VERTEX COVER

- 2-APPRX (MAXIMAL MATCHINGS)
- EXACT ALGORITHM $(O(n^2 2^k))$

MATHEMATICAL PROGRAMMING

LINEAR PROGRAMMING

SEMI-DEFINITE PROGRAMMING

MATHEMATICAL PROGRAM

VARIABLES: $x_1, \dots, x_n$ ($x_i \in \mathbb{R}$ or $x_i \in \mathbb{Q}$)

OBJECTIVE FUNCTION: $f(x_1, \dots, x_n)$ (MAXIMIZE f or MINIMIZE f) or

CONSTRAINT FUNCTIONS: $g_i(x_1 \dots x_n) \le b_i$ (or $g_i(x_1, \dots, x_n) \ge b_i$) FOR $i = 1, \dots, m$

IN A LINEAR PROGRAM, f IS LINEAR AND g_i IS LINEAR $(\forall g_i)$.

$$\begin{cases} \max x_1 + x_2 \\ x_1 \ge 3 \\ x_2 \ge 5 \\ x_1 \le 2.5 \end{cases}$$

NO SOLUTION EXISTS

$$\begin{cases} \max x_1 + x_2 \\ x_2 \ge 5 \\ x_1 \le 2.5 \end{cases}$$

THERE EXIST INFINITELY MANY SOLUTIONS AND THE "OPTIMAL VALUE" OF THIS LP IS UNBOUNDED (∞)

$$\begin{cases} \max x_1 + x_2 \\ 2x_1 + x_2 \le 10 \\ x_1 \ge 0 \\ x_2 \ge 0 \end{cases}$$

$x_1 = 0, x_2 = 10 \Rightarrow$ OBJ $= 10$

(DUALITY)

IF $f(x_1, \dots, x_n) = \sum_{j=1}^{n} c_j x_j$ AND $g_i(x_1, \dots, x_n) = \sum_{j=1}^{n} a_{ij} x_j$,
THEN IF EACH COEFFICIENT c_j, a_{ij} AND EACH TERM b_i, IS A RATIONAL NUMBER REPRESENTABLE WITH AT MOST t BITS, THEN THE LINEAR PROGRAM CAN BE OPTIMIZED IN TIME $O((n \cdot m \cdot t)^c)$ FOR SOME CONSTANT $c > 0$.

(DOES THERE EXIST AN $O((n \cdot m)^c\, t)$ ALGO FOR LP SOLVING?)

VERTEX COVER ON $G(V, E)$

- VARIABLES: $\{x_v \mid v \in V\}$

$$\begin{cases} \min \sum_{v \in V} x_v \\ x_u + x_v \ge 1 & \forall \{u, v\} \in E \\ x_v \in \{0, 1\} & \forall v \in V. \end{cases}$$

(LINEAR) INTEGRAL PROGRAM

BY SOLVING THIS IP (INTEGRAL PROGRAM), WE'D OBTAIN AN OPTIMAL VC.

LET $x^*_{v_1}, x^*_{v_2}, \dots, x^*_{v_n}$ BE AN OPTIMAL SOLUTION TO THE IP.

DEFINE $S^* = \{v \mid v \in V \land x^*_v = 1\}$

THM: S^* IS AN OPTIMAL VC OF $G(V, E)$.

L: S^* IS A VERTEX COVER.

P: S^* BEING " " " MEANS THAT $\forall \{u, v\} \in E$ $u \in S^*$ OR $v \in S^*$ (OR $u, v \in S^*$).

GIVEN THAT $x^*_u + x^*_v \ge 1$ FOR ALL $\{u, v\} \in E$, AND THAT $x^*_u, x^*_v \in \{0, 1\}$, AT LEAST ONE OF x^*_u AND x^*_v IS EQUAL TO 1. THUS $u \in S^*$ OR $v \in S^*$. □

L: IF S IS A VERTEX COVER, THEN $\exists$ FEASIBLE (IT SATISFIES ALL THE CONSTRAINTS) SOLUTION $\{x_v\}_{v \in V}$ TO THE IP, SUCH THAT $\sum_{v \in V} x_v = |S|$.

P: $x_v = 1 \Leftrightarrow v \in S$.

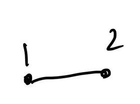
COR: THE OPTIMAL SOLUTION TO THE IP HAS A VALUE EQUAL TO $\min_{\substack{S \subseteq V \\ S \text{ is a VC}}} |S|$.

IPs CANNOT GENERALLY BE SOLVED IN POLY TIME, WHILE - AS WE SAW - LPs CAN.
THUS, WE RELAX THE IP TO A LP.

$$\underbrace{\begin{cases} \min \sum_{v \in V} x_v \\ x_u + x_v \ge 1 & \forall \{u, v\} \in E \\ x_v \in \{0, 1\} & \forall v \in V. \end{cases}}_{IP} \xRightarrow{\text{RELAX}} \underbrace{\begin{cases} \min \sum_{v \in V} x_v \\ x_u + x_v \ge 1 & \forall \{u, v\} \in E \\ 0 \le x_v \le 1 & \forall v \in V. \end{cases}}_{LP}$$

$0 \le x_v \le 1 \Leftrightarrow \begin{cases} x_v \le 1 \\ -x_v \le 0 \end{cases}$

HOW TO CREATE A VC FROM A LP SOLUTION?



$x_1 = x_2 = \frac{1}{2}$

$S = \{v \mid x_v \ge \frac{1}{2} \land v \in V\}$ "ROUNDING RULE"

L: S IS A VC.

P: LET $\{u, v\} \in E$. THE LP CONTAINS THE CONSTRAINT $x_u + x_v \ge 1$.

GIVEN THAT OUR SOLUTION THEN GUARANTEES THAT $x_u + x_v \ge 1$ (THE LP SOLUTION IS FEASIBLE BY DEFINITION)

$x_u + x_v \ge 1 \Rightarrow \max(x_u, x_v) \ge \frac{x_u + x_v}{2} \ge \frac{1}{2}$.

THUS $u \in S$ OR $v \in S$ (OR $u, v \in S$) — THE EDGE $\{u, v\}$ IS COVERED. □

L2: $|S| \le 2 \sum_{v \in V} x_v$.

P: $|S| = \sum_{v \in S} 1 \le \sum_{v \in S} (2 x_v) = 2 \sum_{v \in S} x_v \le 2 \sum_{v \in V} x_v$. □

$v \in S \Rightarrow x_v \ge \frac{1}{2}$

$x_v \ge 0 \;\; \forall v \in V$

RECALL THAT THE LP MINIMIZES $\sum_{v \in V} x_v$.

LET $\{x^*_v\}_{v \in V}$ BE AN OPTIMAL SOLUTION TO THE LP.

LET S^* BE THE RESULT OF THE APPLICATION OF OUR ROUNDING RULE, $S^* = \{v \mid v \in V \land x^*_v \ge \frac{1}{2}\}$.

$$|S^*| \overset{L2}{\le} 2 \sum_{v \in V} x^*_v = 2\, LP^* \le 2\, IP^* = 2 \min_{\substack{S \subseteq V \\ S \text{ is a VC}}} |S| = 2 \cdot OPT.$$

THE LP IS A RELAXATION OF THE IP

IF L2 COULD BE IMPROVED (TO, SAY, $|S| \le 1.9 \sum_{v \in V} x_v$), THEN THE APPROXIMATION RATIO WOULD BE DIRECTLY IMPROVED.

$G(V, E)$

L: EACH OPTIMAL VC FOR $G(V, E)$ CONTAINS TWO NODES.

$$\begin{cases} \min \sum_{v \in V} x_v \\ x_u + x_v \ge 1 & \forall \{u, v\} \in E \\ 0 \le x_v \le 1 & \forall v \in V. \end{cases}$$

LP

$x_1 + x_2 + x_3 = 2$
$x_1 = x_2 = 1$
$x_3 = 0$

$2: x_1 + x_2 \ge 1$
$1: x_1 + x_3 \ge 1$
$1: x_2 + x_3 \ge 1$

L: $x_1 = x_2 = x_3 = \frac{1}{2}$ IS A FEASIBLE SOLUTION OF VALUE $\frac{3}{2}$.

P: $\forall \{i, j\} \in E$, $x_i + x_j = 2 \cdot \frac{1}{2} = 1 \ge 1$, THUS EACH CONSTRAINT IS SATISFIED.

THE OBJ. VALUE IS $3 \cdot \frac{1}{2} = \frac{3}{2}$. □

L: THERE EXISTS NO FEASIBLE LP SOLUTION OF VALUE $< \frac{3}{2}$.

P: FOR $\{i, j\} \in E$, $x_i + x_j \ge 1$.

$$2 \sum_{v \in V} x_v = 2x_1 + 2x_2 + 2x_3 = (x_1 + x_2) + (x_2 + x_3) + (x_1 + x_3) \ge 1 + 1 + 1 = 3$$

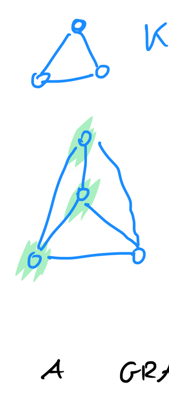
BY FEASIBILITY $(x_i + x_j \ge 1)$

THUS, $\sum_{v \in V} x_v \ge \frac{3}{2}$. □

COR: $LP^* = \frac{3}{2}$

$IP^* = 2$

THUS, FOR △, $\frac{IP^*}{LP^*} = \frac{2}{3/2} = \frac{4}{3}$ INTEGRALITY GAP



K3

$\boxed{\frac{IP^*}{LP^*} \le 2}$ ✓

EX: SHOW THAT $\forall \varepsilon > 0$, THERE EXISTS A GRAPH $G(V, E)$ SUCH THAT $\frac{OPT_{VC}(G(V,E))}{LP^*(G(V,E))} \ge 2 - \varepsilon$.

HINT 1: (K_t, LARGE t)

HINT 2: PROVE THAT THE OPTIMAL VC FOR K_t CONTAINS $t-1$ NODES.

HINT 3: WHAT IS THE OPTIMAL LP SOLUTION FOR K_t?