CSC226 - Assignment 3 Solutions

1. How many ways are there to arrange the letters in the word "**probabilistic**"?

   There are 13 letters with 2 **b**'s and 3 **i**'s

   Number of ways to arrange letters = $\frac{13!}{2!3!}$ = 518918400

2. With reference to the previous problem, how many ways if all the **b**'s have to precede all the **i**'s?

   Because we are interested in solutions where the **b**'s precede the **i**'s, we can consider these characters to be the same. That is, we can replace the **b**'s and **i**'s with another character such as **x**. Thus we are interested in finding the number of ways to arrange "**proxaxxlxstxc**".

   There are 13 letters with 5 **x**'s

   Number of ways to arrange letters so that **b**'s precede **i**'s = $\frac{13!}{5!} = 51891840$

3. How many ways are there to put 100 (unlabelled) balls into 50 labelled boxes?

   $n = 100$
   $k = 50$

   $$\text{Number of ways} = \binom{n+k-1}{n} = \binom{149}{100}$$
   $$= 6{,}709{,}553{,}636{,}577{,}310{,}764{,}746{,}744{,}793{,}643{,}105{,}249{,}380$$

We computed the "mean time to failure" to be $1/p$ if the probability of failure is p. What is the mean time to the second failure? In a sense, the answer is obvious, but prove it from first principles. That is, write down an expression for q_k, the probability that the second failure occurs on the k-th trial, and then compute and simplify the weighted sum $\sum kq_k$.

HINT: Recall that in class we computed the mean time to failure by determining *pk*, the probability that the first failure occurs on the *k*-th trial, to be the quantity in Definition 19.4.6 of the MIT notes. We then simplified $\sum k\, p_k$, the expected number of trials, using the formula $(1-x)^{-2} = 1 + 2x + 3x^2 + \cdots$, which we obtained by differentiating the geometric series $(1-x)^{-1} = 1 + x + x^2 + \cdots$.

For the second failure after *k* trials, there must be 2 failures (each with probability *p*) and *k-2* successes (each with probability *(p-1)*). Also, there are *(k-1)* positions at which the first failure may occur. This gives the following result for q_k:

$$q_k = \text{Second failure on kth trial}$$
$$= (k-1)(1-p)^{k-2}p^2$$

$$E[x] = \sum_{k\geq 1} kq_k = p^2 \sum_{k\geq 1} k(k-1)x^{k-2} \;\; where\; x = 1-p$$

In lecture, we derived $\sum_{k\geq 1} kx^{k-1} = \frac{1}{(1-x)^2}$ by taking the derivative of the geometric series. Take the second derivative of the geometric series to obtain $\sum_{k\geq 1} k(k-1)x^{k-2} = \frac{2}{(1-x)^3}$. Substitute this result into $E[x]$ to obtain the answer:

$$E[x] = p^2 \sum_{k\geq 1} k(k-1)x^{k-2} = \frac{2p^2}{(1-x)^3} = \frac{2p^2}{(1-(1-p))^3} = \frac{2}{p}$$

4. What number do you get if you subtract the binomial coefficients $\binom{n}{k}$with an even *k* from those with an odd *k*, where *n* is fixed?

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k = 0$$

Proof: From the Binomial Theorem we have

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Let *x* = -1 and *y* = 1:

$$(-1+1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k (1)^{n-k}$$

$$0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^k$$

Alternate proofs can be found here:
https://proofwiki.org/wiki/Alternating_Sum_and_Difference_of_Binomial_Coefficients_for_Given_n

5. What difference of binomial coefficients is equal to the sum $\binom{12}{5} + \binom{11}{5} + \binom{10}{5} + \binom{9}{5} + \binom{8}{5}$?

$$\binom{12}{5} + \binom{11}{5} + \binom{10}{5} + \binom{9}{5} + \binom{8}{5} = \sum_{i=5}^{12} \binom{i}{5} - \sum_{i=5}^{7} \binom{i}{5}$$

From the Hockey Stick Identity for Pascal's triangle we have,

$$\sum_{i=r}^{n} \binom{i}{r} = \binom{n+1}{r+1}$$

Applying this identity to our sums, we get:

$$\binom{12}{5} + \binom{11}{5} + \binom{10}{5} + \binom{9}{5} + \binom{8}{5} = \binom{13}{6} - \binom{8}{6}$$

6. Imagine a maze created in an *m* by *n* grid. Assume that there is a *unique* path from any cell to any other cell. What is the total length of the walls in the maze as a function of *m* and *n*? For example, below is a 2 by 2 grid with the required path property and the total length of walls is 9. Explain your answer.

```
+---+---+
|       |
+---+   +
|       |
+---+---+
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The maximum number of walls in an *m* by *n* grid is *m(n+1) +n(m+1).*

Construct the dual graph for the *m* by *n* grid by placing a vertex at the center of each cell and connecting the vertices if there is a wall separating the two cells in the complete grid (see dashed red lines in Fig. 1).

Find a spanning tree for the dual graph (see yellow lines in Fig. 1). For each edge in the spanning tree, remove a wall in the original graph. This will create a maze with a *unique* path from any cell to any other cell.

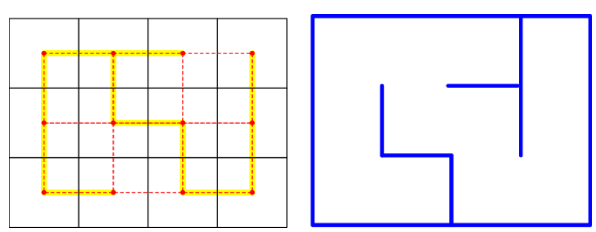

Figure 1: A 3 by 4 grid, its dual graph, a spanning tree and the resulting maze

The number of vertices in the dual graph is *mn*. The number of edges in the spanning tree is therefore *mn-1*. As a result:

Length of walls in the maze = *m(n+1) +n(m+1) –(mn-1) = (m+1)(n+1).*