

Regge finite elements in differential geometry: metric, geodesic, and curvature approximation

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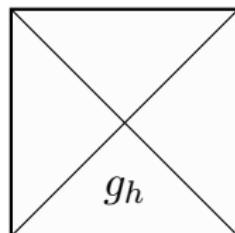
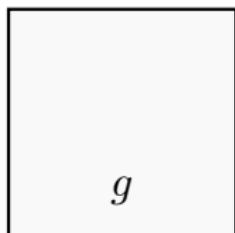
Max Wardetzky (University of Göttingen)



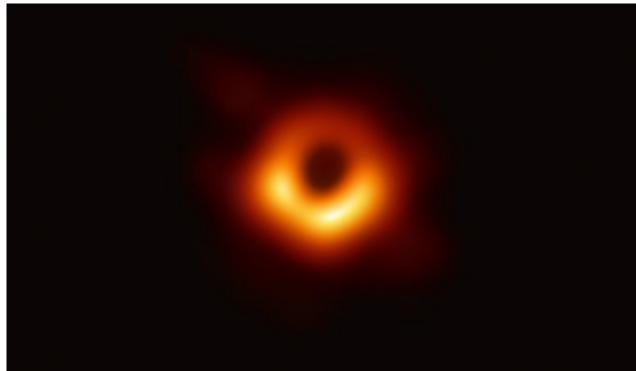
Der Wissenschaftsfonds.



- How and with which FE to approximate the metric of a Riemannian manifold?
- How to compute geodesics on such discrete, non-smooth metrics?
- How to compute curvature quantities like Gauss curvature on non-smooth metrics?



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Regge metric

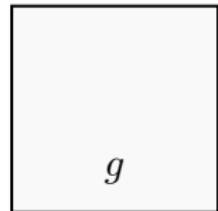
Geodesics

Distributional Gauss and scalar curvature

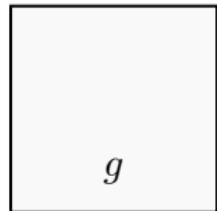
Distributional Riemann curvature

Regge metric

Riemannian manifold (M, g)



Riemannian manifold $(M \subset \mathbb{R}^2, g)$



Riemannian manifold (M, g)

Levi-Civita connection ∇

$$\nabla_X g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

Finite element methods

Goal: Approximate a function u on a manifold M

Approximation space \mathcal{S} (finite dimensional)

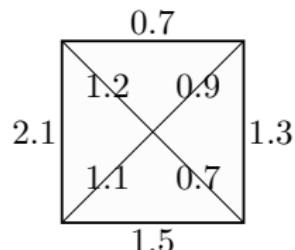
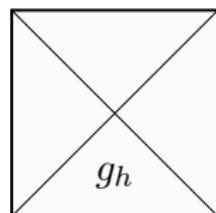
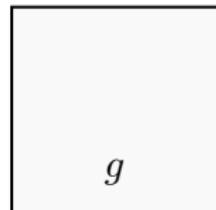
Approximation error $\|u - u_h\|$

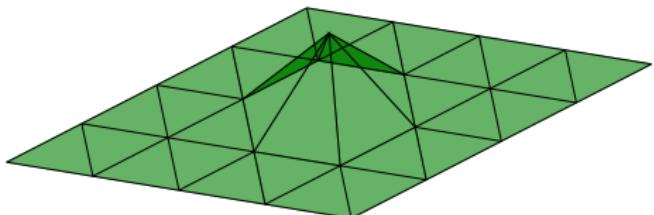
Riemannian manifold (M, g)

Levi-Civita connection ∇

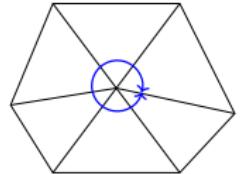
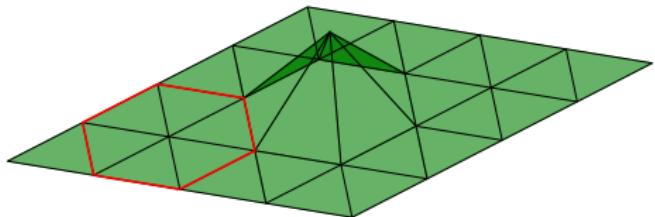
$$\nabla_X g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

- Approximation of g on a triangulation
- Compute lengths and angles
- Regge calculus and finite elements

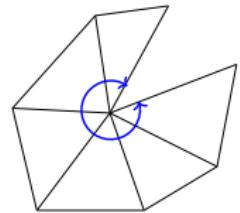
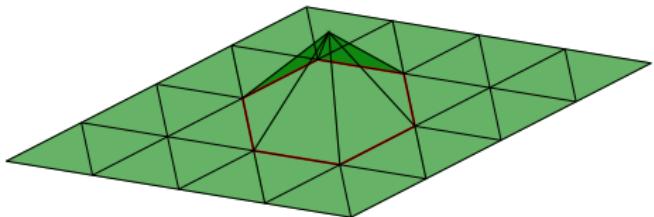




- REGGE: General relativity without coordinates, *Il Nuovo Cimento* (1955-1965), 19 (1961).

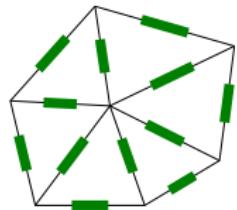
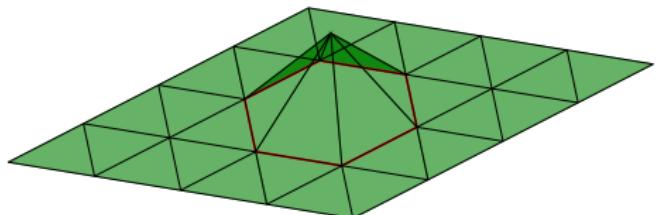


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- angle defect

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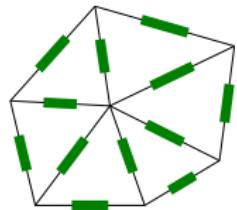
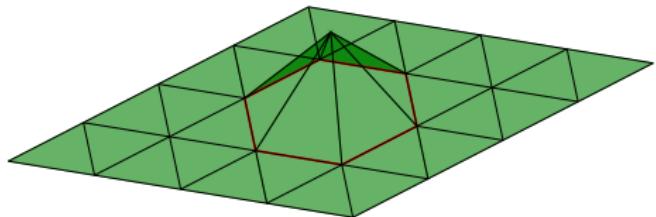
- metric tensor



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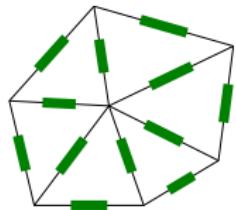
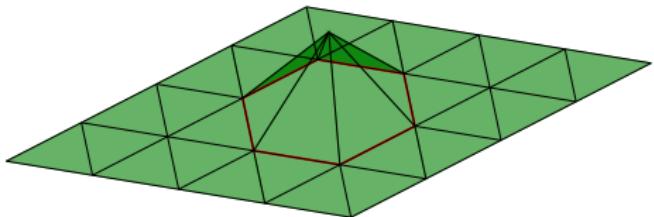


SORKIN: Time-evolution problem in Regge calculus, *Phys. Rev. D* 12 (1975).



- metric tensor

- REGGE: General relativity without coordinates, *Il Nuovo Cimento* (1955-1965), 19 (1961).
- CHEEGER, MÜLLER, SCHRADER: On the curvature of piecewise flat spaces, *Communications in Mathematical Physics*, 92(3) (1984).



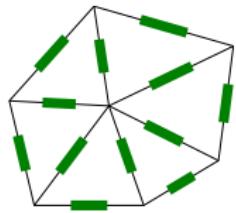
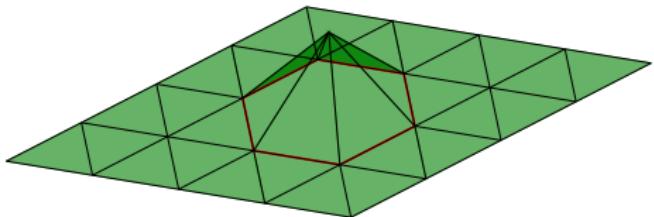
- metric tensor (tangential-tangential continuous)

$$\text{Reg}_h^k = \{\varepsilon \in \mathcal{P}^k(\mathcal{T}, \mathbb{R}_{\text{sym}}^{d \times d}) \mid [\![t^\top \varepsilon t]\!]_E = 0 \text{ for all edges } E\}$$

$$H(\text{curl curl}) = \{\varepsilon \in L^2(\Omega, \mathbb{R}_{\text{sym}}^{d \times d}) \mid \text{curl}^\top \text{curl}(\varepsilon) \in H^{-1}(\Omega, \mathbb{R}^{(2d-3) \times (2d-3)})\}$$



CHRISTIANSEN: On the linearization of Regge calculus, *Numerische Mathematik* 119, 4 (2011).



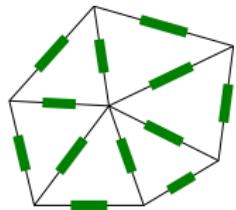
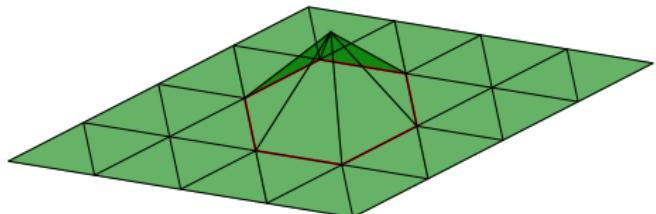
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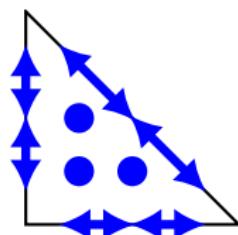
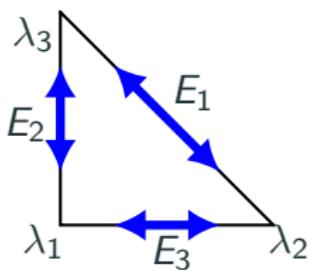
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 N.: Mixed Finite Element Methods for Nonlinear Continuum Mechanics and Shells, *PhD thesis, TU Wien (2021)*.

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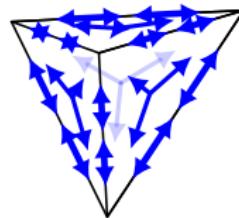
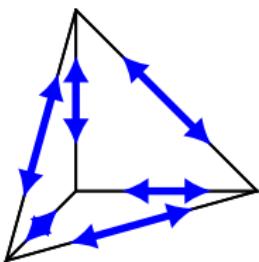


$$\varphi_{E_i} = \nabla \lambda_j \odot \nabla \lambda_k, \quad t_j^\top \varphi_{E_i} t_j = c_i \delta_{ij}, \quad \varphi_{T_i} = \lambda_i \nabla \lambda_j \odot \nabla \lambda_k$$

$\mathcal{R}_h^k : C^0(\Omega) \rightarrow \text{Reg}_h^k$ canonical interpolant

$$\int_E (g - \mathcal{R}_h^k g)_{tt} q \, dl = 0 \text{ for all } q \in \mathcal{P}^k(E)$$

$$\int_T (g - \mathcal{R}_h^k g) : Q \, da = 0 \text{ for all } Q \in \mathcal{P}^{k-1}(T, \mathbb{R}_{\text{sym}}^{2 \times 2})$$



$$\mathcal{R}_h^k : C^0(\Omega) \rightarrow \text{Reg}_h^k$$

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$$\int_F (g - \mathcal{R}_h^k g) : Q_F \, da = 0 \text{ for all } Q_F \in \mathcal{P}^{k-1}(F, TF \odot TF)$$

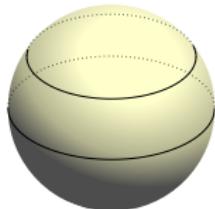
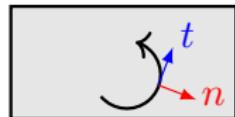
$$\int_T (g - \mathcal{R}_h^k g) : Q_T \, dx = 0 \text{ for all } Q_T \in \mathcal{P}^{k-2}(T, \mathbb{R}_{\text{sym}}^{3 \times 3})$$

Geodesics

Geodesic curvature

$$\kappa(g) = g(\nabla_t t, n)$$

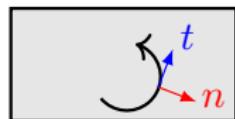
$n = t \times \nu$ (for embedded surface)



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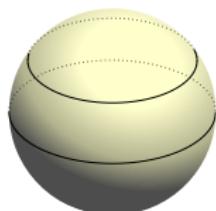
$n = t \times \nu$ (for embedded surface)



Christoffel symbols: $\nabla_{\partial_j} \partial_k = \Gamma_{jk}^l \partial_l$

$$\Gamma_{ij}^k(g) = \frac{1}{2}g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}) = g^{kl} \Gamma_{ijl}(g)$$

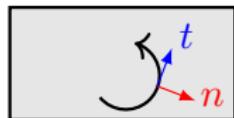
$$\nabla_X Y = (X^i \partial_i Y^k + X^i Y^j \Gamma_{ij}^k) \partial_k$$



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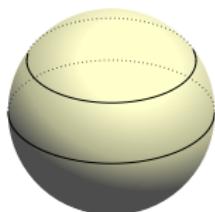
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$$\nabla_X Y = (X^i \partial_i Y^k + X^i Y^j \Gamma_{ij}^k) \partial_k$$

$$\kappa(g) = \frac{\sqrt{\det g}}{g_{\hat{t}\hat{t}}^{3/2}} \left(\partial_{\hat{t}} \hat{t} \cdot \hat{n} + \Gamma_{\hat{t}\hat{t}}^{\hat{n}} \right)$$

$$\Gamma_{\hat{t}\hat{t}}^{\hat{n}} = \Gamma_{ij}^k \hat{t}^i \hat{t}^j \hat{n}_k$$



Geodesic curve

$\gamma : [a, b] \rightarrow M$ is geodesic if $\nabla_{\dot{\gamma}} \dot{\gamma} = 0$, i.e.

$$\ddot{\gamma}^k + \dot{\gamma}^i \dot{\gamma}^j \Gamma_{ij}^k = 0, \quad k = 1, \dots, n$$

Geodesics are critical points of energy

$$E(\gamma) = \frac{1}{2} \int_0^T g_{ij}(\gamma(s)) \dot{\gamma}^i(s) \dot{\gamma}^j(s) ds$$

Geodesic curve

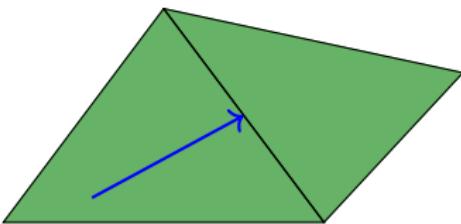
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What to do for non-smooth g ?



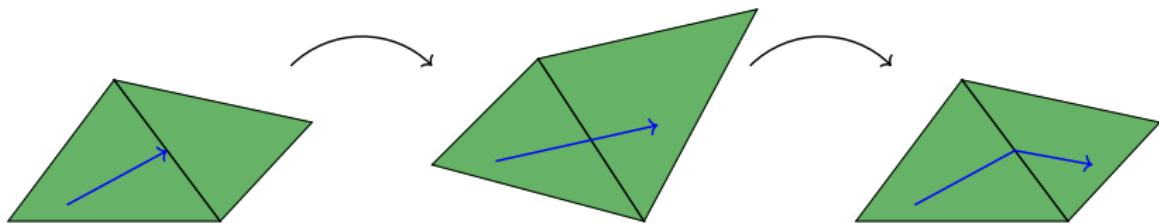
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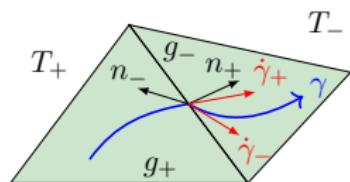
Discrete local geodesic

Let $g \in \text{Reg}_h^k$. A pw smooth $\gamma : [a, b] \rightarrow \mathcal{T}$ not intersecting faces of dimension $\leq (N - 2)$ is a local geodesic iff it satisfies the geodesic equation in T and where γ intersects a facet F

$$g_{ij}^+ \dot{\gamma}_+^i t^j = g_{ij}^- \dot{\gamma}_-^i t^j, \quad g_{ij}^+ \dot{\gamma}_+^i n_+^j = -g_{ij}^- \dot{\gamma}_-^i n_-^j.$$

Its kinematic energy $g_{ij} \dot{\gamma}^i \dot{\gamma}^j$ is constant.

Update formula: $\dot{\gamma}_-^i = \dot{\gamma}_+^i - (g_{ij}^+ \dot{\gamma}_+^j n_+^k)(n_+^i + n_-^i)$



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Convergence geodesic

Assume that $g_h \in \text{Reg}_h^k$ is a family of Regge metrics fulfilling $\|g - g_h\|_{L^\infty} \leq \frac{1}{2}\|g^{-1}\|_{L^\infty}^{-1}$ uniformly in h . Let $\gamma : [0, T] \rightarrow M$ be a smooth geodesic and γ_h a family of geodesics with respect to g_h with the same initial conditions as γ . Then

$$|\dot{\gamma}(t) - \dot{\gamma}_h(t)| \leq C(\|g - g_h\|_{W^{1,\infty}(\mathcal{T}_h)} + h^{-1}\|g - g_h\|_{L^\infty}),$$

$$|\gamma(t) - \gamma_h(t)| \leq C(h\|g - g_h\|_{W^{1,\infty}(\mathcal{T}_h)} + \|g - g_h\|_{L^\infty}).$$



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$$|\dot{\gamma}(t) - \dot{\gamma}_h(t)| \leq C h^k |g|_{W^{k+1,\infty}},$$

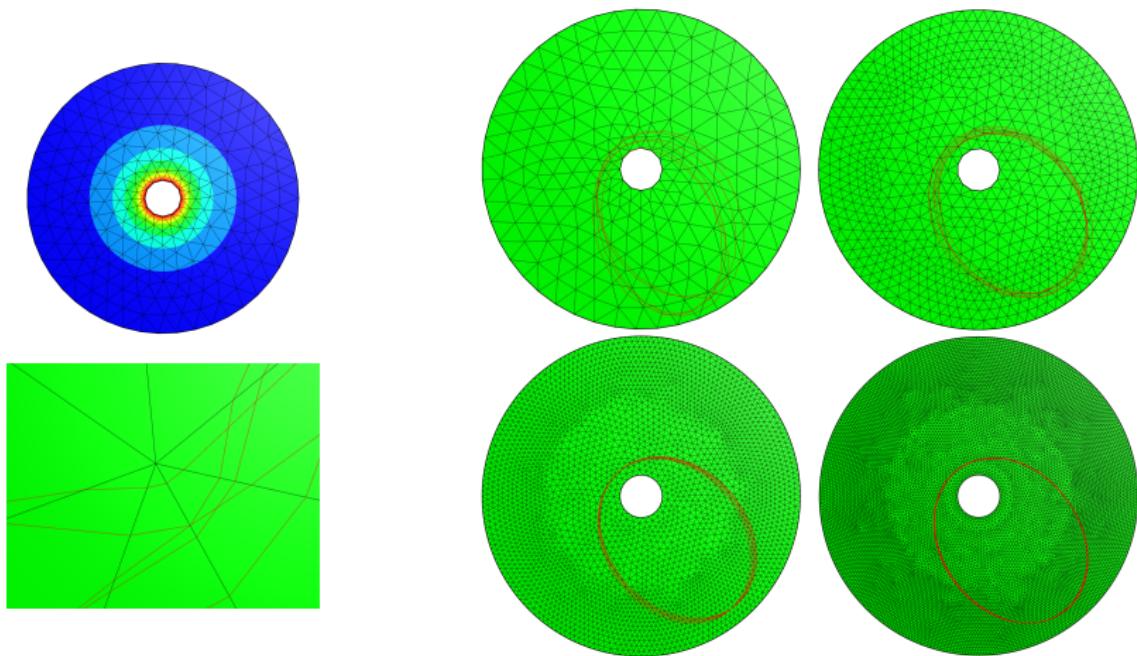
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$$E = E_{\text{kin}} + V, \quad V(q) = -|q|^{-1}$$

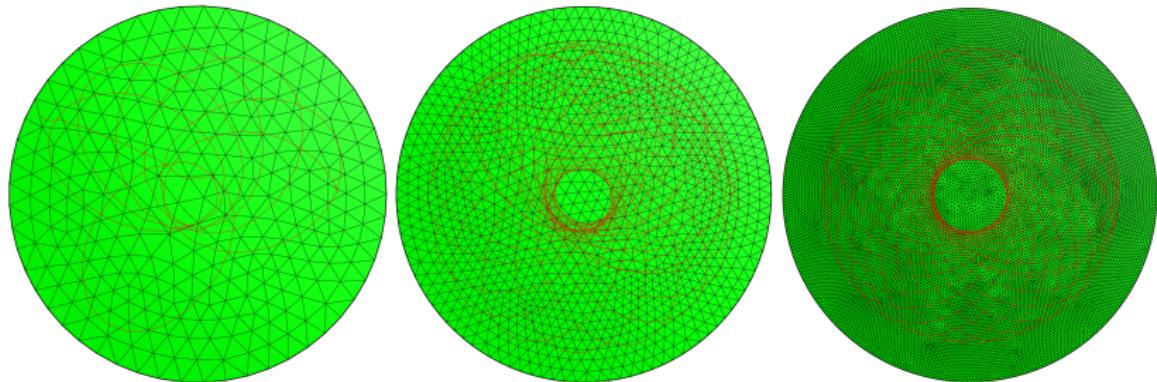
Jacobi metric: $g = 2(E - V)\delta = 2(E + |q|^{-1})\delta$



Taking into account of relativistic effects (Mercury orbit)

Star mass M , particle mass m , total energy $E \leq m$

$$ds = \left(E^2 - m^2 + \frac{2Mm^2}{r} \right) \left(\frac{dr^2}{\left(1 - \frac{2M}{r} \right)^2} + \frac{r^2 d\Phi^2}{1 - \frac{2M}{r}} \right)$$



Distributional Gauss and scalar curvature

Riemann curvature tensor:

$$\mathfrak{R}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z$$

$$R(X, Y, Z, W) = g(\mathfrak{R}(X, Y)Z, W)$$

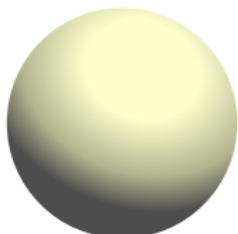
$$R_{ijkl} = \partial_i \Gamma_{jkl} - \partial_j \Gamma_{ikl} + \Gamma_{ik}^p \Gamma_{jpl} - \Gamma_{jk}^p \Gamma_{ipl}$$

Riemann curvature tensor:

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$$R_{ijkl} = \partial_i \Gamma_{jkl} - \partial_j \Gamma_{ikl} + \Gamma_{ik}^p \Gamma_{jpl} - \Gamma_{jk}^p \Gamma_{ipl}$$



- **Gauss curvature**

$$K = \frac{g(R(u, v)v, u)}{\|u\|_g \|v\|_g - g(u, v)^2} = \frac{R_{1221}}{\det g}$$



- **Scalar curvature**

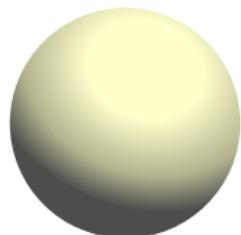
$$S = g^{ik} g^{jl} R_{ijkl}$$

Riemann curvature tensor:

$$\mathfrak{R}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z$$

$$R(X, Y, Z, W) = g(\mathfrak{R}(X, Y)Z, W)$$

$$R_{ijkl} = \partial_i \Gamma_{jkl} - \partial_j \Gamma_{ikl} + \Gamma_{ik}^p \Gamma_{jpl} - \Gamma_{jk}^p \Gamma_{ipl}$$

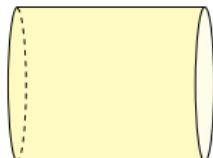


- Ricci tensor

$$\text{Ric}_{ij} = g^{ab} R_{aibj}$$

- Einstein tensor

$$G = \text{Ric} - \frac{1}{2} S g$$

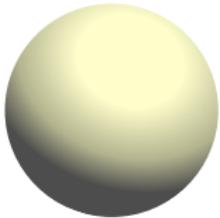


Gauss–Bonnet

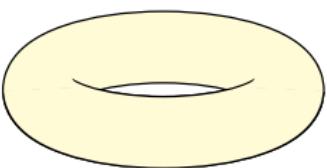
On manifold M :

$$\int_M K(g) + \int_{\partial M} \kappa(g) + \sum_V (\pi - \triangle_V^M(g)) = 2\pi\chi_M$$

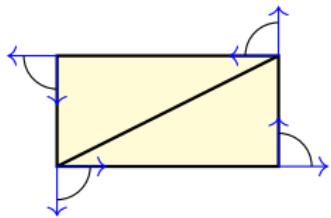
$$\chi_M(\mathcal{T}) = n_V - n_E + n_T$$



$$\chi_M = 2$$



$$\chi_M = 0$$



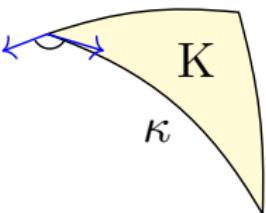
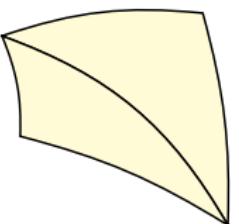
$$\chi_M = 1$$

Gauss–Bonnet

On triangle T :

$$\int_T K(g) + \int_{\partial T} \kappa(g) + \sum_{i=1}^3 (\pi - \sphericalangle_{V_i}^T(g)) = 2\pi$$

$$\chi_T = 3 - 3 + 1 = 1$$



Gauss–Bonnet

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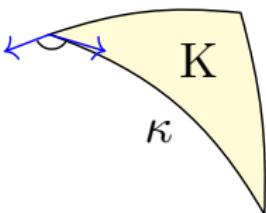
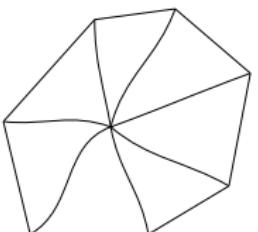


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Distributional densitized Gauss curvature

Let $g \in \text{Reg}_h^k(\mathcal{T})$ and $\varphi \in \mathring{\mathcal{V}}_h^q$

$$\langle (K\omega)(g), \varphi \rangle = \sum_{T \in \mathcal{T}} K_T(\varphi, g) + \sum_{E \in \mathcal{E}} K_E(\varphi, g) + \sum_{V \in \mathcal{V}} K_V(\varphi, g)$$

-  BERCHENKO-KOGAN, GAWLIK: Finite element approximation of the Levi-Civita connection and its curvature in two dimensions, *Found Comput Math* (2022).

Distributional densitized Gauss curvature

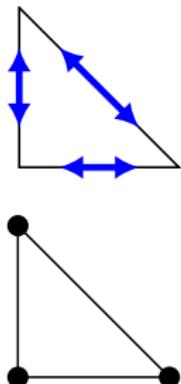
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$$K_T(\varphi, g) = \int_T K(g) \varphi \omega_T$$

$$K_E(\varphi, g) = \int_E [\kappa(g)] \varphi \omega_E$$

$$\begin{aligned} K_V(\varphi, g) &= (2\pi - \sum_{T: V \subset T} \triangleleft_V^T(g)) \varphi(V) \\ &=: \Theta_V(g) \varphi(V) \end{aligned}$$



Distributional densitized Gauss curvature

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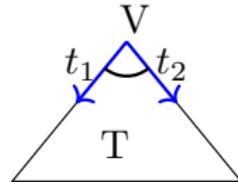
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$$K_V(\varphi, g) = (2\pi - \sum_{T: V \subset T} \triangleleft_V^T(g)) \varphi(V) \quad \triangleleft_V^T(g) = \arccos \left(\frac{t_1^\top g t_2}{\|t_1\|_g \|t_2\|_g} \right)$$

$$=: \Theta_V(g) \varphi(V)$$



- Second fundamental form $\mathbb{II}(X, Y) = g(\nabla_X n, Y)$
2D $\mathbb{II}(t, t) = \kappa(g)$
- mean curvature $H = \text{tr}(\mathbb{II}|_F)$

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Distributional densitized scalar curvature

Let $g \in \text{Reg}_h^k$ and $\varphi \in \mathring{\mathcal{V}}_h^q$

$$\begin{aligned}\langle (S\omega)(g), \varphi \rangle &= \sum_{T \in \mathcal{T}} \int_T S\varphi \omega_T + 2 \sum_{F \in \mathcal{F}} \int_F [\![H]\!] \varphi \omega_F \\ &\quad + 2 \sum_{E \in \mathcal{E}} \int_E \Theta_E(g) \varphi \omega_E\end{aligned}$$



GAWLIK, N.: Finite element approximation of scalar curvature in arbitrary dimension, *arXiv:2301.02159*.

Convergence scalar curvature 2D

Let $\{g_h\}_{h>0}$ Regge metrics on shape-regular $\{\mathcal{T}_h\}_{h>0}$ triangulations of Ω . Assume $\lim_{h \rightarrow 0} \|g_h - g\|_{L^\infty(\Omega)} = 0$ and $C_1 = \sup_{h>0} \max_{T \in \mathcal{T}_h} \|g_h\|_{W^{1,\infty}(T)} < \infty$. Then

$$\begin{aligned} \| (S\omega)(g_h) - (S\omega)(g) \|_{H^{-2}} &\leq C \left(1 + \max_T h_T^{-1} \|g_h - g\|_{L^\infty(T)} \right. \\ &\quad \left. + \max_T |g_h - g|_{W^{1,\infty}(T)} \right) \|g_h - g\| \end{aligned}$$

$$\|\sigma\|^2 = \|\sigma\|_{L^2(\Omega)}^2 + \sum_T h_T^2 |\sigma|_{H^1(T)}^2$$



GAWLIK, N.: Finite element approximation of scalar curvature in arbitrary dimension, *arXiv:2301.02159*.

Convergence scalar curvature 2D

Let $\{g_h\}_{h>0} \in \text{Reg}_h^k$ optimal-order interpolants on shape-regular $\{\mathcal{T}_h\}_{h>0}$ triangulations of Ω . Assume $\lim_{h \rightarrow 0} \|g_h - g\|_{L^\infty(\Omega)} = 0$ and $C_1 = \sup_{h>0} \max_{T \in \mathcal{T}_h} \|g_h\|_{W^{1,\infty}(T)} < \infty$. Then

$$\|(S\omega)(g_h) - (S\omega)(g)\|_{H^{-2}} \leq C h^{k+1} |g|_{H^{k+1}}$$

$$|\!|\!|\sigma|\!|\!|^2 = \|\sigma\|_{L^2(\Omega)}^2 + \sum_T h_T^2 |\sigma|_{H^1(T)}^2$$

Optimal-order interpolant: $|\mathcal{I}_h g - g|_{H^m(T)} \leq Ch_T^{k-m} |g|_{H^m(T)}$



GAWLIK, N.: Finite element approximation of scalar curvature in arbitrary dimension, *arXiv:2301.02159*.

Convergence scalar curvature $N > 2$

Assume additionally $C_2 = \sup_{h>0} \max_{T \in \mathcal{T}_h} |g_h|_{W^{2,\infty}(T)} < \infty$.

$$\begin{aligned} \|(\mathcal{S}\omega)(g_h) - (\mathcal{S}\omega)(g)\|_{H^{-2}(\Omega)} &\leq C \left(1 + \max_T h_T^{-2} \|g_h - g\|_{L^\infty(T)} \right. \\ &\quad \left. + \max_T h_T^{-1} |g_h - g|_{W^{1,\infty}(T)} \right) \|g_h - g\|. \end{aligned}$$

$$\|\sigma\|^2 = \|\sigma\|_{L^2(\Omega)}^2 + \sum_T h_T^2 |\sigma|_{H^1(T)}^2 + \sum_T h_T^4 |\sigma|_{H^2(T)}^2$$

-  GAWLIK, N.: Finite element approximation of scalar curvature in arbitrary dimension, *arXiv:2301.02159*.

Convergence scalar curvature $N > 2$

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$$\|(S\omega)(g_h) - (S\omega)(g)\|_{H^{-2}(\Omega)} \leq C h^{k+1} |g|_{H^{k+1}}$$

for $k > 0$.

$$\|\sigma\|^2 = \|\sigma\|_{L^2(\Omega)}^2 + \sum_T h_T^2 |\sigma|_{H^1(T)}^2 + \sum_T h_T^4 |\sigma|_{H^2(T)}^2$$

-  GAWLIK, N.: Finite element approximation of scalar curvature in arbitrary dimension, *arXiv:2301.02159*.

Lifting of distributional Gauss curvature

For $g \in \text{Reg}_h^k$ find $K_h \in \mathring{\mathcal{V}}_h^q$ such that for all $\varphi \in \mathring{\mathcal{V}}_h^q$

$$\int_{\Omega} K_h \varphi \omega = \langle (K\omega)(g), \varphi \rangle.$$

Convergence Gauss curvature

$$\lim_{h \rightarrow 0} \|g_h - g\|_{L^\infty(\Omega)} = 0, \quad \lim_{h \rightarrow 0} h^{-1} \log h^{-1} \|g_h - g\|_{L^2(\Omega)} = 0$$

$$\|K_h - K\|_{H_h^l} \leq C h^{-l-1} (\|g_h - g\|_{H_h^1} + \inf_{u_h \in \mathring{\mathcal{V}}_h^k} \|K - u_h\|_{L^2})$$



GAWLIK: High-Order Approximation of Gaussian Curvature with Regge Finite Elements, *SIAM J. Numer. Anal.* (2020).

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$g_h \in \text{Reg}_h^k$ opt-order interp, $-1 \leq l \leq k-2$, $q \geq \max\{1, k-2\}$

$$\|K_h - K\|_{H_h^l} \leq C h^{-l+k-1} (|g|_{H^{k+1}} + |K|_{H^k})$$



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GAWLIK: High-Order Approximation of Gaussian Curvature with Regge Finite Elements, *SIAM J. Numer. Anal.* (2020).

Lifting of distributional Gauss curvature

For $g \in \text{Reg}_h^k$ find $K_h \in \mathring{\mathcal{V}}_h^{k+1}$ such that for all $\varphi \in \mathring{\mathcal{V}}_h^{k+1}$

$$\int_{\Omega} K_h \varphi \omega = \langle (K\omega)(g), \varphi \rangle.$$

Convergence Gauss curvature

$g_h \in \text{Reg}_h^k$ canonical interpolant, $-1 \leq l \leq k - 1$

$$\|K_h - K\|_{H^{-1}} \leq C h^1 (|g|_{W^{1,\infty}} + |K|_{L^2})$$

-  GOPALAKRISHNAN, N., SCHÖBERL, WARDETZKY: Analysis of curvature approximations via covariant curl and incompatibility for Regge metrics, arXiv:2206.09343.

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Convergence Gauss curvature

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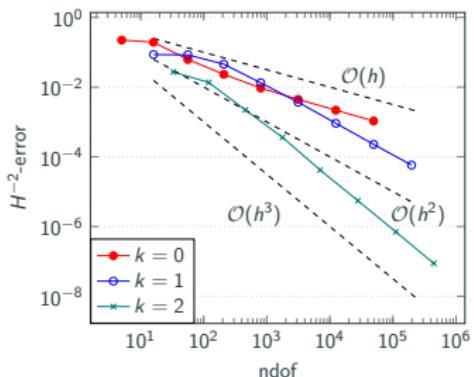
$$\|K_h - K\|_{H_h^l} \leq C h^{-l+k} (|g|_{W^{k+1,\infty}} + |K|_{H^k})$$

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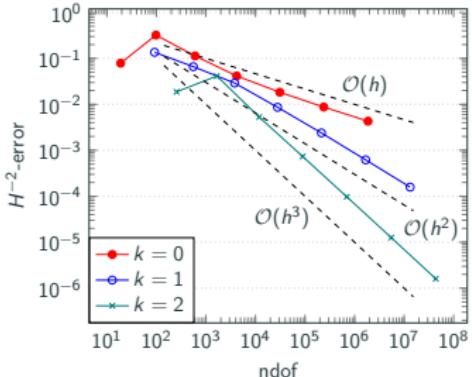
Numerical example (scalar curvature 2D and 3D)



NGSolve

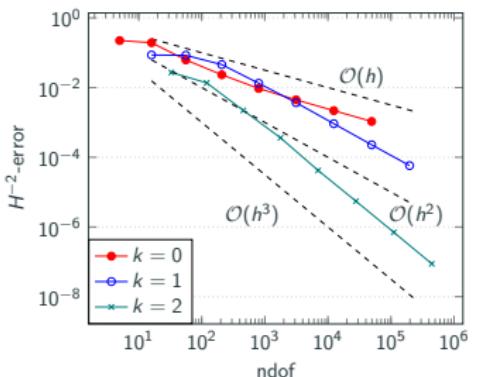
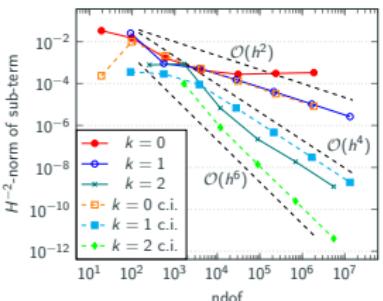


2D

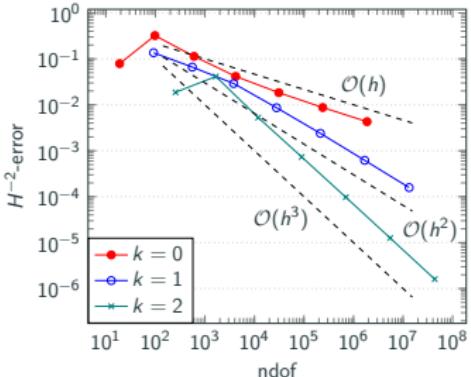


3D

Numerical example (scalar curvature 2D and 3D)



2D



3D

Numerical example (Gauss curvature)

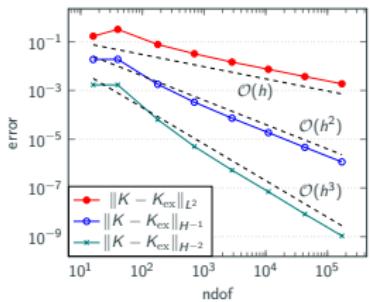
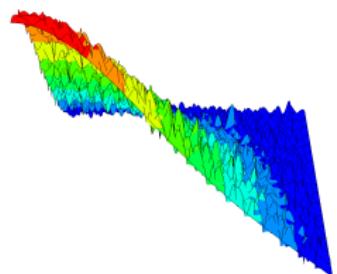
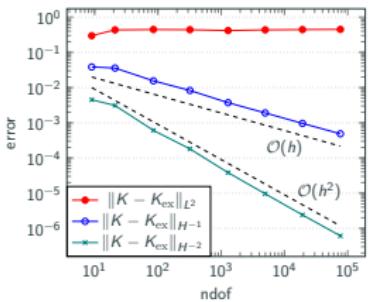
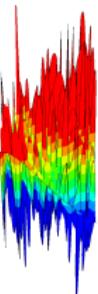
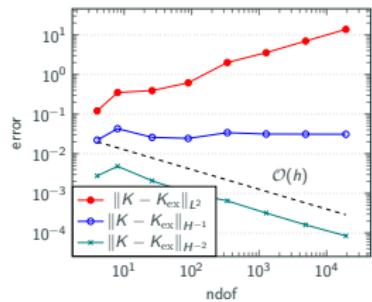
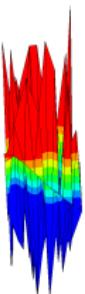


$$g = \begin{pmatrix} 1 + (\partial_x f)^2 & \partial_x f \partial_y f \\ \partial_x f \partial_y f & 1 + (\partial_y f)^2 \end{pmatrix} \quad f = \frac{1}{2}(x^2 + y^2) - \frac{1}{12}(x^4 + y^4)$$

$$K(g) = \frac{81(1-x^2)(1-y^2)}{(9+x^2(x^2-3)^2+y^2(y^2-3)^2)^2}$$

Numerical example (Gauss curvature)

Optimal-order interpolant ($q = k + 1$)



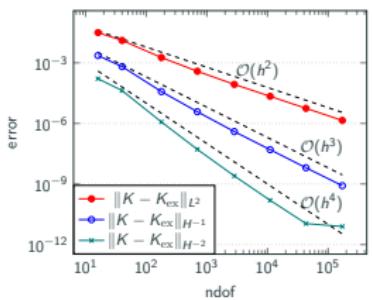
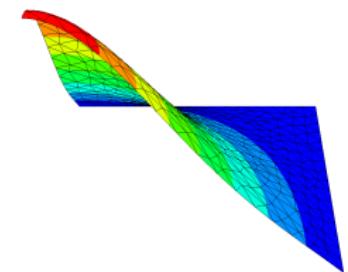
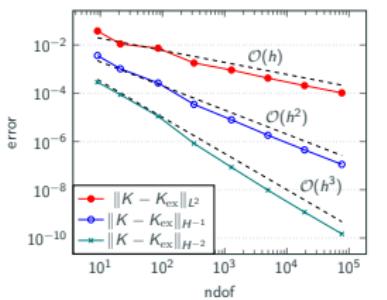
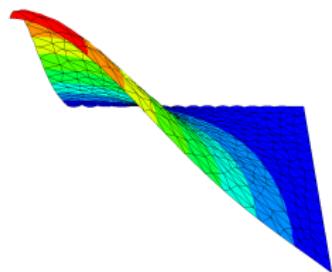
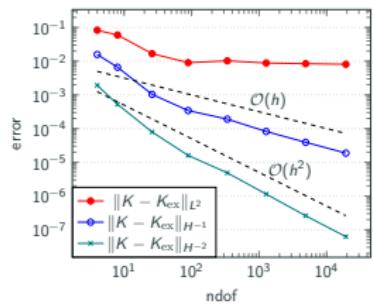
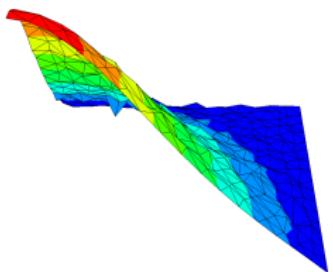
$k = 0$

$k = 1$

22

Numerical example (Gauss curvature)

Canonical interpolant



$k = 0$

$k = 1$

22

$$\langle (K\omega)(g), \varphi \rangle = \sum_{T \in \mathcal{T}} \int_T K(g) \varphi \omega_T + \sum_{E \in \mathcal{E}} \int_E [\kappa(g)] \varphi \omega_E + \sum_{V \in \mathcal{V}} \Theta_V(g) \varphi(V)$$

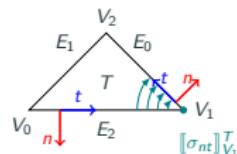
$$\langle (K\omega)(g), \varphi \rangle = \sum_{T \in \mathcal{T}} \int_T K(g) \varphi \omega_T + \sum_{E \in \mathcal{E}} \int_E [\kappa(g)] \varphi \omega_E + \sum_{V \in \mathcal{V}} \Theta_V(g) \varphi(V)$$

- Observation: $(\mathbb{S}_g \sigma = \sigma - \text{tr}_g(\sigma) g)$

$$\frac{d}{dt} ((K\omega)(g + t\sigma))|_{t=0} = \frac{1}{2} \operatorname{div}_g \operatorname{div}_g (\mathbb{S}_g \sigma) \omega_g$$

$$(\operatorname{div}_g A)_i = \partial_j A_{ij} - \Gamma_{ji}^k A_{kj} - \Gamma_{jj}^k A_{ik}$$

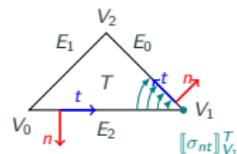
Hellan–Herrmann–Johnson method



$$\begin{aligned} \langle \operatorname{div} \operatorname{div} \mathbb{S} \sigma, u_h \rangle &= \sum_{T \in \mathcal{T}} \int_T \operatorname{div} \operatorname{div} \mathbb{S} \sigma \, u_h \, dx - \sum_{E \in \mathcal{E}} \int_E [(\operatorname{div} \mathbb{S} \sigma)_n + \nabla_t (\sigma_{nt})] u_h \, ds \\ &\quad + \sum_{V \in \mathcal{V}} \sum_{T \supset V} [\sigma_{nt}]_V^T u_h(V) \end{aligned}$$

- M. COMODI: The Hellan-Herrmann-Johnson method: some new error estimates and postprocessing, *Math. Comp.* (1989).

Hellan–Herrmann–Johnson method



$$\begin{aligned} \langle \operatorname{div} \operatorname{div} \mathbb{S} \sigma, u_h \rangle &= \sum_{T \in \mathcal{T}} \int_T \operatorname{div} \operatorname{div} \mathbb{S} \sigma \, u_h \, dx - \sum_{E \in \mathcal{E}} \int_E [(\operatorname{div} \mathbb{S} \sigma)_n + \nabla_t (\sigma_{nt})] u_h \, ds \\ &\quad + \sum_{V \in \mathcal{V}} \sum_{T \supset V} [\sigma_{nt}]_V^T u_h(V) \end{aligned}$$

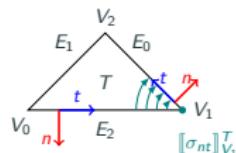
$$G(t) = g_h + t(g - g_h), \quad \sigma_h = G'(t) = g - g_h$$

$$\langle (K\omega)(g), u_h \rangle - \langle (K\omega)(g_h), u_h \rangle = \frac{1}{2} \int_0^1 \langle \operatorname{div}_{G_h(t)} \operatorname{div}_{G_h(t)} \mathbb{S}_{G_h(t)}(\sigma_h), u_h \rangle \, dt$$



GAWLIK: High-Order Approximation of Gaussian Curvature with Regge Finite Elements, *SIAM J. Numer. Anal.* (2020).

Hellan–Herrmann–Johnson method



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$$G(t) = g_h + t(g - g_h), \quad \sigma_h = G'(t) = g - g_h$$

$$\langle (K\omega)(g), u_h \rangle - \langle (K\omega)(g_h), u_h \rangle = \frac{1}{2} \int_0^1 \langle \operatorname{div}_{G_h(t)} \operatorname{div}_{G_h(t)} \mathbb{S}_{G_h(t)}(\sigma_h), u_h \rangle \, dt$$

$$\text{Show } |\langle \operatorname{div}_{G_h(t)} \operatorname{div}_{G_h(t)} \mathbb{S}_{G_h(t)}(\sigma_h), u_h \rangle| \leq Ch^{-1} \|g_h - g\|_{H_h^1} \|u_h\|_{H^1}$$



GAWLIK: High-Order Approximation of Gaussian Curvature with Regge Finite Elements, *SIAM J. Numer. Anal.* (2020).

Integral representation scalar curvature

$$\frac{d}{dt} \langle (S\omega)(g(t)), u \rangle = \langle \operatorname{div}_g \operatorname{div}_g \mathbb{S}_g \sigma, u \rangle - a_h(g; \sigma, u)$$

$$\bar{\mathbb{I}} = \mathbb{I} - H g$$

$$\begin{aligned} a_h(g; \sigma, u) = & \sum_{T \in \mathcal{T}} \int_T \langle G, \sigma \rangle u \omega_T + \sum_{F \in \mathcal{F}} \int_F \langle [\bar{\mathbb{I}}]_F, \sigma|_F \rangle u \omega_F \\ & - \sum_{E \in \mathcal{E}} \int_E \langle \Theta_E g|_E, \sigma|_E \rangle u \omega_E \end{aligned}$$

-  GAWLIK, N.: Finite element approximation of scalar curvature in arbitrary dimension, *arXiv:2301.02159*.

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leads to **distributional Einstein tensor** for $u = 1$

-  GAWLIK, N.: Finite element approximation of scalar curvature in arbitrary dimension, *arXiv:2301.02159*.

Alternative representation: $\langle \operatorname{div}_g \operatorname{div}_g \mathbb{S}_g \sigma, u_h \rangle = -\langle \operatorname{inc}_g \sigma, u_h \rangle$

$$\int_E (\sigma - \mathcal{R}_h^k \sigma)_{tt} q \, dl = 0 \text{ for all } q \in \mathcal{P}^k(E)$$

$$\int_T (\sigma - \mathcal{R}_h^k \sigma) : q \, da = 0 \text{ for all } q \in \mathcal{P}^{k-1}(T, \mathbb{R}^{2 \times 2})$$

$$\langle \operatorname{inc}(\sigma - \mathcal{R}_h^k \sigma), u_h \rangle = 0 \text{ for all } u_h \in \mathring{\mathcal{V}}_h^{k+1}$$

-  GOPALAKRISHNAN, N., SCHÖBERL, WARDETZKY: Analysis of curvature approximations via covariant curl and incompatibility for Regge metrics, *arXiv:2206.09343*.

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$$\langle \operatorname{inc}(\sigma - \mathcal{R}_h^k \sigma), u_h \rangle = 0 \text{ for all } u_h \in \mathring{\mathcal{V}}_h^{k+1}$$

$$|\langle \operatorname{inc}_g(\sigma - \mathcal{R}_h^k \sigma), u_h \rangle| \leq C \|\sigma - \mathcal{R}_h^k \sigma\| \|u_h\|_{H^1} \text{ for all } u_h \in \mathring{\mathcal{V}}_h^{k+1}$$

-  GOPALAKRISHNAN, N., SCHÖBERL, WARDETZKY: Analysis of curvature approximations via covariant curl and incompatibility for Regge metrics, *arXiv:2206.09343*.

Distributional Riemann curvature

- Riemann curvature tensor R_{ijkl} has 6 independent entries
 - Curvature operator $Q : M \rightarrow \Lambda^2(M) \odot \Lambda^2(M)$
- $$\langle Q(u \wedge v), w \wedge z \rangle = \langle R(u, v)z, w \rangle \text{ for all } u, v, w, z \in \mathfrak{X}(M)$$

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- No Gauss–Bonnet theorem in 3D

B fourth order tensor with same (skew-)symmetries as R

$$\begin{aligned}\langle (R\omega)(g), B \rangle &= \sum_{T \in \mathcal{T}} \int_T \langle R, B \rangle \omega_T + \sum_{F \in \mathcal{F}} \int_F \langle [\![\mathbb{I}]\!], B_{n \cdot n \cdot} \rangle \omega_F \\ &\quad + \sum_{E \in \mathcal{E}} \int_E \Theta_E(g) B_{n \nu n \nu} \omega_E\end{aligned}$$

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In 3D: $V \in \text{Reg}_h^k$

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$$(A \times B)^{ij} = \frac{1}{\omega_T^2} \varepsilon^{ikl} \varepsilon^{jmn} A_{km} B_{ln}$$

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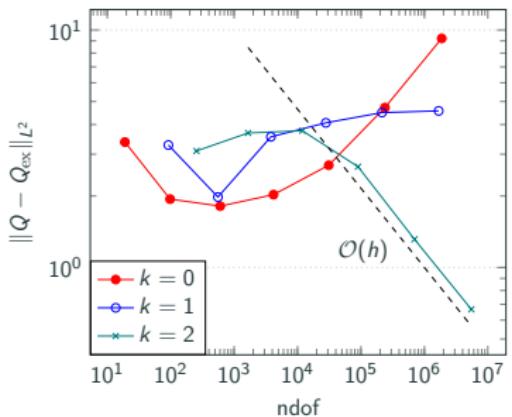
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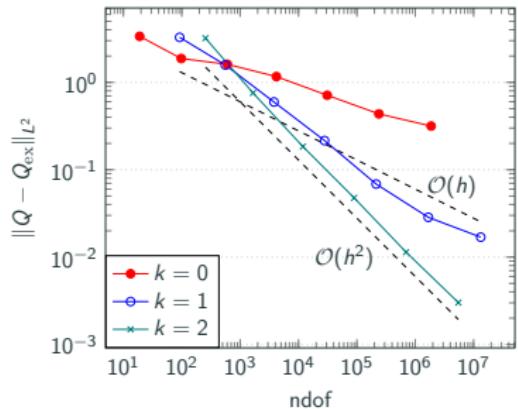
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Analysis: WIP

Numerical example (Curvature operator 3D)



optimal-order interpolant



canonical interpolant

- Regge finite elements approximating metrics
- Approximation of geodesics and curvature quantities

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- Approximation of geodesics and curvature quantities
- Analysis of distributional Riemann curvature tensor
- Analysis of Einstein tensor
- Application to general relativity (Einstein field equation)

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4th NGSolve UserMeeting

July 9-11, Portland, Oregon



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Thank You for Your attention!