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## Systolic Arrays for Matrix Transpose and Other Reorderings

DIANNE P. O'LEARY

**Abstract**—In this correspondence, a systolic array is described for computing the transpose of an  $n \times n$  matrix in time  $3n - 1$  using  $n^2$  switching processors and  $n^2$  bit buffers. A one-dimensional implementation is also described. Arrays are also given to take a matrix in by rows and put it out by diagonals, and vice versa.

**Index Terms**—Matrix transpose, parallel computation, systolic array.

### I. INTRODUCTION

The task of computing the transpose of a matrix arises in many matrix algorithms, for example, in computing congruence transformations  $B = UAU^T$  and in computing a step of the  $QR$  algorithm for finding eigenvalues of a matrix.

Systolic arrays, introduced by Kung and Leiserson [4], are a very popular implement for parallel matrix computation. However, there seems to be no consensus on the best way to transpose a matrix for use in a systolic array algorithm. For example, Bojanczyk, Brent, and

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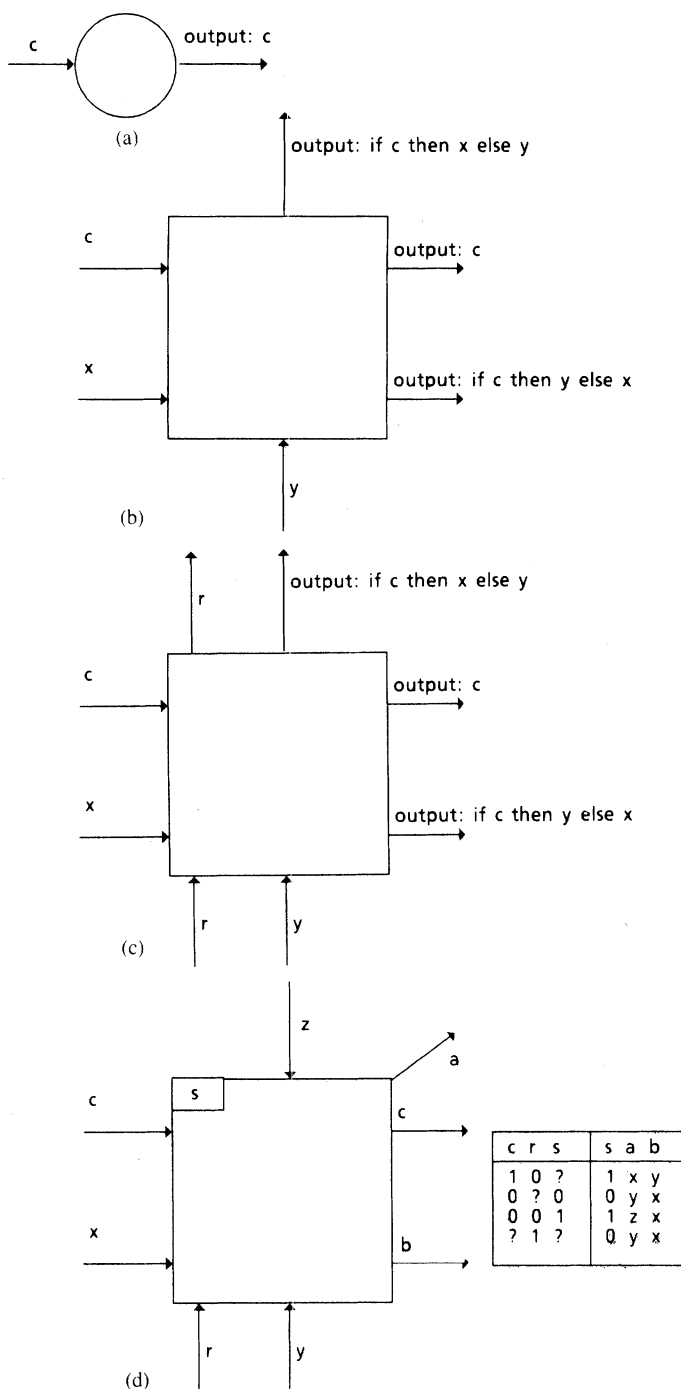
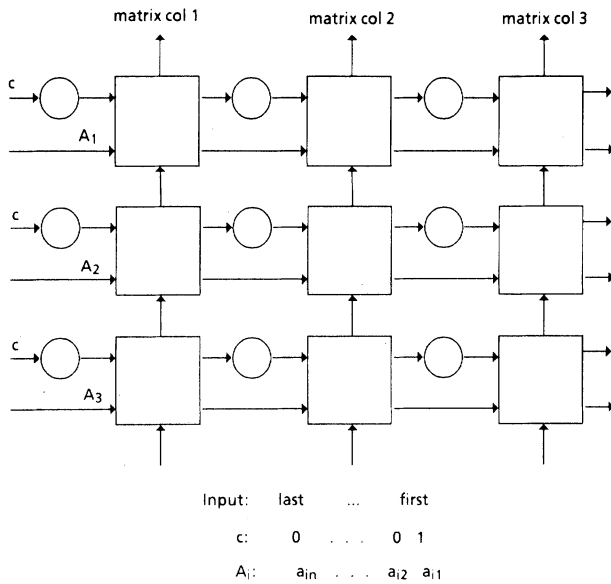


Fig. 1. Four systolic array elements. The outputs and new state definitions for the last one are indicated in the table.

Kung [2] suggest using "a buffer that supports fast two-dimensional addressing," and the speed of their algorithm depends critically on the speed of this hardware. Ullman [5] discusses a systolic algorithm of Atallah and Kosaraju [1] which transposes a matrix in place.

We present in this paper a rather simple systolic array which puts out the transpose of a matrix. The matrix is pumped in systolically, rather than being preloaded as in the above two methods. We also present two modifications of this array; one takes a matrix in by rows and puts it out by diagonals and the other reverses this process. A one-dimensional version of the matrix transpose array is also given. Alternate designs for the diagonal-to-row and row-to-diagonal conversions are given by Ipsen [3] who also presents an algorithm for transposition of a matrix in diagonal format.

To keep diagrams uncluttered, we adopt the convention that unlabeled inputs and outputs are either indeterminate or of no interest. The definitions of the four building blocks of the arrays are given in Fig. 1. The circles represent buffers which cause the control

Fig. 2. A two-dimensional systolic array for matrix transpose ( $n = 3$ ).

bits to travel at the proper velocity. They take a single bit as input and give a single bit as output. The squares are very simple switching processors. The first receives as input a control bit  $c$  from the left, and possible matrix elements  $x$  from the left and  $y$  from below. If  $c$  is zero, then  $x$  is sent to the right and  $y$  is sent up. If  $c$  is one, then  $y$  is sent to the right and  $x$  is sent up. The second square is similar to the first but has an extra single bit input and output called  $r$ . The third is a variant of this switch which includes one bit  $s$  of memory. Its operation is defined by the accompanying table in Fig. 1.

## II. THE SYSTOLIC ARRAY FOR MATRIX TRANSPOSITION

The architecture for the first systolic array is shown in Fig. 2 for the case  $n = 3$ . The matrix is fed into the array from the left, one row per row of processors. A control bit sequence is also fed from the left in each row of processors; a 1 is sent one time unit before the beginning of the row, and  $n - 1$  0's follow. (The first column of bit buffers can be eliminated; then the control bit 1 would be synchronized with the beginning of the row, and one time unit would be saved.)

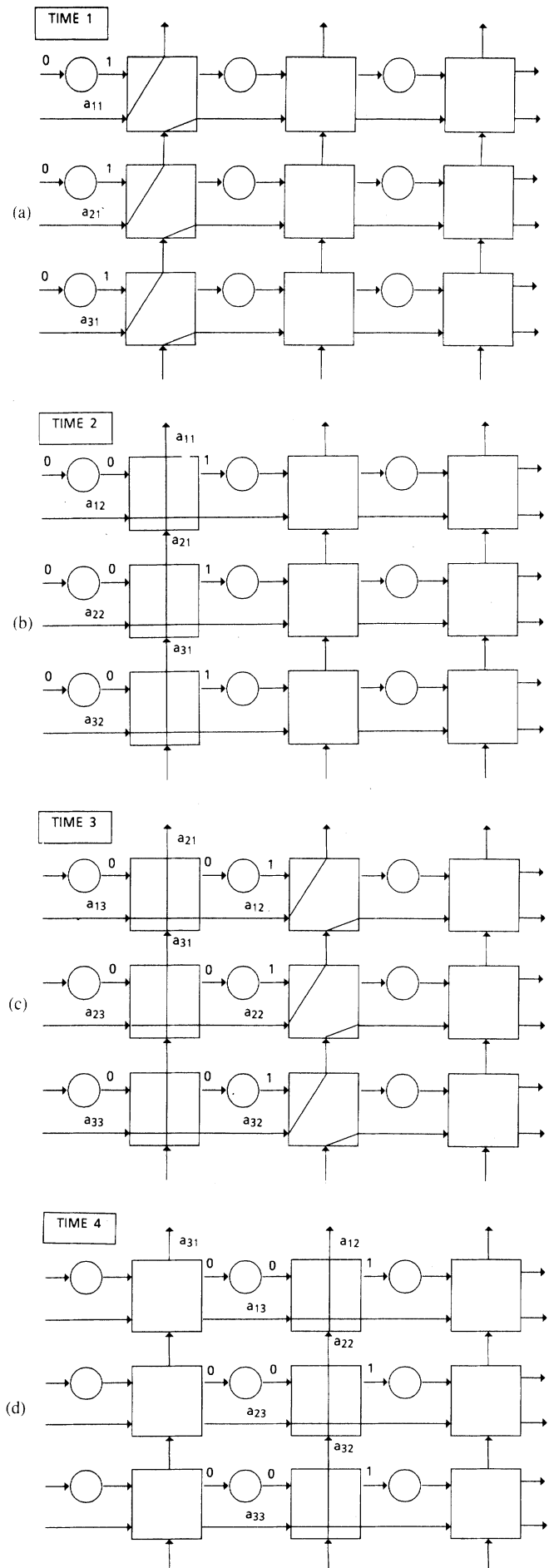
The transpose of the matrix exits from the top, one column per column of processors. A simulation for the  $3 \times 3$  case in which all rows are started simultaneously is given in Fig. 3. Element  $(i, j)$  enters the array at time  $j$  and exits at time  $2j + i - 1$ , giving a total time of  $3n - 1$  for transposing a matrix of size  $n \times n$ .

Elements in a given column come out at consecutive times, just as they were loaded. The array still works if later rows are loaded delayed relative to earlier ones, as long as the earlier sequences of control signals are padded by extra zeroes.

The array can be modified in case the elements in each row are not available at consecutive time units. For example, if elements are loaded as  $(i, 1), *, *, (i, 2), *, *, \dots (i, n)$  where  $*$  represents an indeterminate, then each bit buffer should be replaced by three such buffers.

## III. A ONE-DIMENSIONAL IMPLEMENTATION OF THE TRANSPOSITION ARRAY

The array discussed in the previous section can be easily modified to use  $n$  processors and  $n$  bit buffers, but then the time is increased to  $n^2 + n$ , and elements in a given column exit at intervals of time  $n$ . The array is shown in Fig. 4 for the case  $n = 3$ . The processors are the same as before, although the switching processor can be simplified since the  $y$  input is always indeterminate. The matrix is entered by rows  $(1, 1), (1, 2), \dots, (1, n), (2, 1), (2, 2), \dots, (2, n), \dots, (n, 1), (n, 2), \dots, (n, n)$ , and, as before, the control signal is 0 except at the beginning of a row. A simulation of the array is given in Fig. 5. Element  $(i, j)$  enters the array at time  $(i - 1)n + j$  and exits  $j$  time units later, for a total time of  $n^2 + n$ .

Fig. 3. Transpose of a  $3 \times 3$  matrix using the two-dimensional array.

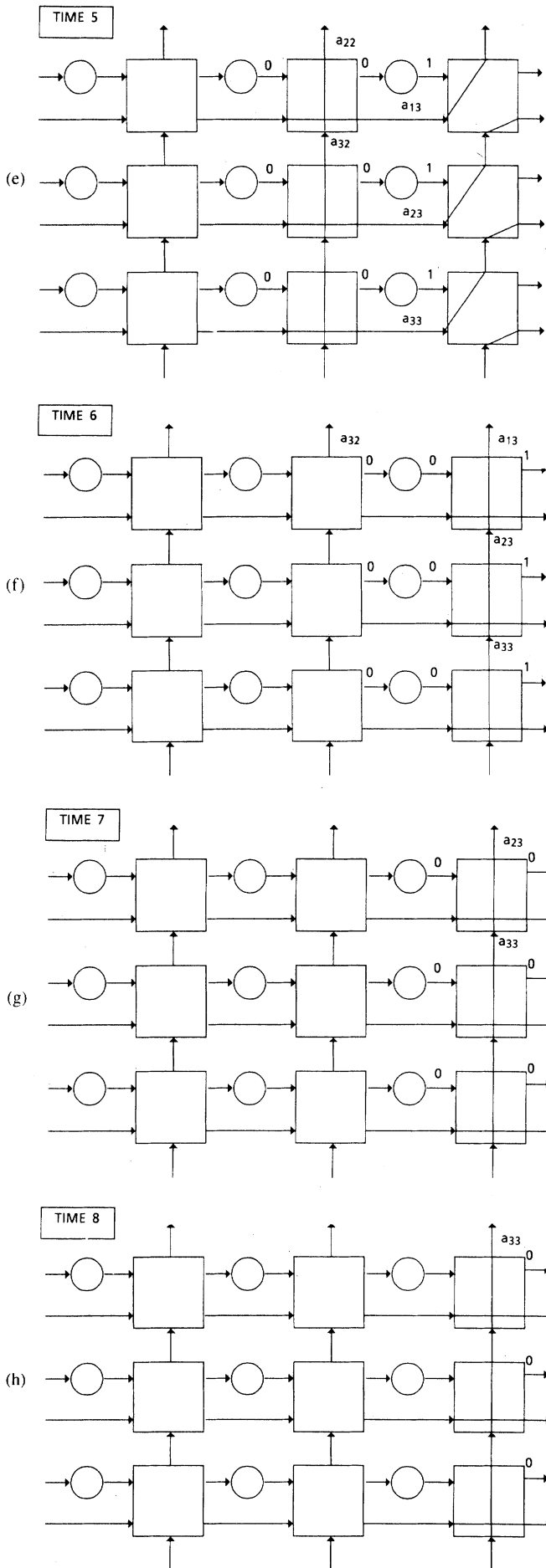


Fig. 3. (Continued).

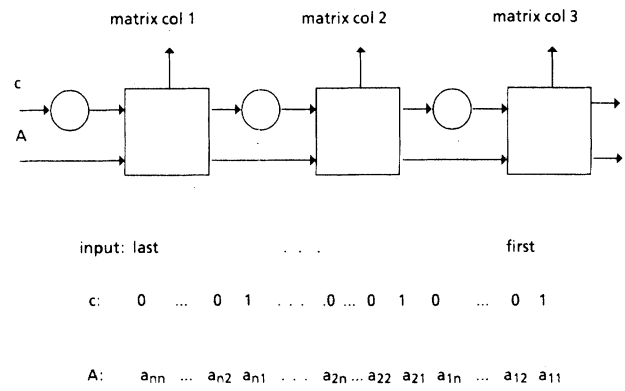


Fig. 4. A one-dimensional systolic array for matrix transpose ( $n = 3$ ).

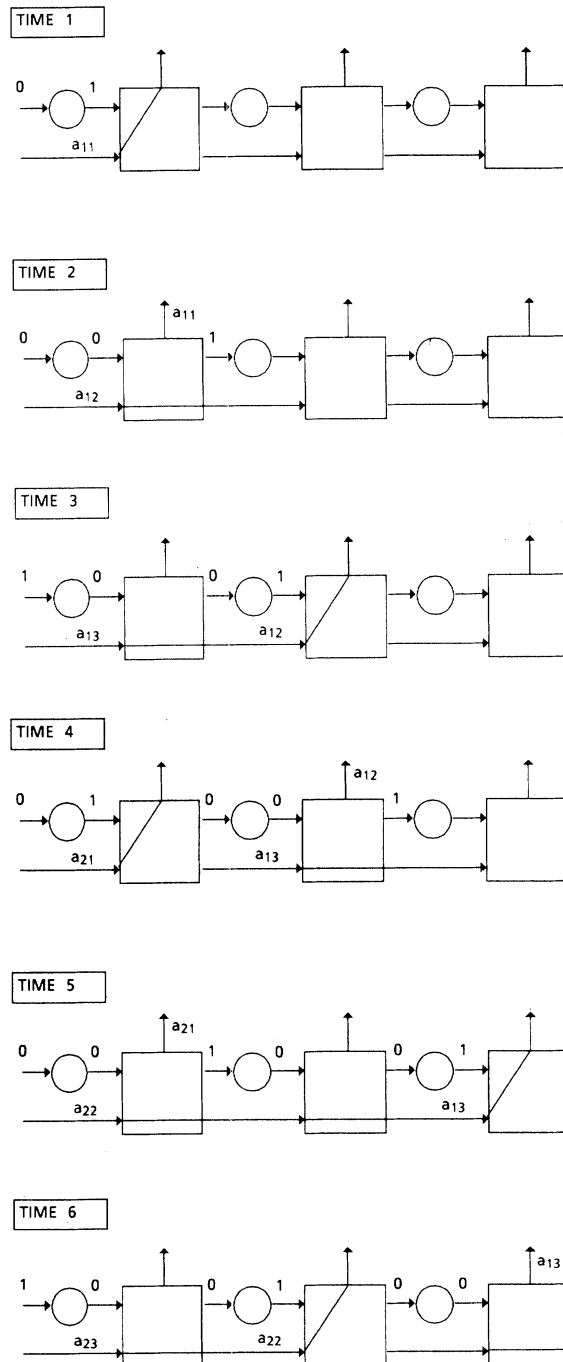


Fig. 5. Transpose of a  $3 \times 3$  matrix using the one-dimensional array.

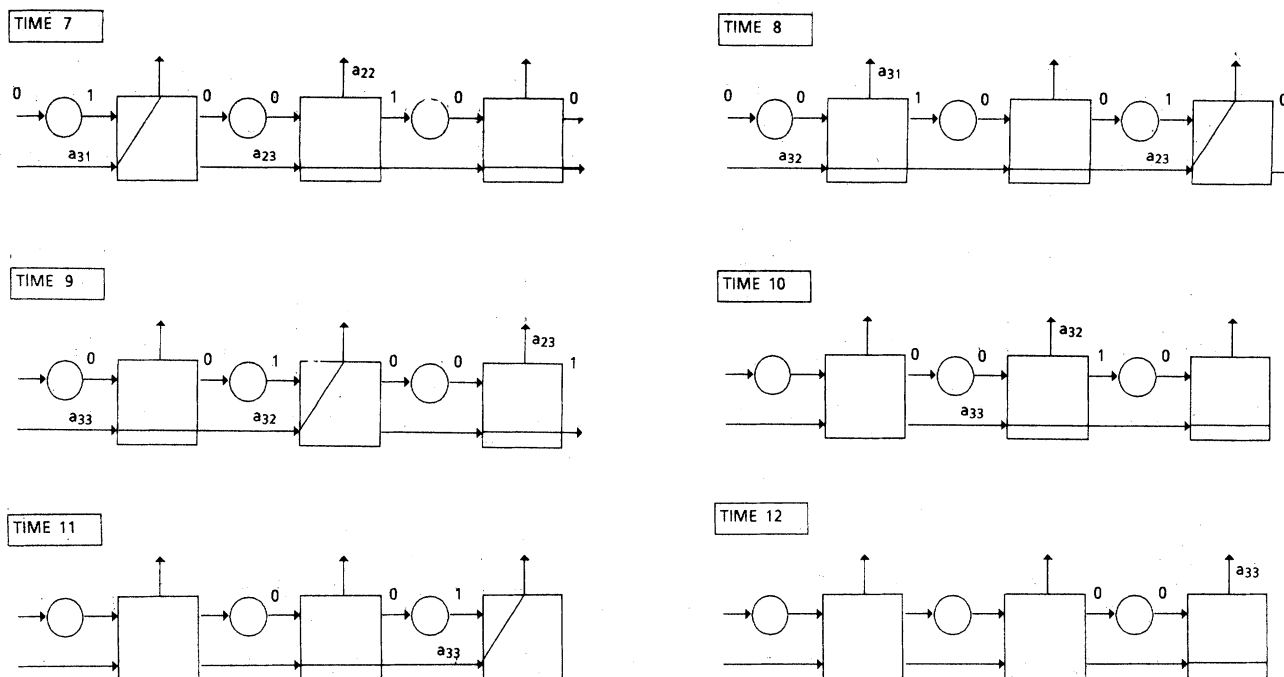
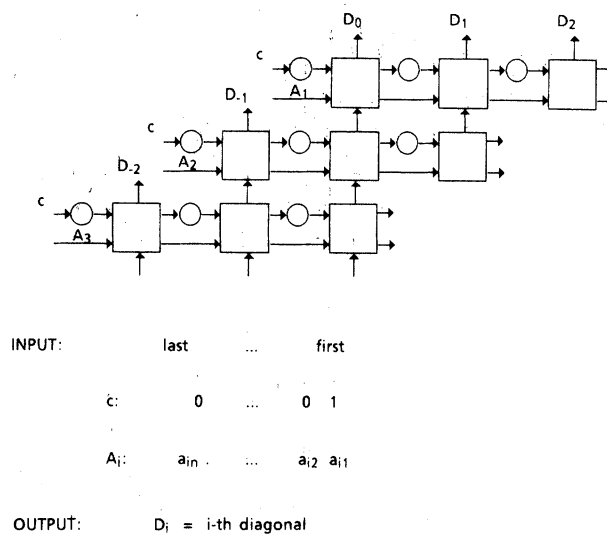


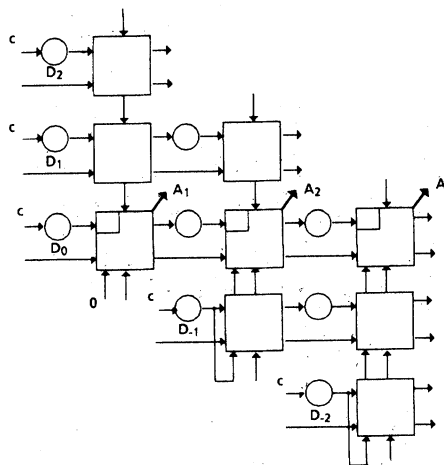
Fig. 5. (Continued).

Fig. 6. An array for converting from row ordering to diagonal ordering ( $n = 3$ ).

#### IV. IN BY ROWS, OUT BY DIAGONALS

We present in this section two other arrays which modify the orderings of the matrix elements. The first, shown in Fig. 6, consists of  $n^2$  processors plus bit buffers. It takes a matrix in by rows and puts it out by diagonals. We use the notation  $D_k$  to denote the  $k$ th diagonal, consisting of those elements  $a_{ij}$  for which  $j - i = k$ . The operation takes time  $4n - 1$  if all rows are started simultaneously, and elements on a given diagonal are put out at intervals of four time units. The operation of the array is similar to the examples above, so a simulation is omitted.

The second array of  $n^2$  processors plus bit buffers, shown in Fig. 7, transforms a matrix from diagonal ordering to row ordering. The time is  $2n$ , again assuming that all diagonals are started simultaneously. Here, the array must be primed with zero control signals before the matrix input is begun, and the processors are slightly more complicated than in the arrays above. The center row of switches perform one of three routings instead of only two, and the processors below have an extra control signal. (See Fig. 1.) A simulation of the



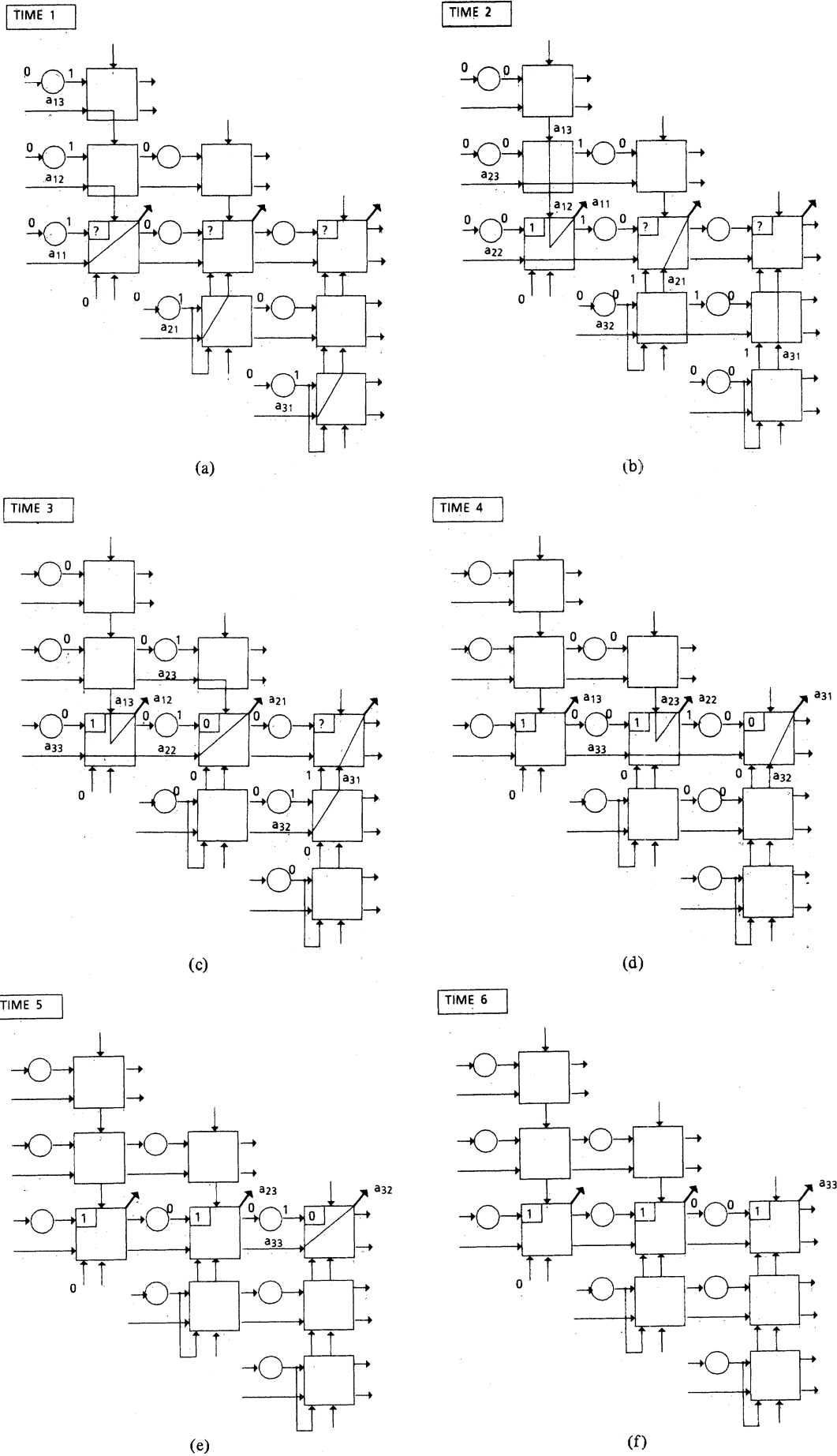


Fig. 8. Reordering of a  $3 \times 3$  matrix using the array of Fig. 7

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## Exact Product Form Solution for Queueing Networks with Blocking

I. F. AKYILDIZ

**Abstract**—This work investigates closed queueing networks with blocking composed of two stations with multiple servers. Blocking occurs when a job wanting to enter a full station is forced to remain in its source station, thus blocking the source station until room is available at the destination station. This type of blocking is known as classical blocking. We show that, for a two-station closed queueing network with blocking, there exists an equivalent nonblocking network with the same state space structure. Utilizing this concept, we demonstrate that two-station closed queueing networks with blocking have product form solutions.

**Index Terms**—Blocking, equilibrium state probabilities, normalization constant, performance analysis, performance measures, queueing networks, state space transformations.

### I. INTRODUCTION

The basic results of product form networks are given in Baskett, Chandy, Muntz and Palacios [4]. They show that queueing networks with different classes of jobs, exponential and nonexponential service time distributions, and different queueing disciplines (FCFS, RR-PS or LCFS-PR) have product form solution. This was a remarkable result for the queueing network theory. The term "product form" means that the equilibrium state probabilities can be expressed as a product of terms for each queue in the network. The product form networks, also known as BCMP or separable networks, are based on the assumption that all stations have infinite capacities. If the stations have finite capacity, blocking can occur in the network. Since blocking causes interdependencies between stations, blocking queueing networks cannot be analyzed by existing product form algorithms.

Formally we distinguish between three types of blocking: *classical*, *rejection*, and *service* blocking.

In the first case, blocking occurs when a job completing service at station  $i$  cannot proceed to station  $j$  because station  $j$  is full. The job is forced to wait in the station  $i$ 's server until it is allowed to enter the destination station  $j$ . The station  $i$ 's server stops processing until station  $j$  releases a job [1], [2], [11], [13], [15], [20]. In the second case, blocking occurs when a job completing service at station  $i$  attempts to join destination station  $j$ . If station  $j$  is full at that moment, the job is refused. The rejected job goes with a certain probability (called rejection probability) back to the station  $i$ 's server and receives a new service. This is repeated until some job completes a service at station  $j$  and a place becomes available [3], [9], [16], [22].

In the third case, blocking occurs when a job in front of queue at station  $i$  declares its destination station  $j$  before it starts its service in station  $i$ 's server. If the destination station  $j$  is full, the  $i$ th server becomes blocked, i.e., it cannot serve jobs. When a departure occurs from destination station  $j$ , the  $i$ th server becomes unblocked and the job begins receiving service [5], [8], [10], [19].

In recent years there has been a growing interest in the development of computational methods to analyze queueing networks with blocking. The interest developed primarily from the realization that these models are useful in the study of subsystem behavior in computers and communication networks, in addition to providing detailed descriptions of several computer-related applications such as flexible manufacturing systems.

Several investigators in recent years have published results on queueing networks with classical, rejection and service blocking. A bibliography concerning queueing network models with blocking is given by Perros [14]. Formal comparisons between the distinct classes of blocking have been carried out by Onvural and Perros [12].

In this paper we introduce the state space transformation concept which provides exact results for two-station closed queueing networks with classical blocking.

### II. PRODUCT FORM SOLUTION

We will consider closed queueing networks with  $N = 2$  stations and  $K$  single-class jobs. Each station consists of a single queue served by ( $m_i \geq 1$ ), servers each with exponentially distributed service time with mean value  $1/\mu_i$  (for  $i = 1, 2$ ). The service discipline in each station is first-come, first-served. Each station has a fixed finite capacity  $M_i$  where  $M_i = (\text{queue capacity} + m_i)$ , (for  $i = 1, 2$ ). Cases in which the stations can have infinite capacity are also allowed. Any station whose capacity exceeds the total number of jobs in the network can be considered to have infinite capacity. It is obvious that the total number of jobs  $K$  must be smaller than the sum of the station capacities, that is,

$$K < \sum_{i=1}^2 M_i.$$

A job which is serviced by the  $i$ th station proceeds to the  $j$ th station with probability  $p_{ij}$ , (for  $i, j = 1, 2$ ), if the  $j$ th station is not full. That is, if the number of jobs in the  $j$ th station  $k_j$  is less or equal to  $M_j$  for  $i, j = 1, 2$ . Otherwise, the job is blocked in the  $i$ th station until a job in the  $j$ th station has completed its servicing and a place becomes available. Note that the case  $i = j$  is allowed. A station can have a transition back to itself, which will be shown by an example in Section IV. The queueing network with the above assumptions is known as the network with classical blocking which we will investigate in this work.

As is generally known, the following binomial coefficient formula is valid for closed queueing networks without station capacity limits. It indicates the number of possible ways that  $K$  jobs can be distributed into  $N$  stations.

$$Z = \binom{N+K-1}{N-1} \quad (1)$$

where  $Z$  is the total number of states in a closed queueing network.

For queueing networks with two stations (1) is simplified to

$$Z = K + 1. \quad (2)$$

Structure for the state space of the two-station network is illustrated in Fig. 1.

The state  $(k, n)$  denotes that  $k$  jobs are in the first station and  $n$  jobs are in the second station. The transition rates from one state to another are equal to the service rates  $\mu_i$  multiplied by the number of servers  $m_i$  in each station. Fig. 1 shows that there must be at least  $(K - m_i)$  waiting places in each queue ( $m_i$  jobs can be in service) to ensure that all states are feasible. Since each station in the network

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