Surface PDEs and curvature approximation in NGSolve

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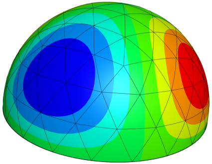
Michael Neunteufel (TU Wien)

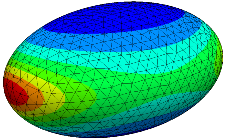
4th NGSolve User-Meeting, Portland, Oregon, July 10th, 2023

Motivation

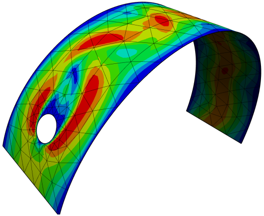
Poisson equation on surface



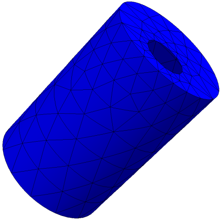
Extrinsic/intrinsic curvature approximation



Navier-Stokes equations on surface



Shells



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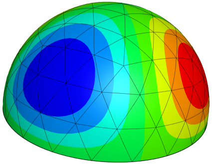

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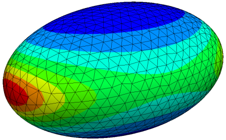

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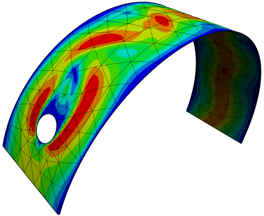

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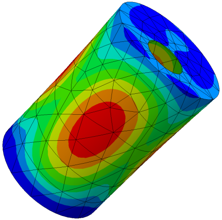
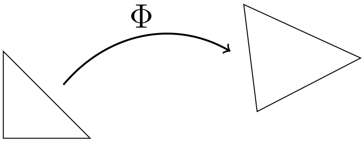


PDEs on surfaces



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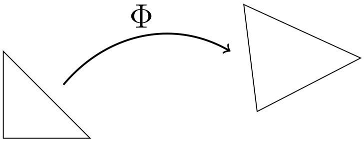

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- $J = \det(\boldsymbol{F})$

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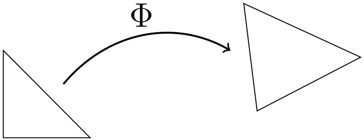

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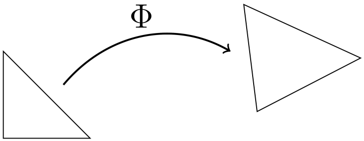

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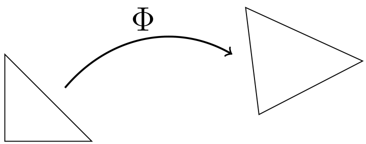

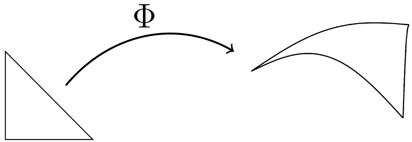

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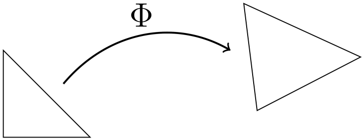

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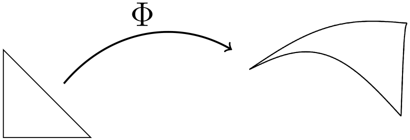

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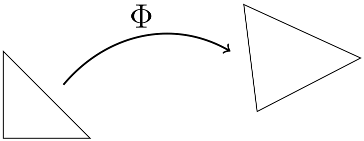

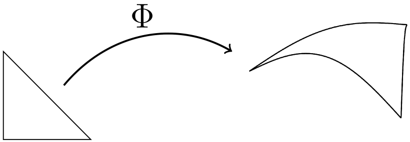

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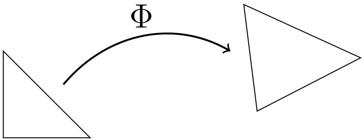

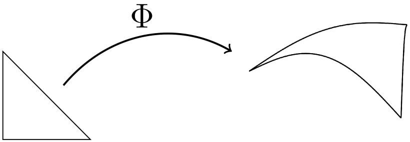

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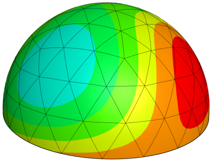
Surface Poisson problem

Find $u \in H^1(S)$ with $u = u_D$ on Γ_D such that for all $v \in H^1_{\Gamma_D}(S)$



$$\int_S \nabla_S u \cdot \nabla_S v \, ds = \int_S f \, v \, ds.$$

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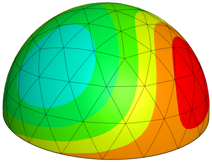

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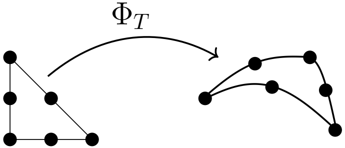

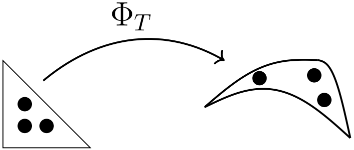

- Mapping (dis-)continuous FE via $u \circ \Phi = \hat{u}$

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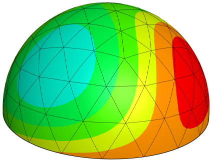
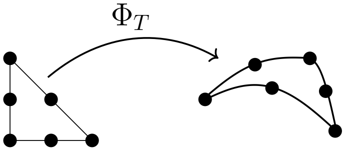
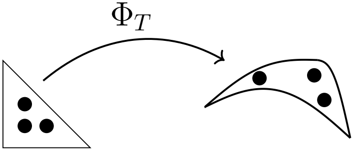
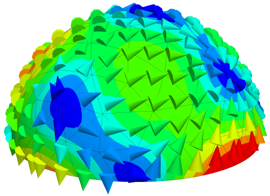


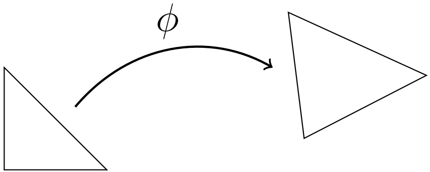




- Mapping (dis-)continuous FE via $u \circ \Phi = \hat{u}$
- How to map RT/BDM or Nédélec elements?
- Can we construct tangential vector fields?



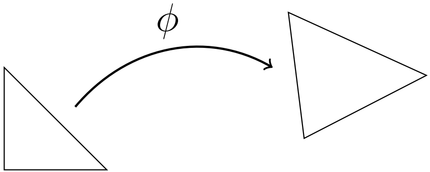
Mapping to the surface



Mapping to the surface

- Covariant transformation (H(curl)-conforming Nédélec elements)

$$u \circ \phi = \boldsymbol{F}^{-T} \hat{u}, \quad \hat{u} \in \mathcal{N}_{I/II}(\hat{T}) \qquad\qquad \boldsymbol{F} = \nabla_{\hat{x}} \phi,$$



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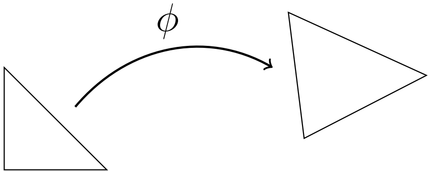

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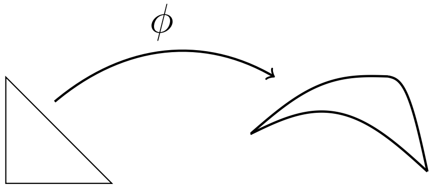

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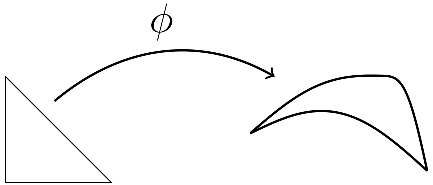

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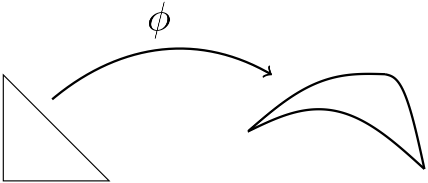
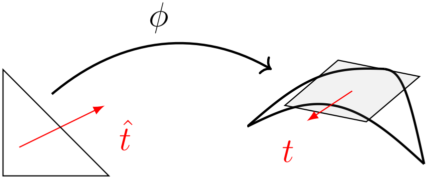




- u and v are tangential vector fields on S! $\qquad u \cdot \nu = v \cdot \nu = 0$

Surface spaces in NGSolve

Taking trace of existing 3D elements

- H1
- H(curl)
- H(curl curl)
- NumberSpace

Map 2D reference element onto surface

- L2
- H(div)
- H(div div)
- FacetSpaces

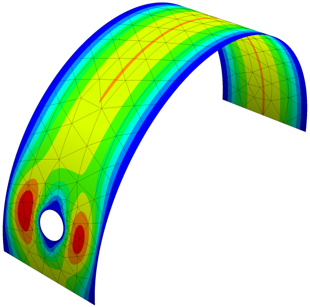
Surface Navier-Stokes

Unsteady surface Navier-Stokes equations

Find $(u, p) \in H^1(S, TS) \times L^2(S)$ such that $(\boldsymbol{P} = \boldsymbol{I} - \nu \otimes \nu,\ \varepsilon_S(u) = 0.5(\boldsymbol{P}\nabla_S u + (\boldsymbol{P}\nabla_S u)^T))$

$$\partial_t u - 2\overline{\nu}\boldsymbol{P}\mathrm{div}_S(\varepsilon_S(u)) + (u \cdot \nabla_S)u + \nabla_S p = f, \qquad \overline{\nu} \ldots \text{viscosity}$$
$$\mathrm{div}_S(u) = 0$$

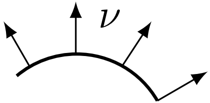
- H(div)-conforming HDG
- Exact divergence-free velocity u



LEDERER, LEHRENFELD, SCHÖBERL: Divergence-free Tangential Finite Element Methods for Incompressible Flows on Surfaces, *Int J Numer Methods Eng* (2020).

Distributional extrinsic curvature

- Change of normal vector measures curvature $\nabla_S \nu$



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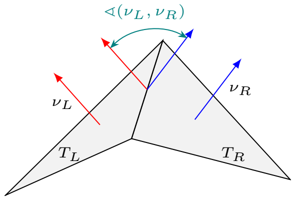
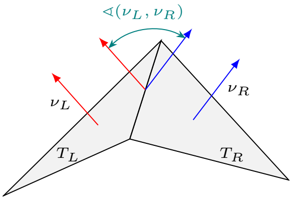
- How to define the Weingartentensor $\nabla_S \nu$ for discrete surface?



Grinspun, Gingold, Reisman, Zorin: Computing discrete shape operators on general meshes, *Computer Graphics Forum 25*, 3 (2006).

Distributional extrinsic curvature



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  - Distributional Weingarten tensor

$$\langle \nabla_S \nu, \boldsymbol{\sigma} \rangle_{\mathcal{T}} = \sum_{T \in \mathcal{T}_h} \int_T \nabla_S \nu|_T : \boldsymbol{\sigma} \, ds + \sum_{E \in \mathcal{E}_h} \int_E \sphericalangle(\nu_L, \nu_R) \boldsymbol{\sigma}_{\mu\mu} \, dl$$

  - Measure jump of normal vector
  - Test function $\boldsymbol{\sigma}$ symmetric, normal-normal continuous $\Rightarrow$ Hellan–Herrmann–Johnson finite elements

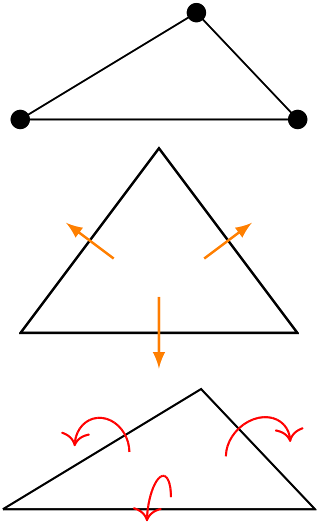
N., Schöberl, Sturm, Numerical shape optimization of Canham-Helfrich-Evans bending energy, *J. Comput. Phys.* (2023).

Hellan–Herrmann–Johnson finite elements

$$H^1(\Omega) = \{u \in L^2(\Omega) \mid \nabla u \in [L^2(\Omega)]^2\}$$
$$V_h^k = \mathcal{P}^k(\mathcal{T}_h) \cap C(\Omega)$$

$$H(\mathrm{div}) = \{\sigma \in [L^2(\Omega)]^2 \mid \mathrm{div}\sigma \in L^2(\Omega)\}$$
$$BDM^k = \{\sigma \in [\mathcal{P}^k(\mathcal{T}_h)]^2 \mid [\![\sigma_n]\!]_F = 0\}$$

$$H(\mathrm{divdiv}) = \{\sigma \in [L^2(\Omega)]^{2\times 2}_{sym} \mid \mathrm{divdiv}\sigma \in H^{-1}(\Omega)\}$$
$$M_h^k = \{\sigma \in [\mathcal{P}^k(\mathcal{T}_h)]^{2\times 2}_{sym} \mid [\![n^T \sigma n]\!]_F = 0\}$$

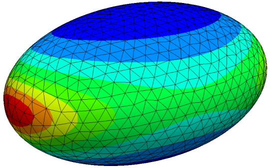


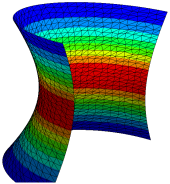
Lifted distributional curvature

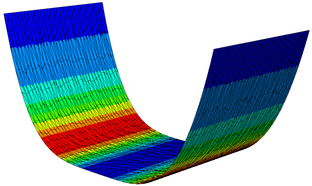
Lifting of distributional Weingarten tensor

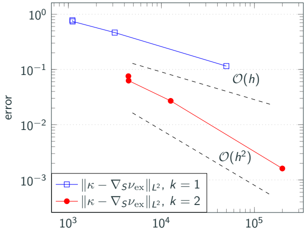
Find $\boldsymbol{\kappa} \in M_h^{k-1}$ for $\mathcal{T}_h$ curving order k s.t. for all $\boldsymbol{\sigma} \in M_h^{k-1}$

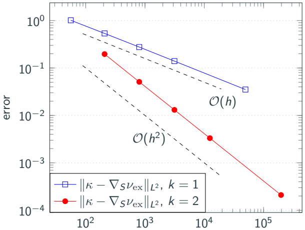
$$\int_{\mathcal{T}_h} \boldsymbol{\kappa} : \boldsymbol{\sigma} \, dx = \langle \nabla_S \nu, \boldsymbol{\sigma} \rangle_{\mathcal{T}}.$$

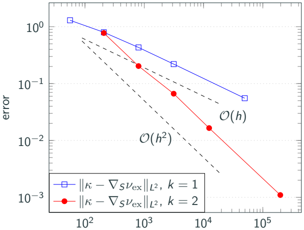












Application: Koiter shell

$$\mathcal{W}(u) = \frac{t}{2}\|\boldsymbol{E}(u)\|^2_{\mathbb{M}} + \frac{t^3}{24}\|\boldsymbol{F}^T\nabla_S(\nu\circ\phi) - \nabla_S\hat{\nu}\|^2_{\mathbb{M}}$$

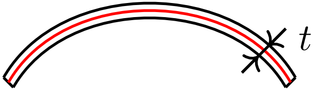
u . . . displacement of mid-surface

t . . . thickness

$\mathbb{M}$ . . . material tensor

$\boldsymbol{F} = \nabla_S u + \boldsymbol{P} = \nabla_S\phi, \qquad \boldsymbol{P} = \boldsymbol{I} - \hat{\nu}\otimes\hat{\nu}$

$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^\top\boldsymbol{F} - \boldsymbol{P}) = \frac{1}{2}(\nabla_S u^\top\nabla_S u + \nabla_S u^\top\boldsymbol{P} + \boldsymbol{P}\nabla_S u)$



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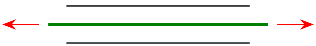

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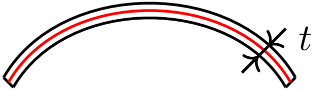

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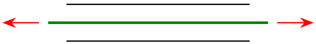


membrane energy

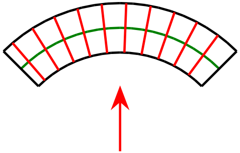
$u \ldots$ displacement of mid-surface

$t \ldots$ thickness

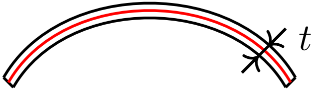
$\mathbb{M} \ldots$ material tensor

$\boldsymbol{F} = \nabla_S u + \boldsymbol{P} = \nabla_S \phi, \qquad \boldsymbol{P} = \boldsymbol{I} - \hat{\nu}\otimes\hat{\nu}$

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^\top\boldsymbol{F} - \boldsymbol{P}) = \frac{1}{2}(\nabla_S u^\top \nabla_S u + \nabla_S u^\top \boldsymbol{P} + \boldsymbol{P}\nabla_S u)$$



bending energy



Distributional curvature for Koiter shell

$$\text{Lifting:}\quad \int_{\mathcal{T}_h} \boldsymbol{\kappa} : \boldsymbol{\sigma}\, ds = \sum_{T\in\mathcal{T}_h}\int_T \nabla_S \nu : \boldsymbol{\sigma}\, ds + \sum_{E\in\mathcal{E}_h}\int_E \sphericalangle(\nu_L,\nu_R)\boldsymbol{\sigma}_{\mu\mu}\, dl$$

- Lifted curvature difference $\boldsymbol{\kappa}^{\mathrm{diff}}$ via three-field formulation

$$\mathcal{L}(u,\boldsymbol{\kappa}^{\mathrm{diff}},\boldsymbol{\sigma}) = \frac{t}{2}\|\boldsymbol{E}(u)\|^2_{\mathbb{M}} + \frac{t^3}{12}\|\boldsymbol{\kappa}^{\mathrm{diff}}\|^2_{\mathbb{M}} - \langle f,u\rangle + \sum_{T\in\mathcal{T}_h}\int_T \left(\boldsymbol{\kappa}^{\mathrm{diff}} - (\boldsymbol{F}^T\nabla_S(\nu\circ\phi) - \nabla_S\hat{\nu})\right) : \boldsymbol{\sigma}\, ds$$
$$+ \sum_{E\in\mathcal{E}_h}\int_E (\sphericalangle(\nu_L,\nu_R) - \sphericalangle(\hat{\nu}_L,\hat{\nu}_R))\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\, dl$$

- Lagrange parameter $\boldsymbol{\sigma}\in M_h^{k-1}$ moment tensor
- Eliminate $\boldsymbol{\kappa}^{\mathrm{diff}} \to$ two-field formulation in $(u,\boldsymbol{\sigma})$

N., Schöberl: The Hellan–Herrmann–Johnson and TDNNS method for linear and nonlinear shells, *arXiv:2304.13806*.

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- Lifted curvature difference $\boldsymbol{\kappa}^{\mathrm{diff}}$ via three-field formulation

$$\mathcal{L}(u, \boldsymbol{\sigma}) = \frac{t}{2}\|\boldsymbol{E}(u)\|^2_{\mathbb{M}} - \frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\mathbb{M}^{-1}} - \langle f, u\rangle + \sum_{T\in\mathcal{T}_h} \int_T \left(\boldsymbol{F}^T \nabla_S(\nu\circ\phi) - \nabla_S\hat{\nu}\right) : \boldsymbol{\sigma}\, ds$$
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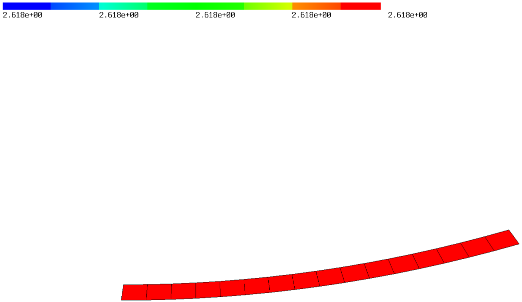
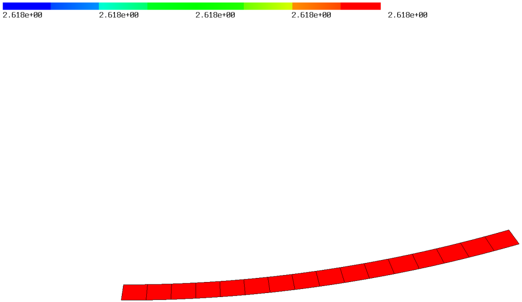
Cantilever subjected to end moment





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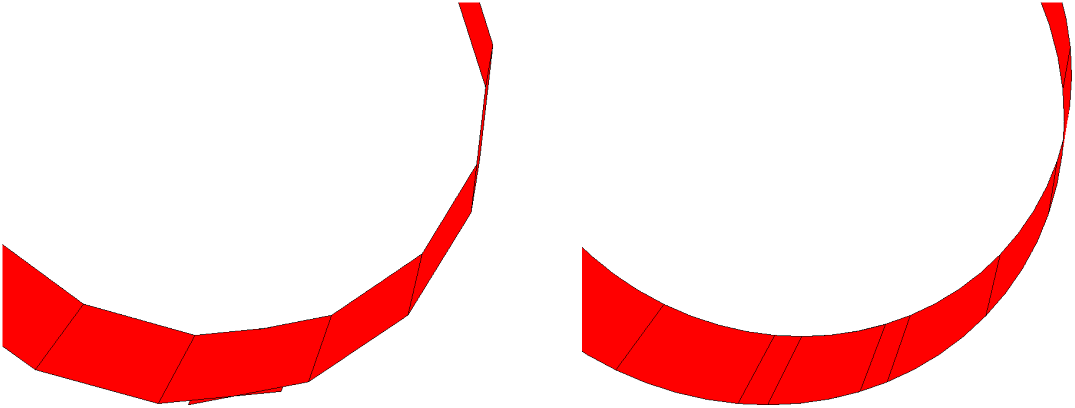


Gaussian curvature and mean curvature

Let τ_1, τ_2 be an orthonormal basis for the tangent space to a surface $\mathcal{S}$ with unit normal ν.

Let the matrix $A_{ij} = \tau_i \cdot \nabla_{\tau_j} \nu$ have eigenvalues κ_1, κ_2 (called the **principal curvatures**).

Gaussian curvature: $K = \det A = \kappa_1 \kappa_2$

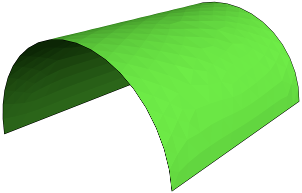
Mean curvature: $H = \operatorname{Tr} A = \kappa_1 + \kappa_2$

Gauss's Theorema Egregium: K is *intrinsic*; it depends only on the metric $g = \nabla\Phi^T \nabla\Phi$.

In contrast, H is *extrinsic*; it cannot be computed from g alone.

$K = 0,\ H = 0$

$\kappa_1 = 0,\ \kappa_2 = 0$



$K = 0,\ H > 0$

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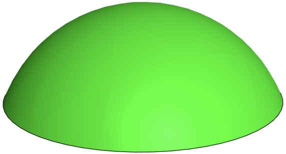


$K > 0,\ H > 0$

$\kappa_1 > 0,\ \kappa_2 > 0$

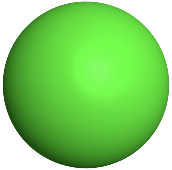
Gaussian curvature

Gauss-Bonnet Theorem on a smooth surface $\mathcal{S}$ without boundary:

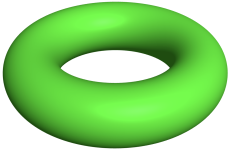
$$\int_{\mathcal{S}} K ds = 2\pi\chi(\mathcal{S})$$

where

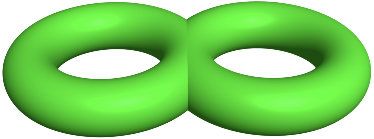
$$\begin{aligned}\chi(\mathcal{S}) &= \text{ Euler characteristic of } \mathcal{S} \\ &= \#V - \#E + \#F \text{ in any triangulation of } \mathcal{S} \\ &= 2 - 2G, \text{where } G = \text{genus of } \mathcal{S} \text{ (number of ``handles'')}\end{aligned}$$



$\chi = 2,\ G = 0$



$\chi = 0,\ G = 1$



$\chi = -2,\ G = 2$

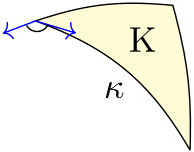
Gauss-Bonnet Theorem on a smooth surface $\mathcal{S}$ with boundary:

$$\int_{\mathcal{S}} K ds + \int_{\partial \mathcal{S}} \kappa_g \, d\ell + \sum_{\text{corners } V} (\pi - \sphericalangle_V) = 2\pi\chi(\mathcal{S})$$

where

$$\begin{aligned} \kappa_g &= \text{ geodesic curvature of } \partial\mathcal{S} \\ &= \mu \cdot \nabla_\tau \tau \;\; (\tau = \text{unit tangent}, \; \mu = \text{unit conormal}) \end{aligned}$$

$$\pi - \sphericalangle_V = \text{ turning angle at } V$$



Distributional Gaussian curvature

Q: What if $\mathcal{S}$ isn't smooth?

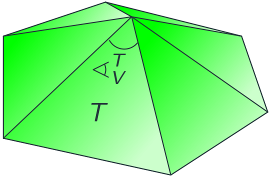
A: Then K is a distribution.

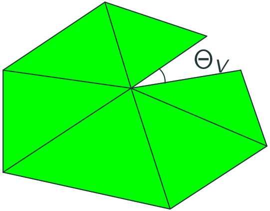
If $\mathcal{S}$ is a *piecewise linear* triangulated surface, then

$$\langle K, \varphi \rangle = \sum_{V \in \mathcal{V}_h} \Theta_V \varphi(V)$$

for every scalar function φ, where

$$\begin{aligned} \Theta_V &= \text{ angle defect at } V \\ &= 2\pi - \sum_{T : V \subset T} \sphericalangle_V^T \end{aligned}$$





Distributional Gaussian curvature

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Berchenko-Kogan & G.: Finite element approximation of the Levi-Civita connection and its curvature in two dimensions, *Found Comput Math* (2022).

Gopalakrishnan, N., Schöberl, Wardetzky: Analysis of curvature approximations via covariant incompatibility and curl for Regge metrics, *arXiv:2206.09343* (2022).

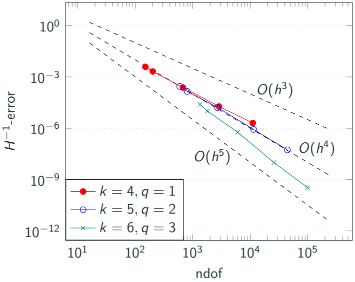
Lifted distributional Gaussian curvature

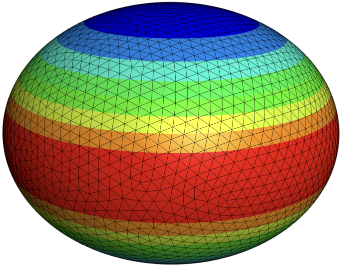
Lifted distributional Gaussian curvature

Given $\mathcal{T}_h$ with curving order k, find $K_h \in V_h^q$ s.t. for all $\varphi \in V_h^q$

$$\int_{\mathcal{S}} K_h \varphi \, ds = \sum_{V \in \mathcal{V}_h} \Theta_V \varphi(V) + \sum_{E \in \mathcal{E}_h} \int_E [\![\kappa_g]\!] \varphi d\ell + \sum_{T \in \mathcal{T}_h} \int_T K|_T \varphi ds$$

Error estimate: $\|K_h - K\|_{H^{-1}} = O(h^{\min\{k-1, q+2\}})$





G.: High-order approximation of Gaussian curvature with Regge finite elements, *SIAM J. Numerical Analysis* (2020).

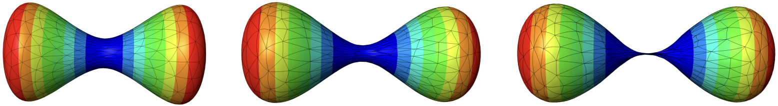
Mean curvature

- Recall: Mean curvature $H = \kappa_1 + \kappa_2$
- Useful identity: $H\nu = -\Delta_{\mathcal{S}}\,\mathrm{id}$, where $\mathrm{id} : \mathcal{S} \to \mathbb{R}^3$ maps each point $(x_1, x_2, x_3) \in \mathcal{S}$ to its position vector $(x_1, x_2, x_3) \in \mathbb{R}^3$
- This motivates the following discretization of **mean-curvature flow** $\dot{x} = -H\nu$:

Discrete mean curvature flow

Find $x(t) \in (V_h^k)^3$ s.t. for all $v \in (V_h^k)^3$,

$$\int_{\mathcal{S}(x)} \dot{x} \cdot v \, ds + \int_{\mathcal{S}(x)} \nabla_{\mathcal{S}(x)}\,\mathrm{id} : \nabla_{\mathcal{S}(x)} v \, ds = 0.$$



DZIUK: An algorithm for evolutionary surfaces, *Numerische Mathematik* (1991).

Tutorials

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- Surface Navier-Stokes
- Extrinsic curvature approximation
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Thank You for Your attention!