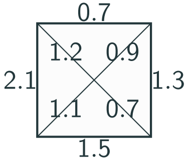
- $\mathcal{R}$ is a nonlinear second-order differential operator in g

$$\mathcal{R}_{ijkl} = \partial_i\Gamma_{jkl} - \partial_j\Gamma_{ikl} - \Gamma_{ilp}\Gamma^p_{jk} + \Gamma_{jlp}\Gamma^p_{ik}, \qquad \Gamma^p_{jk} = g^{pq}\Gamma_{jkq}, \;\; \Gamma_{ijk} = 1/2(\partial_i g_{jl} + \partial_j g_{il} - \partial_k g_{ij})$$

- "$\mathcal{R}(g)$" is a nonlinear distribution. What are the curvature contributions on element interfaces?

- Regge's idea: Approximate metric by assigning squared lengths to edges



REGGE: General relativity without coordinates, *Il Nuovo Cimento* (1955-1965), (1961).

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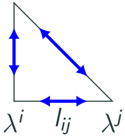
- Regge's idea: Approximate metric by assigning squared lengths to edges
- With barycentric coordinates λ^i



$$g = -\sum_{i \neq j} l_{ij}^2 \nabla\lambda^i \odot \nabla\lambda^j$$

REGGE: General relativity without coordinates, *Il Nuovo Cimento* (1955-1965), (1961).

SORKIN: Time-evolution problem in Regge calculus, *Phys. Rev. D 12* (1975).

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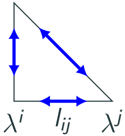

  - g is piecewise constant and tangential-tangential continuous: for all interior facets F the value $g(X, Y)$ coincides from both elements for all tangential $X, Y \in \mathfrak{X}(F)$
  - Regge finite element space

$$\mathcal{R}_h^k = \{g \in \mathcal{S}(\mathscr{T}) : g_{ij} \in \mathcal{P}^k(\mathscr{T}), \, g \text{ is tt-continuous}\}$$

CHRISTIANSEN: On the linearization of Regge calculus, *Numerische Mathematik, (2011).*

LI: Regge Finite Elements with Applications in Solid Mechanics and Relativity, *PhD thesis, University of Minnesota (2018).*

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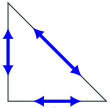

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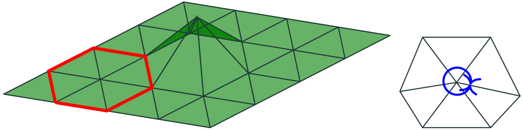
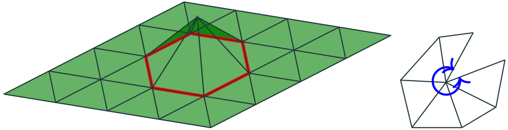
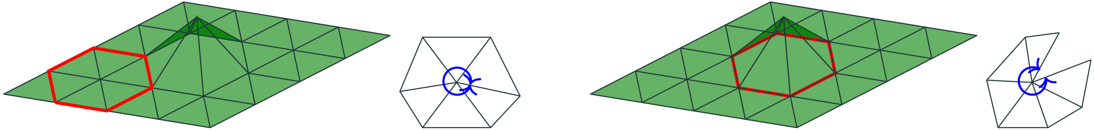
Christiansen: On the linearization of Regge calculus, *Numerische Mathematik, (2011).*

N.: Mixed Finite Element Methods for Nonlinear Continuum Mechanics and Shells, *PhD thesis, TU Wien (2021).*



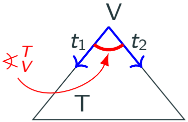


Angle defect



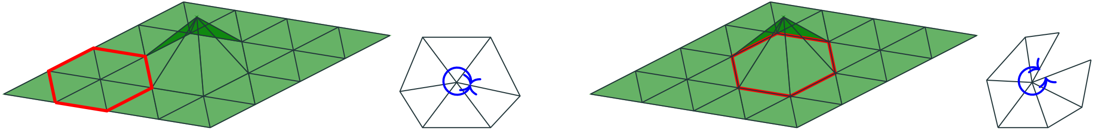
- In 2D, the angle defect Θ_V at vertex V is given by

$$\Theta_V = 2\pi - \sum_{T \supset V} \sphericalangle_V^T,$$



with the interior angle $\sphericalangle_V^T$ is measured with $g|_T$

Angle defect



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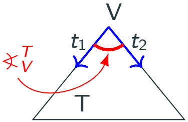
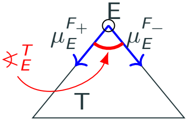


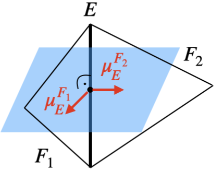
$$\Theta_V = 2\pi - \sum_{T \supset V} \sphericalangle_V^T,$$

with the interior angle $\sphericalangle_V^T$ is measured with $g|_T$

- N dimensions: generalized angle defect at $\mathring{\mathscr{E}} = \{$interior subsimplices of codimension 2$\}$

$$\Theta_E = 2\pi - \sum_{T \supset E} \underbrace{\arccos(g(\mu_E^{F_+}, \mu_E^{F_-}))}_{\sphericalangle_E^T}$$





Plane g-perpendicular to E

- Second fundamental form: For hypersurface F with g-normal ν

$$I\!I^{\nu}(X, Y) = -g(\nabla_X \nu, Y), \qquad X, Y \in \mathfrak{X}(F)$$

- Since the metric g and the g-normal ν jumps across interior facets $F \in \mathring{\mathscr{F}}$, the second fundamental form jumps as well
- Facet contribution: Jump of second fundamental form $[\![I\!I]\!]$
- Motivation via Gauss-Bonnet theorem or mollification argument

- Second fundamental form: For hypersurface F with g-normal ν

$$II^{\nu}(X,Y) = -g(\nabla_X \nu, Y), \qquad X, Y \in \mathfrak{X}(F)$$

- Since the metric g and the g-normal ν jumps across interior facets $F \in \mathring{\mathscr{F}}$, the second fundamental form jumps as well
- Facet contribution: Jump of second fundamental form $[\![II]\!]$
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Summary: The generalized Riemann curvature tensor has the following contributions

$$\mathcal{R}(g) = \begin{cases} \mathcal{R}|_T & \text{on each } T \in \mathscr{T} \\ [\![II]\!] & \text{on each } F \in \mathring{\mathscr{F}} \\ \Theta_E & \text{on each } E \in \mathring{\mathscr{E}} \end{cases}$$

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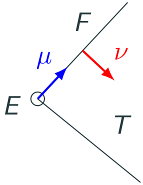
Generalized Riemann curvature (Gopalakrishnan, N., Schöberl, Wardetzky)

$$\widetilde{\mathcal{R}\omega}(A) = \sum_{T\in\mathscr{T}} \int_T \langle \mathcal{R}, A\rangle \omega_T + 4 \sum_{F\in\mathring{\mathscr{F}}} \int_F \langle [\![I\!I]\!], A_{.\nu\nu.}|_F\rangle \omega_F + 4 \sum_{E\in\mathring{\mathscr{E}}} \int_E \Theta_E \, A_{\mu\nu\nu\mu}\, \omega_E$$

$\widetilde{\mathcal{R}\omega}$ is acting on $A \in \mathring{\mathcal{A}}$, ω_D volume form on D

$$A_{.\nu\nu.}(X, Y) = A(X, \nu, \nu, Y), \qquad A_{\mu\nu\nu\mu} = A(\mu, \nu, \nu, \mu).$$



Test space $\mathring{\mathcal{A}}$ (has Riemann curvature tensor symmetries)

$$\begin{aligned}\mathcal{A} = \{A \in \mathcal{T}^4(\mathscr{T}) \;:\; & A_{.\nu\nu.}|_F \text{ is single-valued for all } F \in \mathring{\mathscr{F}}, \text{ and} \\ & A(X, Y, Z, W) = -A(Y, X, Z, W) = -A(X, Y, W, Z) = A(Z, W, X, Y)\}\end{aligned}$$

$$\mathring{\mathcal{A}} = \{A \in \mathcal{A} \;:\; A_{.\nu\nu.}|_F = 0 \text{ on } \partial\Omega\}$$

$$\widetilde{\mathcal{R}\omega}(A) = \sum_{T\in\mathscr{T}} \int_T \langle \mathcal{R}, A\rangle \omega_T + 4 \sum_{F\in\mathring{\mathscr{F}}} \int_F \langle [\![I\!I]\!], A_{.\nu\nu.}|_F\rangle \omega_F + 4 \sum_{E\in\mathring{\mathscr{E}}} \int_E \Theta_E \, A_{\mu\nu\nu\mu} \, \omega_E$$

Set $A = -v\,\omega \otimes \omega$ for $v \in \mathring{\mathcal{V}} = \{u \in \Lambda^0(\mathscr{T}) \;:\; u \text{ continuous}, \; u|_{\partial\Omega} = 0\}$. Then

- On element T: $\langle \mathcal{R}|_T, A\rangle = 4K|_T\, v$, K Gauss curvature

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Gauss curvature:

$$K = \frac{\mathcal{R}_{1221}}{\det g}$$

Specialization to generalized Gauss curvature (2D)

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$$\widetilde{\mathcal{R}\omega}(A) = \sum_{T\in\mathscr{T}} \int_T \langle \mathcal{R}, A\rangle \omega_T + 4 \sum_{F\in\mathring{\mathscr{F}}} \int_F \langle [\![II]\!], A_{\cdot\nu\nu\cdot}|_F \rangle \omega_F + 4 \sum_{E\in\mathring{\mathscr{E}}} \int_E \Theta_E \, A_{\mu\nu\nu\mu}\, \omega_E$$

Set $A = -v\,\omega \otimes \omega$ for $v \in \mathring{\mathcal{V}} = \{u \in \Lambda^0(\mathscr{T}) \;:\; u \text{ continuous}, \; u|_{\partial\Omega} = 0\}$. Then

- On element T: $\langle \mathcal{R}|_T, A\rangle = 4K|_T\, v$, K Gauss curvature
- On edge F: $\langle [\![II]\!], A_{\cdot\nu\nu\cdot}|_F\rangle = [\![\kappa]\!]\, v$, κ geodesic curvature

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Geodesic curvature: with g-unit tangent τ along edge F

$$\kappa = g(\nu, \nabla_\tau \tau) = II^\nu(\tau, \tau)$$

$$\widetilde{\mathcal{R}\omega}(A) = \sum_{T\in\mathscr{T}} \int_T \langle \mathcal{R}, A\rangle \omega_T + 4 \sum_{F\in\mathring{\mathscr{F}}} \int_F \langle [\![I\!I]\!], A_{.\nu\nu.}|_F \rangle \omega_F + 4 \sum_{E\in\mathring{\mathscr{E}}} \int_E \Theta_E\, A_{\mu\nu\nu\mu}\, \omega_E$$

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- On vertex E: $\Theta_E A_{\mu\nu\nu\mu} = \Theta_E v$

Generalized Gauss curvature

$$\widetilde{K\omega}(v) = \frac{1}{4}\widetilde{\mathcal{R}\omega}(A) = \sum_{T\in\mathscr{T}} \int_T K|_T\, v\, \omega_T + \sum_{F\in\mathring{\mathscr{F}}} \int_F [\![\kappa]\!]\, v\, \omega_F + \sum_{E\in\mathring{\mathscr{E}}} \int_E \Theta_E\, v\, \omega_E$$

Berchenko-Kogan, Gawlik: Finite element approximation of the Levi-Civita connection and its curvature in two dimensions, *Found Comput Math* (2022).

Gopalakrishnan, N., Schöberl, Wardetzky: Analysis of curvature approximations via covariant curl and incompatibility for Regge metrics, *SMAI J. Comput. Math.* (2023).

Specialization to generalized scalar curvature

$$\widetilde{\mathcal{R}\omega}(A) = \sum_{T\in\mathscr{T}} \int_T \langle \mathcal{R}, A\rangle \omega_T + 4 \sum_{F\in\mathring{\mathscr{F}}} \int_F \langle [\![II]\!], A_{\cdot\nu\nu\cdot}|_F\rangle \omega_F + 4 \sum_{E\in\mathring{\mathscr{E}}} \int_E \Theta_E \, A_{\mu\nu\nu\mu} \, \omega_E$$

Set $A = v\, g \owedge g$ for $v \in \mathring{\mathcal{V}} = \{u \in \Lambda^0(\mathscr{T}) \,:\, u \text{ continuous},\ u|_{\partial\Omega} = 0\}$. Then

- On element T: $\langle \mathcal{R}|_T, A\rangle = 4S|_T v$, $S = g^{ij}g^{kl}\mathcal{R}_{kijl}$ scalar curvature

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Kulkarni-Nomizu product $\owedge$: produces a 4-tensor from two symmetric 2-tensors with Riemann symmetries

$$(h \owedge k)(X, Y, Z, W) = h(X, W)k(Y, Z) + h(Y, Z)k(X, W) - h(X, Z)k(Y, W) - h(Y, W)k(X, Z)$$

Specialization to generalized scalar curvature

$$\widetilde{\mathcal{R}\omega}(A) = \sum_{T\in\mathscr{T}} \int_T \langle \mathcal{R}, A\rangle \omega_T + 4 \sum_{F\in\mathring{\mathscr{F}}} \int_F \langle [\![II]\!], A_{\cdot\nu\nu\cdot}|_F\rangle \omega_F + 4 \sum_{E\in\mathring{\mathscr{E}}} \int_E \Theta_E\, A_{\mu\nu\nu\mu}\, \omega_E$$

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- On facet F: $\langle [\![II]\!], A_{\cdot\nu\nu\cdot}|_F\rangle = 2[\![H]\!]\, v$, H mean curvature

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Mean curvature: for a facet F

$$H^\nu = \mathrm{tr}(II^\nu) = g^{ij} II^\nu_{ij}$$

$$\widetilde{\mathcal{R}\omega}(A) = \sum_{T\in\mathscr{T}} \int_T \langle \mathcal{R}, A\rangle \omega_T + 4 \sum_{F\in\mathring{\mathscr{F}}} \int_F \langle [\![II]\!], A_{\cdot\nu\nu\cdot}|_F\rangle \omega_F + 4 \sum_{E\in\mathring{\mathscr{E}}} \int_E \Theta_E\, A_{\mu\nu\nu\mu}\, \omega_E$$

Set $A = v\, g \owedge g$ for $v \in \mathring{\mathcal{V}} = \{u \in \Lambda^0(\mathscr{T}) \,:\, u \text{ continuous},\ u|_{\partial\Omega} = 0\}$. Then

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- On E: $\Theta_E A_{\mu\nu\nu\mu} = 2\Theta_E v$

Generalized scalar curvature

$$\widetilde{S\omega}(v) = \frac{1}{4}\widetilde{\mathcal{R}\omega}(A) = \sum_{T\in\mathscr{T}} \int_T S|_T\, v\, \omega_T + 2 \sum_{F\in\mathring{\mathscr{F}}} \int_F [\![H]\!]\, v\, \omega_F + 2 \sum_{E\in\mathring{\mathscr{E}}} \int_E \Theta_E\, v\, \omega_E$$

Gawlik, N.: Finite element approximation of scalar curvature in arbitrary dimension, *Math. Comp.* (2024).

Specialization to generalized Ricci curvature tensor

$$\widetilde{\mathcal{R}\omega}(A) = \sum_{T\in\mathscr{T}} \int_T \langle \mathcal{R}, A\rangle \omega_T + 4 \sum_{F\in\mathring{\mathscr{F}}} \int_F \langle [\![I\!I]\!], A_{\cdot\nu\nu\cdot}|_F\rangle \omega_F + 4 \sum_{E\in\mathring{\mathscr{E}}} \int_E \Theta_E \, A_{\mu\nu\nu\mu}\, \omega_E$$

Set $A = g \owedge \sigma$ for $\sigma \in \mathring{\Sigma} = \{J\rho : \rho \in \mathring{\mathcal{R}}\}$. Then

- On element T: $\langle \mathcal{R}|_T, A\rangle = 4\langle \mathrm{Ric}|_T, \sigma\rangle$, $\mathrm{Ric} = g^{ij}\mathcal{R}_{kijl}$ Ricci curvature tensor

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$J : \mathcal{S}(\mathscr{T}) \to \mathcal{S}(\mathscr{T})$ is a bijective algebraic operator

$$J\rho = \rho - \frac{1}{2}\mathrm{tr}(\rho) g$$

$$\widetilde{\mathcal{R}\omega}(A) = \sum_{T\in\mathscr{T}} \int_T \langle \mathcal{R}, A\rangle \omega_T + 4 \sum_{F\in\mathring{\mathscr{F}}} \int_F \langle [\![I\!I]\!], A_{\cdot\nu\nu\cdot}|_F\rangle \omega_F + 4 \sum_{E\in\mathring{\mathscr{E}}} \int_E \Theta_E\, A_{\mu\nu\nu\mu}\, \omega_E$$

Set $A = g \owedge \sigma$ for $\sigma \in \mathring{\Sigma} = \{J\rho \,:\, \rho \in \mathring{\mathcal{R}}\}$. Then

- On element T: $\langle \mathcal{R}|_T, A\rangle = 4\langle \mathrm{Ric}|_T, \sigma\rangle$, $\mathrm{Ric} = g^{ij}\mathcal{R}_{kijl}$ Ricci curvature tensor
- On facet F: $\langle [\![I\!I]\!], A_{\cdot\nu\nu\cdot}|_F\rangle = \langle [\![I\!I]\!], \sigma|_F + \sigma(\nu,\nu) g|_F\rangle$
- On E: $\Theta_E A_{\mu\nu\nu\mu} = (\sigma(\nu,\nu) + \sigma(\mu,\mu))\,\Theta_E$

Generalized Ricci curvature tensor

$$\widetilde{\mathrm{Ric}\omega}(\sigma) = \sum_{T\in\mathscr{T}} \int_T \langle \mathrm{Ric}|_T, \sigma\rangle \omega_T + \sum_{F\in\mathring{\mathscr{F}}} \int_F \langle [\![I\!I]\!], \sigma|_F + \sigma(\nu,\nu) g|_F\rangle \omega_F + \sum_{E\in\mathring{\mathscr{E}}} \int_E (\sigma(\nu,\nu) + \sigma(\mu,\mu))\,\Theta_E \omega_E$$

Gopalakrishnan, N., Schöberl, Wardetzky: Generalizing Riemann curvature to Regge metrics, *arXiv:2311.01603*.

$$\widetilde{\mathcal{R}\omega}(A) = \sum_{T\in\mathscr{T}} \int_T \langle \mathcal{R}, A\rangle \omega_T + 4 \sum_{F\in\mathring{\mathscr{F}}} \int_F \langle [\![I\!I]\!], A_{.\nu\nu.}|_F\rangle \omega_F + 4 \sum_{E\in\mathring{\mathscr{E}}} \int_E \Theta_E\, A_{\mu\nu\nu\mu}\, \omega_E$$

Use that $G = \mathrm{Ric} - 1/2 S\, g = J\,\mathrm{Ric}$. Set $A = g \owedge J\sigma$ for $\sigma \in \mathring{\mathcal{R}}$. Then

- On element T: $\langle \mathcal{R}|_T, A\rangle = 4\langle G|_T, \sigma\rangle$
- On facet F: $\langle [\![I\!I]\!], A_{.\nu\nu.}|_F\rangle = \langle [\![I\!I]\!], \sigma|_F - \mathrm{tr}(\sigma|_F) g|_F\rangle = \langle [\![I\!I]\!], \mathbb{S}_F\sigma|_F\rangle = \langle [\![\overline{I\!I}]\!], \sigma|_F\rangle$ where $\mathbb{S}_F\sigma = \sigma|_F - \mathrm{tr}(\sigma|_F) g|_F$ and $\overline{I\!I} = \mathbb{S}_F I\!I = I\!I - H\, g|_F$ the trace-reversed second fundamental form
- On E: $\Theta_E A_{\mu\nu\nu\mu} = -\mathrm{tr}(\sigma|_E)\,\Theta_E$

Generalized Einstein tensor

$$\widetilde{G\omega}(\sigma) = \sum_{T\in\mathscr{T}} \int_T \langle G|_T, \sigma\rangle \omega_T + \sum_{F\in\mathring{\mathscr{F}}} \int_F \langle [\![\overline{I\!I}]\!], \sigma|_F\rangle \omega_F - \sum_{E\in\mathring{\mathscr{E}}} \int_E \mathrm{tr}(\sigma|_E)\, \Theta_E\, \omega_E$$

GAWLIK, N.: Finite element approximation of the Einstein tensor, *IMA J. Numer. Anal.* (2025).

- Assume that $g_h \in \mathcal{R}_h^k$ converges to a smooth metric $\bar{g}$ for $h \to 0$. Does $\widetilde{\mathcal{R}\omega}_{g_h} \to (\mathcal{R}\omega)_{\bar{g}}$?
- Approach: Use its **integral representation**: For Gauss curvature

$$\widetilde{K\omega}_{g}(v) - (K\omega)_{\bar{g}}(v) = \int_0^1 \frac{d}{dt}\widetilde{K\omega}_{g(t)}(v)\, dt, \qquad v \in \mathring{\mathcal{V}},$$

where $g(t) = \bar{g} + t(g - \bar{g})$. Extend to N-dimensions

$$\widetilde{\mathcal{R}\omega}_{g}(A) - (\mathcal{R}\omega)_{\bar{g}}(A) = \int_0^1 \frac{d}{dt}\widetilde{\mathcal{R}\omega}_{g(t)}(A)\, dt, \qquad A \in \mathring{\mathcal{A}},$$

GAWLIK: High-order approximation of Gaussian curvature with Regge finite elements, *SIAM J. Numer. Anal.* (2020).

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- Problems:
  1. Test function A depends on the metric tensor g
  2. Need to linearize curvature contributions

GAWLIK: High-order approximation of Gaussian curvature with Regge finite elements, *SIAM J. Numer. Anal.* (2020).

We use an approach inspired by the Uhlenbeck trick: Define the metric independent test space

$$\mathcal{U} = \{U \in \Lambda^{N-2}(\mathscr{T}) \odot \Lambda^{N-2}(\mathscr{T}) \;:\; U(X_1, \ldots, X_{N-2}, Y_1, \ldots, Y_{N-2}) \text{ is single-valued}$$
$$\text{on all } F \in \mathring{\mathscr{F}} \text{ for all } X_1, \ldots X_{N-2}, Y_1, \ldots, Y_{N-2} \in \mathfrak{X}(F)\}$$

Lemma

The map $\mathbb{A}_g : \mathcal{U} \to \mathcal{A}$, $U \mapsto -\star^{\odot^2} U$ is a bijection.

Define $\widetilde{\mathcal{R}\omega\mathbb{A}}_g(U) = \widetilde{\mathcal{R}\omega}(\mathbb{A}_g(U))$ for g-independent $U \in \mathcal{U}$. Then we have

$$\widetilde{\mathcal{R}\omega\mathbb{A}}_g(U) - (\mathcal{R}\omega\mathbb{A})_{\bar{g}}(U) = \int_0^1 \frac{d}{dt} \widetilde{\mathcal{R}\omega\mathbb{A}}_{g(t)}(U)\, dt.$$

We can proceed computing and estimating the right-hand side.

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Theorem (Gopalakrishnan, N., Schöberl, Wardetzky)

Suppose g_h is a collection of Regge metrics such that $g_h \to \bar{g}$ in L^∞ and g_h is uniformly bounded in $W^{2,\infty}$. Then

$$\|\widetilde{\mathcal{R}\omega\mathbb{A}}_{g_h} - \mathcal{R}\omega\mathbb{A}_{\bar{g}}\|_{H^{-2}} \leqslant C_{\bar{g},g_h} |||g_h - \bar{g}|||_2.$$

Here,

$$|||\sigma|||_2^2 = \sum_{T\in\mathscr{T}} \left(\|\sigma\|_{L^2(T)}^2 + h^2\|\sigma\|_{H^1(T)}^2 + h^4\|\sigma\|_{H^2(T)}^2\right)$$

$$C_{\bar{g},g_h} = C\left(1 + \max_{T\in\mathscr{T}} h_T^{-2+\delta_2^N}\|g_h - \bar{g}\|_{L^\infty(T)} + \max_{T\in\mathscr{T}} h_T^{-1}\|g_h - \bar{g}\|_{W^{1,\infty}(T)}\right)$$

Corollary

If additionally $\|g_h - \bar{g}\|_{W^{t,\infty}} \lesssim h^{s-t}\|\bar{g}\|_{W^{s,\infty}}$ for $0 \leqslant t \leqslant s \leqslant k+1$ for some $k \geqslant 1-\delta_2^N$, then

$$\|\widetilde{\mathcal{R}\omega\mathbb{A}}_{g_h} - \mathcal{R}\omega\mathbb{A}_{\bar{g}}\|_{H^{-2}} \lesssim \mathcal{O}(h^{k+1}).$$

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Incompatibility operator

Define for smooth 2-tensor σ the incompatibility operator $\operatorname{Inc}: \mathcal{T}^2 \to \mathcal{T}^4$ by

$$(\operatorname{Inc}\sigma)(X,Y,Z,W) := \frac{1}{4}\big[(\nabla^2_{Y,Z}\sigma)(X,W) + (\nabla^2_{X,W}\sigma)(Y,Z) - (\nabla^2_{X,Y}\sigma)(Z,W) - (\nabla^2_{Y,W}\sigma)(X,Z)\big].$$

In 2D and 3D Inc can be related to the standard incompatibility operator $\operatorname{inc} = \operatorname{curl}^T\operatorname{curl}$.

Lemma (linearization Riemann curvature tensor)

For t-independent vector fields $X, Y, Z, W \in \mathfrak{X}(T)$ there holds

$$\dot{\mathcal{R}}(X,Y,Z,W) = -\frac{1}{2}(\operatorname{Inc}\dot{g})(X,Y,Z,W) + \frac{1}{2}\left[\dot{g}(\mathcal{R}_{X,Y}Z,W) - \dot{g}(\mathcal{R}_{X,Y}W,Z)\right].$$

Generalized incompatibility operator

For tt-continuous σ, a generalized Inc can be defined as

$$\widetilde{\operatorname{Inc}\sigma}(A) = \sum_{T\in\mathscr{T}}\int_T \langle \operatorname{Inc}\sigma, A\rangle\,\omega_T + \sum_{F\in\mathring{\mathscr{F}}}\dots + \sum_{E\in\mathring{\mathscr{E}}}\dots$$

Roadmap of the analysis

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1. $\widetilde{\mathcal{R}\omega\mathbb{A}}_{g_h}(U) - (\mathcal{R}\omega\mathbb{A})_{\bar{g}}(U) = \int_0^1 \frac{d}{dt}\widetilde{\mathcal{R}\omega\mathbb{A}}_{g(t)}(U)\,dt$ with $g(t) = \bar{g} + t(g_h - \bar{g})$.

2. $\frac{d}{dt}\widetilde{\mathcal{R}\omega\mathbb{A}}_{g(t)}(U) = a(g;\dot{g},U) + b(g;\dot{g},U)$ sum of the bilinear forms a and b.

3. $b(g;\dot{g},U) = -2\,\widetilde{\mathrm{Inc}\,\dot{g}}(A)$, with $\dot{g} = g_h - \bar{g}$ and $A = \mathbb{A}_g(U)$.
   - Analyze the adjoint: $\widetilde{\mathrm{Inc}\,\dot{g}}(A) = (\widetilde{\mathrm{Inc}^* A})(\dot{g})$
     Then all spatial derivatives are applied on the test function A, not $\dot{g}$.
   - $b(g;\dot{g},U) \leqslant C_{g_h,\bar{g}} \|\!|g_h - \bar{g}|\!\|_2 \|U\|_{H^2}$

4. $a(g;\dot{g},U)$ has no spatial derivatives of $\dot{g}$
   - $a = 0$ in 2D, but $a \neq 0$ in higher dimensions
   - $a(g;\dot{g},U) \leqslant C_{g_h,\bar{g}} \|\!|g_h - \bar{g}|\!\|_2 \|U\|_{H^2}$

Numerical results (2D)

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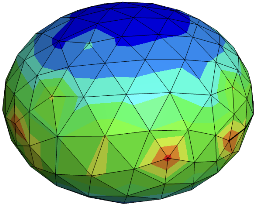
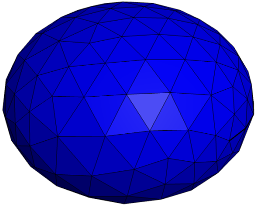
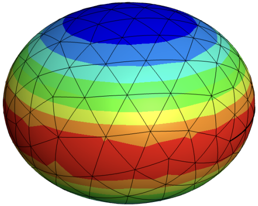
$k = 0$

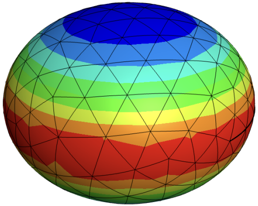


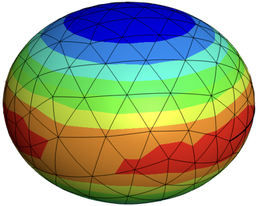


NGSolve

$k = 1$







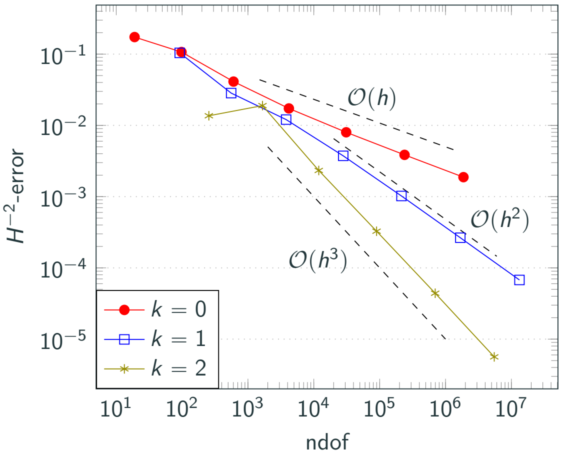
all terms | no angle defect | no geodesic curvature

$$\widetilde{K\omega}(v) = \sum_{T\in\mathscr{T}} \int_T K|_T \, v \, \omega_T \quad + \quad \sum_{E\in\mathring{\mathscr{E}}} \int_E \Theta_E \, v \, \omega_E \quad + \quad \sum_{F\in\mathring{\mathscr{F}}} \int_F [\![\kappa]\!] \, v \, \omega_F$$

$$\Phi(x,y,z) = (x,y,z,f(x,y,z)),$$
$$f(x,y,z) = \tfrac{1}{2}(x^2+y^2+z^2) - \tfrac{1}{12}(x^4+y^4+z^4)$$
$$\bar{g} = \nabla\Phi^T\nabla\Phi, \qquad q(x) = x^2(x^2-3)^2$$

$$\mathcal{R}_{ijkl} = \varepsilon_{ijr}\varepsilon_{kls}\delta^{rs}\frac{9\prod_{m\neq r}(x_m^2-1)}{q(x)+q(y)+q(z)+9}$$

- Confirms theory for $k \geqslant 1$
- For $k = 0$ linear convergence is observed?!

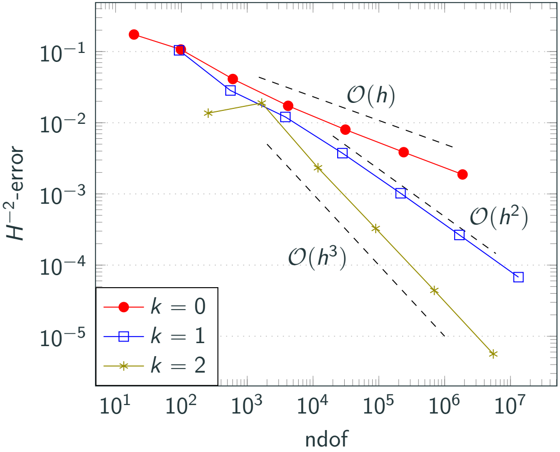


Numerical results (3D)

$$\Phi(x,y,z) = (x,y,z,f(x,y,z)),$$
$$f(x,y,z) = \tfrac{1}{2}(x^2+y^2+z^2) - \tfrac{1}{12}(x^4+y^4+z^4)$$
$$\bar{g} = \nabla\Phi^T\nabla\Phi, \qquad q(x) = x^2(x^2-3)^2$$

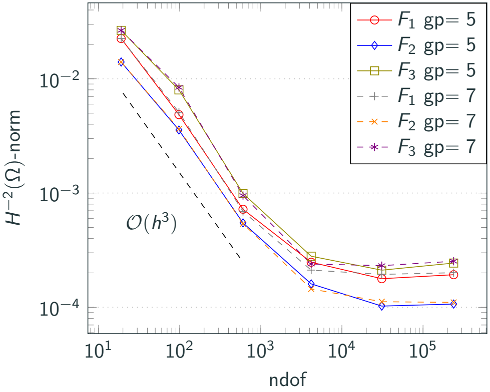
$$\mathcal{R}_{ijkl} = \varepsilon_{ijr}\varepsilon_{kls}\delta^{rs}\frac{9\prod_{m\neq r}(x_m^2-1)}{q(x)+q(y)+q(z)+9}$$

- Confirms theory for $k \geqslant 1$
- For $k=0$ linear convergence is observed?!
- Test only parts where theory indicates no convergence

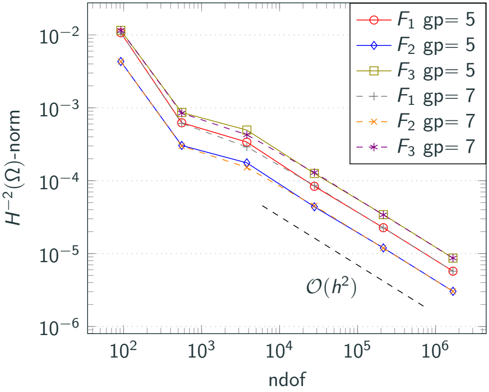


Test convergence of theoretical sub-optimal terms (F_1, F_2, $F_1 + F_2 =: F_3$)

We observe rapid convergence, then stagnation of error → pre-asymptotic



$k = 0$



$k = 1$

Summary & Outlook

- Definition of generalized Riemann curvature tensor (Gauss, scalar, Ricci, Einstein)
- Numerical analysis with integral representation
- Uhlenbeck trick for test functions
- Generalized incompatibility operator and adjoint

Summary & Outlook

- Extrinsic curvature of embedded submanifolds & connection 1-form
- Framework combining discrete differential geometry and distributional FEM
- NGSDiffGeo: `https://github.com/MichaelNeunteufel/NGSDiffGeo`
- Application to (nonlinear) shell analysis, geometric flows, and numerical relativity

Summary & Outlook

- Extrinsic curvature of embedded submanifolds & connection 1-form
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Thank You for Your attention!

Gopalakrishnan, N., Schöberl, Wardetzky: Analysis of curvature approximations via covariant curl and incompatibility for Regge metrics, *SMAI J. Comput. Math.* (2023).

Gopalakrishnan, N., Schöberl, Wardetzky: On the improved convergence of lifted distributional Gauss curvature from Regge elements, *RINAM* (2024).

Gawlik, N.: Finite element approximation of scalar curvature in arbitrary dimension, *Math. Comp.* (2024).

Gawlik, N.: Finite element approximation of the Einstein tensor, *IMA J. Numer. Anal.* (2025).

Gopalakrishnan, N., Schöberl, Wardetzky: Generalizing Riemann curvature to Regge metrics, *arXiv:2311.01603*.

Lemma

Let $\sigma := \dot{g}(t)$, $A \in \mathcal{A}$, $(SA)(X, Y, Z, W) = A(X, Z, Y, W)$ swaps second with third argument. There holds

$$\dot{A}(X, Y, Z, W) = -\mathrm{tr}(\sigma)A(X, Y, Z, W) + A(\sigma(X, \cdot)^\sharp, Y, Z, W) + A(X, \sigma(Y, \cdot)^\sharp, Z, W) + A(X, Y, \sigma(Z, \cdot)^\sharp, W) + A(X, Y, Z, \sigma(W, \cdot)^\sharp),$$

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$$\begin{aligned}
\dot{A}(X,Y,Z,W) &= -\mathrm{tr}(\sigma)A(X,Y,Z,W) + A(\sigma(X,\cdot)^\sharp,Y,Z,W) + A(X,\sigma(Y,\cdot)^\sharp,Z,W)\\
&\qquad + A(X,Y,\sigma(Z,\cdot)^\sharp,W) + A(X,Y,Z,\sigma(W,\cdot)^\sharp),\\
\frac{d}{dt}\big(\langle \mathcal{R},A\rangle\omega_T\big) &= \big(2\langle\nabla^2\sigma,S(A)\rangle + \langle\mathcal{R}(\sigma(\cdot,\cdot)^\sharp,\cdot,\cdot,\cdot),A\rangle - \frac{1}{2}\mathrm{tr}(\sigma)\langle\mathcal{R},A\rangle\big)\omega_T,\\
\frac{d}{dt}\big(\langle[\![I\!I]\!],A_{\cdot\nu\nu\cdot}|_F\rangle\omega_F\big) &= \frac{1}{2}\langle[\![(\sigma(\nu,\nu)-\mathrm{tr}(\sigma|_F))I\!I + 2(\nabla_F\sigma)(\nu,\cdot)|_F - (\nabla_\nu\sigma)|_F]\!],A_{\cdot\nu\nu\cdot}|_F\rangle\,\omega_F,\\
\frac{d}{dt}\big(\Theta_E A_{\mu\nu\nu\mu}\,\omega_E\big) &= -\frac{1}{2}\Big(\sum_{F\supset E}[\![\sigma(\nu,\mu)]\!]_F^E + \mathrm{tr}(\sigma|_E)\Theta_E\Big)A_{\mu\nu\nu\mu}\,\omega_E.
\end{aligned}$$

Proposition

Let $\sigma := \dot{g}$ and $A \in \mathring{\mathcal{A}}$ with corresponding $U = \mathbb{A}_g^{-1} A \in \mathring{\mathcal{U}}$. Then there holds

$$\frac{d}{dt}\widetilde{\mathcal{R}\omega\mathbb{A}_g}(U) = a(g;\sigma,U) + b(g;\sigma,U),$$

$$\begin{aligned} a(g;\sigma,U) = &\sum_{T\in\mathscr{T}}\int_T \left(\langle \mathcal{R}(\sigma(\cdot,\cdot)^\sharp,\cdot,\cdot,\cdot),\mathbb{A}_g U\rangle - \frac{1}{2}\mathrm{tr}(\sigma)\langle\mathcal{R},\mathbb{A}_g U\rangle\right)\omega_T \\ &- 2\sum_{F\in\mathring{\mathscr{F}}}\int_F \Big(\mathrm{tr}(\sigma|_F)\langle [\![II]\!], (\mathbb{A}_g U)_{.\nu\nu.}|_F\rangle - [\![II]\!] : \sigma|_F : (\mathbb{A}_g U)_{.\nu\nu.}|_F\Big)\omega_F \\ &- 2\sum_{E\in\mathring{\mathscr{E}}}\int_E \mathrm{tr}(\sigma|_E)\Theta_E(\mathbb{A}_g U)_{\mu\nu\nu\mu}\,\omega_E \end{aligned}$$

$[\![II]\!] : \sigma|_F : (\mathbb{A}_g U)_{.\nu\nu.} = [\![II]\!]_{ij}(\sigma|_F)^{jk}((\mathbb{A}_g U)_{.\nu\nu.})_k{}^i.$

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$$\begin{aligned} b(g;\sigma,U) &= 2\sum_{T\in\mathscr{T}}\int_T \langle \nabla^2\sigma, S\mathbb{A}_g U\rangle \omega_T \\ &+ 2\sum_{F\in\mathring{\mathscr{F}}}\int_F \langle [\![\sigma(\nu,\nu)II + (\nabla_F\sigma)(\nu,\cdot)|_F + \nabla_F(\sigma(\nu,\cdot))|_F - (\nabla_\nu\sigma)|_F]\!], (\mathbb{A}_g U)_{\cdot\nu\nu\cdot}|_F\rangle \omega_F \\ &- 2\sum_{E\in\mathring{\mathscr{E}}}\int_E \sum_{F\supset E}[\![\sigma(\nu,\mu)]\!]_F^E (\mathbb{A}_g U)_{\mu\nu\nu\mu}\,\omega_E. \end{aligned}$$