The Hellan–Herrmann–Johnson Method for Nonlinear Shells

Michael Neunteufel, Joachim Schöberl

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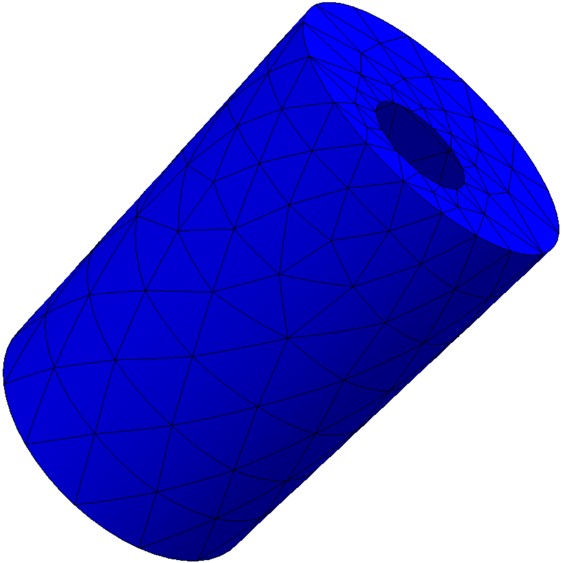
Motivation



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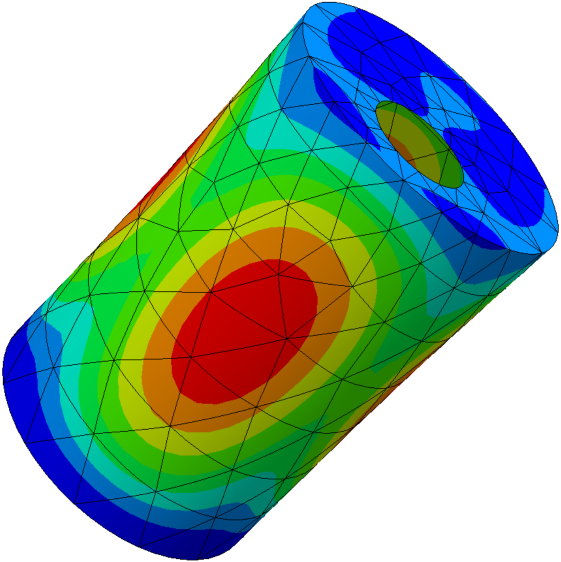

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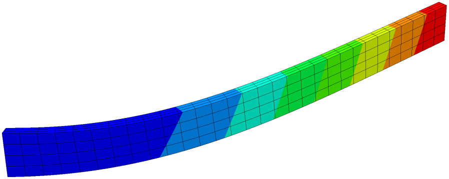
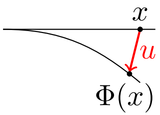
Notation

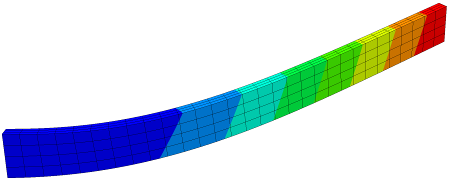
Deformation $\Phi : \Omega \to \mathbb{R}^3$



Deformation $\Phi : \Omega \to \mathbb{R}^3$

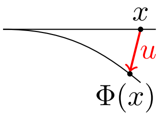
Displacement $u := \Phi - id$

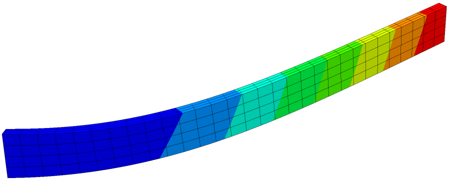




Continuum mechanics

Deformation	$\Phi : \Omega \to \mathbb{R}^3$
Displacement	$u := \Phi - id$
Deformation gradient	$\boldsymbol{F} := \nabla\Phi$





Continuum mechanics

Deformation $\Phi : \Omega \to \mathbb{R}^3$

Displacement $u := \Phi - id$

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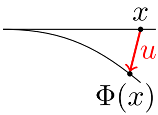
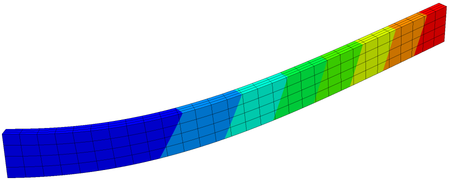
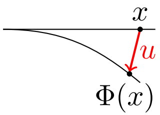
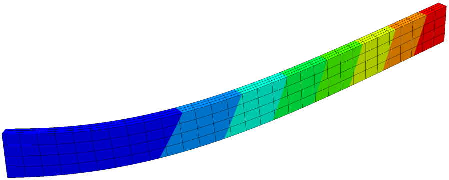




Deformation	$\Phi : \Omega \to \mathbb{R}^3$
Displacement	$u := \Phi - id$
Deformation gradient	$\boldsymbol{F} := \boldsymbol{I} + \nabla u$
Cauchy-Green strain tensor	$\boldsymbol{C} := \boldsymbol{F}^T \boldsymbol{F}$





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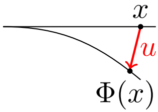
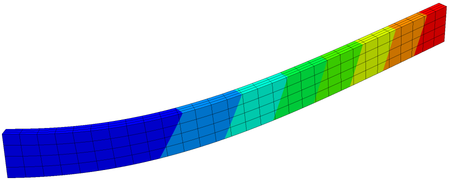
Green strain tensor	$\boldsymbol{E} := \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I})$





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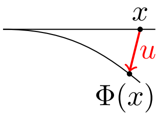
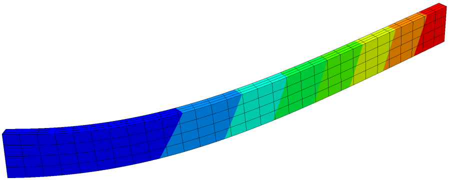


Elasticity

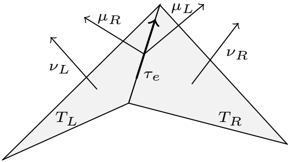
$$\mathcal{W}(u) = \frac{1}{2}\|\boldsymbol{E}\|^2_{\boldsymbol{M}} - \langle f, u \rangle$$



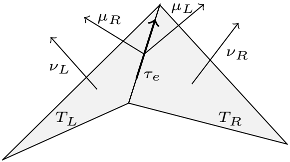
- Normal vector ν

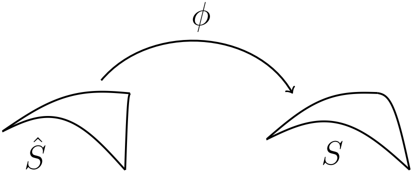
  Tangent vector τ_e

  Element normal vector $\mu = \pm\nu \times \tau_e$



- Normal vector $\hat{\nu}$
  Tangent vector $\hat{\tau}_e$
  Element normal vector $\hat{\mu} = \pm\hat{\nu} \times \hat{\tau}_e$





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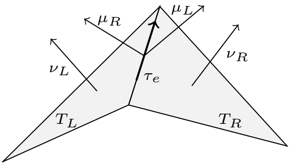
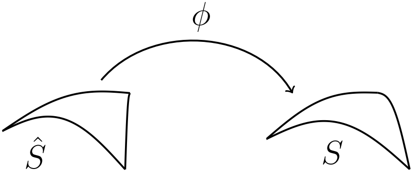


- $\boldsymbol{F} = \nabla_{\hat{\tau}}\phi$, $J = \|\operatorname{cof}(\boldsymbol{F})\|_F$



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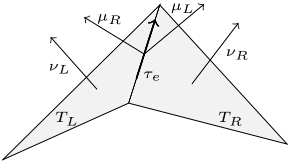
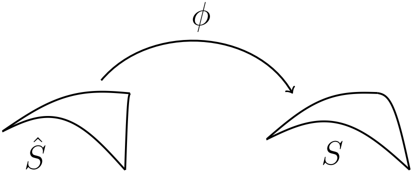
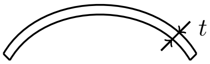
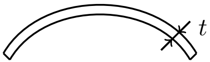


- $\boldsymbol{F} = \nabla_{\hat{\tau}}\phi,\ J = \|\operatorname{cof}(\boldsymbol{F})\|_F$
- $\nu \circ \phi = \frac{1}{J}\operatorname{cof}(\boldsymbol{F})\hat{\nu}$
  $\tau_e \circ \phi = \frac{1}{J_B}\boldsymbol{F}\hat{\tau}_e$
  $\mu \circ \phi = \pm\nu \times \tau_e$



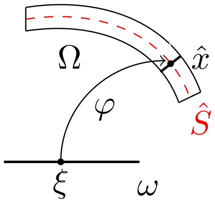


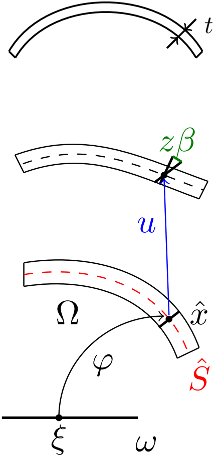
- Model of reduced dimensions



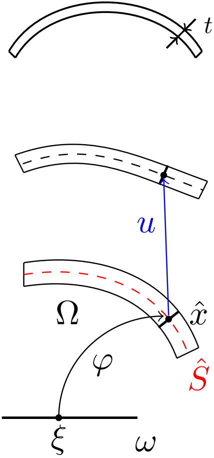
- Model of reduced dimensions

- $\Omega = \left\{ \varphi(\xi) + z\hat{\nu}(\xi) : \xi \in \omega, z \in \left[-\frac{t}{2}, \frac{t}{2}\right] \right\}$





- Model of reduced dimensions
- $\Omega = \left\{\varphi(\xi) + z\hat{\nu}(\xi) :\ \xi \in \omega, z \in \left[-\frac{t}{2}, \frac{t}{2}\right]\right\}$
- $\Phi(\hat{x} + z\hat{\nu}(\xi)) = \phi(\hat{x}) + z\ (\nu + \beta) \circ \phi(\hat{x})$



- Model of reduced dimensions
- $\Omega = \left\{\varphi(\xi) + z\hat{\nu}(\xi) : \xi \in \omega, z \in \left[-\frac{t}{2}, \frac{t}{2}\right]\right\}$
- $\Phi(\hat{x} + z\hat{\nu}(\xi)) = \phi(\hat{x}) + z\,\nu \circ \phi(\hat{x})$

Shell energy (Kirchhoff–Love)

$$\mathcal{W}(u) = \frac{t}{2}\|\boldsymbol{E}_{\tau\tau}(u)\|^2_{\boldsymbol{M}} + \frac{t^3}{24}\|\boldsymbol{F}^T\nabla(\nu\circ\phi) - \nabla\hat{\nu}\|^2_{\boldsymbol{M}}$$

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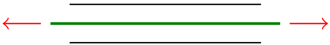
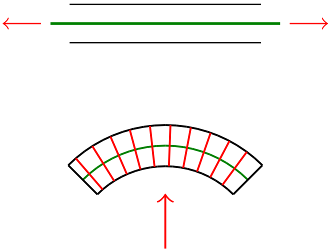


- Membrane energy

$$\mathcal{W}(u) = \frac{t}{2}\|\boldsymbol{E}_{\tau\tau}(u)\|^2_{\boldsymbol{M}} + \frac{t^3}{24}\|\boldsymbol{F}^T\nabla(\nu\circ\phi) - \nabla\hat{\nu}\|^2_{\boldsymbol{M}}$$



- Membrane energy
- Bending energy

Shell energy (Kirchhoff–Love)

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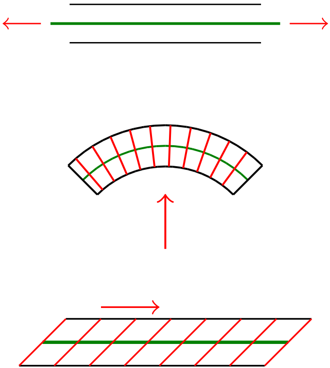
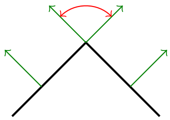


- Membrane energy
- Bending energy
- Shearing energy

Method and Shell Element

Moment tensor

$$\mathcal{W}(u) = \frac{t}{2}\|\boldsymbol{E}_{\tau\tau}(u)\|^2_{\boldsymbol{M}} + \frac{t^3}{24}\|\boldsymbol{F}^T\nabla\nu - \nabla\hat{\nu}\|^2_{\boldsymbol{M}} + \frac{t^3}{24}\sum_{\hat{E}\in\hat{\mathcal{E}}_h}\|\sphericalangle(\nu_L,\nu_R) - \sphericalangle(\hat{\nu}_L,\hat{\nu}_R)\|^2_{\boldsymbol{M},\hat{E}}$$



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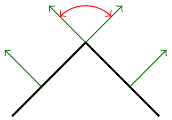
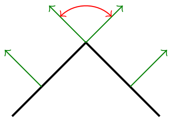


- Measure change of angles

$$\mathcal{W}(u) = \frac{t}{2}\|\boldsymbol{E}_{\tau\tau}(u)\|^2_{\boldsymbol{M}} + \frac{t^3}{24}\|\boldsymbol{F}^T\nabla\nu - \nabla\hat{\nu}\|^2_{\boldsymbol{M}} + \frac{t^3}{24}\sum_{\hat{E}\in\hat{\mathcal{E}}_h}\|\sphericalangle(\nu_L,\nu_R) - \sphericalangle(\hat{\nu}_L,\hat{\nu}_R)\|^2_{\boldsymbol{M},\hat{E}}$$



- Measure change of angles

$$\mathcal{L}(u,\boldsymbol{\sigma}) = \frac{t}{2}\|E_{\tau\tau}(u)\|^2_{\boldsymbol{M}} - \frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\boldsymbol{M}^{-1}} + \langle\boldsymbol{F}^T\nabla\nu - \nabla\hat{\nu},\boldsymbol{\sigma}\rangle + \sum_{\hat{E}\in\hat{\mathcal{E}}_h}\langle\sphericalangle(\nu_L,\nu_R) - \sphericalangle(\hat{\nu}_L,\hat{\nu}_R),\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\rangle_{\hat{E}}$$

Moment tensor

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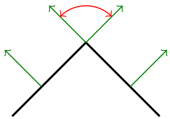

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- $\boldsymbol{\sigma}$ has physical meaning of moment

Moment tensor

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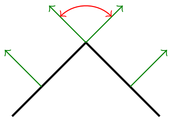

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- $\boldsymbol{\sigma}$ has physical meaning of moment
- Fourth order problem $\to$ second order problem

Shell problem

Find $u \in [H^1(\hat{S})]^3$ and $\boldsymbol{\sigma} \in H(\mathrm{divdiv}, \hat{S})$ for the saddle point problem

$$\mathcal{L}(u, \boldsymbol{\sigma}) = \frac{t}{2}\|E_{\tau\tau}(u)\|^2_{\boldsymbol{M}} - \frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\boldsymbol{M}^{-1}} + G(u, \boldsymbol{\sigma}) - \langle f, u\rangle,$$

with

$$G(u, \boldsymbol{\sigma}) = \sum_{\hat{T}\in\hat{\mathcal{T}}_h} \int_{\hat{T}} \boldsymbol{\sigma} : (\boldsymbol{H}_\nu + (1 - \hat{\nu}\cdot\nu)\nabla\hat{\nu})\, d\hat{x}$$
$$- \sum_{\hat{E}\in\hat{\mathcal{E}}_h} \int_{\hat{E}} (\sphericalangle(\nu_L, \nu_R) - \sphericalangle(\hat{\nu}_L, \hat{\nu}_R))\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\, d\hat{s}.$$

$\boldsymbol{H}_\nu := \sum_i (\nabla^2 u_i)\nu_i$

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$$- \sum_{\hat{E}\in\hat{\mathcal{E}}_h} \int_{\hat{E}} (\sphericalangle(\nu_L, \nu_R) \qquad\qquad) \boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}} \, d\hat{s}.$$

$\boldsymbol{H}_\nu := \sum_i (\nabla^2 u_i)\nu_i$

Shell problem (Hybridization)

Find $u \in [H^1(\hat{S})]^3$, $\boldsymbol{\sigma} \in H(\mathrm{divdiv}, \hat{S})^{dc}$ and $\alpha \in \Gamma(\hat{S})$ for

$$\mathcal{L}(u, \boldsymbol{\sigma}) = \frac{t}{2}\|E_{\tau\tau}(u)\|^2_{\boldsymbol{M}} - \frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\boldsymbol{M}^{-1}} + G(u, \boldsymbol{\sigma}, \alpha) - \langle f, u\rangle,$$

with

$$\begin{aligned} G(u, \boldsymbol{\sigma}, \alpha) = & \sum_{\hat{T}\in\hat{\mathcal{T}}_h} \int_{\hat{T}} \boldsymbol{\sigma} : (\boldsymbol{H}_\nu + (1 - \hat{\nu}\cdot\nu)\nabla\hat{\nu})\, d\hat{x} \\ & - \sum_{\hat{E}\in\hat{\mathcal{E}}_h} \int_{\hat{E}} (\sphericalangle(\nu_L, \nu_R) - \sphericalangle(\hat{\nu}_L, \hat{\nu}_R))\frac{1}{2}(\boldsymbol{\sigma}_{\hat{\mu}_L\hat{\mu}_L} + \boldsymbol{\sigma}_{\hat{\mu}_R\hat{\mu}_R})\, d\hat{s} \\ & + \int_{\hat{E}} \alpha_{\hat{\mu}} [\![\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}]\!]\, d\hat{s}. \end{aligned}$$

The space H(divdiv)

$$H^1(\Omega) := \{u \in L^2(\Omega) \,|\, \nabla u \in [L^2(\Omega)]^d\}$$

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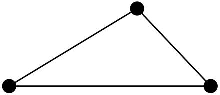
$$V_k := \Pi^k(\mathcal{T}_h) \cap C(\Omega)$$



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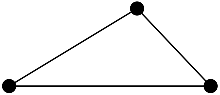
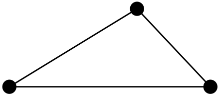
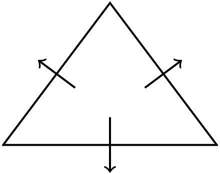
$$H(\mathrm{div}) := \{\sigma \in [L^2(\Omega)]^d \mid \mathrm{div}(\sigma) \in L^2(\Omega)\}$$



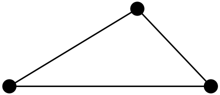
$$H(\mathrm{div}) := \{\sigma \in [L^2(\Omega)]^d \,|\, \mathrm{div}(\sigma) \in L^2(\Omega)\}$$

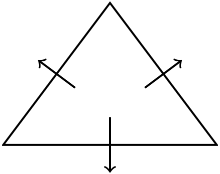
$$BDM_k := \{\sigma \in [\Pi^k(\mathcal{T}_h)]^d \,|\, \sigma_n \text{ is continuous over elements}\}$$





$$H(\mathrm{divdiv}) := \{\boldsymbol{\sigma} \in [L^2(\Omega)]^{d\times d}_{sym} \mid \mathrm{div}(\mathrm{div}(\boldsymbol{\sigma})) \in H^{-1}(\Omega)\}$$

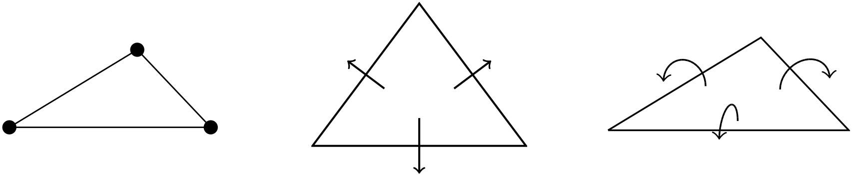




The space H(divdiv)

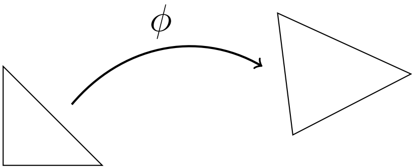
$$H(\mathrm{divdiv}) := \{\boldsymbol{\sigma} \in [L^2(\Omega)]^{d\times d}_{sym} \mid \mathrm{div}(\mathrm{div}(\boldsymbol{\sigma})) \in H^{-1}(\Omega)\}$$

$$M_h^k := \{\boldsymbol{\sigma} \in [\Pi^k(\mathcal{T}_h)]^{d\times d}_{sym} \mid n^T \boldsymbol{\sigma} n \text{ is continuous over elements}\}$$



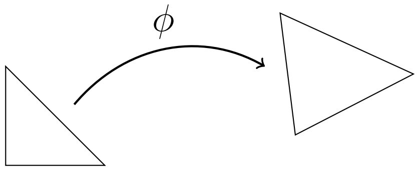
A. PECHSTEIN AND J. SCHÖBERL: The TDNNS method for Reissner-Mindlin plates, *J. Numer. Math. (2017) 137, pp. 713-740.*

Mapping to the surface



- Piola transformation

$$u \circ \phi = P[\hat{u}] = \frac{1}{J}\boldsymbol{F}\hat{u} \qquad\qquad \boldsymbol{F} = \nabla_{\hat{x}}\phi,\ J = \det(\boldsymbol{F})$$



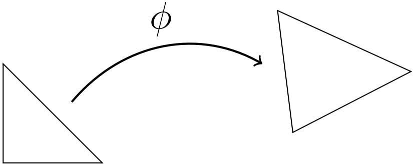
Mapping to the surface

- Piola transformation

$$u \circ \phi = P[\hat{u}] = \frac{1}{J} \boldsymbol{F} \hat{u} \qquad \boldsymbol{F} = \nabla_{\hat{x}} \phi, \; J = \det(\boldsymbol{F})$$

- Preserve normal-normal continuity

$$\boldsymbol{\sigma} \circ \phi = \frac{1}{J^2} \boldsymbol{F} \hat{\boldsymbol{\sigma}} \boldsymbol{F}^T$$



Mapping to the surface

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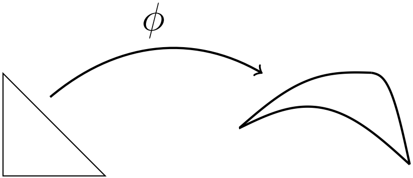
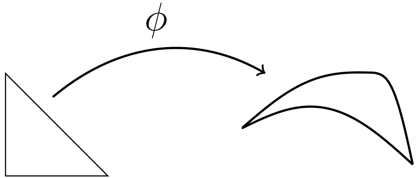


- Piola transformation

$$u \circ \phi = P[\hat{u}] = \frac{1}{J}\boldsymbol{F}\hat{u} \qquad \boldsymbol{F} = \nabla_{\hat{x}}\phi,\ J = \sqrt{\det(\boldsymbol{F}^T\boldsymbol{F})}$$

- Preserve normal-normal continuity

$$\boldsymbol{\sigma} \circ \phi = \frac{1}{J^2}\boldsymbol{F}\hat{\boldsymbol{\sigma}}\boldsymbol{F}^T$$



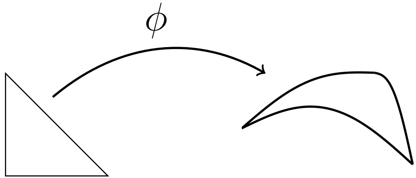
Mapping to the surface

- Piola transformation

$$u \circ \phi = P[\hat{u}] = \frac{1}{J}\boldsymbol{F}\hat{u} \qquad \boldsymbol{F} = \nabla_{\hat{x}}\phi,\ J = \|\operatorname{cof}(\boldsymbol{F})\|$$

- Preserve normal-normal continuity

$$\boldsymbol{\sigma} \circ \phi = \frac{1}{J^2}\boldsymbol{F}\hat{\boldsymbol{\sigma}}\boldsymbol{F}^T$$



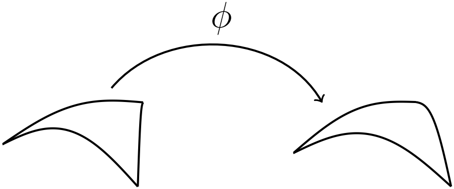
Mapping to the surface

- Piola transformation

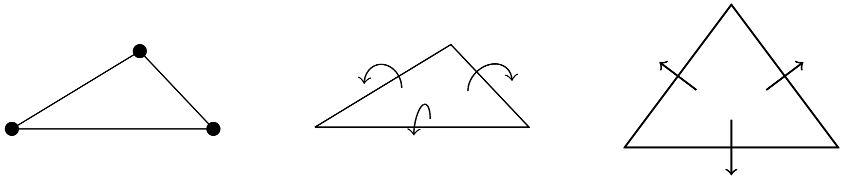
$$u \circ \phi = P[\hat{u}] = \frac{1}{J} \boldsymbol{F} \hat{u} \qquad \boldsymbol{F} = \nabla_{\hat{x}} \phi, \; J = \| \operatorname{cof}(\boldsymbol{F}) \|$$

- Preserve normal-normal continuity

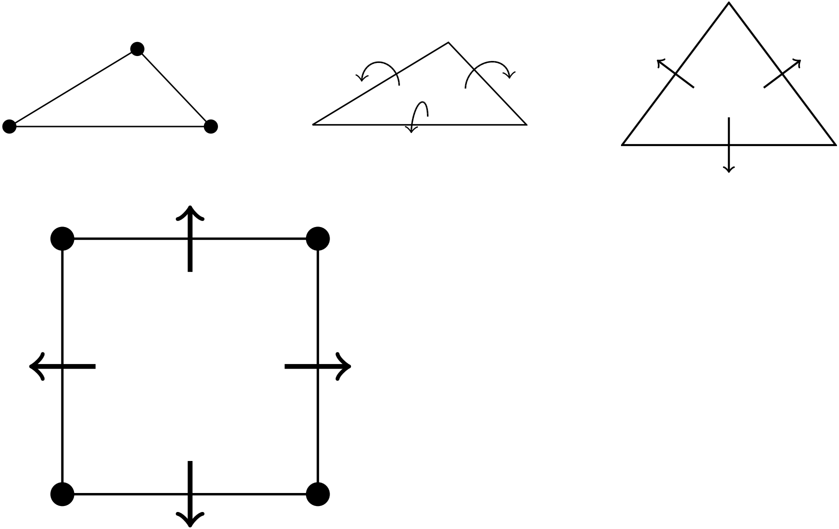
$$\boldsymbol{\sigma} \circ \phi = \frac{1}{J^2} \boldsymbol{F} \hat{\boldsymbol{\sigma}} \boldsymbol{F}^T$$

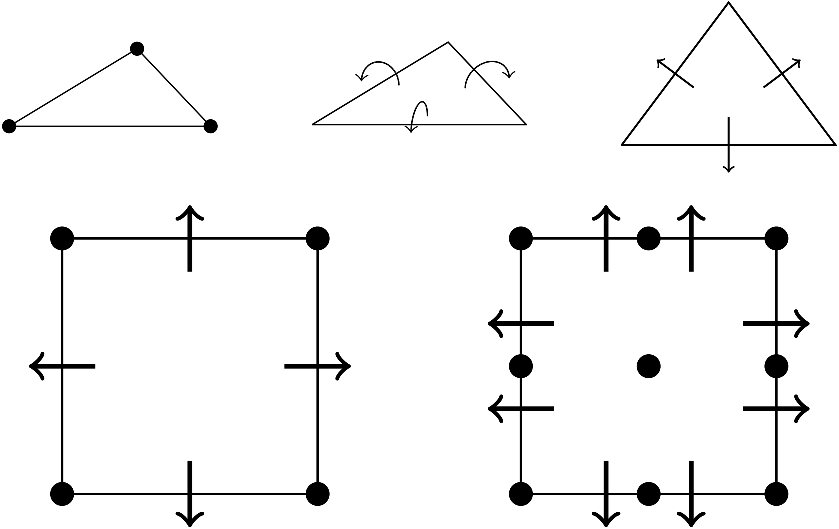


Shell element



Shell element



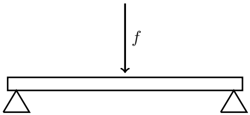


Relation to HHJ

Hellan–Herrmann–Johnson method

- Discretization method for 4th order elliptic problems

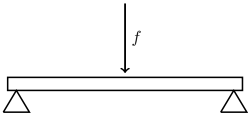
$$\mathrm{div}(\mathrm{div}(\nabla^2 u)) = f$$



Hellan–Herrmann–Johnson method

- Discretization method for 4th order elliptic problems

$$\operatorname{div}(\operatorname{div}(\nabla^2 u)) = f \quad \Rightarrow u \in H^2(\Omega)$$



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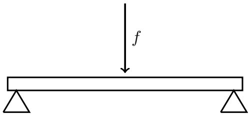
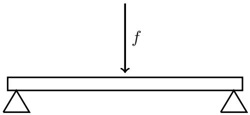


$$\begin{aligned} \boldsymbol{\sigma} &= \nabla^2 u, \\ \mathrm{div}(\mathrm{div}(\boldsymbol{\sigma})) &= f, \end{aligned}$$

- Discretization method for 4th order elliptic problems

$$\mathrm{div}(\mathrm{div}(\nabla^2 u)) = f \quad \Rightarrow u \in H^2(\Omega)$$



$$\begin{aligned} \boldsymbol{\sigma} &= \nabla^2 u, && \Rightarrow u \in H^1(\Omega) \\ \mathrm{div}(\mathrm{div}(\boldsymbol{\sigma})) &= f, && \Rightarrow \boldsymbol{\sigma} \in H(\mathrm{divdiv}, \Omega) \end{aligned}$$

Hellan–Herrmann–Johnson

Find $u \in H^1(\Omega)$ and $\boldsymbol{\sigma} \in H(\mathrm{divdiv}, \Omega)$ for the saddle point problem

$$\begin{aligned}\mathcal{L}(u, \boldsymbol{\sigma}) = &-\frac{1}{2}\|\boldsymbol{\sigma}\|^2 + \sum_{T \in \mathcal{T}_h} \int_T \nabla u \cdot \mathrm{div}(\boldsymbol{\sigma})\, dx - \int_{\partial T} (\nabla u)_\tau \boldsymbol{\sigma}_{\mu\tau}\, ds \\ &- \langle f, u \rangle.\end{aligned}$$

M. Comodi: The Hellan-Herrmann-Johnson method: some new error estimates and postprocessing, *Math. Comp. 52 (1989) pp. 17–29.*

Hellan–Herrmann–Johnson

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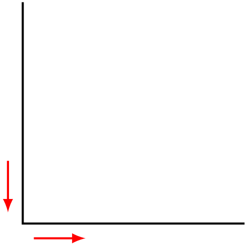
$$- \langle f, u \rangle.$$

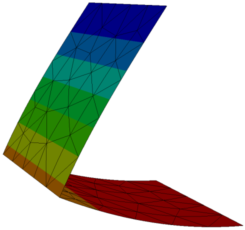
Linearization

If the undeformed configuration is a flat plane and f works orthogonal on it, the HHJ method is the linearization of the bending energy of our method.

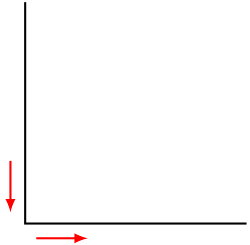
Kinks & Membrane Locking

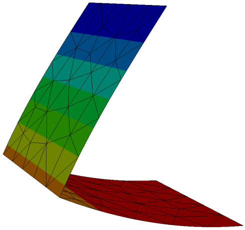
Structures with kinks



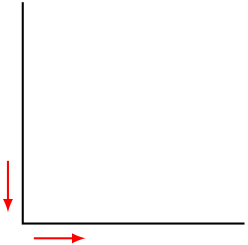


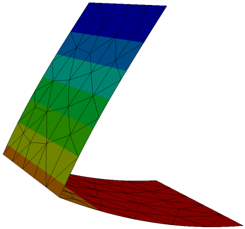
- Normal-normal continuous moment $\boldsymbol{\sigma}$

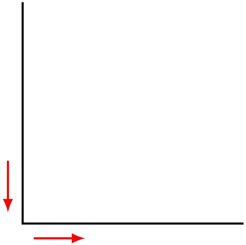




- Normal-normal continuous moment $\boldsymbol{\sigma}$
- Preserve kinks



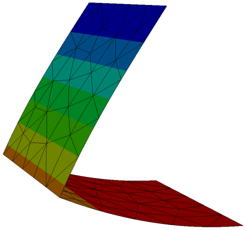




- Normal-normal continuous moment $\boldsymbol{\sigma}$
- Preserve kinks
- Variation of $\mathcal{L}(u, \boldsymbol{\sigma})$ in direction $\delta\boldsymbol{\sigma}$

$$\int_{\hat{E}} (\sphericalangle(\nu_L, \nu_R) - \sphericalangle(\hat{\nu}_L, \hat{\nu}_R))\delta\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\, d\hat{s} \stackrel{!}{=} 0$$

$$\Rightarrow \sphericalangle(\nu_L, \nu_R) - \sphericalangle(\hat{\nu}_L, \hat{\nu}_R) = 0$$



Membrane locking

$$\|\boldsymbol{E}_{\tau\tau}\|^2_{\boldsymbol{M}}$$

$$\|\mathrm{sym}(\boldsymbol{P}_\tau \nabla_\tau u)\|^2_{\boldsymbol{M}} = \|\frac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha}) - b_{\alpha\beta} u_3\|^2_{\boldsymbol{M}}$$

Membrane locking

$$\|\Pi_{L^2}^k \boldsymbol{E}_{\tau\tau}\|_{\boldsymbol{M}}^2$$

$$\|\mathrm{sym}(\boldsymbol{P}_\tau \nabla_\tau u)\|_{\boldsymbol{M}}^2 = \|\frac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha}) - b_{\alpha\beta} u_3\|_{\boldsymbol{M}}^2$$

- Reduced integration for quadrilateral meshes

TECHNISCHE UNIVERSITÄT WIEN

$$\|\mathcal{I}_{\mathcal{R}}^k \boldsymbol{E}_{\tau\tau}\|_{\boldsymbol{M}}^2$$

$$\|\mathrm{sym}(\boldsymbol{P}_\tau \nabla_\tau u)\|_{\boldsymbol{M}}^2 = \|\frac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha}) - b_{\alpha\beta} u_3\|_{\boldsymbol{M}}^2$$

- Reduced integration for quadrilateral meshes
- Regge interpolant for triangles

$$\|\mathcal{I}_{\mathcal{R}}^k \boldsymbol{E}_{\tau\tau}\|_{\boldsymbol{M}}^2$$

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- Reduced integration for quadrilateral meshes
- Regge interpolant for triangles
- $\boldsymbol{R} \in \mathrm{Reg}_h^k$, $\boldsymbol{Q} \in [\mathrm{Reg}_h^k]^*$

$$\|\boldsymbol{R}\|_{\boldsymbol{M}}^2 + \langle \boldsymbol{Q}, \boldsymbol{R} - \boldsymbol{E}_{\tau\tau}\rangle$$

Regge elements

$$\mathrm{Reg}_h^k := \{\boldsymbol{\sigma} \in [\Pi^k(\mathcal{T}_h)]^{d\times d}_{sym} \mid t^T \boldsymbol{\sigma}\, t \text{ is continuous over elements}\}$$

S. H. Christiansen: On the linearization of Regge calculus, *Numerische Mathematik 119, 4 (2011), pp. 613–640.*

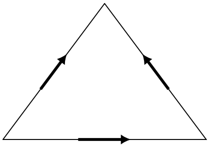
$$\mathrm{Reg}_h^k := \{\boldsymbol{\sigma} \in [\Pi^k(\mathcal{T}_h)]^{d\times d}_{sym} \mid t^T\boldsymbol{\sigma}\, t \text{ is continuous over elements}\}$$

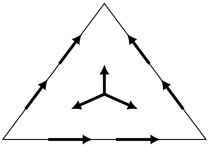
L. Li: Regge Finite Elements with Applications in Solid Mechanics and Relativity, *PhD thesis, University of Minnesota (2018).*

Regge elements

$$\mathrm{Reg}_h^k := \{\boldsymbol{\sigma} \in [\Pi^k(\mathcal{T}_h)]^{d\times d}_{sym} \mid t^T \boldsymbol{\sigma} t \text{ is continuous over elements}\}$$

$$H(\mathrm{curlcurl}) := \{\boldsymbol{\sigma} \in [L^2(\Omega)]^{d\times d}_{sym} \mid \mathrm{curl}\ (\mathrm{curl}\ \boldsymbol{\sigma})^T \in [H^{-1}(\Omega)]^{d\times d}\}$$





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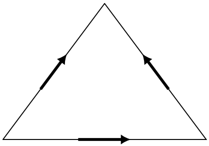

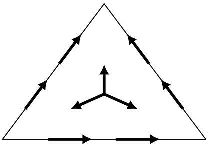

- Covariant mapping: $\boldsymbol{\sigma} \circ \Phi = \boldsymbol{F}^{-T} \hat{\boldsymbol{\sigma}} \boldsymbol{F}^{-1}$

Regge elements

$$\mathrm{Reg}_h^k := \{\boldsymbol{\sigma} \in [\Pi^k(\mathcal{T}_h)]_{sym}^{d\times d} \mid t^T \boldsymbol{\sigma} t \text{ is continuous over elements}\}$$
$$H(\mathrm{curlcurl}) := \{\boldsymbol{\sigma} \in [L^2(\Omega)]_{sym}^{d\times d} \mid \mathrm{curl}\,(\mathrm{curl}\,\boldsymbol{\sigma})^T \in [H^{-1}(\Omega)]^{d\times d}\}$$

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Membrane locking

$$\|\mathcal{I}_{\mathcal{R}}^k \boldsymbol{E}_{\tau\tau}\|_{\boldsymbol{M}}^2$$

$$\|\mathrm{sym}(\boldsymbol{P}_\tau \nabla_\tau u)\|_{\boldsymbol{M}}^2 = \|\frac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha}) - b_{\alpha\beta} u_3\|_{\boldsymbol{M}}^2$$

- Reduced integration for quadrilateral meshes
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- $\boldsymbol{R} \in \mathrm{Reg}_h^k$, $\boldsymbol{Q} \in [\mathrm{Reg}_h^k]^*$

$$\|\boldsymbol{R}\|_{\boldsymbol{M}}^2 + \langle \boldsymbol{Q}, \boldsymbol{R} - \boldsymbol{E}_{\tau\tau} \rangle$$

Dual space

$$\begin{aligned}\langle \cdot, \cdot \rangle : [\mathrm{Reg}_h^k]^* \times \mathrm{Reg}_h^k &\to \mathbb{R}\\ (\Psi, \varphi) &\mapsto \Psi(\varphi)\end{aligned}$$

Dual space

$$\langle \cdot, \cdot \rangle : [\mathrm{Reg}_h^k]^* \times \mathrm{Reg}_h^k \to \mathbb{R}$$
$$(\Psi, \varphi) \mapsto \Psi(\varphi)$$

- Edge functionals

$$\Psi_{E_{\alpha\beta},l} : \boldsymbol{\sigma} \mapsto \int_{E_{\alpha\beta}} \boldsymbol{\sigma}_{\tau_E \tau_E} q_{E,l} \, d\lambda_1, \qquad \{q_{E,l}\} \text{ basis of } \Pi^k(E_{\alpha\beta})$$

Dual space

$$\begin{aligned}\langle \cdot, \cdot \rangle : [\mathrm{Reg}_h^k]^* \times \mathrm{Reg}_h^k &\to \mathbb{R}\\ (\Psi, \varphi) &\mapsto \Psi(\varphi)\end{aligned}$$

- Edge functionals

$$\Psi_{E_{\alpha\beta},l} : \boldsymbol{\sigma} \mapsto \int_{E_{\alpha\beta}} \boldsymbol{\sigma}_{\tau_E \tau_E} q_{E,l} \, d\lambda_1, \qquad \{q_{E,l}\} \text{ basis of } \Pi^k(E_{\alpha\beta})$$

- Element functionals

$$\Psi_{T,l} : \boldsymbol{\sigma} \mapsto \int_T \boldsymbol{\sigma} : \boldsymbol{q}_{T,l} \, d\lambda_2, \qquad \{\boldsymbol{q}_{T,l}\} \text{ basis of } [\Pi^{k-1}(T)]^{2\times 2}_{sym}$$

Dual space

$$\langle \cdot, \cdot \rangle : [\mathrm{Reg}_h^k]^* \times \mathrm{Reg}_h^k \to \mathbb{R}$$
$$(\Psi, \varphi) \mapsto \Psi(\varphi)$$

$$\mathcal{I}_{\mathcal{R}}^k : [C^\infty(\Omega)]^{2\times 2} \to \mathrm{Reg}_h^k$$
$$\boldsymbol{\sigma} \mapsto \sum_{i=0}^{N_k} \Psi_i(\boldsymbol{\sigma})\varphi_i$$

- Edge functionals

$$\Psi_{E_{\alpha\beta},l} : \boldsymbol{\sigma} \mapsto \int_{E_{\alpha\beta}} \boldsymbol{\sigma}_{\tau_E\tau_E} q_{E,l}\, d\lambda_1, \qquad \{q_{E,l}\} \text{ basis of } \Pi^k(E_{\alpha\beta})$$

- Element functionals

$$\Psi_{T,l} : \boldsymbol{\sigma} \mapsto \int_T \boldsymbol{\sigma} : \boldsymbol{q}_{T,l}\, d\lambda_2, \qquad \{\boldsymbol{q}_{T,l}\} \text{ basis of } [\Pi^{k-1}(T)]_{sym}^{2\times 2}$$

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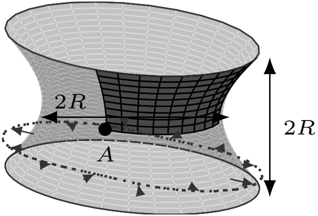
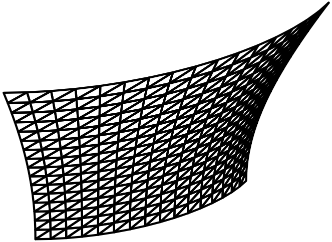
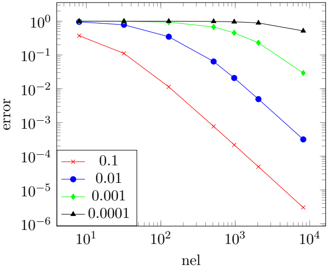
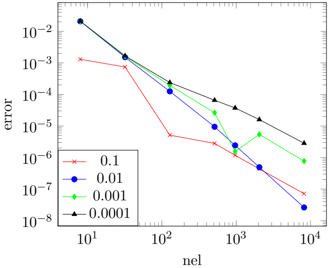
$$\|\boldsymbol{R}\|_{\boldsymbol{M}}^2 + \langle \boldsymbol{Q}, \boldsymbol{R} - \boldsymbol{E}_{\tau\tau}\rangle$$

Hyperboloid with free ends



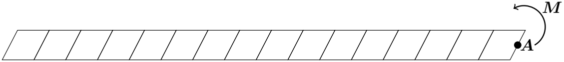


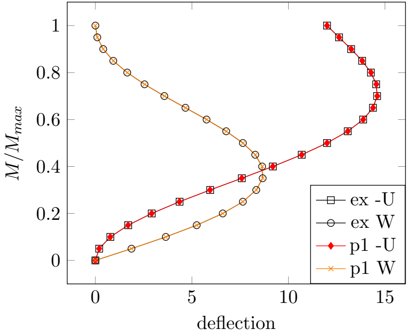




Numerical Examples

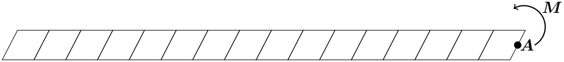
Cantilever subjected to end moment

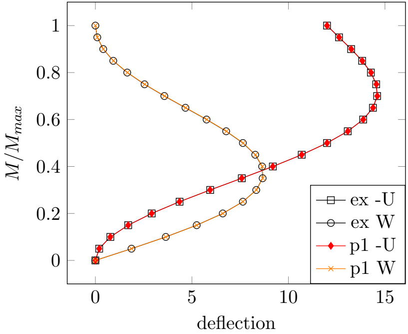




- $E = 1.2 \times 10^6$
  $\nu = 0$
  $L = 12$
  $W = 1$
  $t = 0.1$
  $M = 50\frac{\pi}{3}$

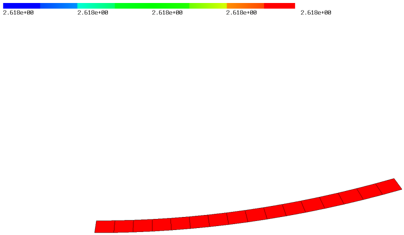
Cantilever subjected to end moment

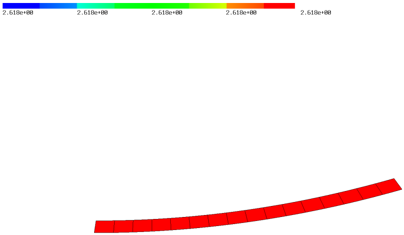




NGSolve

Cantilever subjected to end moment





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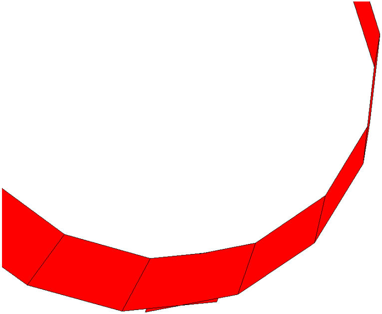

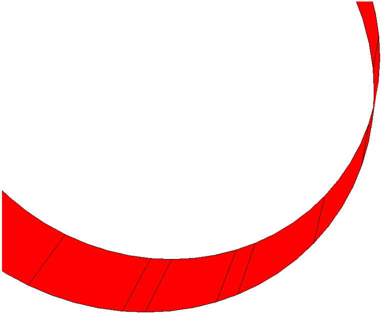

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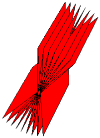
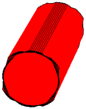
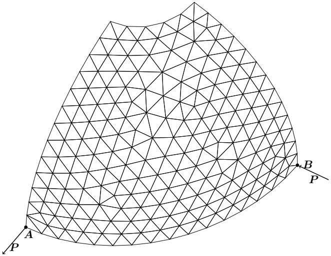


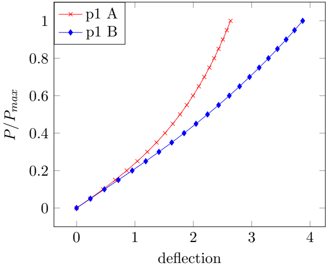


Hemispherical Shell

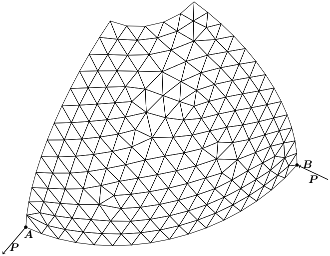


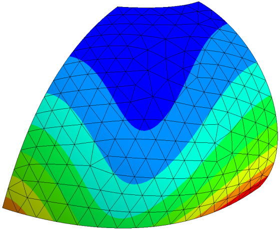
- $t = 0.04$
  $P = 50$
  $E = 6.825 \times 10^7$
  $\nu = 0.3$
  $R = 10$

h	2	1	0.5	0.25
p1	4.1218	3.8811	3.8560	3.8735
p3	3.8319	3.8781	3.8796	3.8796



Hemispherical Shell





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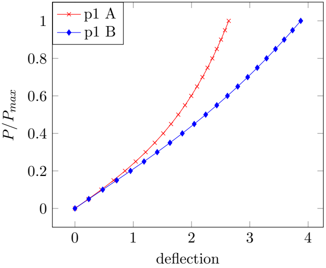
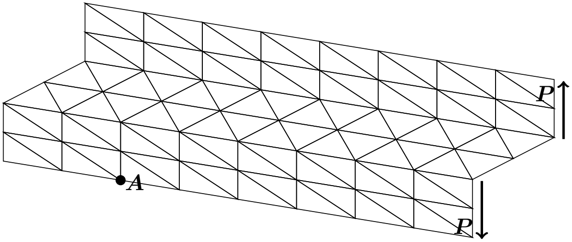
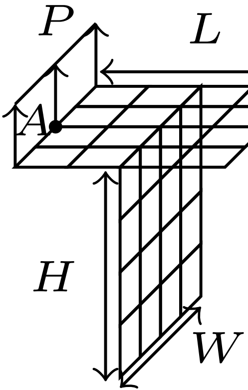




- $P = 6 \times 10^5$
  $E = 2.1 \times 10^{11}$
  $\nu = 0.3$
  $t = 0.1$
  $L = 10$
  $W = 2$
  $H = 1$

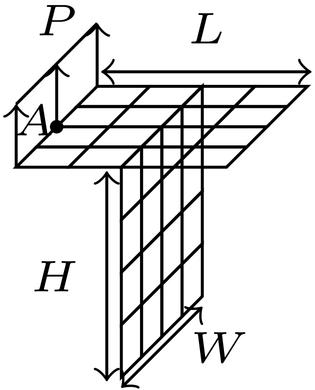
- Membrane stress $\boldsymbol{\Sigma}_{xx}$ at point $\boldsymbol{A}$

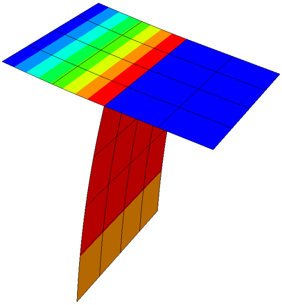
	p1	p3
8x6	-0.7620×10^8	-1.0929×10^8
32x15	-1.0777×10^8	-1.0933×10^8
64x30	-1.0989×10^8	-1.0933×10^8



- $P = 2 \times 10^3$
  $E = 6 \times 10^6$
  $\nu = 0$
  $t = 0.1$
  $L = 1$
  $W = 1$
  $H = 1$

T-Section Cantilever





Summary

- Kirchhoff–Love shell element

Summary

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- Moment tensor

Summary

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- Generalization of HHJ to shells

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- Kinks without extra treatment

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- Possible extension to Reissner–Mindlin shells

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Thank you for your attention!

Literature

M. Neunteufel and J. Schöberl: The Hellan–Herrmann–Johnson Method for Nonlinear Shells, http://arxiv.org/abs/1904.04714

M. Neunteufel and J. Schöberl: Avoiding Membrane Locking with Regge Interpolation, http://arxiv.org/abs/1907.06232