1. OFFLINE / ONLINE ALGORITHMS
2. DETERMINISTIC / RANDOMIZED / (QUANTUM) ALGORITHMS
3. * WORST-CASE INPUT / AVERAGE-CASE " ALGOS ✓
4. SINGLE PROCESSOR / PARALLEL /DISTRIBUTED ALGOS

IF S IS A SET,

$\binom{S}{k} = \{T \mid T \subseteq S \land |T| = k\}$

$\left|\binom{S}{k}\right| = \frac{|S|!}{k!(|S|-k)!} = \binom{|S|}{k}$

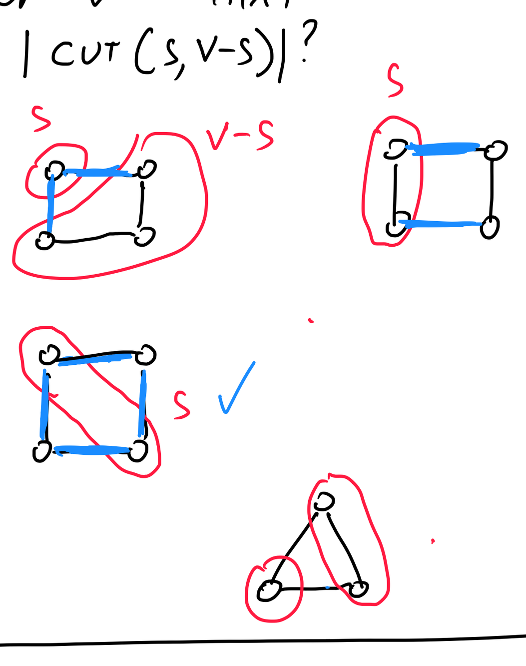
THE MAX-CUT PROBLEM

INPUT: UNDIRECTED GRAPH $G(V,E)$ $\quad \left(E \subseteq \{\{v,w\} \mid v,w \in V \land v \neq w\} = \binom{V}{2}\right)$

Q: WHAT IS THE BIPARTITION $(S, V-S)$ OF V THAT MAXIMIZES $|CUT(S, V-S)|$?

$CUT(S,V-S) = \{e \mid e \in E \land e \cap S \neq \emptyset \land e \cap (V-S) \neq \emptyset\}$

$= \{e \mid e \in E \land |e \cap S| = 1\}$

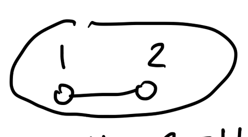


THE MAX-CUT PROBLEM IS NP-HARD

A $(0{,}878\ldots)$-APPROXIMATION EXISTS. (THIS IS BASED ON SDP).

TODAY, WE'RE GOING TO CONSIDER A "$\frac{1}{2}$-APPRX"

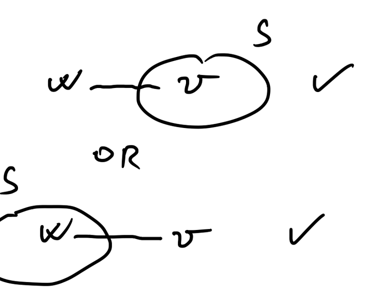
RANDOM-CUT(V, E):
- $S \leftarrow \emptyset$
- FOR EACH $v \in V$:
  - FLIP AN INDEPENDENT/FAIR COIN C_v
  - IF C_v IS HEADS:
    - $S \leftarrow S \cup \{v\}$
- RETURN $(S, V-S)$



$\frac{1}{4}$	$C_1 = H$	$C_2 = H$	$S = \{1,2\}$	0	EDGES	CUT
$\frac{1}{4}$	$C_1 = T$	$C_2 = T$	$S = \emptyset$	0	"	"
$\frac{1}{4}$	$C_1 = H$	$C_2 = T$	$S = \{1\}$	1	"	"
$\frac{1}{4}$	$C_1 = T$	$C_2 = H$	$S = \{2\}$	1	"	"

THM: IF S^* IS AN OPTIMAL SOLUTION TO MAX-CUT ON $G(V,E)$, AND IF S IS THE SET RETURN BY RANDOM-CUT, $E[|CUT(S,V-S)|] \geq \frac{1}{2}|CUT(S^*, V-S^*)|$

L1: LET S BE THE SET RETURNED BY RANDOM-CUT. THEN, $\forall e \in E$, $\Pr\{e \in CUT(S,V-S)\} = \frac{1}{2}$



P: $e \in CUT(S,V-S)$ IFF $|S \cap e| = 1$

IF $e = \{v,w\}$, $e \in CUT(S, V-S)$

IFF $((v \in S \land w \notin S) \lor (v \notin S \land w \in S))$

LET $\xi_1 :=$ "$v \in S \land w \notin S$" AND $\xi_2 =$ "$v \notin S \land w \in S$".

$\Pr\{\xi_1\} = \Pr\{C_v = H \land C_w = T\} \overset{\text{BY IND. OF } C_v, C_w}{=} \Pr\{C_v = H\} \cdot \Pr\{C_w = T\}$

$= \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

$\Pr\{\xi_2\} = \Pr\{C_v = T \land C_w = H\} = \frac{1}{4}$

$\Pr\{e \in CUT(S,V-S)\} = \Pr\{\xi_1 \lor \xi_2\} = \underbrace{\Pr\{\xi_1\}}_{=\frac{1}{4}} + \underbrace{\Pr\{\xi_2\}}_{=\frac{1}{4}} - \underbrace{\Pr\{\xi_1 \land \xi_2\}}_{=0}$

$= \frac{1}{4} + \frac{1}{4} + 0 = \frac{1}{2}$. ∎

L2: $\forall T \subseteq V$, $CUT(T, V-T) \subseteq E$, $|CUT(T, V-T)| \leq |E|$.

P: TRIVIAL

COR 3: LET S BE THE SET OF NODES RETURNED BY RAND-CUT. $\mathbb{E}[|CUT(S,V-S)|] = \frac{|E|}{2}$. ∎

P: BY L1, $\forall e \in E$ $\Pr\{e \in CUT(S,V-S)\} = \frac{1}{2}$.

$\mathbb{E}[|CUT(S,V-S)|] \overset{\text{LIN. OF EXP.}}{=} \sum_{e \in E} \Pr\{e \in CUT(S,V-S)\} = \sum_{e \in E} \frac{1}{2} = \frac{|E|}{2}$. ∎

THE THEOREM FOLLOWS FROM L2 AND COR 3.

"TAIL INEQUALITIES"
1. MARKOV INEQ.
2. CHEBYSHEV INEQ.
3. CHERNOFF BOUND.

MARKOV INEQ. (THM): IF Y IS A NON-NEG. RANDOM VARIABLE, THEN $\forall c \geq 1$, $\Pr\{Y \geq c \cdot E[Y]\} \leq \frac{1}{c}$.

P: LET ξ BE AN EVENT, AND LET
$X_\xi = \begin{cases} 1 & \text{IF } \xi \text{ HAPPENS,} \\ 0 & \text{O/W.} \end{cases}$

FOR $e > 0$, CONSIDER $X_{Y \geq e} = \begin{cases} 1 & \text{IF } Y \geq e, \\ 0 & \text{O/W.} \end{cases}$

THEN, $\Pr\{e \cdot X_{Y \geq e} \leq Y\} = 1$

NOW, $\mathbb{E}[e\, X_{Y \geq e}] \leq \mathbb{E}[Y]$

$\mathbb{E}[e\, X_{Y \geq e}] = e \cdot \mathbb{E}[X_{Y \geq e}] = e \cdot \Pr\{Y \geq e\}$

$e \cdot \Pr\{Y \geq e\} \leq \mathbb{E}[Y]$

$\Pr\{Y \geq e\} \leq \frac{\mathbb{E}[Y]}{e}$ ✓

LET US CHOOSE $e = c \cdot \mathbb{E}[Y]$. THEN,

$\Pr\{Y \geq c \cdot \mathbb{E}[Y]\} \leq \frac{\cancel{\mathbb{E}[Y]}}{c \cdot \cancel{\mathbb{E}[Y]}} = \frac{1}{c}$. ∎

(ALG.)
- FOR $i = 1$ TO t
  - RUN RANDOM-CUT(V, E) INDEPENDENTLY
  - LET $S_i, V-S_i$ BE THE RESULTING CUT
  - LET $C_i = |CUT(S_i, V-S_i)|$
- RETURN A LARGEST CUT.

FOR $i \in [t] = \{1, 2, \ldots, t\}$, LET $N_i = |E| - C_i$.

THEN, $N_i \geq 0$. LET $0 < \varepsilon < 1$,

$\Pr\{N_i \geq (1+\varepsilon)\mathbb{E}[N_i]\} \overset{\text{MARKOV}}{\leq} \frac{1}{1+\varepsilon} = \frac{1+\varepsilon}{1+\varepsilon} - \frac{\varepsilon}{1+\varepsilon} = 1 - \frac{\varepsilon}{1+\varepsilon} \leq 1 - \frac{\varepsilon}{2}$

$= \Pr\{|E| - C_i \geq (1+\varepsilon)(|E| - \mathbb{E}[C_i])\}$

$= \Pr\{|E| - (1+\varepsilon)|E| \geq C_i - (1+\varepsilon)\mathbb{E}[C_i]\}$

$= \Pr\{-\varepsilon|E| \geq C_i - (1+\varepsilon)\mathbb{E}[C_i]\}$ $\qquad |E| = 2\mathbb{E}[C_i]$

$= \Pr\{-\cancel{2}\varepsilon\mathbb{E}[C_i] + (1+\cancel{\varepsilon})\mathbb{E}[C_i] \geq C_i\}$

$= \Pr\{C_i \leq (1-\varepsilon)\mathbb{E}[C_i]\}$

"BAD EVENT"

$\Pr\{C_i > (1-\varepsilon)\mathbb{E}[C_i]\} \geq \frac{\varepsilon}{2}$

"GOOD EVENT"

SUPPOSE THAT WE RUN RANDOM-CUT FOR $t = \left\lceil \frac{2}{\varepsilon} \ln \frac{1}{\delta} \right\rceil$ (δ WILL BE THE PROB. OF ERROR)

$\Pr\{\forall i \in [t] : C_i \leq (1-\varepsilon)\mathbb{E}[C_i]\} \overset{\text{BY IND. OF THE } t \text{ RUNS}}{=} \prod_{i=1}^{t} \Pr\{C_i \leq (1-\varepsilon)\mathbb{E}[C_i]\}$

$\leq \prod_{i=1}^{t}\left(1 - \frac{\varepsilon}{2}\right) = \left(1 - \frac{\varepsilon}{2}\right)^t$

$\leq \left(e^{-\frac{\varepsilon}{2}}\right)^t = e^{-\frac{\varepsilon t}{2}} \leq e^{-\frac{\cancel{\varepsilon}}{\cancel{2}} \frac{\cancel{2}}{\cancel{\varepsilon}} \ln\frac{1}{\delta}}$

$= e^{-\ln\frac{1}{\delta}} = \delta$. ∎

$\forall x \in [0,1) : 1 - x \leq e^{-x}$