# VERTEX COVER

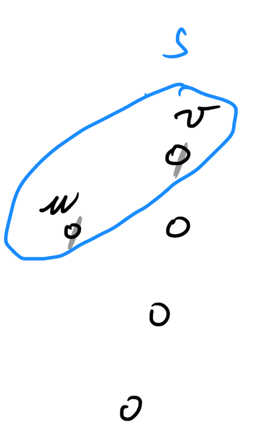
INPUT: UNDIRECTED $G(V,E)$

OUTPUT: A SUBSET $S \subseteq V$, OF SMALLEST CARDINALITY, S.T. $\forall e \in E$ $e \cap S \neq \phi$. ($e \in E$ IS A PAIR OF NODES $e \in \binom{V}{2}$)

$G(V,E)$

HAS A MINIMUM VC OF SIZE 2.

VC IS NP-COMPLETE



MAXIMAL MATCHING$(V,E)$:

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S <- ϕ
WHILE E ≠ ϕ:
    PICK e = {w, v} ∈ E
    S <- S ∪ {w, v}
    REMOVE ALL THE EDGES INCIDENT ON w, OR ON v, FROM E
RETURN S.
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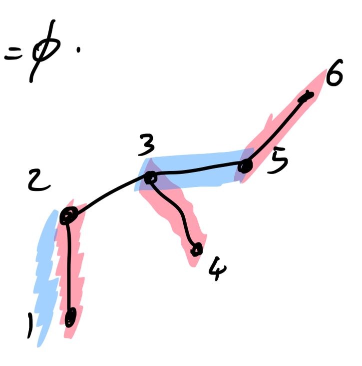
THM: MAXIMAL MATCHING RETURNS A 2-APPRX TO VERTEX COVER. (IF $S$ IS THE SOLUTION RETURNED BY MAXIMAL MATCHING AND IF $S^*$ IS THE OPTIMAL SOLUTION, $|S| \leq 2|S^*|$).

DEF: A MATCHING OF $G(V,E)$ IS A SUBSET $A \subseteq E$ S.T. $\forall \{e,e'\} \in \binom{A}{2}$ : $e \cap e' = \phi$.

$M = \{\{1,2\},\{3,4\}\}$ NOT A MAXIMAL MATCHING



DEF: A MAXIMAL MATCHING OF $G(V,E)$ IS A SUBSET $A \subseteq E$ THAT (a) IS A MATCHING, AND (b) SUCH THAT $\forall e \in E - A$, $A \cup \{e\}$ IS NOT A MATCHING.

L1: IF $e_1, e_2, \dots, e_t$ ARE THE EDGES SELECTED BY MAXIMAL MATCHING, THEN $\{e_1, e_2, \dots, e_t\}$ IS A MAXIMAL MATCHING.

P: EXERCISE.

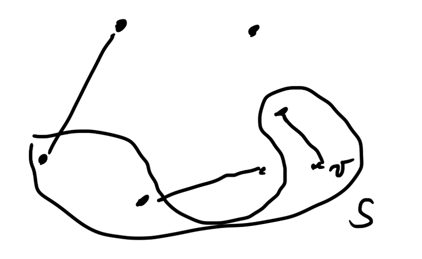
L2: IF MAXIMAL MATCHING SELECTS $t$ EDGES, THEN $|S| = 2t$.

P: TRIVIAL. ($\{e_1, \dots, e_t\}$ IS A MATCHING).

L3: LET $A$ BE ANY MATCHING OF $G(V,E)$. THEN, IF $S$ IS A VERTEX COVER OF $G(V,E)$, $|S| \geq |A|$.

P: A VERTEX COVER FOR $G(V,E)$ IS ALSO A " " FOR $G(V,B)$ FOR EACH $B \subseteq E$.

NOW, $A \subseteq E$. THUS, THE VERTEX COVER $S$ MUST COVER EACH EDGE IN $A$ (IT IS, INDEED, A VERTEX COVER FOR $G(V,A)$).



$G(V,A)$

IN $G(V,A)$ NO NODE HAS DEGREE LARGER THAN 1 (FOR OTHERWISE $A$ WOULD NOT BE A MATCHING).

ANY NODE $v \in V$ CAN COVER AT MOST ONE EDGE OF $G(V,A)$, SINCE $\deg_{G(V,A)}(v) \leq 1$.

THUS, A SET OF, SAY, $K$ NODES CAN COVER AT MOST $K$ EDGES OF $G(V,A)$.

NOW, $G(V,A)$ HAS $|A|$ EDGES — THUS, EACH VERTEX COVER OF $G(V,A)$ HAS TO HAVE AT LEAST $|A|$ NODES.

T: MAXIMAL MATCHING RETURNS A 2-APPROXIMATION TO VERTEX COVER

P: LET $A$ IS THE MAXIMAL MATCHING PRODUCED BY MAXIMAL MATCHING (IN L1, WE DENOTED IT AS $\{e_1, e_2, \dots, e_t\} = A$).

THEN, THE SOLUTION $S$ RETURNED BY MAXIMAL MATCHING CONTAINS $|S| = 2|A|$. (L2).

GIVEN THAT $A$ IS A MATCHING, IF $S^*$ IS AN OPTIMAL (SMALLEST) VERTEX COVER, $|S^*| \geq |A|$.

$$|S^*| \geq |A| = \frac{|S|}{2}$$

$$|S| \leq 2|S^*|. \quad \square$$

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VC IS CONJECTURED TO BE NP-HARD TO APPROXIMATE TO $2-\varepsilon$ ($\forall$ CONSTANT $\varepsilon > 0$).

VERTEX COVER IS FIXED-PARAMETER TRACTABLE.

IF $G(V,E)$ HAS A VERTEX COVER OF $K$ NODES, THEN AN OPTIMAL " " OF $G(V,E)$ CAN BE FOUND IN TIME $O(n^2 \cdot 2^K)$

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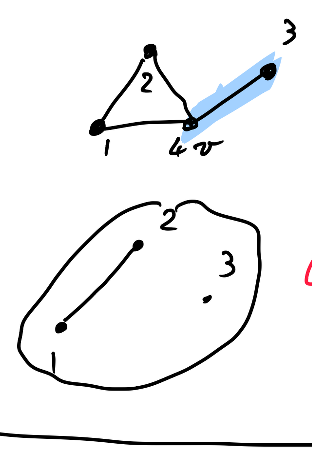
VC(G(V,E), K):                                  (K = ℓ+1)
    IF E = ϕ:
        RETURN TRUE
    ELSE IF K = 0:
        RETURN FALSE
    ELSE:
        FIX {w, v} ∈ E.
        IF VC(G[V-{w}], K-1):
            RETURN TRUE
        ELSE IF VC(G[V-{v}], K-1):
            RETURN TRUE
        ELSE:
            RETURN FALSE
```

$G(V,E)$ IS A GRAPH

$G[S]$ (SUBGRAPH OF $G$ INDUCED BY $S$), FOR $S \subseteq V$, IS THE GRAPH HAVING $S$ AS ITS SET OF VERTICES AND $\{e \mid e \in E \wedge e \subseteq S\}$.



$G(V,E)$

$G[S]$

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L1: $VC(G(V,E),K)$ TAKES TIME $O(n^2 2^K)$

L2: $VC(G(V,E),K) == \text{TRUE}$ IFF $G(V,E)$ HAS A VC OF SIZE $K$.

P OF L1:

BY INDUCTION: "RUNNING $VC(G(V,E),\ell)$ CAUSES AT MOST $2^{\ell+1}-1$ CALLS TO THE FUNCTION $VC(\dots)$"

BASE CASE ($\ell = 0$): ONLY THE FIRST CALL TO $VC(\dots,0)$ WILL BE GENERATED.

$$2^{\ell+1}-1 = 2^{0+1}-1 = 2-1 = 1, \quad \checkmark$$

IND. STEP ($\ell+1$): WE ASSUME THAT THE CLAIM IS TRUE FOR $\ell$, AND WE PROVE IT FOR $\ell+1$.

THE NUMBER OF CALLS GENERATED BY $VC(\dots,\ell+1)$ IS NO MORE THAN TWICE THE NUMBER OF CALLS GENERATED BY $VC(\dots,\ell)$ PLUS 1

THE TOTAL NUMBER OF CALLS, BY INDUCTION, IS THEN

$$\leq 1 + 2 \cdot (2^{\ell+1}-1)$$
$$= 1 + 2^{\ell+2} - 2 = 2^{\ell+2} - 1. \quad \checkmark$$

SINCE EACH SINGLE CALL TAKES TIME AT MOST $O(n^2)$, THE CLAIM FOLLOWS. $\square$

P OF L2: EXERCISE.