MAX-CUT

$n = |V|$

$$QP: \begin{cases} \max \sum_{\{i,j\}\in E} \frac{1-x_i \cdot x_j}{2} & \\ x_i^2 = 1 & \forall i \in V \\ x_i \in \mathbb{R} & \forall i \in V \end{cases} \implies SDP: \begin{cases} \max \sum_{\{i,j\}\in E} \frac{1-\underline{v}_i \cdot \underline{v}_j}{2} & \\ \underline{v}_i \cdot \underline{v}_i = 1 & \forall i \in V \\ \underline{v}_i \in \mathbb{R}^n & \forall i \in V \end{cases}$$

L: $QP^* = \overset{\text{MAX-CUT'S}}{OPT}$

L: $SDP^* \geq QP^* (= OPT)$

P: LET $x_1, x_2, \dots, x_n$ BE A FEASIBLE SOLUTION TO THE QP.

SET $\underline{v}_i = \begin{pmatrix} x_i \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $\forall i = 1 \dots n$.

WE PROVE:

- THAT THE SDP SOLUTION IS FEASIBLE.

$\forall i \in [n]$, $1 \geq \underline{v}_i \cdot \underline{v}_i = \sum_{j=1}^{n} \underline{v}_i(j) \cdot \underline{v}_i(j) = \sum_{j=1}^{n} \underline{v}_i(j)^2 = \underline{v}_i(1)^2 = x_i^2 = 1$

- THAT THE VALUE OF THE SDP SOLUTION IS NOT SMALLER THAN THE VALUE OF THE QP SOLUTION:

$$\sum_{\{i,j\}\in E} \frac{1-\underline{v}_i \cdot \underline{v}_j}{2} = \sum_{\{i,j\}\in E} \frac{1-x_i x_j}{2},$$

THAT IS, THEY ARE ACTUALLY EQUAL. □

THEN, WE HAVE $SDP^* \geq \overset{\text{MAX-CUT'S}}{OPT}$.
CAN WE BOUND THE IG? CAN WE FIND A CONSTANT c S.T. $OPT \geq c \cdot SDP^*$?

GOEMANS - WILLIAMSON'S (ROUNDING) ALGORITHM

$GW(G(V,E))$:

- SOLVE THE $\overset{\text{MAX-CUT}}{SDP}$, OBTAINING VECTORS $\underline{v}_1, \dots, \underline{v}_n$ (FOR $n = |V|$)
- SAMPLE A UNIFORM-AT-RANDOM VECTOR $\underline{y}$ ON THE n-DIMENSIONAL HYPERSPHERE $H_n = \{ \underline{x} \mid \|\underline{x}\|_2 = 1, \underline{x} \in \mathbb{R}^n \}$.
- $S \leftarrow \{ i \mid \underline{v}_i \cdot \underline{y} \geq 0 \}$
- RETURN $(S, V-S)$.

$\underline{z} = \begin{pmatrix} N(0,1) \\ N(0,1) \\ \vdots \\ N(0,1) \end{pmatrix}$

$\boxed{\underline{y} = \frac{1}{\|\underline{z}\|_2} \underline{z}}$ IS UAR IN H_n

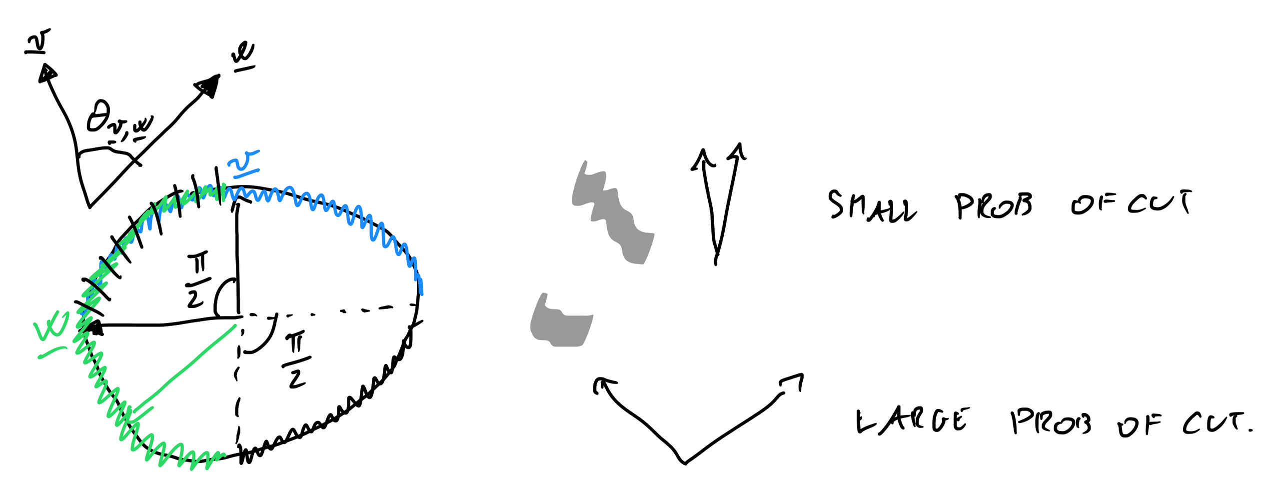
$\underline{v} \cdot \underline{w} = \sum_{t=1}^{n} \underline{v}(t)\, \underline{w}(t)$, FOR $\underline{v}, \underline{w} \in \mathbb{R}^n$

NOW, SUPPOSE THAT $\|\underline{v}\|_2 = \|\underline{w}\|_2 = 1$. THEN,

- $\|\underline{v}\|_2^2 = \|\underline{w}\|_2^2 = 1^2 = 1$
- $\underline{v} \cdot \underline{v} = \underline{w} \cdot \underline{w} = 1$
- $\underline{v} \cdot \underline{w} = \cos(\theta_{\underline{v},\underline{w}})$ (COSINE SIMILARITY)

$\|\underline{v}\|_2 = \sqrt{\sum_t \underline{v}(t)^2} = \sqrt{\underline{v} \cdot \underline{v}}$

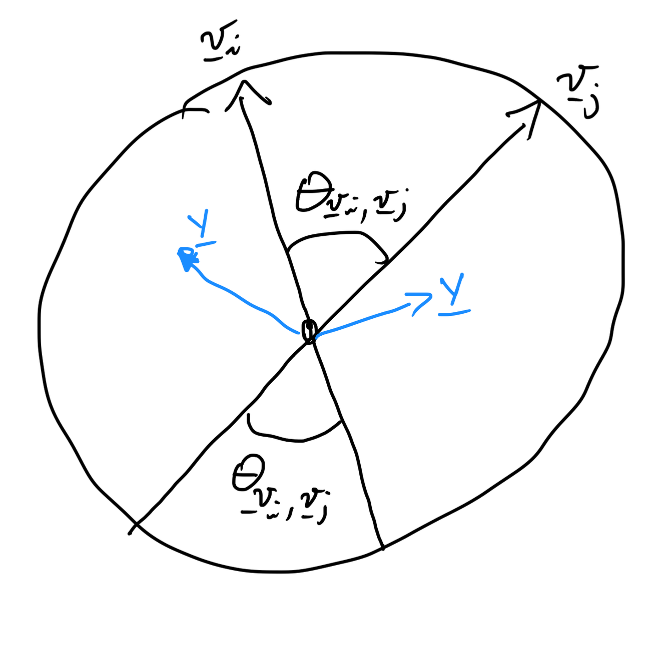
$\|\underline{v}\|_2^2 = \underline{v} \cdot \underline{v}$



L1: LET $\{i,j\} \in E$. THEN, $\Pr\{ |\{i,j\} \cap S| = 1 \} = \Pr\{ \{i,j\} \text{ IS CUT} \} = \frac{\theta_{\underline{v}_i, \underline{v}_j}}{\pi}$

P(SKETCH): CONSIDER A "PLANE" (A 2-DIMENSIONAL FLAT) PASSING THROUGH THE ORIGIN $\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $\underline{v}_i$ AND $\underline{v}_j$.

THE HYPERSPHERE (WHICH IS CENTERED ON $\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$) WILL BE CUT BY THIS PLANE AS FOLLOWS:



IN PARTICULAR THE ANGLE BETWEEN THE PROJECTIONS OF THE VECTORS REMAINS $\theta_{\underline{v}_i, \underline{v}_j}$.

SINCE $\underline{y}$ IS SAMPLED U-A-R ON THE HYPERSPHERE, THE PROJECTION OF $\underline{y}$ ON THE PLANE WILL BE A U-A-R VECTOR / DIRECTION FROM $\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$.

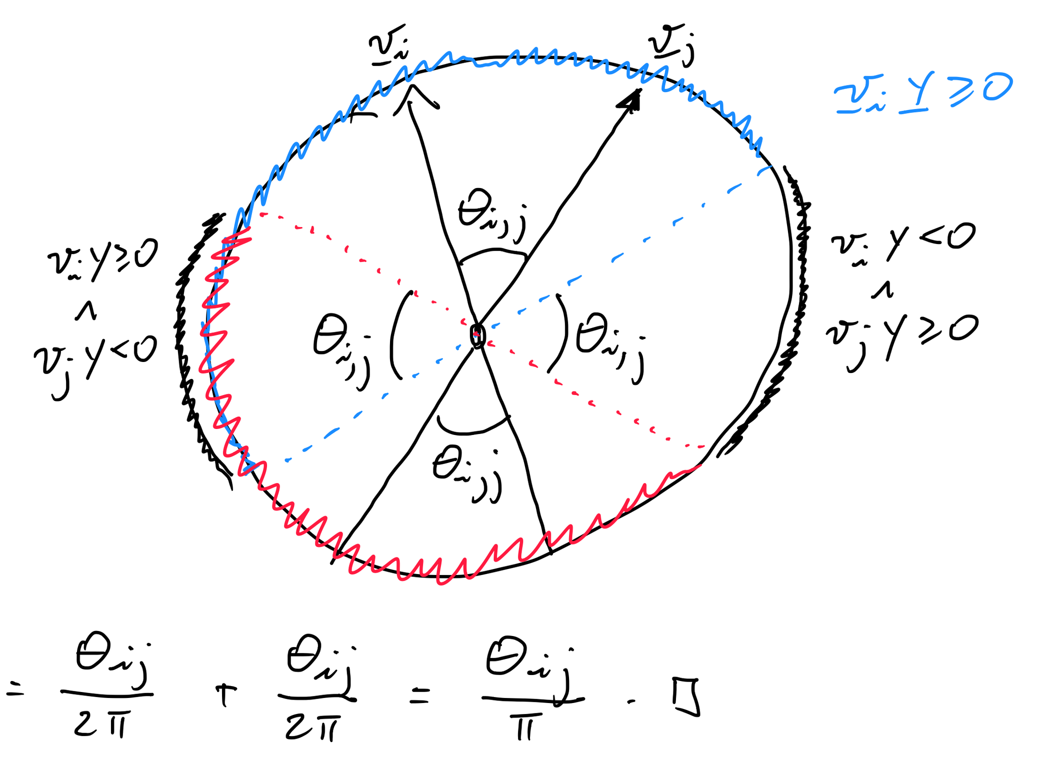
THEN,

$\Pr\{ \{i,j\} \text{ IS CUT} \} =$

$\Pr\{ (i \in S \wedge j \notin S) \vee (i \notin S \wedge j \in S) \} =$

$\Pr\{ i \in S \wedge j \notin S \} + \Pr\{ i \notin S \wedge j \in S \} =$

$\Pr\{ \underline{v}_i \cdot \underline{y} \geq 0 \wedge \underline{v}_j \cdot \underline{y} < 0 \} + \Pr\{ \underline{v}_i \cdot \underline{y} < 0 \wedge \underline{v}_j \cdot \underline{y} \geq 0 \} = ⊛$



$⊛ = \frac{\theta_{ij}}{2\pi} + \frac{\theta_{ij}}{2\pi} = \frac{\theta_{ij}}{\pi}$. □

(OBSERVE THAT G-W REDUCES TO OUR ORIGINAL UAR-CUT SOLUTION IF $\theta_{ij} = \frac{\pi}{2}$ $\forall \{i,j\} \in \binom{V}{2}$.
IN PARTICULAR, G-W USES THE SDP SOLUTION TO INDUCED CONTROLLED BIAS ON THE RANDOM-CUT ALGORITHM).

THM: IF $(S, V-S)$ IS THE CUT RETURNED BY G-W, THEN

$$\mathbb{E}[|CUT(S, V-S)|] \geq \alpha_{GW} \cdot SDP^* \quad (\geq \alpha_{GW} \cdot OPT)$$

FOR $\alpha_{GW} = \frac{2}{\pi} \min_{x \in (-1,1)} \frac{\arccos(x)}{1-x} \approx 0{,}878\dots$

P: $\mathbb{E}[|CUT(S, V-S)|] = \sum_{\{i,j\}\in E} \Pr\{ \{i,j\} \text{ IS CUT BY } S \}$

$\overset{(L1)}{=} \sum_{\{i,j\}\in E} \frac{\theta_{\underline{v}_i, \underline{v}_j}}{\pi}$ ($\underline{v}_i$ IS THE SDP VECTOR FOR NODE i)

$= \sum_{\{i,j\}\in E} \frac{\arccos(\underline{v}_i \cdot \underline{v}_j)}{\pi}$

NOW, SUPPOSE THAT α IS SUCH THAT

$$\frac{\arccos(x)}{\pi} \geq \alpha \left(\frac{1}{2} - \frac{x}{2} \right), \text{ FOR EACH } x \in (-1,1).$$

$\frac{1 - \underline{v}_i \cdot \underline{v}_j}{2}$

THEN,

$$\mathbb{E}[|CUT(S, V-S)|] \geq \alpha \cdot \sum_{\{i,j\}\in E} \frac{1 - \underline{v}_i \cdot \underline{v}_j}{2} = \alpha \cdot SDP^*.$$

WHAT IS THE LARGEST α FOR WHICH

$\frac{\arccos(x)}{\pi} \geq \alpha \left(\frac{1}{2} - \frac{x}{2} \right)$ HOLDS $\forall x \in (-1,1)$?

$\alpha \frac{1-x}{2} \leq \frac{\arccos(x)}{\pi}$ $\forall x \in (-1,1)$

$\alpha (1-x) \leq \frac{2}{\pi} \arccos(x)$ $\forall x \in (-1,1)$

$\alpha \leq \frac{2}{\pi} \frac{\arccos(x)}{1-x}$ $\forall x \in (-1,1)$

IF WE THEN SET $\alpha = \alpha_{GW} \triangleq \min_{x \in (-1,1)} \left(\frac{2}{\pi} \frac{\arccos(x)}{1-x} \right) \approx 0{,}878\dots$ □

$$0{,}878\dots = \alpha_{GW} \leq IG\left(\frac{\text{MAX-CUT}}{\text{SDP}} \right) \leq ?$$