DENSEST SUBGRAPH

$$\begin{cases} \max \sum_{\{i,j\}\in E} x_{\{i,j\}} \\ x_{\{i,j\}} \le x_i & \forall \{i,j\}\in E \\ x_{\{i,j\}} \le x_j & \forall \{i,j\}\in E \\ \sum_{i\in V} x_i \le 1 \\ x_{\{i,j\}} \ge 0 & \forall \{i,j\}\in E \\ x_i \ge 0 & \forall i\in V \end{cases} \iff$$

"STANDARD FORM"

PRIMAL LP

$$\begin{cases} \max \sum_{\{i,j\}\in E} x_{\{i,j\}} \\ y_{ij}: \ x_{\{i,j\}} - x_i \le 0 & \forall \{i,j\}\in E \\ y_{ji}: \ x_{\{i,j\}} - x_j \le 0 & \forall \{i,j\}\in E \\ y^*: \ \sum_{i\in V} x_i \le 1 \\ x_{\{i,j\}} \ge 0 & \forall \{i,j\}\in E \\ x_i \ge 0 & \forall i\in V \end{cases}$$

DUAL LP

$$\begin{cases} \min y^* \\ x_{\{i,j\}}: \ y_{ij} + y_{ji} \ge 1 \\ x_i: \ y^* - \sum_{j:\{i,j\}\in E} y_{ij} \ge 0 \\ y^* \ge 0 \\ y_{ij}, y_{ji} \ge 0 & \forall \{i,j\}\in E \end{cases}$$

OUR GREEDY 2-APPRX FOR DS IS —ESSENTIALLY— A DUAL LP SOLUTION:

$$y^* = \Delta_{\vec{\Phi}_{GR}} \qquad y_{ij} = \begin{cases} 1 & \text{IF } \{i,j\} \text{ IS DIRECTED TOWARDS } i \text{ IN } \vec{\Phi}_{GR} \\ 0 & \text{O/W.} \end{cases}$$

$x_{\{i,j\}}$: $y_{ij} + y_{ji} \ge 1$ ✓

$1 + 0 \ge 1$ OR $0 + 1 \ge 1$

x_i: $y^* \overset{?}{\ge} \sum_{j:\{i,j\}\in E} y_{ij}$

$$\Delta_{\vec{\Phi}_{GR}} = y^* \overset{\checkmark}{\ge} \sum_{j:\{i,j\}\in E} y_{ij} = \deg_{\vec{\Phi}_{GR}}(i)$$

THEN, EACH CONSTRAINT IS SATISFIED BY OUR SOLUTION.

THE VALUE OF OUR SOLUTION IS $\Delta_{\vec{\Phi}_{GR}}$.

$$\frac{\Delta_{\vec{\Phi}_{GR}}}{2} \le \text{GREEDY SOL} \le \text{PRIMAL}^* \le \text{DUAL}^* \le \Delta_{\vec{\Phi}_{GR}}$$

VERTEX COVER IS A MINIMIZATION PROBLEM (DS IS A MAXIMIZATION PROBLEM)

DUAL
$$\begin{cases} \min \sum_{v\in V} x_v \\ y_{\{u,v\}}: \ x_u + x_v \ge 1 & \forall \{u,v\}\in E \\ x_v \ge 0 & \forall v\in V \end{cases}$$

PRIMAL | DUAL OF THE DUAL
$$\begin{cases} \max \sum_{\{u,v\}\in E} y_{\{u,v\}} \\ x_v: \ \sum_{u:\{u,v\}\in E} y_{\{u,v\}} \le 1 \\ y_{\{u,v\}} \ge 0 & \forall \{u,v\}\in E \end{cases}$$

$\forall$ MAXIMAL MATCHING $M \subseteq E$

- FOR THE DUAL, WE SET $x_u = \begin{cases} 1 & \text{IF } \exists e \in M \text{ ST. } u \in e \\ 0 & \text{O/W} \end{cases}$

  IS THIS DUAL SOL. FEASIBLE?

  YES, BY CONTRADICTION IF $x_u + x_v < 1$, FOR SOME $\{u,v\}\in E$, THEN WE COULD ADD THE EDGE $\{u,v\}$ TO M — M, THUS, IS NOT A MAXIMAL MATCHING.

- WHAT IS THE VALUE OF THIS SOLUTION? IT IS $\sum_{v\in V} x_v = 2|M|$.

- FOR THE PRIMAL, WE SET $y_{\{u,v\}} = \begin{cases} 1 & \text{IF } \{u,v\}\in M \\ 0 & \text{O/W.} \end{cases}$

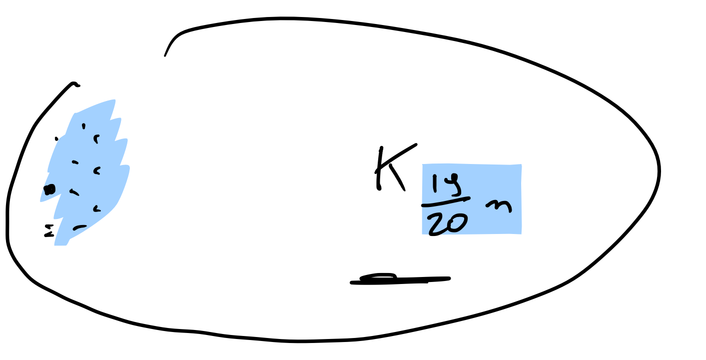
  (IN THE ALGORITHM, IF $\exists$ AN EDGE e, THE TWO ENDPOINTS OF e END UP IN THE SOLUTION, AND ARE THEN REMOVED FROM THE GRAPH.)

  THE SOLUTION IS FEASIBLE BECAUSE M IS A MATCHING.

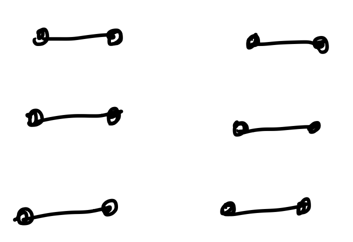
$$|M| \le \text{PRIMAL}^* \le \text{DUAL}^* \le 2|M|.$$

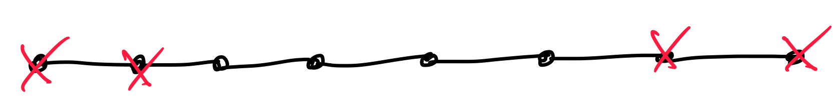
GREEDY $(G(V,E))$:
- $S_0 \leftarrow V$
- $i \leftarrow 0$
- WHILE $S_i \ne \emptyset$:
  - LET v_i BE A NODE OF MINIMUM DEGREE IN $G[S_i]$
  - $S_{i+1} \leftarrow S_i - \{v_i\}$
  - $i \leftarrow i+1$
- RETURN A S_i MAXIMIZING $\rho(S_i) = \frac{|E(S_i)|}{|S_i|}$.

THIS ALGORITHM REQUIRES n ITERATIONS.

$\frac{n}{20}$  $K_{\frac{19}{20}n}$ ✗

GREEDY1 $(G(V,E))$:
- $S_0 \leftarrow V$
- $i \leftarrow 0$
- WHILE $S_i \ne \emptyset$:
  - $A_i \leftarrow \{v \mid v \in S_i \wedge \deg_{S_i}(v) = \min_{w\in S_i} \deg_{S_i}(w)\}$
  - $S_{i+1} \leftarrow S_i - A_i$
  - $i \leftarrow i+1$
- RETURN A S_i MAXIMIZING $\rho(S_i) = \frac{|E(S_i)|}{|S_i|}$.

 ✓ (1 ITERATION)

 $(\approx \frac{n}{2}$ ITERAT.)