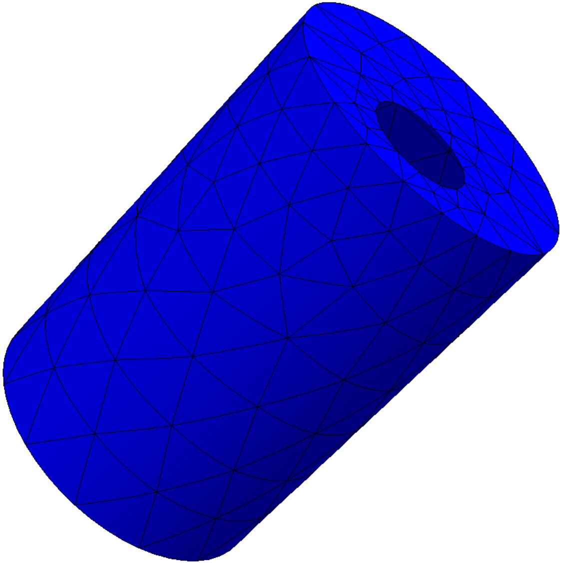
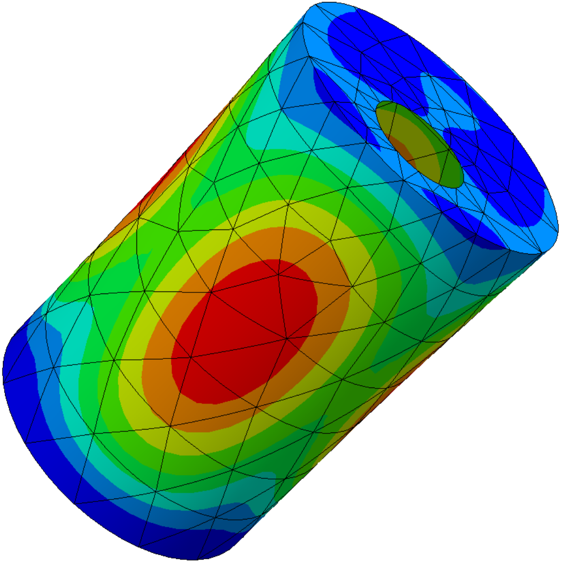
Nonlinear Shells - Avoiding Locking by the Hellan–Herrmann–Johnson Method and Regge Interpolation

Michael Neunteufel, Joachim Schöberl

Stuttgart, January 16, 2020



Motivation



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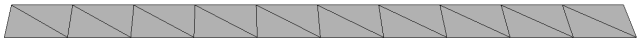
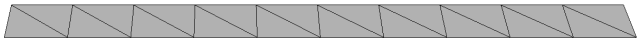
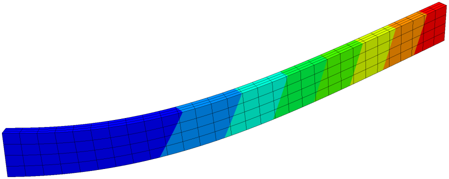




Contents

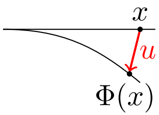
Notation

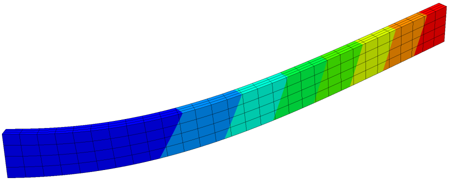
Deformation $\Phi : \Omega \to \mathbb{R}^3$



Deformation $\Phi : \Omega \to \mathbb{R}^3$

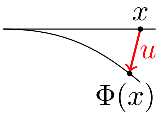
Displacement $u := \Phi - id$

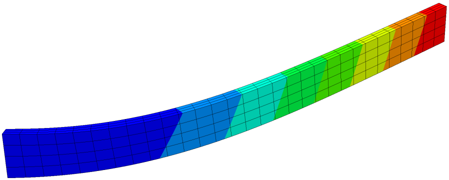




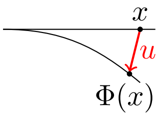
Continuum mechanics

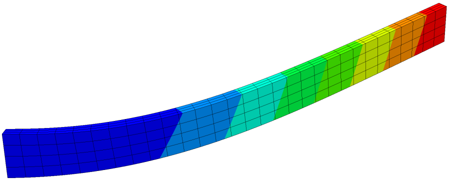
Deformation	$\Phi : \Omega \to \mathbb{R}^3$
Displacement	$u := \Phi - id$
Deformation gradient	$\boldsymbol{F} := \nabla \Phi$



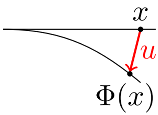


Deformation	$\Phi : \Omega \to \mathbb{R}^3$
Displacement	$u := \Phi - id$
Deformation gradient	$\boldsymbol{F} := \boldsymbol{I} + \nabla u$



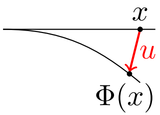


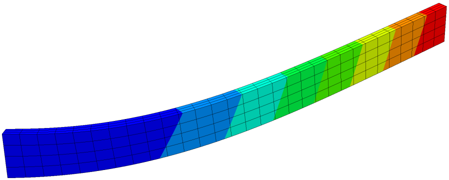
Deformation	$\Phi : \Omega \to \mathbb{R}^3$
Displacement	$u := \Phi - id$
Deformation gradient	$\boldsymbol{F} := \boldsymbol{I} + \nabla u$
Cauchy-Green strain tensor	$\boldsymbol{C} := \boldsymbol{F}^T \boldsymbol{F}$



$$\frac{||\Phi(x + \Delta x) - \Phi(x)||^2}{||\Delta x||^2} = \frac{\Delta x^T F^T F \Delta x}{||\Delta x||^2} + \mathcal{O}(||\Delta x||)$$

Deformation	$\Phi : \Omega \to \mathbb{R}^3$
Displacement	$u := \Phi - id$
Deformation gradient	$\boldsymbol{F} := \boldsymbol{I} + \nabla u$
Cauchy-Green strain tensor	$\boldsymbol{C} := \boldsymbol{F}^T \boldsymbol{F}$
Green strain tensor	$\boldsymbol{E} := \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I})$





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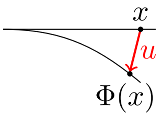
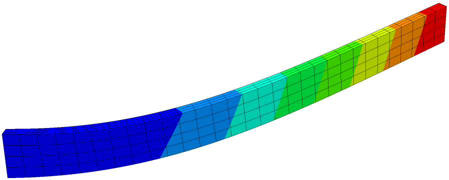
Green strain tensor	$\boldsymbol{E} := \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I})$
Linearized strain tensor	$\epsilon(u) := \frac{1}{2}(\nabla u^T + \nabla u)$





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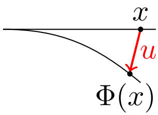
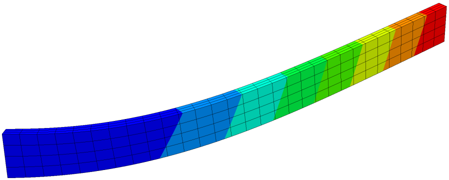
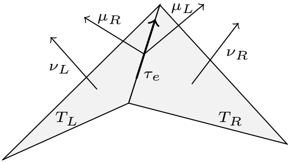


Elasticity

$$\mathcal{W}(u) = \frac{1}{2}\|\boldsymbol{E}\|_{\boldsymbol{M}}^2 - \langle f, u \rangle$$



- Normal vector ν
  Tangent vector τ_e
  Element normal vector $\mu = \pm\nu \times \tau_e$



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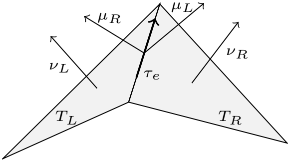
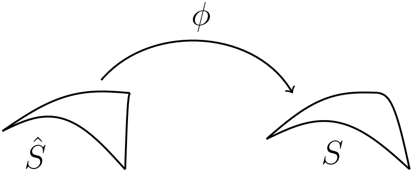
- Normal vector $\hat{\nu}$

  Tangent vector $\hat{\tau}_e$

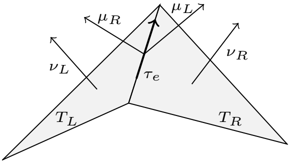
  Element normal vector $\hat{\mu} = \pm\hat{\nu} \times \hat{\tau}_e$



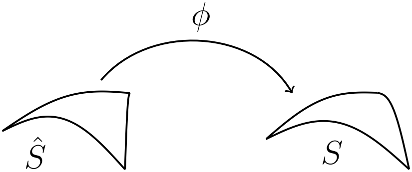


Differential geometry

- Normal vector $\hat{\nu}$
  Tangent vector $\hat{\tau}_e$
  Element normal vector $\hat{\mu} = \pm\hat{\nu} \times \hat{\tau}_e$



- $\boldsymbol{F} = \nabla_{\hat{\tau}}\phi$, $J = \sqrt{\det(\boldsymbol{F}^T\boldsymbol{F})}$



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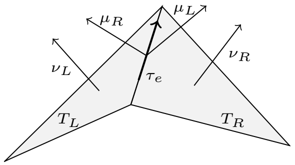
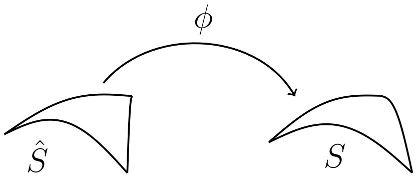
  Tangent vector $\hat{\tau}_e$
  Element normal vector $\hat{\mu} = \pm\hat{\nu} \times \hat{\tau}_e$



- $\boldsymbol{F} = \nabla_{\hat{\tau}}\phi$, $J = \|\operatorname{cof}(\boldsymbol{F})\|_F$



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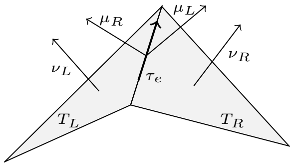
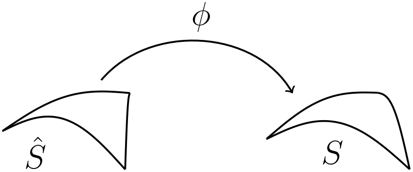
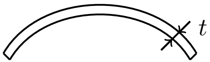
  Tangent vector $\hat{\tau}_e$
  Element normal vector $\hat{\mu} = \pm\hat{\nu} \times \hat{\tau}_e$

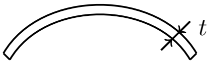


- $\boldsymbol{F} = \nabla_{\hat{\tau}}\phi$, $J = \|\operatorname{cof}(\boldsymbol{F})\|_F$
- $\nu \circ \phi = \frac{1}{J}\operatorname{cof}(\boldsymbol{F})\hat{\nu}$
  $\tau_e \circ \phi = \frac{1}{J_B}\boldsymbol{F}\hat{\tau}_e$
  $\mu \circ \phi = \pm\nu \times \tau_e$





- Model of reduced dimensions



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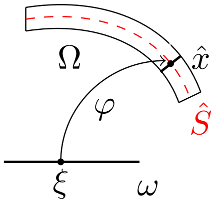
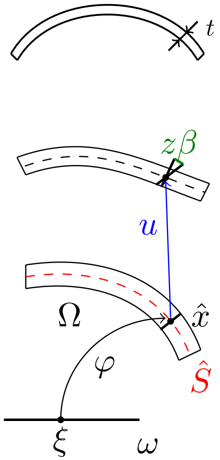
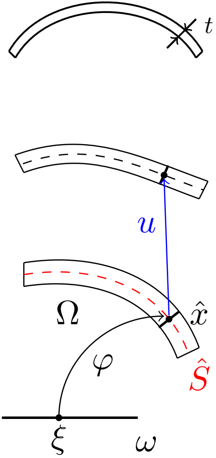


- $\Omega = \left\{\varphi(\xi) + z\hat{\nu}(\xi) : \xi \in \omega, z \in \left[-\frac{t}{2}, \frac{t}{2}\right]\right\}$



- Model of reduced dimensions
- $\Omega = \left\{\varphi(\xi) + z\hat{\nu}(\xi) : \xi \in \omega, z \in \left[-\frac{t}{2}, \frac{t}{2}\right]\right\}$
- $\Phi(\hat{x} + z\hat{\nu}(\xi)) = \phi(\hat{x}) + z\ (\nu + \beta) \circ \phi(\hat{x})$



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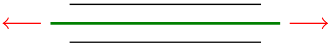
- $\Omega = \left\{\varphi(\xi) + z\hat{\nu}(\xi) : \xi \in \omega, z \in \left[-\frac{t}{2}, \frac{t}{2}\right]\right\}$
- $\Phi(\hat{x} + z\hat{\nu}(\xi)) = \phi(\hat{x}) + z\,\nu \circ \phi(\hat{x})$

Shell energy (Kirchhoff–Love)

$$\mathcal{W}(u) = \frac{t}{2}\|\boldsymbol{E}_{\tau\tau}(u)\|^2_{\boldsymbol{M}} + \frac{t^3}{24}\|\boldsymbol{F}^T\nabla(\nu\circ\phi) - \nabla\hat{\nu}\|^2_{\boldsymbol{M}}$$

Shell energy (Kirchhoff–Love)

$$\mathcal{W}(u) = \frac{t}{2}\|\boldsymbol{E}_{\tau\tau}(u)\|_{\boldsymbol{M}}^2 + \frac{t^3}{24}\|\boldsymbol{F}^T\nabla(\nu\circ\phi) - \nabla\hat{\nu}\|_{\boldsymbol{M}}^2$$



- Membrane energy

Shell energy (Kirchhoff–Love)

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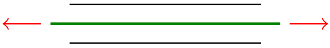

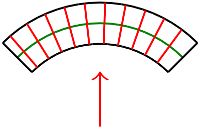

- Membrane energy
- Bending energy

Shell energy (Kirchhoff–Love)

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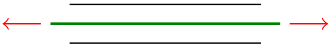
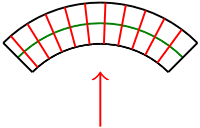
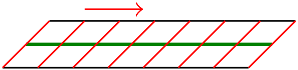
$$\mathcal{W}(u) = \frac{t}{2}\|\boldsymbol{E}_{\tau\tau}(u)\|^2_{\boldsymbol{M}} + \frac{t^3}{24}\|\boldsymbol{F}^T\nabla(\nu\circ\phi) - \nabla\hat{\nu}\|^2_{\boldsymbol{M}}$$



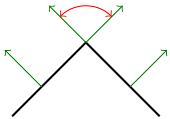




- Membrane energy
- Bending energy
- Shearing energy

Method and Shell Element

$$\mathcal{W}(u) = \frac{t}{2}\|\boldsymbol{E}_{\tau\tau}(u)\|^2_{\boldsymbol{M}} + \frac{t^3}{24}\|\boldsymbol{F}^T\nabla\nu - \nabla\hat{\nu}\|^2_{\boldsymbol{M}}$$
$$+ \frac{t^3}{24}\sum_{\hat{E}\in\hat{\mathcal{E}}_h}\|\sphericalangle(\nu_L,\nu_R) - \sphericalangle(\hat{\nu}_L,\hat{\nu}_R)\|^2_{\boldsymbol{M},\hat{E}}$$



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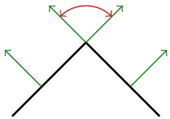
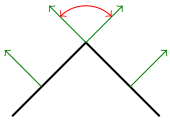


- Measure change of angles

$$\mathcal{W}(u) = \frac{t}{2}\|\boldsymbol{E}_{\tau\tau}(u)\|^2_{\boldsymbol{M}} + \frac{t^3}{24}\|\boldsymbol{F}^T\nabla\nu - \nabla\hat{\nu}\|^2_{\boldsymbol{M}} + \frac{t^3}{24}\sum_{\hat{E}\in\hat{\mathcal{E}}_h}\|\sphericalangle(\nu_L,\nu_R) - \sphericalangle(\hat{\nu}_L,\hat{\nu}_R)\|^2_{\boldsymbol{M},\hat{E}}$$



- Measure change of angles

$$\mathcal{L}(u,\boldsymbol{\sigma}) = \frac{t}{2}\|E_{\tau\tau}(u)\|^2_{\boldsymbol{M}} - \frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\boldsymbol{M}^{-1}} + \langle\boldsymbol{F}^T\nabla\nu - \nabla\hat{\nu},\boldsymbol{\sigma}\rangle + \sum_{\hat{E}\in\hat{\mathcal{E}}_h}\langle\sphericalangle(\nu_L,\nu_R) - \sphericalangle(\hat{\nu}_L,\hat{\nu}_R),\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\rangle_{\hat{E}}$$

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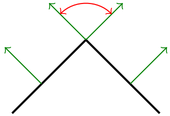

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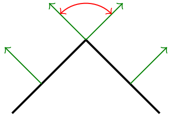
- $\boldsymbol{\sigma}$ has physical meaning of moment

$$\mathcal{W}(u) = \frac{t}{2}\|\boldsymbol{E}_{\tau\tau}(u)\|^2_{\boldsymbol{M}} + \frac{t^3}{24}\|\boldsymbol{F}^T\nabla\nu - \nabla\hat{\nu}\|^2_{\boldsymbol{M}} + \frac{t^3}{24}\sum_{\hat{E}\in\hat{\mathcal{E}}_h}\|\sphericalangle(\nu_L,\nu_R) - \sphericalangle(\hat{\nu}_L,\hat{\nu}_R)\|^2_{\boldsymbol{M},\hat{E}}$$



- Measure change of angles

$$\mathcal{L}(u,\boldsymbol{\sigma}) = \frac{t}{2}\|E_{\tau\tau}(u)\|^2_{\boldsymbol{M}} - \frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\boldsymbol{M}^{-1}} + \langle\boldsymbol{F}^T\nabla\nu - \nabla\hat{\nu},\boldsymbol{\sigma}\rangle + \sum_{\hat{E}\in\hat{\mathcal{E}}_h}\langle\sphericalangle(\nu_L,\nu_R) - \sphericalangle(\hat{\nu}_L,\hat{\nu}_R),\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\rangle_{\hat{E}}$$

- $\boldsymbol{\sigma}$ has physical meaning of moment
- Fourth order problem $\to$ second order problem

Shell problem

Find $u \in [H^1(\hat{S})]^3$ and $\boldsymbol{\sigma} \in H(\mathrm{divdiv}, \hat{S})$ for the saddle point problem

$$\mathcal{L}(u,\boldsymbol{\sigma}) = \frac{t}{2}\|E_{\tau\tau}(u)\|^2_{\boldsymbol{M}} - \frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\boldsymbol{M}^{-1}} + G(u,\boldsymbol{\sigma}) - \langle f, u\rangle,$$

with

$$G(u,\boldsymbol{\sigma}) = \sum_{\hat{T}\in\hat{\mathcal{T}}_h}\int_{\hat{T}} \boldsymbol{\sigma} : (\boldsymbol{H}_\nu + (1-\hat{\nu}\cdot\nu)\nabla\hat{\nu})\,d\hat{x}$$
$$- \sum_{\hat{E}\in\hat{\mathcal{E}}_h}\int_{\hat{E}} (\sphericalangle(\nu_L,\nu_R) - \sphericalangle(\hat{\nu}_L,\hat{\nu}_R))\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\,d\hat{s}.$$

$\boldsymbol{H}_\nu := \sum_i (\nabla^2 u_i)\nu_i$

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Find $u \in [H^1(\hat{S})]^3$ and $\boldsymbol{\sigma} \in H(\mathrm{divdiv}, \hat{S})$ for the saddle point problem

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with

$$G(u, \boldsymbol{\sigma}) = \sum_{\hat{T}\in\hat{\mathcal{T}}_h} \int_{\hat{T}} \boldsymbol{\sigma} : (\boldsymbol{H}_\nu \qquad\qquad) \, d\hat{x}$$

$$- \sum_{\hat{E}\in\hat{\mathcal{E}}_h} \int_{\hat{E}} (\sphericalangle(\nu_L, \nu_R) \qquad\qquad) \boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}} \, d\hat{s}.$$

$\boldsymbol{H}_\nu := \sum_i (\nabla^2 u_i)\nu_i$

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Shell problem

Find $u \in [H^1(\hat{S})]^3$ and $\boldsymbol{\sigma} \in H(\text{divdiv}, \hat{S})$ for the saddle point problem

$$\mathcal{L}(u, \boldsymbol{\sigma}) = \frac{t}{2} \| \frac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha}) - b_{\alpha\beta} u_3 \|^2_{\boldsymbol{M}} - \frac{6}{t^3} \| \boldsymbol{\sigma} \|^2_{\boldsymbol{M}^{-1}} + G(u, \boldsymbol{\sigma}) - \langle f, u \rangle,$$

with

$$G(u, \boldsymbol{\sigma}) = \sum_{\hat{T} \in \hat{\mathcal{T}}_h} \int_{\hat{T}} \boldsymbol{\sigma} : u_{3|\alpha\beta} \, d\hat{x} - \sum_{\hat{E} \in \hat{\mathcal{E}}_h} \int_{\hat{E}} [\![u_{3|\hat{\mu}}]\!] \boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}} \, d\hat{s}.$$

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Shell problem (Hybridization)

Find $u \in [H^1(\hat S)]^3$, $\boldsymbol{\sigma} \in H(\mathrm{divdiv}, \hat S)^{dc}$ and $\alpha \in \Gamma(\hat S)$ for

$$\mathcal{L}(u,\boldsymbol{\sigma}) = \frac{t}{2}\|E_{\tau\tau}(u)\|^2_{\boldsymbol{M}} - \frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\boldsymbol{M}^{-1}} + G(u,\boldsymbol{\sigma},\alpha) - \langle f, u\rangle,$$

with

$$\begin{aligned} G(u,\boldsymbol{\sigma},\alpha) = &\sum_{\hat T\in\hat{\mathcal{T}}_h}\int_{\hat T}\boldsymbol{\sigma} : (\boldsymbol{H}_\nu + (1-\hat\nu\cdot\nu)\nabla\hat\nu)\,d\hat x \\ &- \sum_{\hat E\in\hat{\mathcal{E}}_h}\int_{\hat E}(\sphericalangle(\nu_L,\nu_R) - \sphericalangle(\hat\nu_L,\hat\nu_R))\frac{1}{2}(\boldsymbol{\sigma}_{\hat\mu_L\hat\mu_L} + \boldsymbol{\sigma}_{\hat\mu_R\hat\mu_R})\,d\hat s \\ &+ \int_{\hat E}\alpha_{\hat\mu}[\![\boldsymbol{\sigma}_{\hat\mu\hat\mu}]\!]\,d\hat s. \end{aligned}$$

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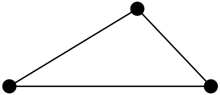
$$H^1(\Omega) := \{u \in L^2(\Omega) \,|\, \nabla u \in [L^2(\Omega)]^d\}$$

$$H^1(\Omega) := \{u \in L^2(\Omega) \mid \nabla u \in [L^2(\Omega)]^d\}$$
$$V_k := \Pi^k(\mathcal{T}_h) \cap C(\Omega)$$



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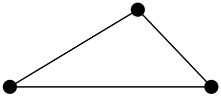
$$H(\mathrm{div}) := \{\sigma \in [L^2(\Omega)]^d \mid \mathrm{div}(\sigma) \in L^2(\Omega)\}$$



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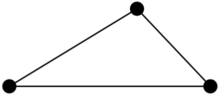
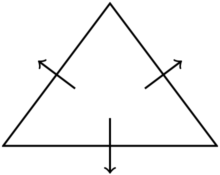
$$H(\mathrm{div}) := \{\sigma \in [L^2(\Omega)]^d \mid \operatorname{div}(\sigma) \in L^2(\Omega)\}$$

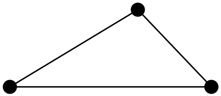
$$BDM_k := \{\sigma \in [\Pi^k(\mathcal{T}_h)]^d \mid \sigma_n \text{ is continuous over elements}\}$$

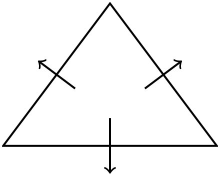




The space H(divdiv)

$$H(\mathrm{divdiv}) := \{\boldsymbol{\sigma} \in [L^2(\Omega)]^{d\times d}_{sym} \mid \mathrm{div}(\mathrm{div}(\boldsymbol{\sigma})) \in H^{-1}(\Omega)\}$$





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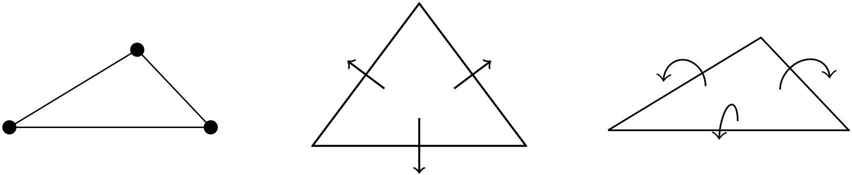
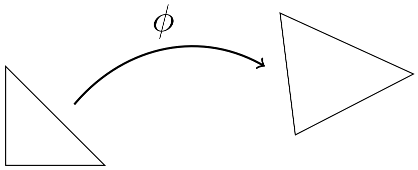
$$M_h^k := \{\boldsymbol{\sigma} \in [\Pi^k(\mathcal{T}_h)]^{d\times d}_{sym} \mid n^T\boldsymbol{\sigma} n \text{ is continuous over elements}\}$$



A. Pechstein and J. Schöberl: The TDNNS method for Reissner-Mindlin plates, *J. Numer. Math. (2017) 137, pp. 713-740.*

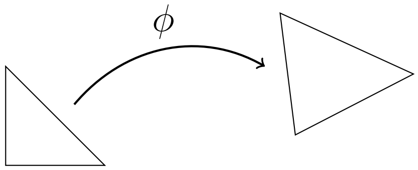
Mapping to the surface



Mapping to the surface

- Piola transformation

$$u \circ \phi = P[\hat{u}] = \frac{1}{J} \boldsymbol{F} \hat{u} \qquad \boldsymbol{F} = \nabla_{\hat{x}} \phi, \; J = \det(\boldsymbol{F})$$



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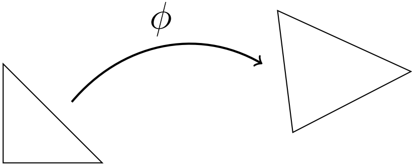
- Preserve normal-normal continuity

$$\boldsymbol{\sigma} \circ \phi = \frac{1}{J^2} \boldsymbol{F} \hat{\boldsymbol{\sigma}} \boldsymbol{F}^T$$



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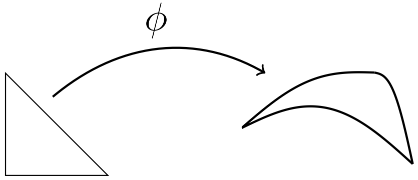

- Piola transformation

$$u \circ \phi = P[\hat{u}] = \frac{1}{J}\boldsymbol{F}\hat{u} \qquad \boldsymbol{F} = \nabla_{\hat{x}}\phi,\ J = \sqrt{\det(\boldsymbol{F}^T\boldsymbol{F})}$$

- Preserve normal-normal continuity

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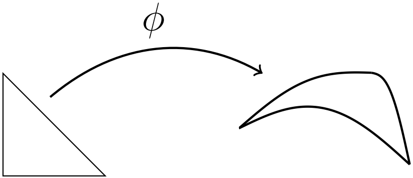

Mapping to the surface

- Piola transformation

$$u \circ \phi = P[\hat{u}] = \frac{1}{J}\boldsymbol{F}\hat{u} \qquad \boldsymbol{F} = \nabla_{\hat{x}}\phi,\ J = \|\operatorname{cof}(\boldsymbol{F})\|$$

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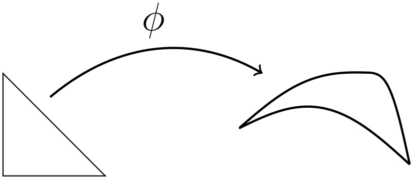

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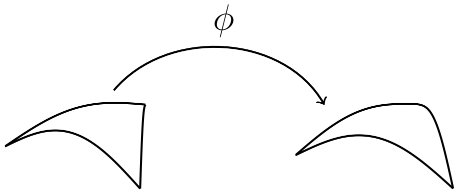
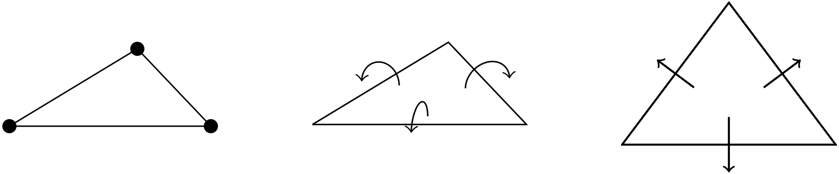
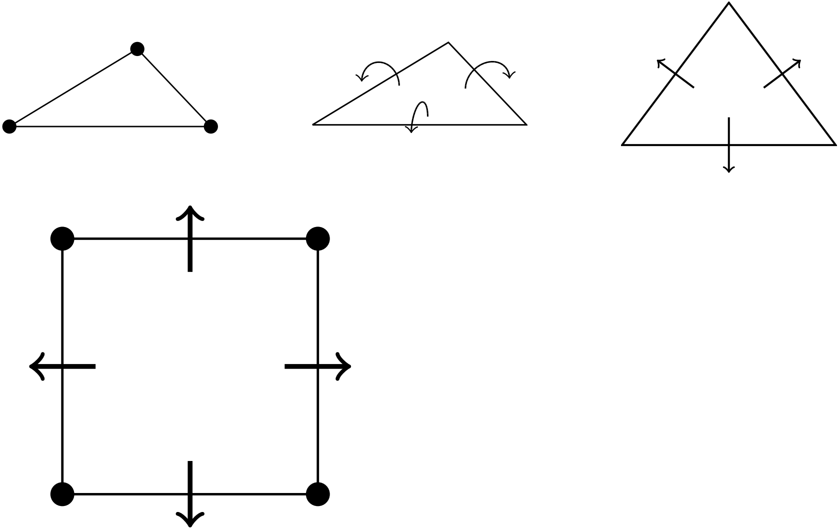
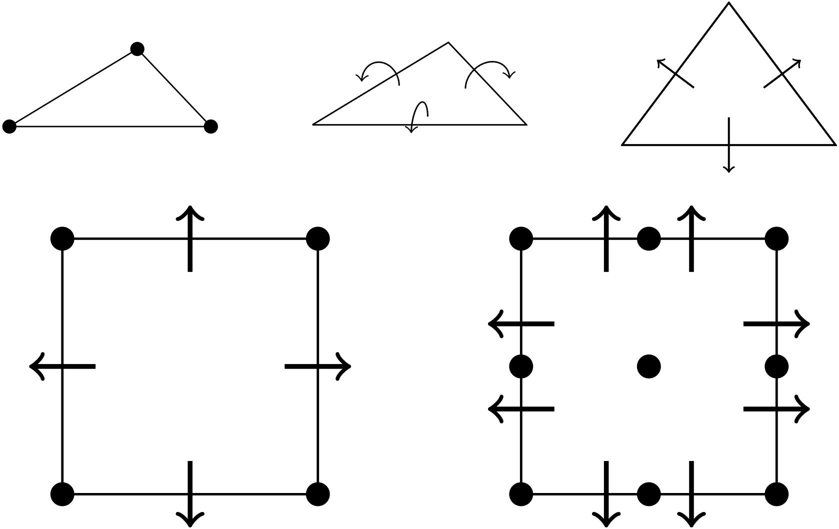




Shell element

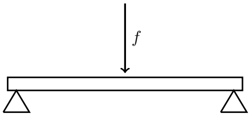




Relation to HHJ

- Discretization method for 4th order elliptic problems

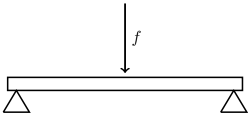
$$\mathrm{div}(\mathrm{div}(\nabla^2 w)) = f$$



Hellan–Herrmann–Johnson method

- Discretization method for 4th order elliptic problems

$$\mathrm{div}(\mathrm{div}(\nabla^2 w)) = f \quad \Rightarrow w \in H^2(\Omega)$$



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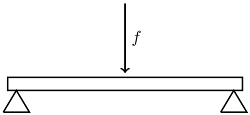

$$\begin{aligned} \boldsymbol{\sigma} &= \nabla^2 w, \\ \mathrm{div}(\mathrm{div}(\boldsymbol{\sigma})) &= f, \end{aligned}$$

Hellan–Herrmann–Johnson method

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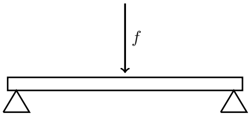

$$\begin{aligned} \boldsymbol{\sigma} &= \nabla^2 w, &\quad \Rightarrow w \in H^1(\Omega) \\ \mathrm{div}(\mathrm{div}(\boldsymbol{\sigma})) &= f, &\quad \Rightarrow \boldsymbol{\sigma} \in H(\mathrm{divdiv}, \Omega) \end{aligned}$$

Hellan–Herrmann–Johnson

Find $w \in H^1(\Omega)$ and $\boldsymbol{\sigma} \in H(\mathrm{divdiv}, \Omega)$ for the saddle point problem

$$\mathcal{L}(w, \boldsymbol{\sigma}) = -\frac{1}{2}\|\boldsymbol{\sigma}\|^2 - \sum_{T \in \mathcal{T}_h} \int_T \nabla w \cdot \operatorname{div}(\boldsymbol{\sigma})\, dx + \int_{\partial T} (\nabla w)_\tau \boldsymbol{\sigma}_{\mu\tau}\, ds$$
$$- \langle f, w \rangle.$$

M. Comodi: The Hellan-Herrmann-Johnson method: some new error estimates and postprocessing, *Math. Comp. 52 (1989) pp. 17–29.*

Hellan–Herrmann–Johnson

Find $w \in H^1(\Omega)$ and $\boldsymbol{\sigma} \in H(\mathrm{divdiv}, \Omega)$ for the saddle point problem

$$\mathcal{L}(w, \boldsymbol{\sigma}) = -\frac{1}{2}\|\boldsymbol{\sigma}\|^2 + \sum_{T \in \mathcal{T}_h} \int_T w_{|\alpha\beta} : \boldsymbol{\sigma}\, dx - \int_{\partial T} w_{|\mu} \boldsymbol{\sigma}_{\mu\mu}\, ds - \langle f, w \rangle.$$

M. Comodi: The Hellan-Herrmann-Johnson method: some new error estimates and postprocessing, *Math. Comp. 52 (1989) pp. 17–29.*

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Hellan–Herrmann–Johnson

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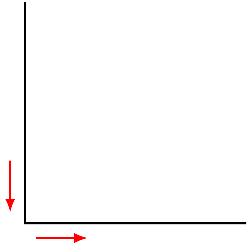
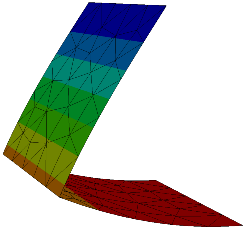
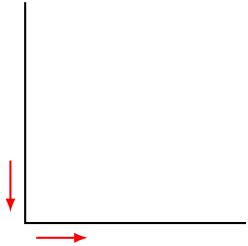
Linearization

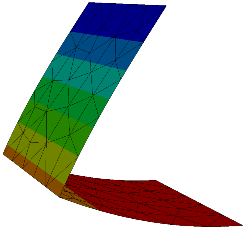
If the undeformed configuration is a flat plane and f works orthogonal on it, the HHJ method is the linearization of the bending energy of our method.



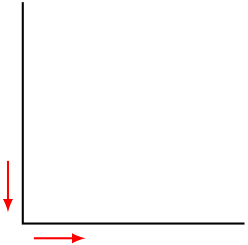


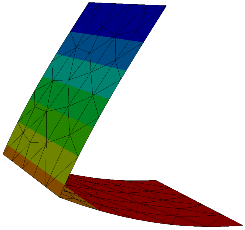
- Normal-normal continuous moment $\boldsymbol{\sigma}$





- Normal-normal continuous moment $\boldsymbol{\sigma}$
- Preserve kinks



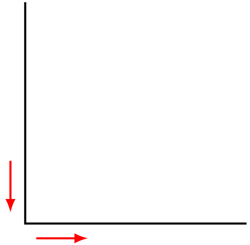


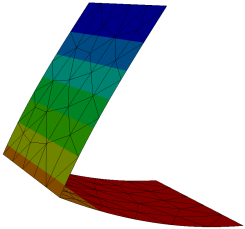
Structures with kinks

- Normal-normal continuous moment $\boldsymbol{\sigma}$
- Preserve kinks
- Variation of $\mathcal{L}(u, \boldsymbol{\sigma})$ in direction $\delta\boldsymbol{\sigma}$

$$\int_{\hat{E}} (\sphericalangle(\nu_L, \nu_R) - \sphericalangle(\hat{\nu}_L, \hat{\nu}_R))\delta\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\, d\hat{s} \stackrel{!}{=} 0$$

$$\Rightarrow \sphericalangle(\nu_L, \nu_R) - \sphericalangle(\hat{\nu}_L, \hat{\nu}_R) = 0$$





Membrane Locking

Membrane locking

$$\mathcal{W}(u) = \frac{t}{2}\|\boldsymbol{E}_{\tau\tau}(u)\|^2_{\boldsymbol{M}} + \frac{t^3}{24}\|\boldsymbol{F}^T\nabla(\nu\circ\phi) - \nabla\hat{\nu}\|^2_{\boldsymbol{M}} - f\cdot u$$

Membrane locking

$$\mathcal{W}(u) = t\, E_{\text{mem}}(u) + t^3\, E_{\text{bend}}(u) - f \cdot u$$

$$\mathcal{W}(u) = \frac{1}{t^2} E_{\text{mem}}(u) + E_{\text{bend}}(u) - \tilde{f} \cdot u$$

- Enforces $E_{\text{mem}}(u) = 0$ in the limit $t \to 0$

Membrane locking

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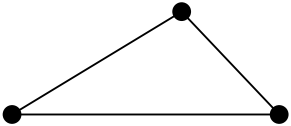

$V_h = \Pi(\mathcal{T}_h) \cap C(\Omega) \subset H^1(\Omega)$

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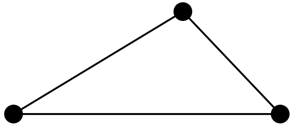
$$E_{\text{mem}}(u) = 0 \quad \not\Rightarrow \quad E_{\text{mem}}(u_h) = 0$$



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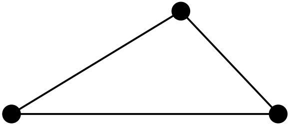

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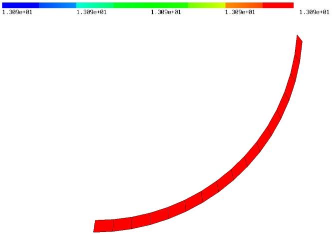
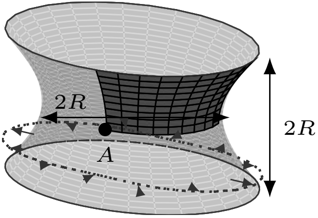
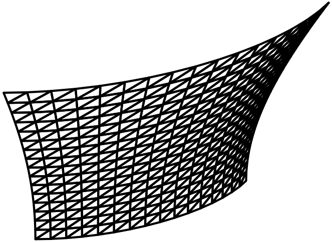
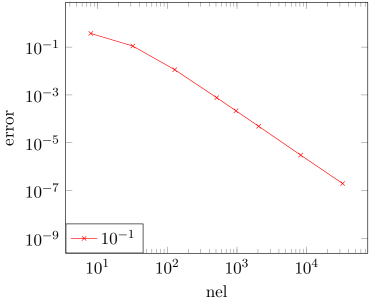


Hyperboloid with free ends

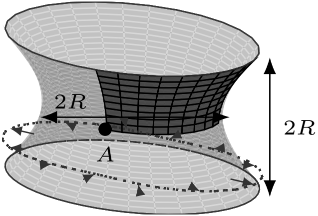


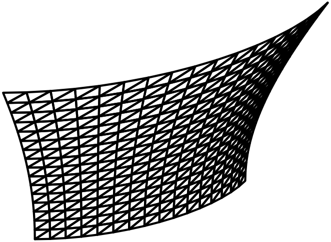


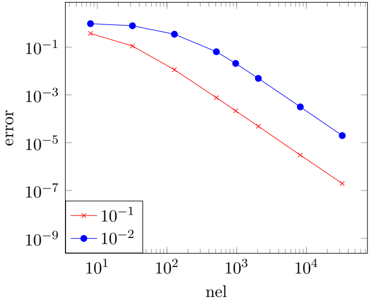


NGSolve

Hyperboloid with free ends

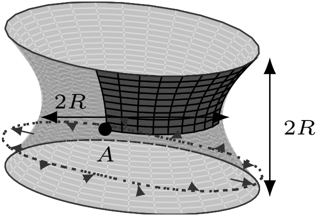


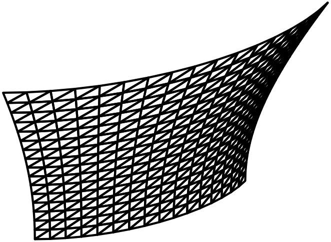


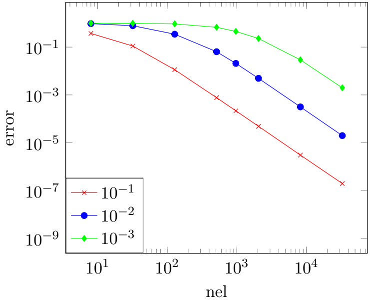


- Pre-asymptotic regime

Hyperboloid with free ends

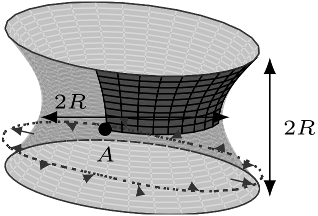


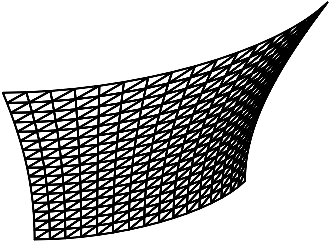


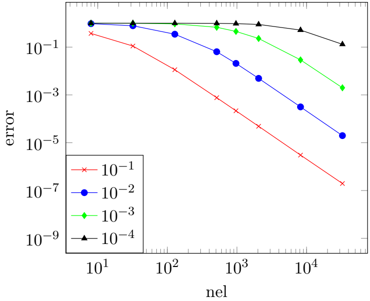


- Pre-asymptotic regime

Hyperboloid with free ends

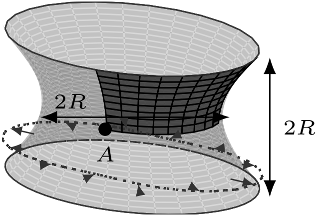


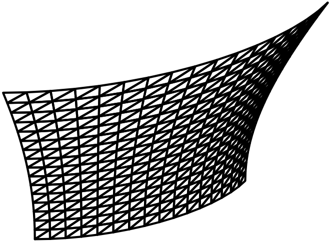


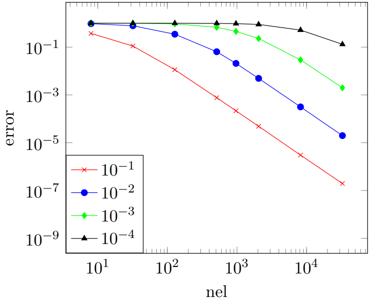


- Pre-asymptotic regime

Hyperboloid with free ends

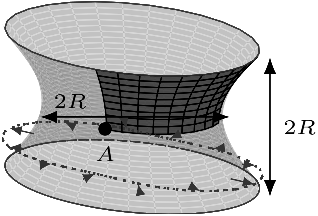


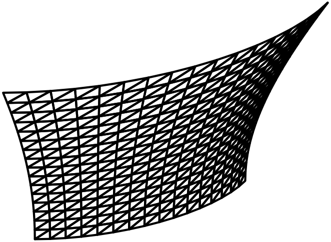


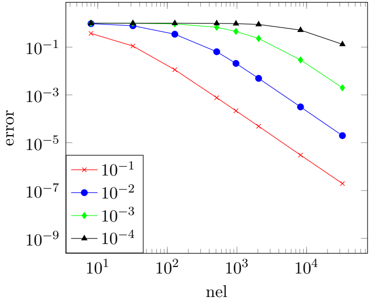


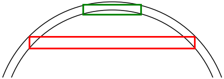
- Pre-asymptotic regime
- $h \prec \sqrt{Rt}$

Hyperboloid with free ends



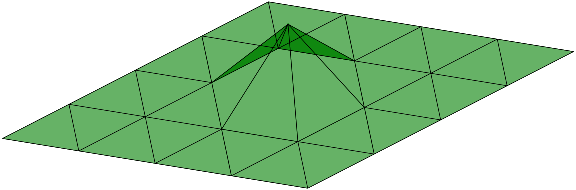




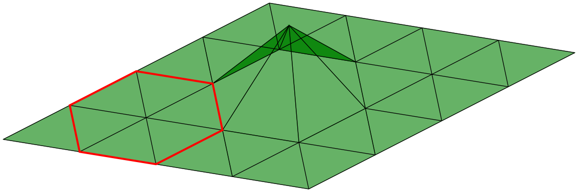


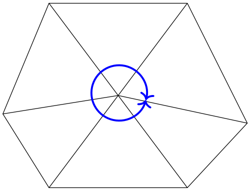
Regge elements

Regge calculus

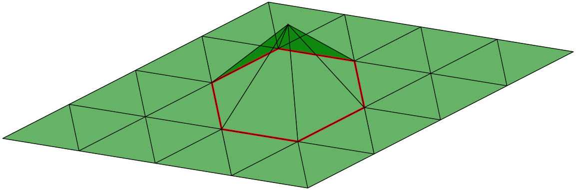


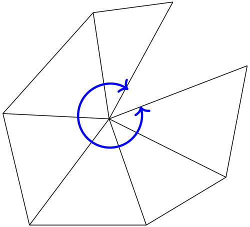
Regge calculus

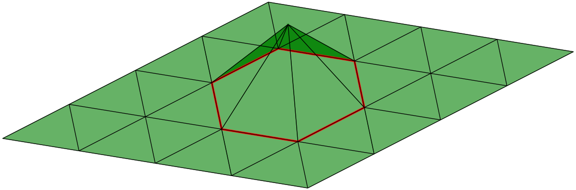


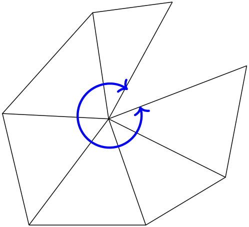


Regge calculus

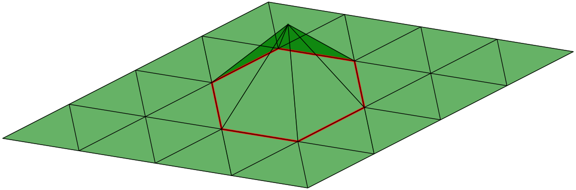


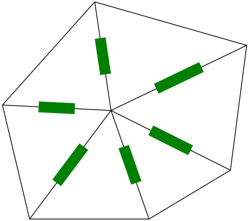






T. Regge: General relativity without coordinates, *Il Nuovo Cimento (1955-1965), 19 (1961), pp. 558–571*.

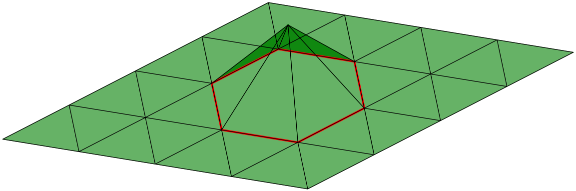


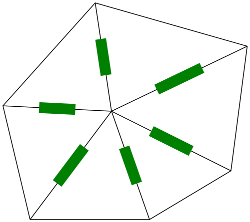


- Metric tensor

T. Regge: General relativity without coordinates, *Il Nuovo Cimento (1955-1965), 19 (1961), pp. 558–571.*

Regge calculus





- Metric tensor
- tangential-tangential continuous

T. Regge: General relativity without coordinates, *Il Nuovo Cimento (1955-1965), 19 (1961), pp. 558–571.*

Regge elements

$$\mathrm{Reg}_h^k := \{\boldsymbol{\sigma} \in [\Pi^k(\mathcal{T}_h)]_{sym}^{d\times d} \mid t^T\boldsymbol{\sigma}\, t \text{ is continuous over elements}\}$$

S. H. CHRISTIANSEN: On the linearization of Regge calculus, *Numerische Mathematik 119, 4 (2011), pp. 613–640.*

Regge elements

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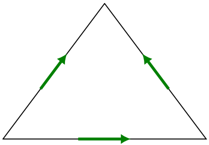
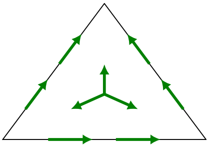
L. Li: Regge Finite Elements with Applications in Solid Mechanics and Relativity, *PhD thesis, University of Minnesota (2018).*

$$\mathrm{Reg}_h^k := \{\boldsymbol{\sigma} \in [\Pi^k(\mathcal{T}_h)]^{d\times d}_{sym} \mid t^T \boldsymbol{\sigma} t \text{ is continuous over elements}\}$$

$$H(\mathrm{curlcurl}) := \{\boldsymbol{\sigma} \in [L^2(\Omega)]^{d\times d}_{sym} \mid \mathrm{curl}\ (\mathrm{curl}\ \boldsymbol{\sigma})^T \in [H^{-1}(\Omega)]^{d\times d}\}$$





Membrane locking

$$\frac{1}{t^2}\|\boldsymbol{E}_{\tau\tau}(u)\|^2_{\boldsymbol{M}}$$

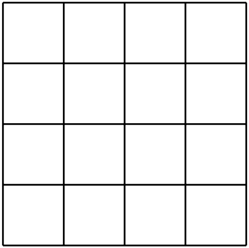
$$\|\mathrm{sym}(\boldsymbol{P}_\tau \nabla_\tau u)\|^2_{\boldsymbol{M}} = \|\frac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha}) - b_{\alpha\beta} u_3\|^2_{\boldsymbol{M}}$$

Membrane locking

$$\frac{1}{t^2}\|\Pi^k_{L^2}\boldsymbol{E}_{\tau\tau}(u)\|^2_{\boldsymbol{M}}$$

$$\|\mathrm{sym}(\boldsymbol{P}_\tau\nabla_\tau u)\|^2_{\boldsymbol{M}} = \|\frac{1}{2}(u_{\alpha|\beta}+u_{\beta|\alpha}) - b_{\alpha\beta}u_3\|^2_{\boldsymbol{M}}$$

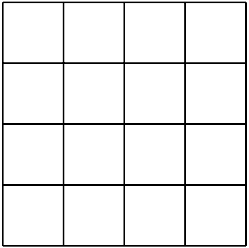
- Reduced integration for quadrilateral meshes

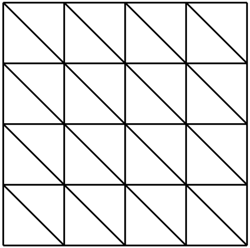


$$\frac{1}{t^2}\|\mathcal{I}_{\mathcal{R}}^k \boldsymbol{E}_{\tau\tau}(u)\|_{\boldsymbol{M}}^2$$

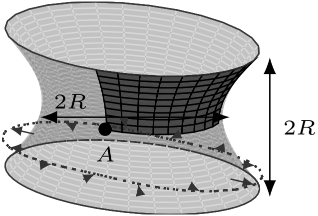
$$\|\mathrm{sym}(\boldsymbol{P}_\tau \nabla_\tau u)\|_{\boldsymbol{M}}^2 = \|\frac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha}) - b_{\alpha\beta} u_3\|_{\boldsymbol{M}}^2$$

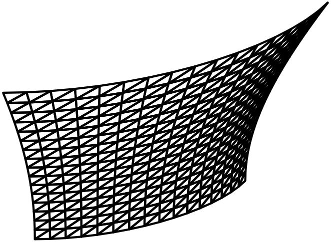
- Reduced integration for quadrilateral meshes
- Regge interpolant for triangles

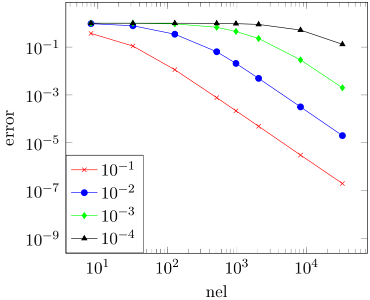


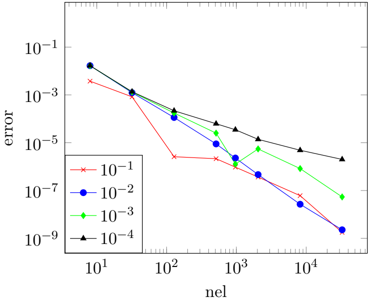


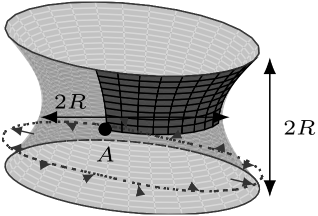
Hyperboloid with free ends







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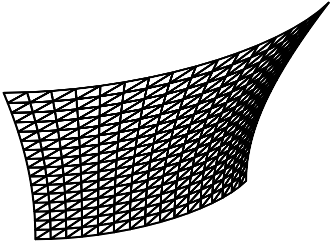

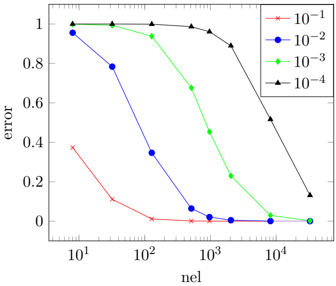
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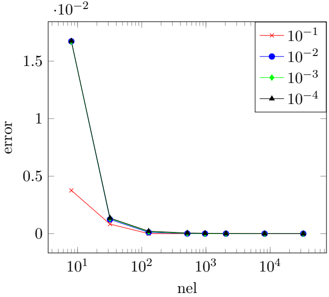
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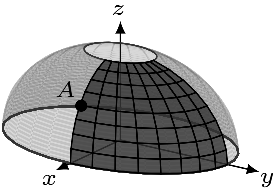
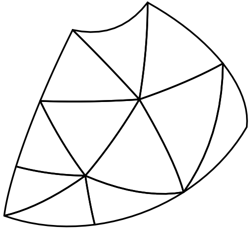
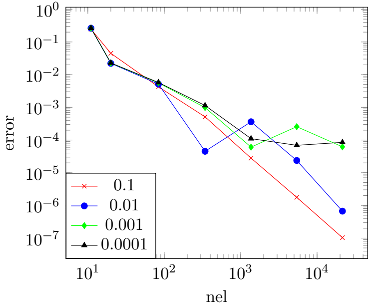
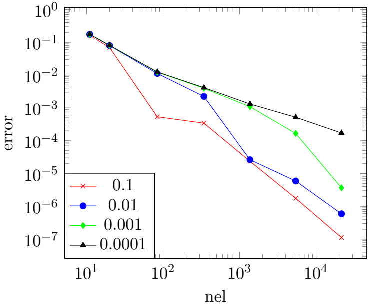
Open hemisphere with clamped ends









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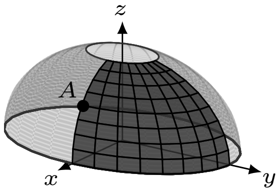
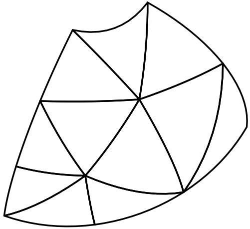
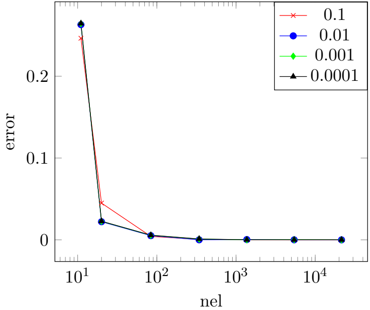
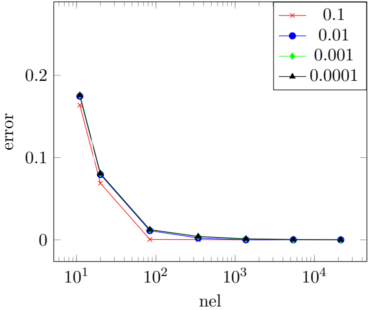
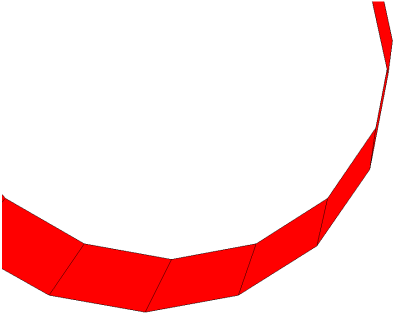


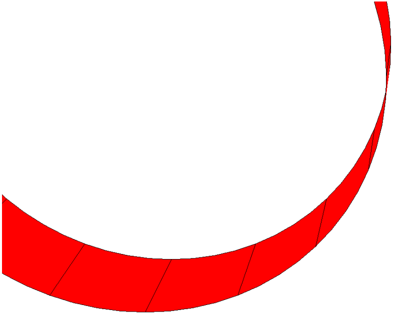






Cantilever subjected to end moment





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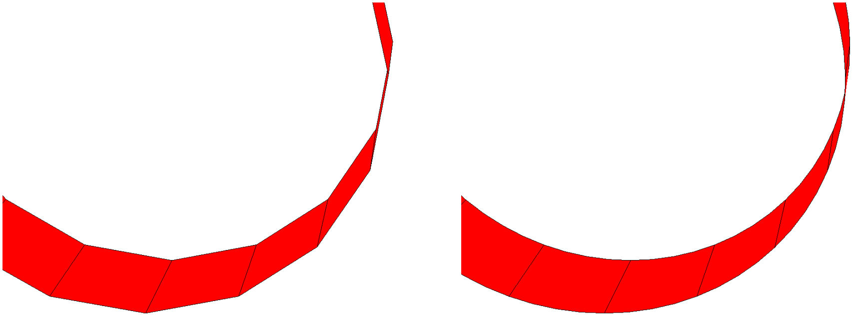
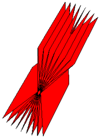
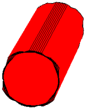
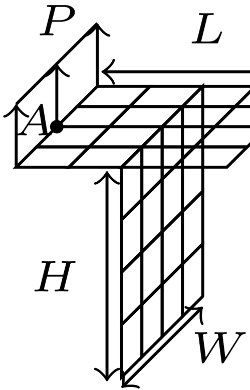


Cantilever subjected to end moment



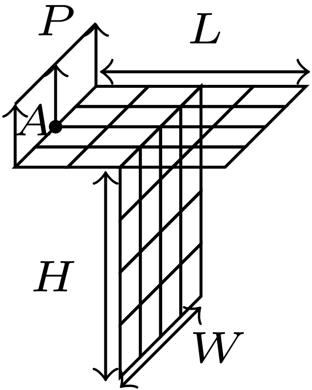


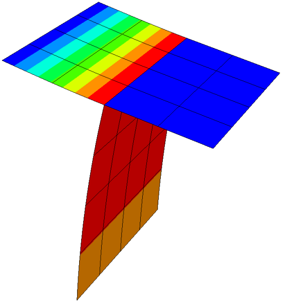
T-Section Cantilever



- $P = 2 \times 10^3$
  $E = 6 \times 10^6$
  $\nu = 0$
  $t = 0.1$
  $L = 1$
  $W = 1$
  $H = 1$

T-Section Cantilever





Summary

- Kirchhoff–Love shell element

Summary

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- Moment tensor

Summary

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Thank You for Your attention!

Literature

M. Neunteufel and J. Schöberl: The Hellan–Herrmann–Johnson Method for Nonlinear Shells, *Computers & Structures (2019) 225.*

M. Neunteufel and J. Schöberl: Avoiding Membrane Locking with Regge Interpolation, http://arxiv.org/abs/1907.06232