Fluid-Structure Interaction with H(div)-Conforming HDG and a new H(curl)-Conforming Method for Non-Linear Elasticity

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Contents

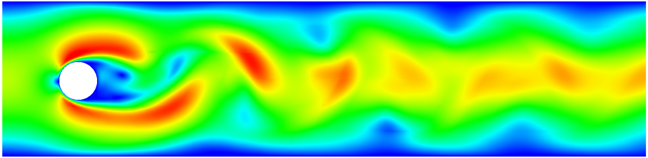
H(div)-conforming HDG for Navier-Stokes equations

Navier-Stokes equations

$$u(x,t)\dots \text{ velocity}$$
$$p(x,t)\dots \text{ pressure}$$

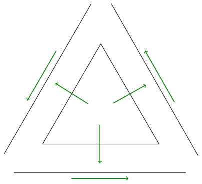
Navier-Stokes

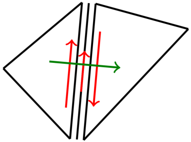
$$\begin{aligned}\frac{\partial u}{\partial t} + (u\cdot\nabla)u - \nu\Delta u + \nabla p &= f \\ \operatorname{div}(u) &= 0\end{aligned}$$



- Normal continuous elements for velocity and discontinuous pressure
- Facet variables for tangential part in a hybrid fashion
- Solution exact divergence free

$$\int_\Omega \mathrm{div}(u_h)\; q_h \, dx = 0 \quad \forall q_h \in Q_h \Rightarrow \mathrm{div}(u_h) = 0$$





Arbitrary Lagrangian Eulerian for HDG

- Standard ALE with deformation $\Phi = id + d$

$$\frac{\partial \hat{u}}{\partial t} + ((\hat{u} - \dot{d}^f) \cdot \nabla)\hat{u} - \nu \Delta \hat{u} + \nabla p = 0$$

- Mesh velocity $\dot{d}$ from differentiating $\hat{u}(\hat{x}, t) = u(\Phi(\hat{x}, t), t)$

Arbitrary Lagrangian Eulerian for HDG

- Standard ALE with deformation $\Phi = id + d$

$$\frac{\partial \hat{u}}{\partial t} + ((\hat{u} - \dot{d}^f) \cdot \nabla)\hat{u} - \nu \Delta \hat{u} + \nabla p = 0$$

- Mesh velocity $\dot{d}$ from differentiating $\hat{u}(\hat{x}, t) = u(\Phi(\hat{x}, t), t)$
- Piola transformation to ensure normal continuity

$$P[u] := \frac{1}{\det(F)} F u, \qquad F = I + \nabla d$$

- Second additional term from differentiating $\hat{u} = \frac{1}{\det(F)} F u \circ \Phi$

$$(\nabla \dot{d}^f - \mathrm{div}(\dot{d}^f)) P[\hat{u}]$$

[Masterthesis M.N. 2017]

H(curl)-conforming discretization for elastic wave equation

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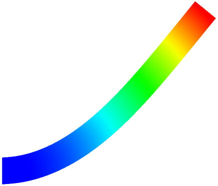
$$
\begin{aligned}
F &= I + \nabla d \\
C &= F^T F \\
\Sigma &= \mu(C - I) + \tfrac{\lambda}{2}\mathrm{tr}(C - I)I
\end{aligned}
$$

Elastic wave

$$\rho \frac{\partial^2 d}{\partial t^2} - \mathrm{div}(F\Sigma) = g$$



H(curl)-conforming discretization for elastic waves

Find $(d, u) \in [H^1(\Omega)]^n \times H(curl, \Omega)$ such that

$$\int_\Omega \frac{\partial d}{\partial t} \cdot v \, dx = \int_\Omega u \cdot v \, dx \qquad \forall v \in H(curl, \Omega)$$

$$\int_\Omega \rho \frac{\partial u}{\partial t} \cdot w \, dx = -\int_\Omega (F\Sigma) : \nabla w \, dx \quad \forall w \in [H^1(\Omega)]^n$$

H(curl)-conforming discretization for elastic waves

Find $(d, u, p) \in [H^1(\Omega)]^n \times H(curl, \Omega) \times P$ such that

$$
\begin{aligned}
\int_\Omega \frac{\partial d}{\partial t} \cdot q \, dx &= \int_\Omega u \cdot q \, dx && \forall q \in P \\
\int_\Omega \rho \frac{\partial u}{\partial t} \cdot v \, dx &= \int_\Omega \frac{\partial p}{\partial t} \cdot v \, dx && \forall v \in H(curl, \Omega) \\
\int_\Omega \frac{\partial p}{\partial t} \cdot w \, dx &= -\int_\Omega (F\Sigma) : \nabla w \, dx && \forall w \in [H^1(\Omega)]^n
\end{aligned}
$$

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\end{aligned}$$

$$P = ?$$

H(curl)-conforming discretization for elastic waves

Find $(d, u, p) \in [H^1(\Omega)]^n \times H(curl, \Omega) \times P$ such that

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\int_\Omega p \cdot w \, dx &= -\int_\Omega (F\Sigma) : \nabla w \, dx && \forall w \in [H^1(\Omega)]^n
\end{aligned}
$$

$$P = H(curl, \Omega)^*$$

H(curl)-conforming discretization for elastic waves

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Find $(d, u, p) \in [H^1(\Omega)]^n \times H(curl, \Omega) \times P$ such that

$$\begin{aligned}
\int_\Omega \frac{\partial d}{\partial t} \cdot q \, dx &= \int_\Omega u \cdot q \, dx && \forall q \in P \\
\int_\Omega \rho \frac{\partial u}{\partial t} \cdot v \, dx &= \int_\Omega p \cdot v \, dx && \forall v \in H(curl, \Omega) \\
\int_\Omega p \cdot w \, dx &= -\int_\Omega (F\Sigma) : \nabla w \, dx && \forall w \in [H^1(\Omega)]^n
\end{aligned}$$

$$P = H(curl, \Omega)^*$$

$p = \text{H(Curl)-TrialFunction()}$

$p = p.Operator("dual")$

NGSolve

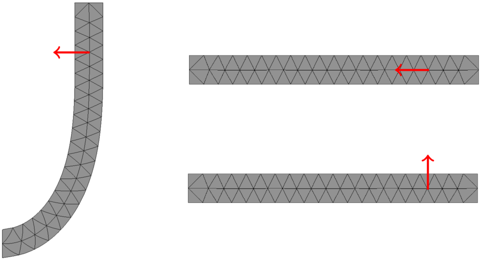
Transformation to material coordinates

- Covariant transformation from global to material velocity

$$u = F^{-T}\hat{u}$$

- Dual transformation for p

$$p = F\hat{p}$$



Find $(d, \hat{u}, \hat{p}) \in [H^1(\Omega)]^n \times H(curl, \Omega) \times H(curl, \Omega)^*$ such that

$$
\begin{aligned}
\int_\Omega \frac{\partial d}{\partial t} \cdot (Fq)\, dx &= \int_\Omega \hat{u} \cdot q \, dx && \forall q \in H(curl, \Omega)^* \\
\int_\Omega \rho \frac{\partial}{\partial t}(F^{-T} \hat{u}) \cdot (F^{-T} v)\, dx &= \int_\Omega \hat{p} \cdot v \, dx && \forall v \in H(curl, \Omega) \\
\int_\Omega (F\hat{p}) \cdot w \, dx &= -\int_\Omega (F\Sigma) : \nabla w \, dx && \forall w \in [H^1(\Omega)]^n
\end{aligned}
$$

H(curl)-conforming discretization for elastic waves

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Find $(d, \hat{u}, \hat{p}) \in [H^1(\Omega)]^n \times H(curl, \Omega)^{dc} \times H(curl, \Omega)^{*,dc}$ such that

$$
\begin{aligned}
\int_\Omega \frac{\partial d}{\partial t} \cdot (Fq)\, dx &= \int_\Omega \hat{u} \cdot q \, dx && \forall q \in H(curl, \Omega)^{*,dc} \\
\int_\Omega \rho \frac{\partial}{\partial t}(F^{-T}\hat{u}) \cdot (F^{-T} v)\, dx &= \int_\Omega \hat{p} \cdot v \, dx && \forall v \in H(curl, \Omega)^{dc} \\
\int_\Omega (F\hat{p}) \cdot w \, dx &= -\int_\Omega (F\Sigma) : \nabla w \, dx && \forall w \in [H^1(\Omega)]^n
\end{aligned}
$$

- Static condensation for discontinuous $\hat{u}$ and $\hat{p}$
- Further discretisation in 2d and 3d
- Optimal energy conservation in space discretization

Find $(d, \hat{u}, \hat{p}) \in [H^1(\Omega)]^n \times H(curl, \Omega) \times H(curl, \Omega)^*$ such that

$$
\begin{aligned}
\int_\Omega \frac{\partial d}{\partial t} \cdot (Fq)\, dx &= \int_\Omega \hat{u} \cdot q \, dx \quad && \forall q \in H(curl, \Omega)^* \\
\int_\Omega \rho\Big(F^{-T} \dot{\hat{u}} \cdot F^{-T} v - \frac{1}{2} C^{-1} \dot{C} C^{-1} \hat{u} \cdot v & \\
+ \frac{1}{2J} \mathrm{curl}(\hat{u}) \times (F^{-T} \hat{u}) \cdot v\Big)\, dx &= \int_\Omega \hat{p} \cdot v \, dx \quad && \forall v \in H(curl, \Omega) \\
\int_\Omega (F\hat{p}) \cdot w + (F\Sigma) : \nabla w \, dx &= 0 && \forall w \in [H^1(\Omega)]^n
\end{aligned}
$$

H(curl)-conforming discretization for elastic waves 2d

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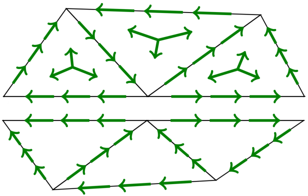
Find $(d, \hat{u}, \hat{p}) \in [H^1(\Omega)]^n \times H(curl, \Omega) \times H(curl, \Omega)^*$ such that

$$
\begin{aligned}
\int_\Omega \frac{\partial d}{\partial t} \cdot (Fq)\,dx &= \int_\Omega \hat{u} \cdot q\,dx \quad && \forall q \in H(curl, \Omega)^* \\
\int_\Omega \rho(F^{-T}\dot{\hat{u}} \cdot F^{-T} v - \frac{1}{2} C^{-1}\dot{C}C^{-1}\hat{u} \cdot v & \\
- \frac{1}{2J^2}\mathrm{curl}(\hat{u})\mathrm{rot}(\hat{u}) \cdot v)\,dx &= \int_\Omega \hat{p} \cdot v\,dx \quad && \forall v \in H(curl, \Omega) \\
\int_\Omega (F\hat{p}) \cdot w + (F\Sigma) : \nabla w\,dx &= 0 && \forall w \in [H^1(\Omega)]^n
\end{aligned}
$$

Interface conditions

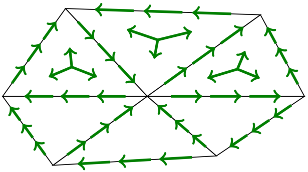
Interface conditions



- Continuity of displacement and tangential continuity of velocity fulfilled

$$d^s = d^f, \quad u^s_\tau = u^f_\tau \qquad \text{on } \Gamma_I$$

Interface conditions



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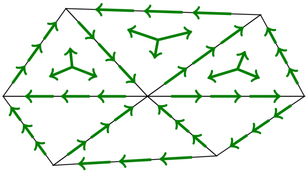
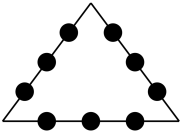




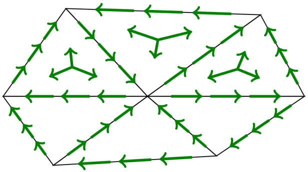
- Continuity of displacement and tangential continuity of velocity fulfilled

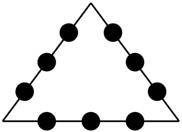
$$d^s = d^f, \quad u^s_\tau = u^f_\tau \qquad \text{on } \Gamma_I$$

- Normal continuity by Lagrange multiplier

$$\int_{\Gamma_I} (u^f - u^s)_n \lambda = 0 \quad \forall \lambda \in L^2(\Gamma_I)$$

Interface conditions





- Continuity of displacement and tangential continuity of velocity fulfilled

$$d^s = d^f, \quad u^s_\tau = u^f_\tau \qquad \text{on } \Gamma_I$$

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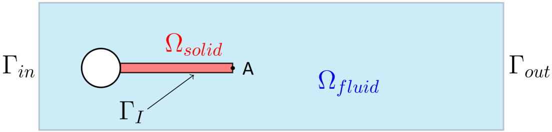
$$\int_{\Gamma_I} (u^f - \frac{\partial d^s}{\partial t})_n \lambda = 0 \quad \forall \lambda \in L^2(\Gamma_I)$$

Numerical results

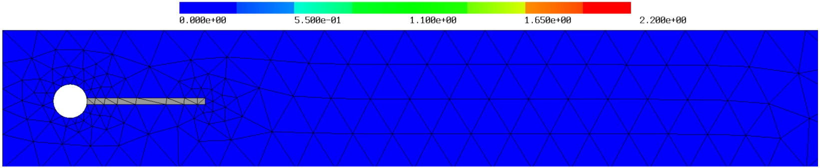
Benchmark (Turek/Hron)



- Parabolic inflow
- Y-displacement of A

[Turek + Hron, 2010]

Video



Benchmark (Turek/Hron)

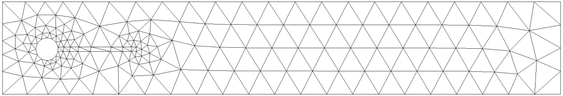


Figure 1: Coarsest mesh level

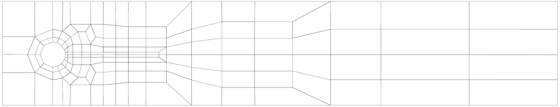
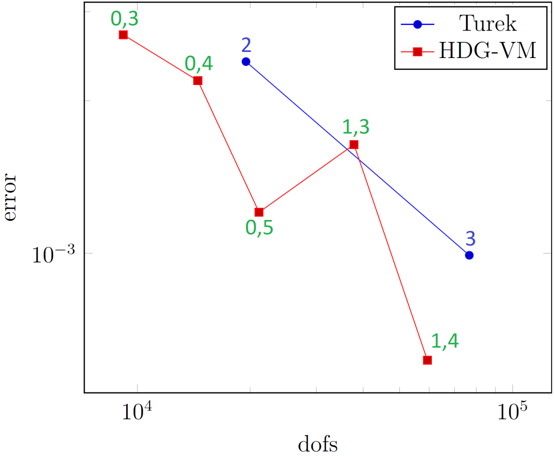


Figure 2: Coarsest mesh level in benchmark

Benchmark (Turek/Hron)



- Uniform h,p refinement
- Faster convergent with p refinement

Summary

- ALE for H(div)-conforming HDG Navier-Stokes
- New spatial discretization for elastic wave equation
- Coupling of both equations

Current work

- Appropriate time discretization for elastic wave equation
- Preconditioner
- Splitting methods

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- Appropriate time discretization for elastic wave equation
- Preconditioner
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THANK YOU FOR YOUR ATTENTION!