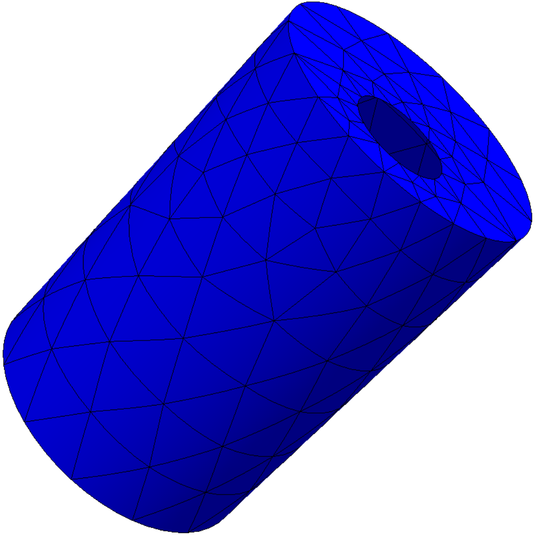
The Hellan-Herrmann-Johnson and TDNNS Method for Nonlinear Koiter and Naghdi Shells

Michael Neunteufel (TU Wien)
Joachim Schöberl (TU Wien)

Jena-Augsburg-Meeting on Numerical Analysis, Augsburg, June 9th, 2023

Motivation



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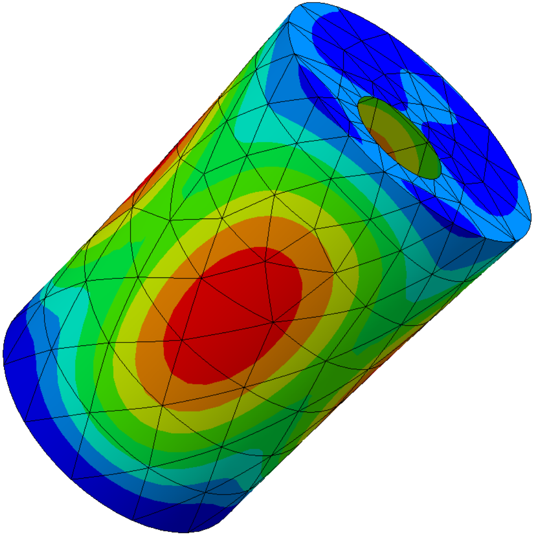
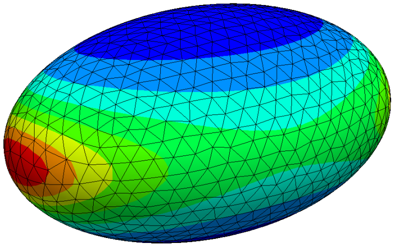
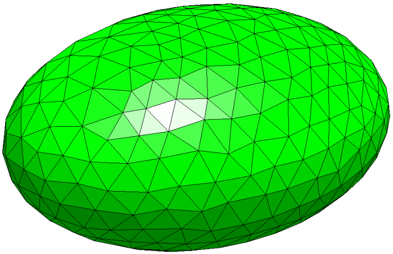


Approximate extrinsic/intrinsic curvature of non-smooth surfaces





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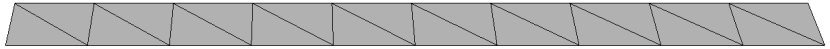
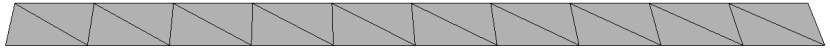




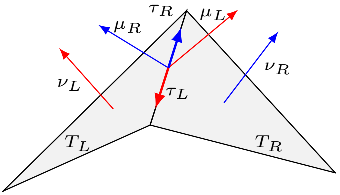
Contents

Distributional extrinsic and intrinsic curvature

- Normal vector ν

  Tangent vector τ

  Element normal vector $\mu = \nu \times \tau$

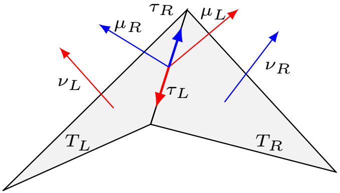


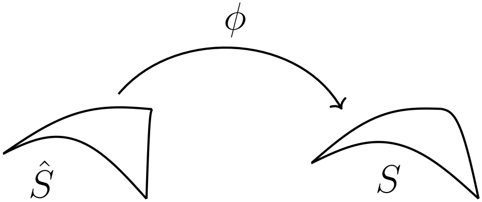
Differential geometry

- Normal vector $\hat{\nu}$

  Tangent vector $\hat{\tau}$

  Element normal vector $\hat{\mu} = \hat{\nu} \times \hat{\tau}$

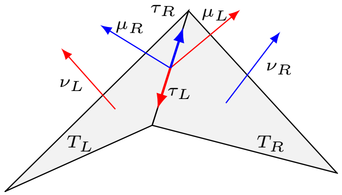


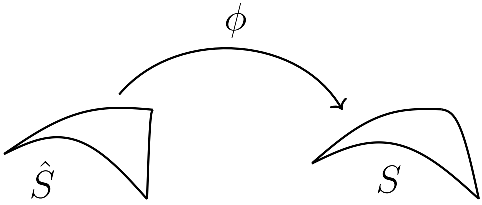


Differential geometry

- Normal vector $\hat{\nu}$
  Tangent vector $\hat{\tau}$
  Element normal vector $\hat{\mu} = \hat{\nu} \times \hat{\tau}$

- $\boldsymbol{F} = \nabla_{\hat{\tau}} \phi$, $J = \sqrt{\det(\boldsymbol{F}^\top \boldsymbol{F})}$





- Normal vector $\hat{\nu}$
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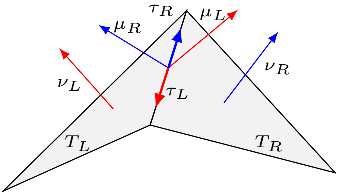
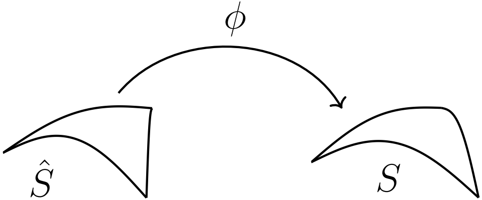


- $\boldsymbol{F} = \nabla_{\hat{\tau}}\phi$, $J = \|\operatorname{cof}(\boldsymbol{F})\|_F$



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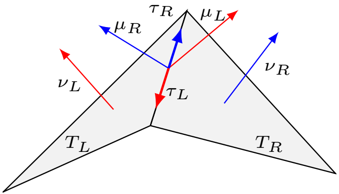
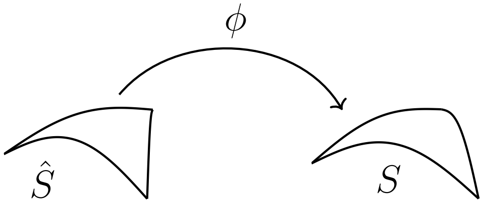
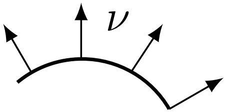


- $\boldsymbol{F} = \nabla_{\hat{\tau}}\phi$, $J = \|\operatorname{cof}(\boldsymbol{F})\|_F$
- $\nu \circ \phi = \frac{1}{J}\operatorname{cof}(\boldsymbol{F})\hat{\nu}$
  $\tau \circ \phi = \frac{1}{J_B}\boldsymbol{F}\hat{\tau}$
  $\mu \circ \phi = \nu \circ \phi \times \tau \circ \phi$



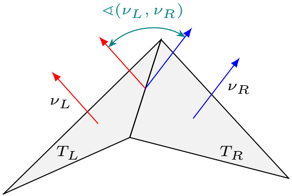
Distributional extrinsic curvature



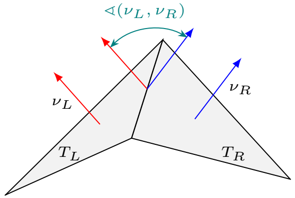
- Change of normal vector measures curvature $\nabla\nu$

Distributional extrinsic curvature

- Change of normal vector measures curvature $\nabla\nu$
- How to define $\nabla\nu$ for discrete surface?



Grinspun, Gingold, Reisman, Zorin: Computing discrete shape operators on general meshes, *Computer Graphics Forum 25*, 3 (2006).



- Change of normal vector measures curvature $\nabla\nu$
- How to define $\nabla\nu$ for discrete surface?

  - Distributional Weingarten tensor

$$\langle \nabla\nu, \boldsymbol{\sigma}\rangle_{\mathscr{T}} = \sum_{T\in\mathscr{T}_h} \int_T \nabla\nu|_T : \boldsymbol{\sigma}\, dx + \sum_{E\in\mathscr{E}_h} \int_E \sphericalangle(\nu_L, \nu_R)\boldsymbol{\sigma}_{\mu\mu}\, ds$$

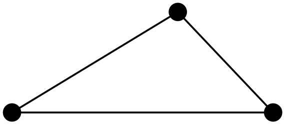
  - Measure jump of normal vector
  - Test function $\boldsymbol{\sigma}$ symmetric, normal-normal continuous $\Rightarrow$ Hellan–Herrmann–Johnson finite elements

N., Schöberl, Sturm, Numerical shape optimization of Canham-Helfrich-Evans bending energy, *J. Comput. Phys.* (2023).

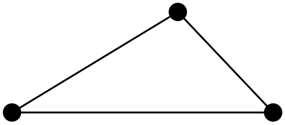
$$H^1(\Omega) := \{u \in L^2(\Omega) \mid \nabla u \in [L^2(\Omega)]^d\}$$

Hellan–Herrmann–Johnson finite elements

$$H^1(\Omega) := \{u \in L^2(\Omega) \mid \nabla u \in [L^2(\Omega)]^d\}$$
$$V_h^k := \mathcal{P}^k(\mathscr{T}_h) \cap C(\Omega)$$

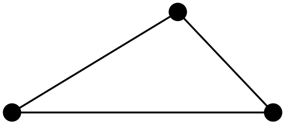


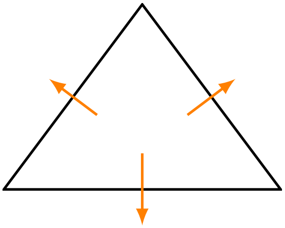
$$H(\mathrm{div}) := \{\sigma \in [L^2(\Omega)]^d \mid \mathbf{div}\sigma \in L^2(\Omega)\}$$



$$H(\mathrm{div}) := \{\sigma \in [L^2(\Omega)]^d \mid \mathbf{div}\sigma \in L^2(\Omega)\}$$

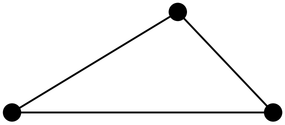
$$BDM^k := \{\sigma \in [\mathcal{P}^k(\mathscr{T}_h)]^d \mid \sigma_n \text{ is continuous over elements}\}$$

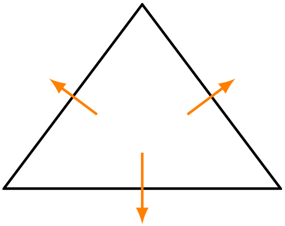




Hellan–Herrmann–Johnson finite elements

$$H(\mathrm{divdiv}) := \{\sigma \in [L^2(\Omega)]^{d\times d}_{sym} \mid \mathbf{divdiv}\sigma \in H^{-1}(\Omega)\}$$





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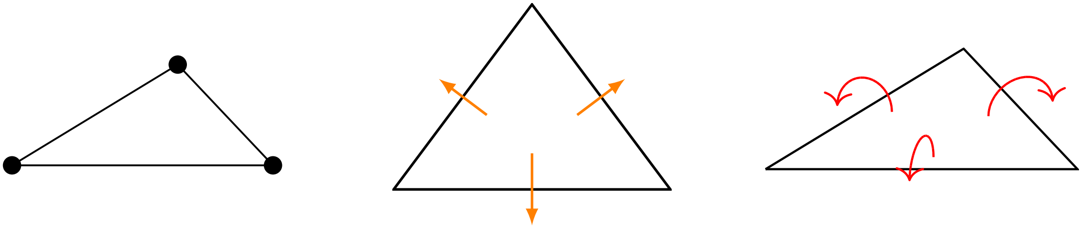
$$H(\mathrm{divdiv}) := \{\sigma \in [L^2(\Omega)]^{d\times d}_{sym} \,|\, \mathbf{divdiv}\sigma \in H^{-1}(\Omega)\}$$

$$M_h^k := \{\sigma \in [\mathcal{P}^k(\mathscr{T}_h)]^{d\times d}_{sym} \,|\, n^T\sigma n \text{ is continuous over elements}\}$$



A. Pechstein and J. Schöberl: The TDNNS method for Reissner-Mindlin plates, *J. Numer. Math. (2017) 137, pp. 713-740.*

Lifted distributional curvature

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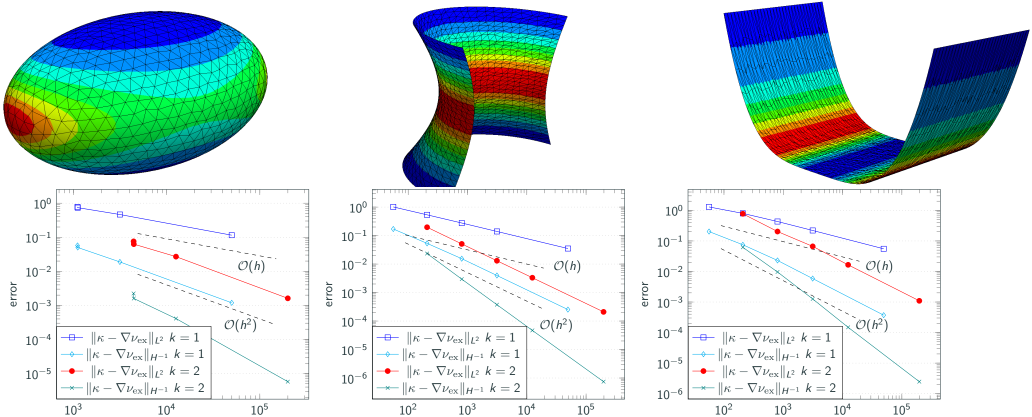
Lifting of distributional Weingarten tensor

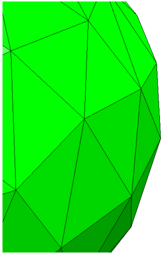
Find $\boldsymbol{\kappa} \in M_h^{k-1}$ for $\mathscr{T}_h$ curving order k s.t. for all $\boldsymbol{\sigma} \in M_h^{k-1}$

$$\int_{\mathscr{T}_h} \boldsymbol{\kappa} : \boldsymbol{\sigma}\, dx = \langle \nabla \nu, \boldsymbol{\sigma} \rangle_{\mathscr{T}}.$$



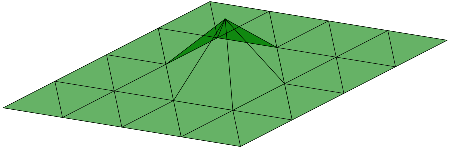
Distributional intrinsic curvature

Gauss Theorema Egregium: Gauss curvature depends on metric, $K = \kappa_1\kappa_2$



angle defect (DDG, Regge calculus)

metric $g = \nabla\Phi^\top\nabla\Phi$



REGGE: General relativity without coordinates, *Il Nuovo Cimento* (1955-1965), 19 (1961).

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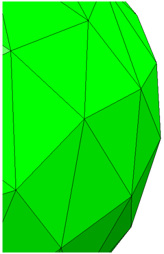

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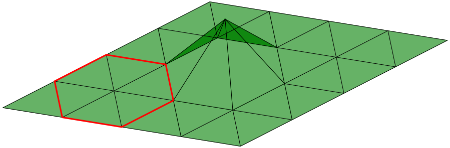

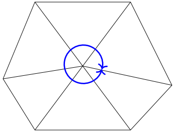

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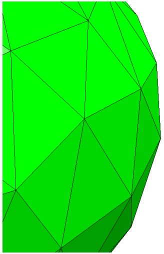

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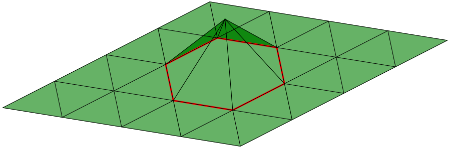

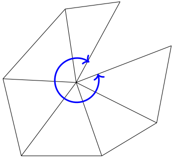

Regge: General relativity without coordinates, *Il Nuovo Cimento* (1955-1965), 19 (1961).

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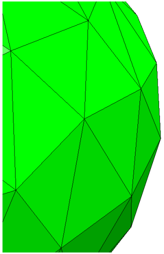
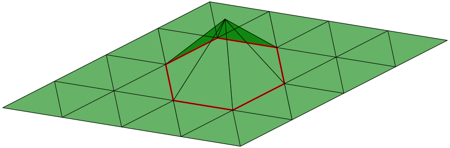
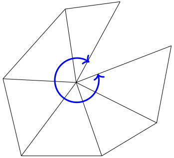
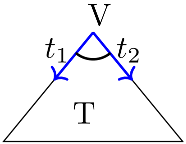


angle defect (DDG, Regge calculus)

metric $g = \nabla\Phi^\top\nabla\Phi \qquad \omega = \sqrt{\det g}$







Let $g \in \mathrm{Reg}_h^0(\mathscr{T}_h)$ and $\varphi \in V_h^1$

$$\langle (K\omega)(g), \varphi\rangle = \sum_{V\in\mathscr{V}} K_V(\varphi, g), \quad K_V(\varphi, g) = \big(2\pi - \sum_{T:V\subset T} \sphericalangle_V^T(g)\big)\varphi(V)$$

Distributional intrinsic curvature

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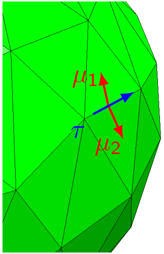


angle defect (DDG, Regge calculus)

metric $g = \nabla\Phi^\top\nabla\Phi$ $\qquad \Gamma_{ij}^k(g) = \frac{1}{2}g^{kl}\left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right) = g^{kl}\Gamma_{ijl}(g)$

$$\kappa_g = g(\nabla_\tau\tau, \mu) = \frac{\sqrt{\det g}}{g_{\hat\tau\hat\tau}^{3/2}}\left(\partial_{\hat\tau}\hat\tau\cdot\hat\mu + \Gamma^{\hat\mu}_{\hat\tau\hat\tau}\right)$$

$$K(g) = \frac{1}{\det g}\left(\partial_1\Gamma_{212} - \partial_2\Gamma_{112} + \Gamma_{11}^p\Gamma_{22p} - \Gamma_{22}^p\Gamma_{11p}\right)$$

Let $g \in \mathrm{Reg}_h^k(\mathscr{T}_h)$ and $\varphi \in V_h^{k+1}$

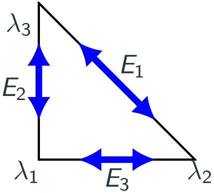
$$\langle (K\omega)(g), \varphi\rangle = \sum_{V\in\mathscr{V}} K_V(\varphi, g) + \int_{\mathscr{T}_h} K(g)\,\varphi\,\omega_T + \sum_{E\in\mathscr{E}_h}\int_E [\![\kappa(g)]\!]\varphi\,\omega_E$$

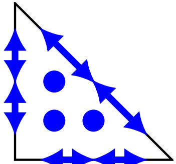
Berchenko-Kogan, Gawlik: Finite element approximation of the Levi-Civita connection and its curvature in two dimensions, *Found Comput Math* (2022).

Regge elements

$$H(\mathrm{curl\,curl}) := \{\sigma \in [L^2(\Omega)]^{2\times 2}_{sym} \mid \mathrm{curl\,curl}\,\sigma \in H^{-1}(\Omega)\}$$

$$\mathrm{Reg}_h^k := \{\varepsilon \in [\mathcal{P}^k(\mathscr{T}_h)]^{d\times d}_{\mathrm{sym}} \mid [\![t^\top \varepsilon\, t]\!]_E = 0 \text{ for all edges } E\}$$





$$\varphi_{E_i} = \nabla\lambda_j \odot \nabla\lambda_k, \qquad t_j^\top \varphi_{E_i} t_j = c_i \delta_{ij}, \qquad \varphi_{T_i} = \lambda_i \nabla\lambda_j \odot \nabla\lambda_k$$

- CHRISTIANSEN: On the linearization of Regge calculus, *Numerische Mathematik 119, 4 (2011).*
- LI: Regge Finite Elements with Applications in Solid Mechanics and Relativity, *PhD thesis, University of Minnesota (2018).*
- N.: Mixed Finite Element Methods For Nonlinear Continuum Mechanics And Shells, *PhD thesis, TU Wien (2021).*

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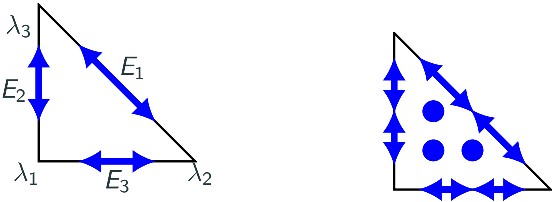

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$\mathcal{R}_h^k : C^0(\Omega) \to \mathrm{Reg}_h^k$ canonical interpolant

$$\int_E (g - \mathcal{R}_h^k g)_{tt}\, q\, dl = 0 \text{ for all } q \in \mathcal{P}^k(E)$$

$$\int_T (g - \mathcal{R}_h^k g) : Q\, da = 0 \text{ for all } Q \in \mathcal{P}^{k-1}(T, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$$

Lifting of distributional Gauss curvature

For $g \in \mathrm{Reg}_h^k$ find $K_h \in V_h^{k+1}$ such that for all $\varphi \in V_h^{k+1}$

$$\int_\Omega K_h \varphi\, \omega = \langle (K\omega)(g), \varphi \rangle.$$

Theorem

Let $g_h \in \mathrm{Reg}_h^k$ optimal-order interpolant, $-1 \leq l \leq k-2$

$$\|K_h - K\|_{H_h^l} \leq C\, h^{-l+k-1}(|g|_{W^{k+1,\infty}} + |K|_{H^k})$$

Gawlik: High-Order Approximation of Gaussian Curvature with Regge Finite Elements, *SIAM J. Numer. Anal.* (2020).

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GAWLIK: High-Order Approximation of Gaussian Curvature with Regge Finite Elements, *SIAM J. Numer. Anal.* (2020).

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Theorem (Gopalakrishnan, N., Schöberl, Wardetzky 2022)

Let $g_h = \mathcal{R}_h^k g \in \mathrm{Reg}_h^k$, $-1 \le l \le k-1$

$$\|K_h - K\|_{H_h^l} \le C\, h^{-l+k}\left(|g|_{W^{k+1,\infty}} + |K|_{H^k}\right)$$

Gopalakrishnan, N., Schöberl, Wardetzky: Analysis of curvature approximations via covariant curl and incompatibility for Regge metrics, *arXiv:2206.09343*.

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Let $g_h = \mathcal{R}_h^k g \in \mathrm{Reg}_h^k$, $-1 \leq l \leq k-1$

$$\|K_h - K\|_{H^{-1}} \leq C\, h^{k+1}\big(|g|_{W^{k+1,\infty}} + |K|_{H^k}\big)$$

Gopalakrishnan, N., Schöberl, Wardetzky: Analysis of curvature approximations via covariant curl and incompatibility for Regge metrics, *arXiv:2206.09343*.

Idea: Integral representation orthogonality properties of interpolant

$$\frac{d}{dt}((K\omega)(g+t\sigma))|_{t=0} = -\frac{1}{2}\,\mathrm{curl}_g\,\mathrm{curl}_g\,\sigma\,\omega_g \qquad\qquad \mathrm{inc}_g := \mathrm{curl}_g\,\mathrm{curl}_g$$

Integral representation $(G(t) = g_h + t(g-g_h),\ \sigma = G'(t) = g - g_h)$

$$\langle (K\omega)(g), u_h\rangle - \langle (K\omega)(g_h), u_h\rangle = -\frac{1}{2}\int_0^1 \langle \mathrm{inc}_{G(t)}\,\sigma, u_h\rangle\,dt$$

$$\langle \mathrm{inc}_g\,\sigma, u\rangle = \sum_{T\in\mathscr{T}_h}\int_T \mathrm{inc}_g\,\sigma\,u\,\omega_T - \int_{\partial T} u\,g(\mathrm{curl}_g\,\sigma - \mathrm{grad}_g\,\sigma(n_g,t_g),t_g)\,\omega_{\partial T} - \sum_{V\in\mathscr{V}_T}[\![\sigma(n_g,t_g)]\!]_V^T u(V)$$

Gawlik: High-Order Approximation of Gaussian Curvature with Regge Finite Elements, *SIAM J. Numer. Anal.* (2020).

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$$\int_E (\sigma - \mathcal{R}_h^k\sigma)_{tt}\, q\, dl = 0 \text{ for all } q\in\mathcal{P}^k(E), \qquad \int_T (\sigma - \mathcal{R}_h^k\sigma) : q\, da = 0 \text{ for all } q \in \mathcal{P}^{k-1}(T,\mathbb{R}^{2\times 2})$$

$$\langle \mathrm{inc}(\sigma - \mathcal{R}_h^k\sigma), u_h\rangle = 0 \text{ for all } u_h \in V_h^{k+1}$$

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$$|\langle \mathrm{inc}_g(\sigma-\mathcal{R}_h^k\sigma), u_h\rangle| \le C \quad \|\sigma-\mathcal{R}_h^k\sigma\|\,\|u_h\|_{H^1} \text{ for all } u_h\in V_h^{k+1}$$

Gopalakrishnan, N., Schöberl, Wardetzky: Analysis of curvature approximations via covariant curl and incompatibility for Regge metrics, *arXiv:2206.09343*.

Idea: Integral representation orthogonality properties of interpolant

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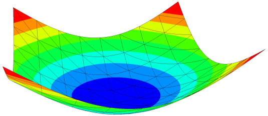
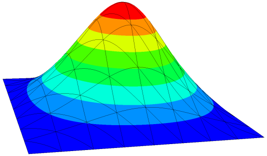
$$|\langle \mathrm{inc}_g(\sigma - \mathcal{I}_k\sigma), u_h\rangle| \le C h^{-1} |\!|\!|\sigma - \mathcal{R}_h^k\sigma|\!|\!|\, \|u_h\|_{H^1} \text{ for all } u_h \in V_h^{k+1}$$

GOPALAKRISHNAN, N., SCHÖBERL, WARDETZKY: Analysis of curvature approximations via covariant curl and incompatibility for Regge metrics, *arXiv:2206.09343*.

Numerical example





NGSolve

$$g = \begin{pmatrix} 1 + (\partial_x f)^2 & \partial_x f \partial_y f \\ \partial_x f \partial_y f & 1 + (\partial_y f)^2 \end{pmatrix} \qquad f = \frac{1}{2}(x^2 + y^2) - \frac{1}{12}(x^4 + y^4)$$

$$K(g) = \frac{81(1 - x^2)(1 - y^2)}{(9 + x^2(x^2 - 3)^2 + y^2(y^2 - 3)^2)^2}$$

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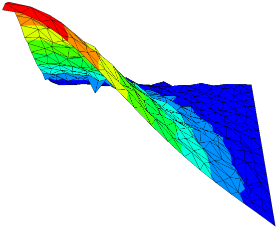
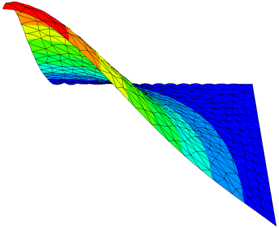
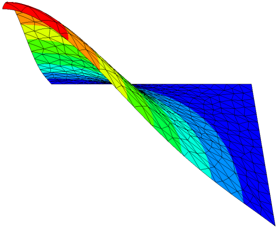
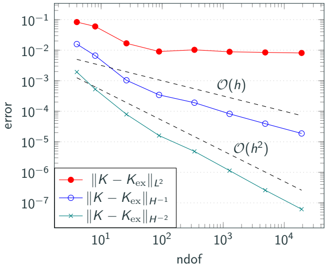
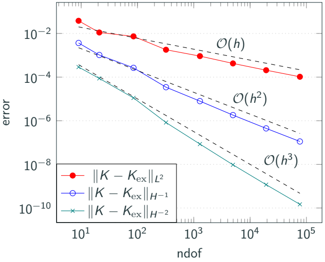
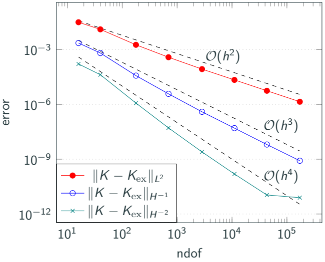










Nonlinear shells

Koiter shell

$$\mathcal{W}(u) = \frac{t}{2}\|\boldsymbol{E}(u)\|_{\mathbb{M}}^2 + \frac{t^3}{24}\|\boldsymbol{F}^T\nabla(\nu\circ\phi) - \nabla\hat{\nu}\|_{\mathbb{M}}^2$$

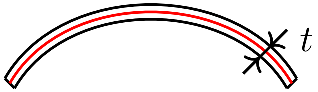
$u\dots$ displacement of mid-surface

$t\dots$ thickness

$\mathbb{M}\dots$ material tensor

$\boldsymbol{F} = \nabla u + \boldsymbol{P} = \nabla\phi, \qquad \boldsymbol{P} = \boldsymbol{I} - \hat{\nu}\otimes\hat{\nu}$

$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^\top\boldsymbol{F} - \boldsymbol{P}) = \frac{1}{2}(\nabla u^\top\nabla u + \nabla u^\top\boldsymbol{P} + \boldsymbol{P}\nabla u)$



Koiter shell

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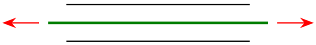


membrane energy

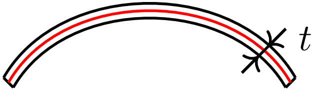
$u\ldots$ displacement of mid-surface

$t\ldots$ thickness

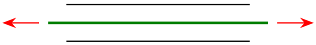
$\mathbb{M}\ldots$ material tensor

$$\boldsymbol{F} = \nabla u + \boldsymbol{P} = \nabla\phi, \qquad \boldsymbol{P} = \boldsymbol{I} - \hat{\nu}\otimes\hat{\nu}$$

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^\top\boldsymbol{F} - \boldsymbol{P}) = \frac{1}{2}(\nabla u^\top\nabla u + \nabla u^\top\boldsymbol{P} + \boldsymbol{P}\nabla u)$$



$$\mathcal{W}(u) = \frac{t}{2}\|\boldsymbol{E}(u)\|^2_{\mathbb{M}} + \frac{t^3}{24}\|\boldsymbol{F}^T\nabla(\nu\circ\phi) - \nabla\hat{\nu}\|^2_{\mathbb{M}}$$



membrane energy

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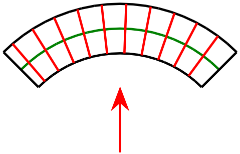
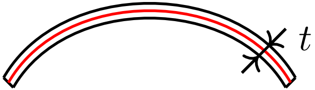


bending energy



Naghdi shell

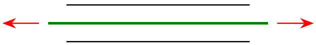
$$\mathcal{W}(u,\gamma) = \frac{t}{2}\|\boldsymbol{E}(u)\|^2_{\mathbb{M}} + \frac{t^3}{24}\|\mathrm{sym}(\boldsymbol{F}^T\nabla(\tilde{\nu}\circ\phi)) - \nabla\hat{\nu}\|^2_{\mathbb{M}} + \frac{t\kappa G}{2}\|\boldsymbol{F}^T\tilde{\nu}\circ\phi\|^2$$

$\gamma \ldots$ shearing

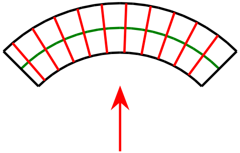
$\tilde{\nu} = \dfrac{\nu+\gamma}{\|\nu+\gamma\|} \ldots$ director

$G \ldots$ shearing modulus

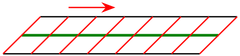
$\kappa = 5/6 \ldots$ shear correction factor



membrane energy



bending energy



shearing energy

Distributional curvature for Koiter shell

$$\text{Lifting:} \quad \int_{\mathscr{T}_h} \boldsymbol{\kappa} : \boldsymbol{\sigma}\, dx = \sum_{T\in\mathscr{T}_h} \int_T \nabla\nu : \boldsymbol{\sigma}\, dx + \sum_{E\in\mathscr{E}_h} \int_E \sphericalangle(\nu_L, \nu_R)\boldsymbol{\sigma}_{\mu\mu}\, ds$$

- Lifted curvature difference $\boldsymbol{\kappa}^{\mathrm{diff}}$ via three-field formulation

$$\mathcal{L}(u, \boldsymbol{\kappa}^{\mathrm{diff}}, \boldsymbol{\sigma}) = \frac{t}{2}\|\boldsymbol{E}(u)\|^2_{\mathbb{M}} + \frac{t^3}{12}\|\boldsymbol{\kappa}^{\mathrm{diff}}\|^2_{\mathbb{M}} - \langle f, u\rangle + \sum_{T\in\mathscr{T}_h} \int_T \left(\boldsymbol{\kappa}^{\mathrm{diff}} - (\boldsymbol{F}^T\nabla(\nu\circ\phi) - \nabla\hat{\nu})\right) : \boldsymbol{\sigma}\, dx$$
$$+ \sum_{E\in\mathscr{E}_h} \int_E (\sphericalangle(\nu_L, \nu_R) - \sphericalangle(\hat{\nu}_L, \hat{\nu}_R))\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\, ds$$

- Lagrange parameter $\boldsymbol{\sigma} \in M_h^{k-1}$ moment tensor
- Eliminate $\boldsymbol{\kappa}^{\mathrm{diff}} \to$ two-field formulation in $(u, \boldsymbol{\sigma})$

N., Schöberl: The Hellan–Herrmann–Johnson and TDNNS method for linear and nonlinear shells, *arXiv:2304.13806*.

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Shell problem

Find $u \in [V_h^k]^3$ and $\boldsymbol{\sigma} \in M_h^{k-1}$ for $(\boldsymbol{H}_\nu := \sum_i (\nabla^2 u_i)\nu_i)$

$$\begin{aligned}\mathcal{L}(u,\boldsymbol{\sigma}) =& \frac{t}{2}\|\boldsymbol{E}(u)\|^2_{\mathbb{M}} - \frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\mathbb{M}^{-1}} - \langle f, u\rangle \\ &+ \sum_{T\in\mathscr{T}_h}\int_T \boldsymbol{\sigma} : (\boldsymbol{H}_\nu + (1-\hat{\nu}\cdot\nu)\nabla\hat{\nu})\,dx \\ &+ \sum_{E\in\mathscr{E}_h}\int_E (\sphericalangle(\nu_L,\nu_R) - \sphericalangle(\hat{\nu}_L,\hat{\nu}_R))\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\,ds\end{aligned}$$

N., Schöberl: The Hellan–Herrmann–Johnson method for nonlinear shells, *Comput. Struct. 225 (2019).*

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Find $u \in [V_h^k]^3$ and $\boldsymbol{\sigma} \in M_h^{k-1}$ for $(\boldsymbol{H}_\nu := \sum_i (\nabla^2 u_i)\nu_i)$

$$
\begin{aligned}
\mathcal{L}(u,\boldsymbol{\sigma}) =& \frac{t}{2}\|\boldsymbol{E}(u)\|^2_{\mathbb{M}} - \frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\mathbb{M}^{-1}} - \langle f, u\rangle \\
&+ \sum_{T\in\mathscr{T}_h} \int_T \boldsymbol{\sigma} : (\boldsymbol{H}_\nu \qquad\qquad\qquad\qquad) \, dx \\
&+ \sum_{E\in\mathscr{E}_h} \int_E (\sphericalangle(\nu_L, \nu_R) \qquad\qquad\qquad\qquad) \boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}} \, ds
\end{aligned}
$$

N., Schöberl: The Hellan–Herrmann–Johnson method for nonlinear shells, *Comput. Struct. 225 (2019).*

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&+ \sum_{T\in\mathscr{T}_h}\int_T \boldsymbol{\sigma} : (\boldsymbol{H}_\nu \qquad\qquad) \, dx \\
&+ \sum_{E\in\mathscr{E}_h}\int_E (\sphericalangle(\nu_L,\nu_R) \qquad\qquad)\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}} \, ds
\end{aligned}
$$

Use hybridization to eliminate $\boldsymbol{\sigma}$ $\to$ recover minimization problem

N., Schöberl: The Hellan–Herrmann–Johnson method for nonlinear shells, *Comput. Struct. 225 (2019).*

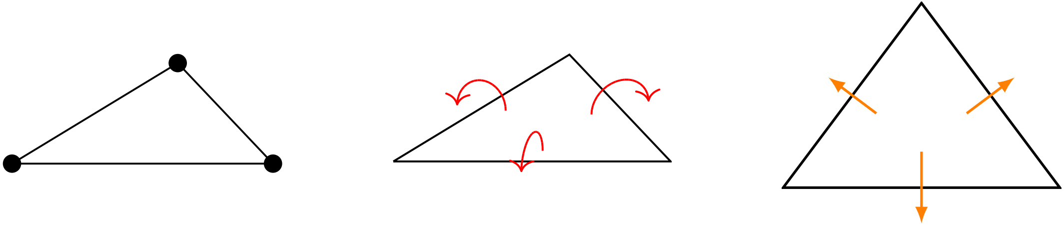
Shell problem

Find $u \in [V_h^k]^3$, $\boldsymbol{\sigma} \in M_h^{\mathrm{dc},k-1}$, and $\alpha \in \Gamma_h^{k-1}$ for

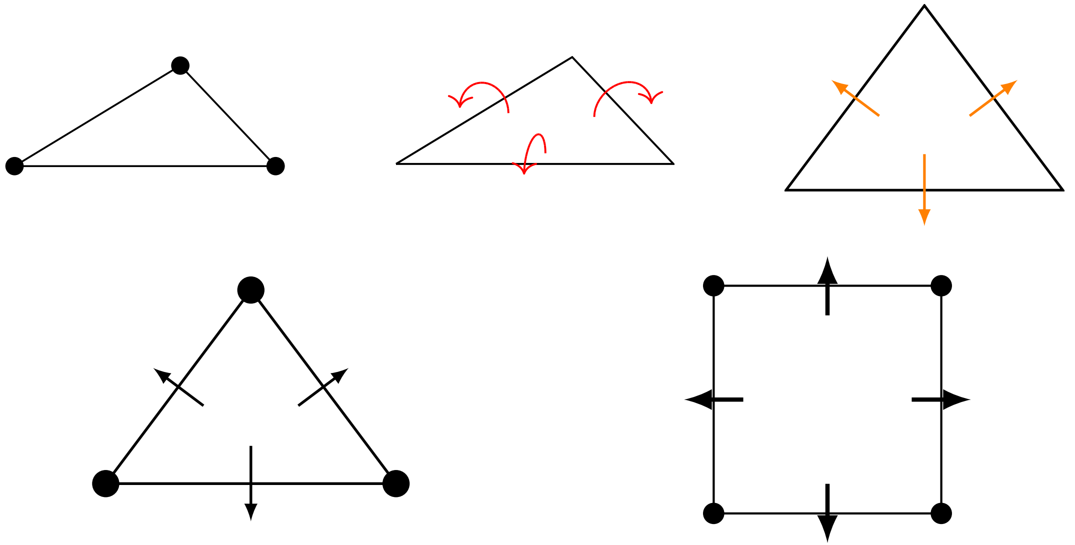
$$\mathcal{L}(u,\boldsymbol{\sigma},\alpha) = \frac{t}{2}\|\boldsymbol{E}(u)\|^2_{\mathbb{M}} - \frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\mathbb{M}^{-1}} - \langle f, u\rangle$$
$$+ \sum_{T\in\mathscr{T}_h}\int_T \boldsymbol{\sigma} : (\boldsymbol{H}_\nu + (1-\hat{\nu}\cdot\nu)\nabla\hat{\nu})\,dx$$
$$+ \sum_{E\in\mathscr{E}_h}\int_E (\sphericalangle(\nu_L,\nu_R) - \sphericalangle(\hat{\nu}_L,\hat{\nu}_R))\{\{\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\}\} + [\![\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}]\!]\alpha_{\hat{\mu}}\,ds$$

$\{\{\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\}\} = \frac{1}{2}((\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}})|_{T_L} + (\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}})|_{T_R})$, $[\![\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}]\!] = (\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}})|_{T_L} - (\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}})|_{T_R}$

N., Schöberl: The Hellan–Herrmann–Johnson method for nonlinear shells, *Comput. Struct. 225 (2019).*



Shell element (Koiter)



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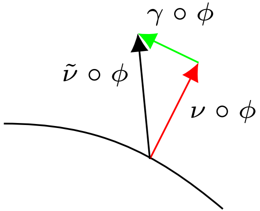
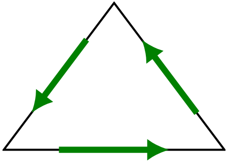
- Use hierarchical shell model
- Additional shearing dofs γ in $H(\mathrm{curl})$
- $\tilde{\nu}\circ\phi = \frac{\nu\circ\phi+\gamma\circ\phi}{\|\nu\circ\phi+\gamma\circ\phi\|}$
- Free of shear locking



$$H(\mathrm{curl}) := \{u \in [L^2(\Omega)]^d \mid \mathrm{curl}\, u \in [L^2(\Omega)]^{2d-3}\}$$
$$\mathcal{N}_{II}^k := \{u \in [\mathcal{P}^k(\mathscr{T}_h)]^d \mid u_t \text{ is continuous over elements}\}$$



Echter, R. and Oesterle, B. and Bischoff, M.: A hierarchic family of isogeometric shell finite elements, *Comput. Methods Appl. Mech. Engrg* (2013) 254, pp. 170–180.

- Use hierarchical shell model
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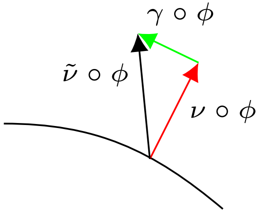
- $\tilde{\nu}\circ\phi = \nu\circ\phi + \gamma\circ\phi = \frac{1}{J}\mathrm{cof}(\boldsymbol{F})\hat{\nu} + (\boldsymbol{F}^{\dagger})^{\top}\hat{\gamma}$
- Free of shear locking



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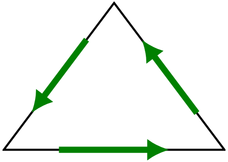

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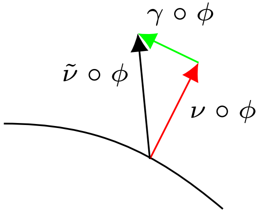
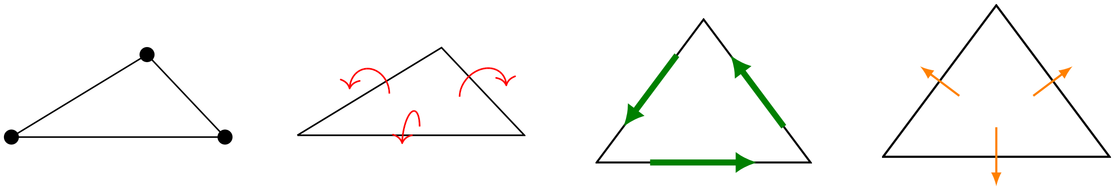
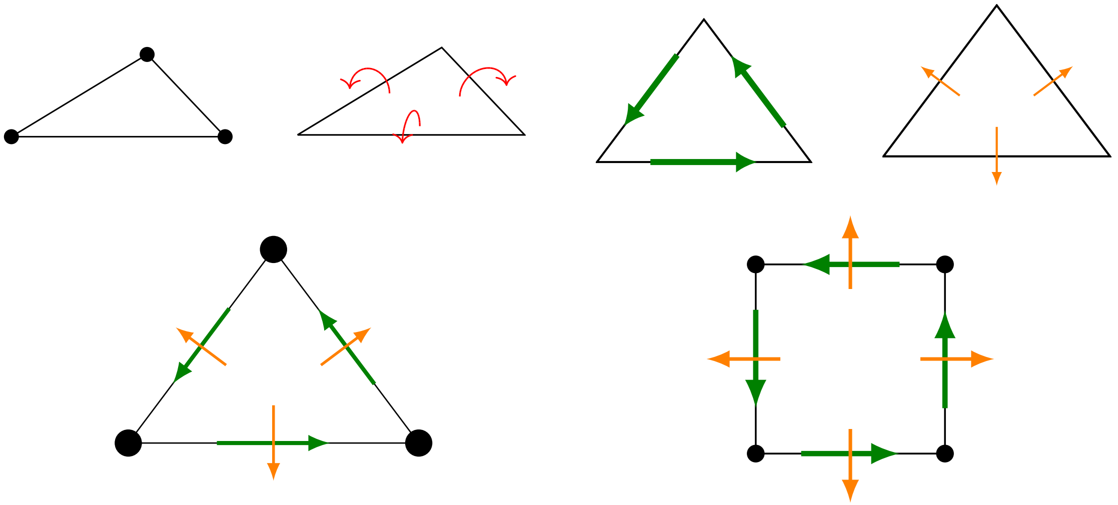


$$
\begin{aligned}
\mathcal{L}(u,\boldsymbol{\sigma},\hat{\gamma}) = \frac{t}{2}\|\boldsymbol{E}(u)\|^2_{\mathbb{M}} + \frac{t\kappa G}{2}\|\hat{\gamma}\|^2 - \frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\mathbb{M}^{-1}} \\
+ \sum_{T\in\mathscr{T}_h}\int_T \left(\boldsymbol{H}_{\tilde{\nu}} + (1-\tilde{\nu}\cdot\hat{\nu})\,\nabla\hat{\nu} - \nabla\hat{\gamma}\right) : \boldsymbol{\sigma}\,dx \\
+ \sum_{E\in\mathscr{E}_h}\int_E \left(\sphericalangle(\nu_L,\nu_R) - \sphericalangle(\hat{\nu}_L,\hat{\nu}_R) + [\![\hat{\gamma}_{\hat{\mu}}]\!]\right)\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\,ds
\end{aligned}
$$

Shell element (Naghdi)



Shell element (Naghdi)



Linear Kirchhoff–Love shell and plate

$$\mathcal{L}^{\text{shell}}_{\text{lin}}(u,\boldsymbol{\sigma}) = \frac{t}{2}\|\mathrm{sym}(\nabla^{\mathrm{cov}} u)\|^2_{\mathbb{M}} - \frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\mathbb{M}^{-1}} + \sum_{T\in\mathscr{T}_h}\Big(\int_T \boldsymbol{H}_{\hat{\nu}} : \boldsymbol{\sigma}\,dx - \int_{\partial T}(\nabla u^\top \hat{\nu})_{\hat{\mu}}\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\,ds\Big)$$

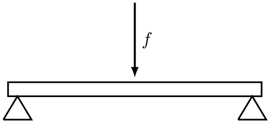
$$\mathcal{L}^{\text{plate}}_{\text{lin}}(w,\boldsymbol{\sigma}) = -\frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\mathbb{M}^{-1}} + \sum_{T\in\mathscr{T}_h}\Big(\int_T \nabla^2 w : \boldsymbol{\sigma}\,dx - \int_{\partial T}\frac{\partial w}{\partial\hat{\mu}}\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\,ds\Big)$$

Linear Kirchhoff–Love shell and plate

$$\mathcal{L}^{\mathrm{shell}}_{\mathrm{lin}}(u,\boldsymbol{\sigma}) = \frac{t}{2}\|\mathrm{sym}(\nabla^{\mathrm{cov}} u)\|^2_{\mathbb{M}} - \frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\mathbb{M}^{-1}} + \sum_{T\in\mathscr{T}_h}\left(\int_T \boldsymbol{H}_{\hat{\nu}} : \boldsymbol{\sigma}\,dx - \int_{\partial T}(\nabla u^\top \hat{\nu})_{\hat{\mu}}\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\,ds\right)$$

$$\mathcal{L}^{\mathrm{plate}}_{\mathrm{lin}}(w,\boldsymbol{\sigma}) = -\frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\mathbb{M}^{-1}} + \sum_{T\in\mathscr{T}_h}\left(\int_T \nabla^2 w : \boldsymbol{\sigma}\,dx - \int_{\partial T}\frac{\partial w}{\partial\hat{\mu}}\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\,ds\right)$$

$$\mathbf{div}\mathbf{div}\nabla^2 w = f \Leftrightarrow \begin{cases} \boldsymbol{\sigma} = \nabla^2 w, \\ \mathbf{div}\mathbf{div}\boldsymbol{\sigma} = f, \end{cases}$$



M. COMODI: The Hellan-Herrmann-Johnson method: some new error estimates and postprocessing, *Math. Comp. 52 (1989) pp. 17–29.*

Linear Reissner–Mindlin shell and plate

$$\mathcal{L}_{\text{lin}}^{\text{shell}}(u,\boldsymbol{\sigma},\hat{\gamma}) = \frac{t}{2}\|\text{sym}(\nabla^{\text{cov}}u)\|_{\mathbb{M}}^2 + \frac{t\kappa G}{2}\|\hat{\gamma}\|^2 - \frac{6}{t^3}\|\boldsymbol{\sigma}\|_{\mathbb{M}^{-1}}^2 + \sum_{T\in\mathcal{T}_h}\Big(\int_T(\boldsymbol{H}_{\hat{\nu}} - \nabla\hat{\gamma}):\boldsymbol{\sigma}\,dx - \int_{\partial T}((\nabla u^\top\hat{\nu})_{\hat{\mu}} - \hat{\gamma}_{\hat{\mu}})\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\,ds\Big)$$

$$\mathcal{L}_{\text{lin}}^{\text{plate}}(w,\boldsymbol{\sigma},\hat{\gamma}) = \frac{t\kappa G}{2}\|\hat{\gamma}\|^2 - \frac{6}{t^3}\|\boldsymbol{\sigma}\|_{\mathbb{M}^{-1}}^2 + \sum_{T\in\mathcal{T}_h}\Big(\int_T(\nabla^2 w - \nabla\hat{\gamma}):\boldsymbol{\sigma}\,dx - \int_{\partial T}(\frac{\partial w}{\partial\hat{\mu}} - \hat{\gamma}_{\hat{\mu}})\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\,ds\Big)$$

A. Pechstein and J. Schöberl: The TDNNS method for Reissner-Mindlin plates, *J. Numer. Math.* (2017) 137, pp. 713–740.

Membrane locking

Membrane locking

$$\mathcal{W}(u) = t\,E_{\text{mem}}(u) + t^3\,E_{\text{bend}}(u) - f \cdot u, \qquad f = t^3 \tilde{f}$$

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$$\mathcal{W}(u) = t^{-2} E_{\text{mem}}(u) + E_{\text{bend}}(u) - \tilde{f} \cdot u, \qquad f = t^3 \tilde{f}$$

Enforces $E_{\text{mem}}(u) = 0$ in the limit $t \to 0$

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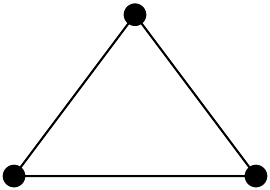

$$V_h = \mathcal{P}^k(\mathscr{T}_h) \cap C(\Omega) \subset H^1(\Omega)$$

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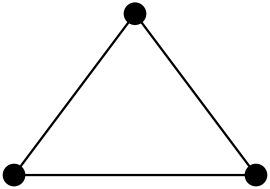
$$E_{\mathrm{mem}}(u) = 0 \quad \not\Rightarrow \quad E_{\mathrm{mem}}(u_h) = 0$$



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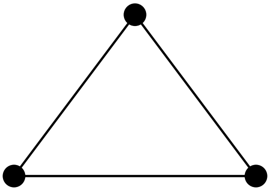

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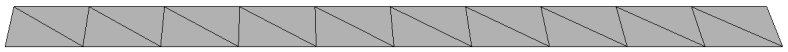
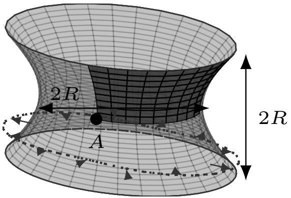
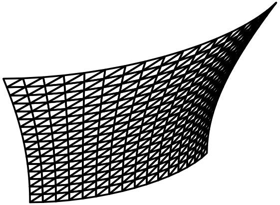
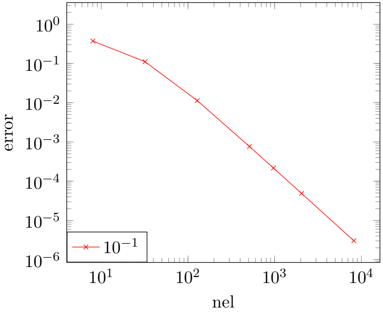


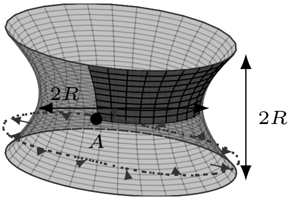
Hyperboloid with free ends

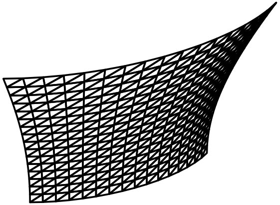


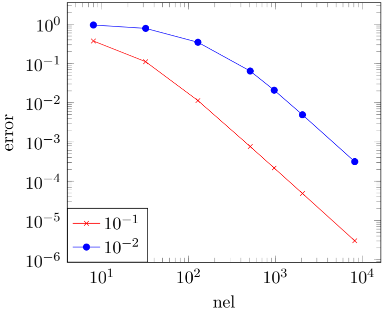




Hyperboloid with free ends

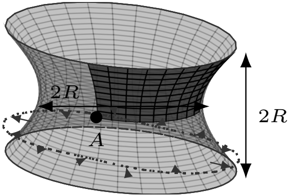


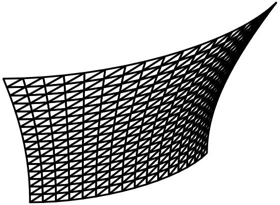


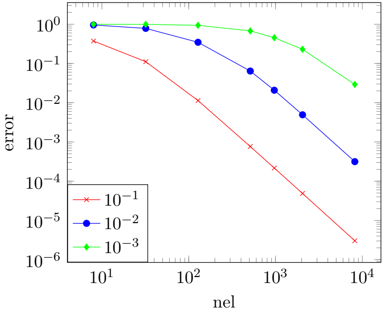


- Pre-asymptotic regime

Hyperboloid with free ends

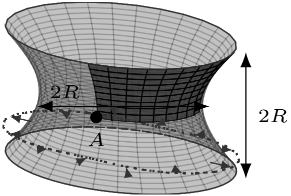


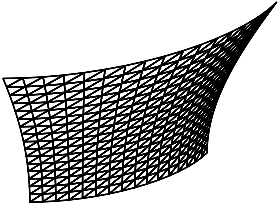


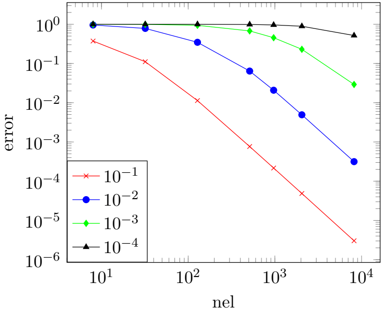


- Pre-asymptotic regime

Hyperboloid with free ends







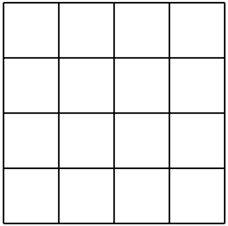
- Pre-asymptotic regime

Membrane locking

$$\frac{1}{t^2}\| \quad \boldsymbol{E}(u_h)\|^2_{\mathbb{M}}$$

Membrane locking

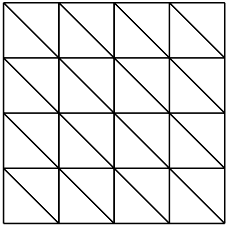
$$\frac{1}{t^2}\|\Pi^k_{L^2}\boldsymbol{E}(u_h)\|^2_{\mathbb{M}}$$

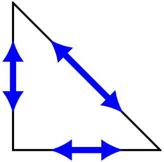


- Reduced integration for quadrilateral meshes

Membrane locking

$$\frac{1}{t^2}\|\mathcal{I}_{\mathcal{R}}^k \boldsymbol{E}(u_h)\|_{\mathbb{M}}^2$$

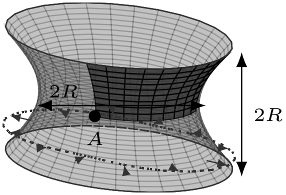


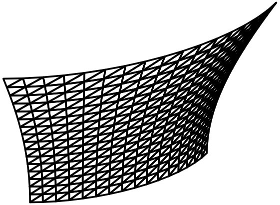


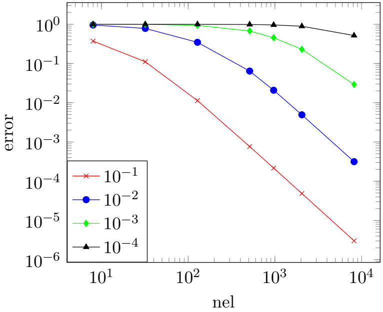
- Reduced integration for quadrilateral meshes
- Regge interpolant for triangles
- Connection to MITC shell elements

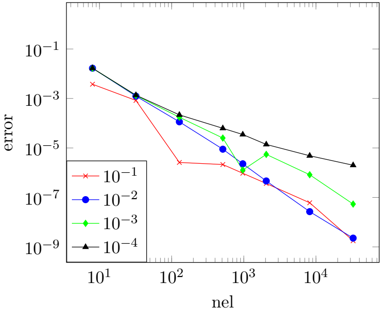
N., Schöberl: Avoiding membrane locking with Regge interpolation, *Comput. Methods Appl. Mech. Engrg 373 (2021).*

Hyperboloid with free ends

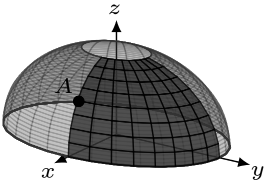


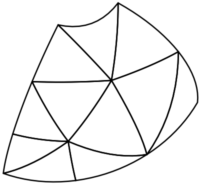


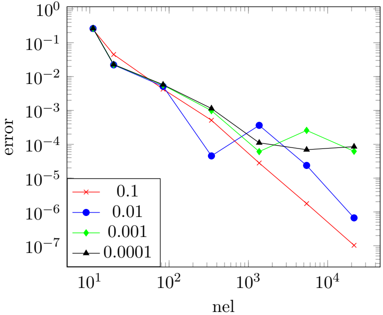


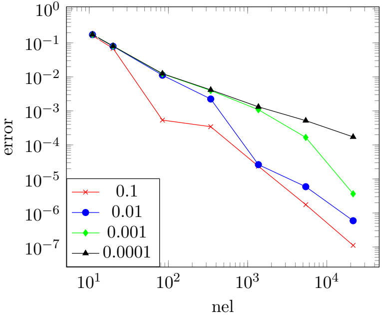


Open hemisphere with clamped ends



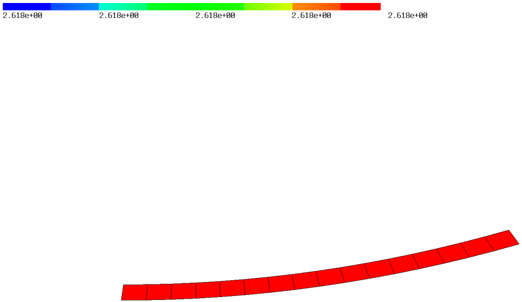


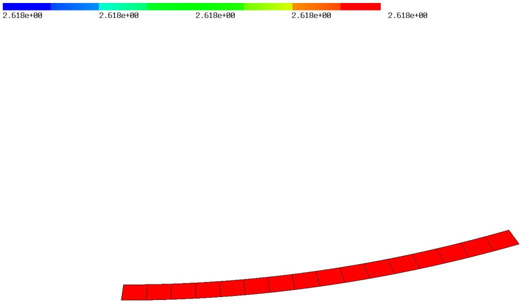




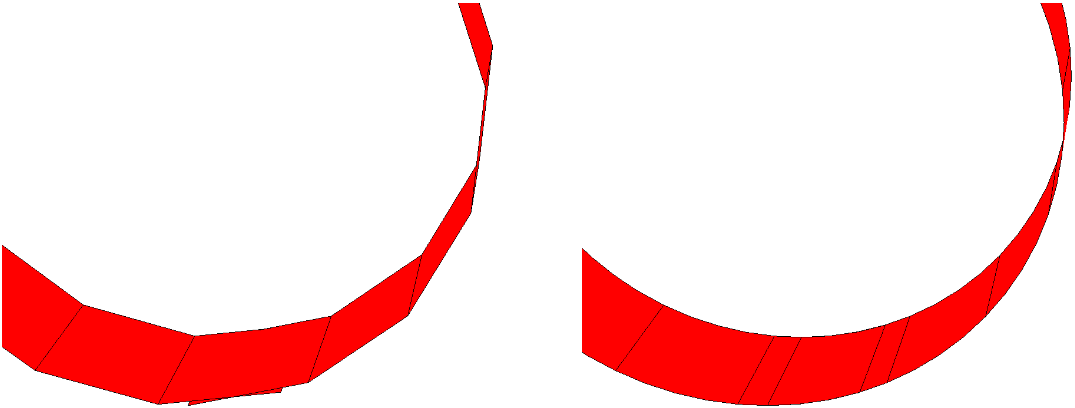
Numerical examples

Cantilever subjected to end moment

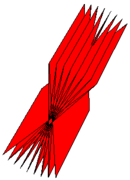


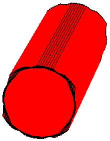


Cantilever subjected to end moment

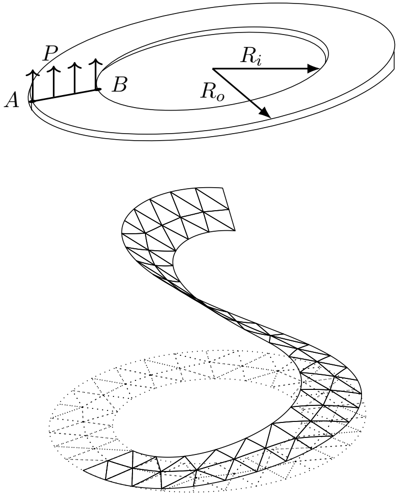


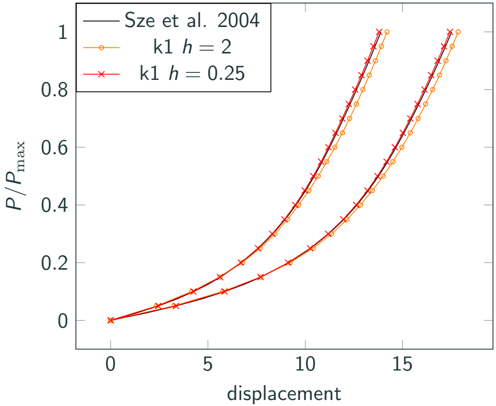
Cantilever subjected to end moment



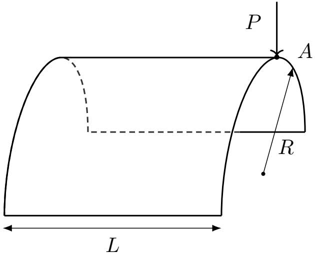


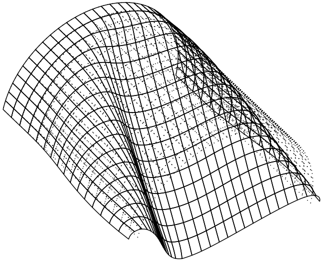
Slit annular plate

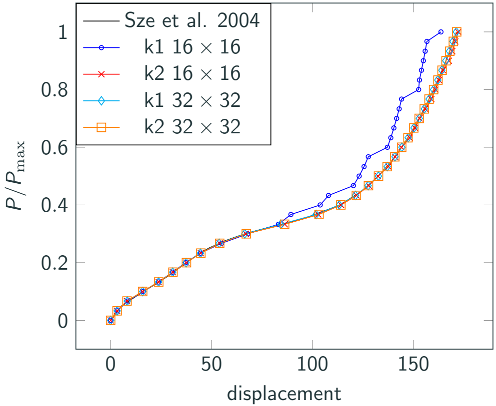




Pinched cylinder

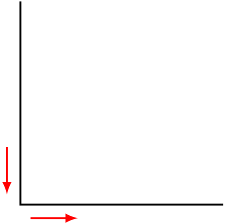




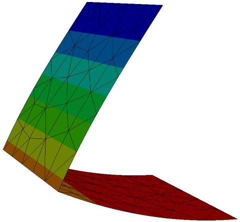


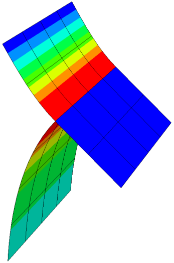
Structures with kinks and branched shells

- Normal-normal continuous moment $\boldsymbol{\sigma}$
- Preserve kinks
- Variation of $\mathcal{L}(u, \boldsymbol{\sigma})$ in direction $\delta\boldsymbol{\sigma}$



$$\int_E (\sphericalangle(\nu_L, \nu_R) - \sphericalangle(\hat{\nu}_L, \hat{\nu}_R))\delta\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\, ds \stackrel{!}{=} 0 \quad \Rightarrow \quad \sphericalangle(\nu_L, \nu_R) - \sphericalangle(\hat{\nu}_L, \hat{\nu}_R) = 0$$





Summary & outlook

TECHNISCHE UNIVERSITÄT WIEN

- Distributional extrinsic/intrinsic curvature
- Application bending energy of nonlinear shells
- Hellan–Herrmann–Johnson and Regge finite elements for stress and strain/metric fields

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- Application bending energy of nonlinear shells
- Hellan–Herrmann–Johnson and Regge finite elements for stress and strain/metric fields

- Coupling for 3D elasticity
- NGSolve Add-On
- Distributional curvature higher dimension $\to$ general relativity

Literature

- N., Schöberl: The Hellan–Herrmann–Johnson and TDNNS method for linear and nonlinear shells, *arXiv:2304.13806*.
- N., Schöberl: The Hellan–Herrmann–Johnson method for nonlinear shells, *Comput. Struct. 225 (2019)*.
- N., Schöberl: Avoiding membrane locking with Regge interpolation, *Comput. Methods Appl. Mech. Engrg 373 (2021)*.
- N.: Mixed Finite Element Methods For Nonlinear Continuum Mechanics And Shells, *PhD thesis, TU Wien (2021)*.
- Gopalakrishnan, N., Schöberl, Wardetzky: Analysis of curvature approximations via covariant curl and incompatibility for Regge metrics, *arXiv:2206.09343*.
- N., Schöberl, Sturm, Numerical shape optimization of Canham-Helfrich-Evans bending energy, *J. Comput. Phys.* (2023).

Thank You for Your attention!

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