THE UNIVERSITY *of* EDINBURGH
School of Physics
and Astronomy

Computation of $U(n)$ and $SU(N)$ Invariant Scalars

MPhys Project Report

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Abstract

An outline of an efficent algorithm for the computation of $U(n)$ and $SU(n)$ invariant scalars utilizing diagrammatic techniques and the representation theory of the symmetric group.

Supervisor: Professor A. D. Kennedy

Personal statement

I began my project by first becoming familar with the diagrammatic birdtrack notation. To do this I used Predrag Cvitanović's book in which the notation was first published. [1] This also involved becoming comfortable with more advanced representation theory than I was previously familar with. Throughout this period I met with my supervisor every week. We mostly discussed the theoretical aspects of the project and possible goals for the project.

At first there were two distinct paths the project could have taken. One option was to look into the graph theoretic aspect of the notation, and to try to write a program for the reduction of particulars kinds of diagrams into simpler ones. The second option was slightly more mathematical, and less computation oriented. This involved the calculation of $U(n)$ and $SU(n)$ invariant scalars using diagrammatic techniques in conjuction with the representation theory of the symmetric group, with the end goal of producing an efficent program to do this. In week four I decided to pursue the second option. I chose this option as the mathematics involved in it sounded very interesting to me.

Before I could attempt to produce a program I had to learn about several aspects of the symmetric group, namely the construction of irreducible representations of S_n using Specht modules and the Garnir relations. Initially I found it difficult to find literature on this subject which described well how to use this theory for practical calculation, but eventually I came across several useful sources. These included Fulton and Sagan's books on Young tableaux and their applications. [2] [3]

I began writing my program during week seven. The program style uses recursive algorithm extensivley. I had little expierience with programming of this type so it took a bit of time for me to get the hang of it. During my weekly meetings I mostly asked for supervisor for advice as to the structure of the program. I continued to develop my program for the rest of the semester. Although I occassionaly got stuck, as when trying to devise a method of generating all standard tableaux of a particular shape, my progress was mostly steady.

I had a working program by the start of January. I spent the first couple weeks of semester two writing tests for my program and improving certain aspects of it. After producing a satisfactory program I decided to spend the remainder of the semester looking in more detail at the mathematical background of the project and the posistion of representation theory in physics in general. I found books by Littlewood and Weyl particularly useful for these purposes. [4] [5] I also looked at some of the relevant contemporary literature in order to explore the potential for alternative methods, mainly the paper by Alcock-Zerilinger and Weiget on the construction of hermitian Young projection operators. Although I still had some technical troubles with the theory, the weekly discussions with my supervisor were generally broader in scope than the first semester meetings.

I began writing my report in week seven of semester two, and began learning how to use the program Jaxodraw, originally intended for Feynman diagrams but equally applicable to birdtrack notation. I had the bulk of the report finished by week ten and spent the remaining time rewriting earlier sections and redrawing earlier diagrams, now that I was more adept with the software.

Contents

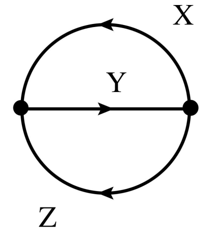
1 Introduction

Since the concept of Lie groups was first developed by mathematician Sophus Lie in the 1870's, they have become an integral part of theoretical physics. Lie originally studied these objects as part of an attempt to find an analougous Galois theory for partail differential equations, rather than polynomails, with a view to applications in classical mechanics. [6] Although of great importance in classical physics Lie groups found even greater utility with the advent of quantum mechanics at the start of the 20th century. A deep connection between group theory and quantum mechanics was quickly found by physicists and mathematicians, notably Hermann Weyl and Eugene Wigner, and a large body of work rapidly grew around the topic. [5] [7]

The most obvious example of this connection is angular momentum, where the representation theory of $SU(2)$ can be used to answer physical questions. Consider the case of two particles, for example, one with total angular momentum j_1, the other with j_2. What are the possible measurements of total angular momentum, j, for the system composed of these two particles. This was the question answered by Clebsch and Gordan, albeit in the context of invariant theory, in the 1870's. [8] The result is of course $|j_1 - j_2| \leq j \leq (j_1 + j_2)$, with j ranging across all intermediate integer steps. Simple though it is, this provides a paradigmatic example of the power of group theory in physics.

$SU(2)$ was the main group of interest for physicists working with quantum mechanics for much of it's early history. $SU(3)$ however increased in importance when it was shown independently by Yuval Ne'eman and Murray Gell-Mann that the relations between different hadrons could be described by an approximate $SU(3)$ symmetry, represnting interchange of quark flavours. [9] The signifigance of $SU(3)$ was increased further by the development of Quantum chromodynamics, which has an exact $SU(3)$ symmetry representing interchange of quark color charge. This symmetry is manifested when calculating physical outcomes, for example, by the association of a group theory factor to every QCD Feynman diagram.

A representation of group is essentially a way of describing a group with a set of matricies which satisfy the same algebraic relations as the group elements. In particle physics we associate to irreducible representations of a group, ie. a representation with no invariant subspaces, a vector space of possible particle states, called a multiplet. To $SU(2)$ irreps for example, we associate particle states with a given total angular momentum, the muliplet states being distinguished by the angular momentum in a particular direction, usually denoted z. Often we find ourselves in a position when we have a multiparticle state, composed of single particles in irreps, the whole system however existing in a reducible representation. To determine physical properties of the sytsem it is neccessary to understand how this reducible representation can be decomposed into a direct sum of irreps, as exemplified by the Clebsch Gordan series mentioned above. Wigner 3n-j coefficents are complex numbers intimatley related to this process. These are essentially produced by tracing over products of operators, the operators projecting from one group representation to another. More generally any scalar produced by contracting over products of Lie algebra generators, Clebsch-Gordan coefficents or structure constants can be considered as a 3n-j coefficent. In the birdtrack notation which is described in the next section, a 3n-j coefficent is denoted by a closed graph with 3n internal lines. For example we write a 3-j coefficent as,



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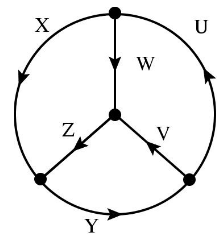
where X, Y, Z label different representations of $SU(N)$. If we consider three more representations, U, W, V, then we denote a 6-j coefficent by,



It will be shown later that any 3n-j coefficent can in terms of the 3-j and 6-j coefficents.

The usual method of calculating these numbers often requires the explicit construction of the relevant matricies in the representations and determining the values directly. This has been done for all commonly occuring cases, and many tabulations of the results can be found online or in books. These results however are almost exclusivley concerned with $SU(2)$ and $SU(3)$. Although this is sufficent for the time being, as these are the groups of most importance in the standard model, the development of gauge theories with different symmetries is increasing the relevance of higher order groups. [10] This is one of the main failings of convential calulation methods. The dimension of the group cannot be left as an unknown during the calculation, as the process of explicitly constructing the relevant matricies requires specification of the group being worked with. In QCD the number of quark colours is sometimes left as an unknown, which also requires the particular N in $SU(N)$ to be left as a variable. [11] The construction of these matricies also becomes increasingly more difficult as the dimension of the group increases.

This is one of the problems which is elegantly solved by the birdtrack notation outlined in the next section. It allows the dimension of the group to be left as an unknown for the entirety of the calculation. A general 3n-j coefficent for $SU(N)$ can therefore be expressed as a polynomial in N using these methods.

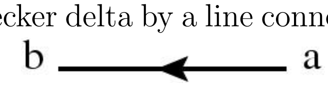
The main purpose of this report is to outline how the methods described by Elvang, Cvitanović and Kennedy [12], can be used in conjuction with results from the representation theory of the symmetric group, to produce an efficent algorithm for the calculation of Wigner 3n-j coefficents for $U(N)$ and $SU(N)$. I have also included some of the results produced by my own implementation of this algorithm. But before I do so it is first neccesary to describe in greater detail some of the topics mentioned here.

2 Background

2.1 Birdtrack Notation

First systematized by Pedrag Cvitanović in his book "Group Theory: Birdtracks, Lies, and Exceptional Groups," the birdtrack formalism provides an alternative to traditional tensor notation for group theory. The main advantage of the notation is that many previously abtruse and unintuitive identities and relations take on very natural appearances, eg. Lie Algebra commutation relations or invariance conditions on tensors. Although in many ways extremely novel, it is not without antecedents, as outlined by Cvitanović himself. [1] A large debt in owed to Roger Penrose inparticular, who in the 70's used very similar notation in the context of general relativity, to study the symmetry properties tensors. In fact the birdtrack notation for symmetrization tensors is virtually the same as Penrose's. [13] There are many other examples of similar techniques, eg. [14], [15], but the lack of widely used conventions has been a great hinderance to the development of these methods. This is one of problems Cvitanović has tried to address in his his book, and seems to have been somewhat successful. Many papers and articles published in the last few years have made use of birdtrack notation, see [11] and [16] for example. Cvitanović also acknowlegdes the influence Feynman diagrams on birdtracks, and superficially there is strong resemblence. It must be stressed however that while Feynman diagrams represent an integral in an series expansion that still has to be evaluated by other means, birdtrack notation is exact, and calculations performed with it are non-approximate.

In birdtrack notation we denote a tensor, invariant under whatever group we happen to be dealing with, by a box with directed lines going in and out of it. The lines represent tensor indicies. We denote the Kronecker delta by a line connecting two indicies. For δ^a_b we have,



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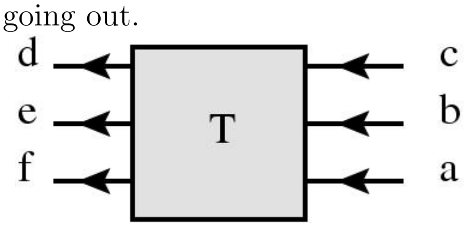
For a general tensor, T^{abc}_{def}, we represent the upper indicies by arrows going into the tensor, and lower indices by arrows going out.



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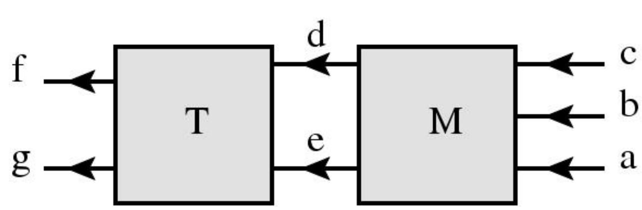
Tensor contractions are denoted by joining the lines between tensors. $T^{de}_{\ \ fg}M^{abc}_{\ \ \ de}$ for instance is written as,



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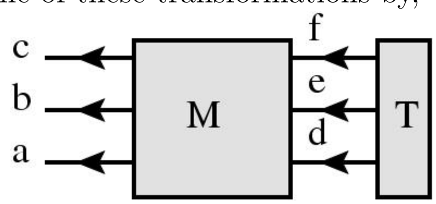
If a tensor has the same number of upper and lower indicies we can view it as a linear transformation acting on the vector of space of totally covariant or contravarient tensors with the correct number indicies. To make this more obvious we can group indicies together and relabel them with a single symbol. The tensor M^{abc}_{def}, could be wirtten as M^{α}_{β} for example. We could then consider this as acting on tensors of the form $T_{abc} = T_{\alpha}$, or $T^{def} = T^{\beta}$. As an example we could represent one of these transformations by,



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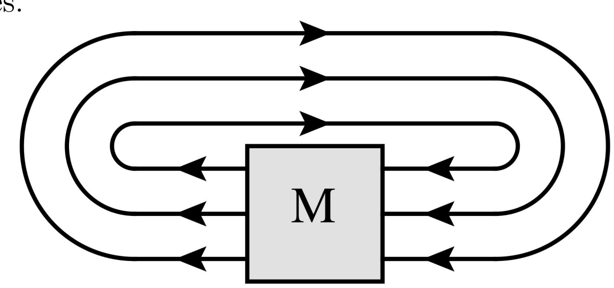
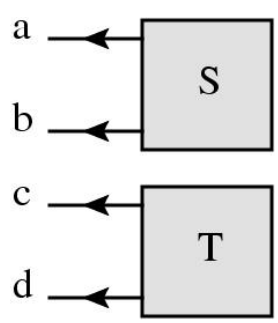
We draw the trace of matrix by simply joining the lines going in and out, ie. contracting upper and lower indices.



We represent a product of tensors by simply drawing them adjacent to one another. Take for example two tensor, S and T, acting on vectorspaces labelled by indicies ab and cd respectivley. We represent the the product, $S \otimes T$, as



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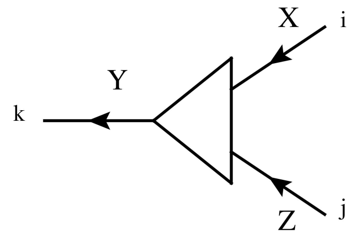
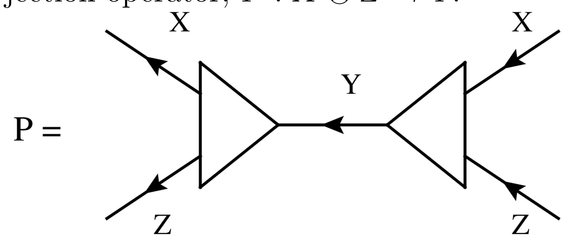
2.2 Clebsch-Gordan Coefficents and Projectors

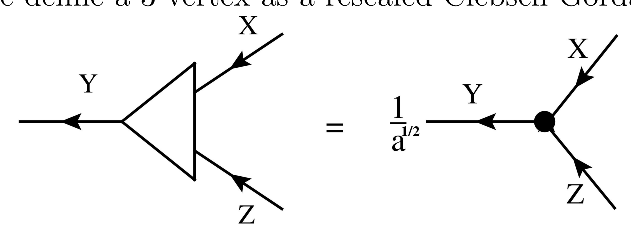
For two irreducible representations of a group, X and Z, consider the product representation, $X \otimes Z$. This representation will in general be reducible. If this product rep contains an irrep Y, then we denote the Clebsch-Gordan coefficents C^{XZY}_{ijk}, by



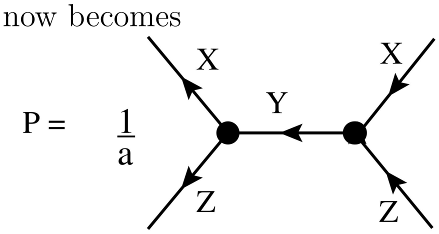
Taking the hermitian conguate of this tensor and then contracting with the original over the Y indices produces a projection operator, $P : X \otimes Z \rightarrow Y$.



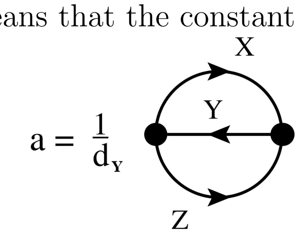
It is important to note that this operator projects into Y as it is imbedded as an invariant subspace in $X \otimes Z$. We define a 3-vertex as a rescaled Clebsch Gordan coefficent.



Altough this definition might look strange 3-verticies are actually ubiqitous in physics. Lie algebra generators for example, such as Pauli spin, or Gell-mann lambda matricies can be considered as 3-verticies. Dirac gamma matrics are also an example of 3-verticies. [1] In terms of the 3-vertex the projector now becomes

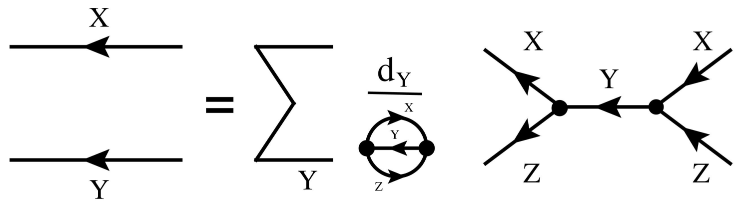


To ensure that these projectors are normalized correctly we must have $Tr(P) = d_Y$, where d_Y is the dimension of Y. This means that the constant a must satisfy,

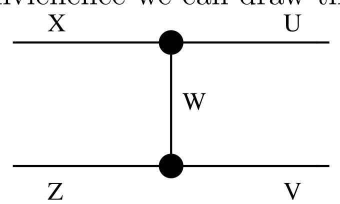


We call the number produced by tracing a product of 3-verticies, a Wigner 3-j coefficent. The utility of 3-verticies is that they are often easier to explicitly realise than Clebsch-Gordan coefficents, and often arise more naturally, as will be seen for $U(n)$ and $SU(n)$ later in the report. In addition to this, if we decide to work with Clebsch-Gordan coefficents then we must include $\sqrt{a}$ in calculations. This means any program built around this must be equipped to deal with algebraic numbers, which will reduce the efficency.

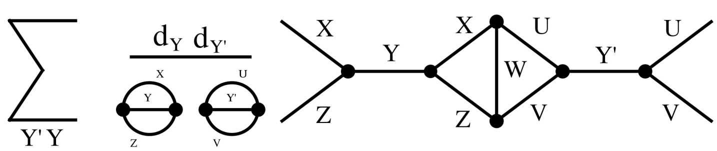
The projectors satisfy a completeness relation, so summing over all irreducible representations Y, we have,



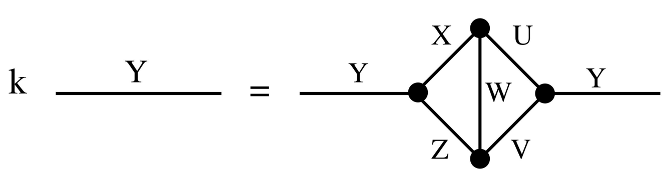
To see how these relations can be utilised in calculation consider the following example from Cvitanović's book. [1] We have a particle in rep U and a particle in rep V interacting by the exchange of a particle in rep W. The final state is two particles, one in the X rep, the other in the Z. Dropping arrows for convienence we can draw this process by



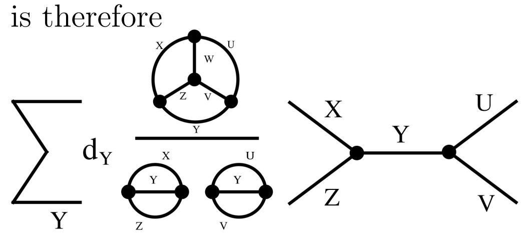
Now inserting two completness relations, one between U and V, the other between X and Z, we get



The middle section of this diagram can be thought of as linear map between Y and Y', which by assumption are irreps. By Schur's lemma we therefore know that this is zero unless $Y = Y'$, in which case it is proportional to the identity.

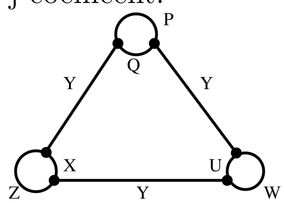


To determine the value of the constant k we trace both sides. This produces a Wigner 6-j coefficent. The final result is therefore

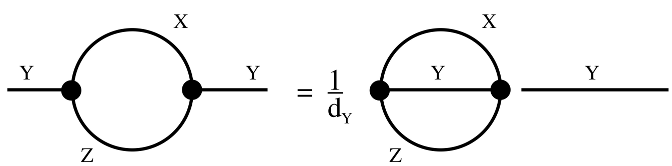


In the terminology of Feynman diagrams we have essentially recoupled from the t-channel into the s-channel.

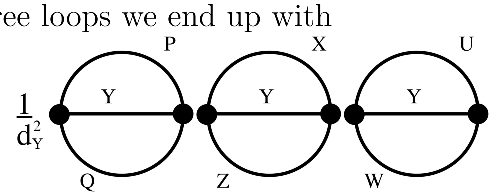
More generally we call any contraction of 3-verticies a Wigner 3n-j coefficent, where 3n is the number of internal lines. A contribution to a two particle scattering process of abitrary order has associated to it a group theory factor in the form of a Wigner 3n-j coefficent. Physical information like cross-sections and energy levels is contained in these 3n-j's. The 3-j's and 6-j's above are especially important as any 3n-j coefficent can be expressed as a sum of products of these two coefficents. [1] To understand why this is it useful to see some examples. Consider the following 9-j coefficent.



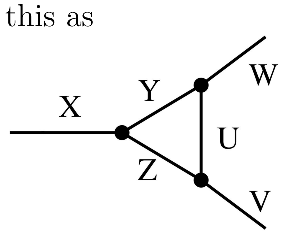
Now using



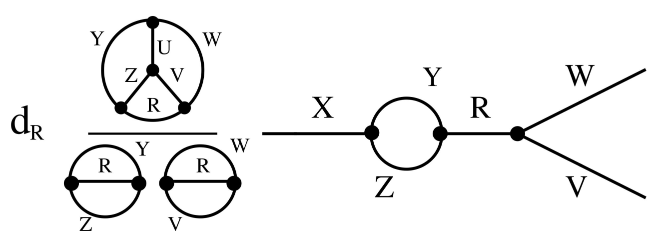
to eliminate each of the three loops we end up with



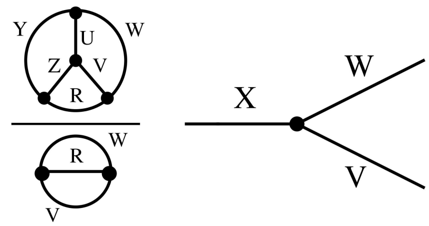
More generally we can eliminate any internal loop with more than two edges in a diagram by applying the recoupling relation above. The simplest case is a three vertex loop. Dropping rep labels for convienence we draw this as



Inserting the recoupling relation on the right most verticies and summing over R implicitly we get



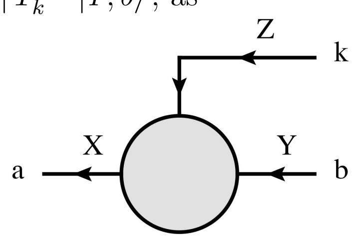
Again using Schur's lemma we can say that this is zero unless $X = R$. Furthermore the middle section containing a loop must therefore be proportional to the identity on X. Using the previous relation this therfore gives



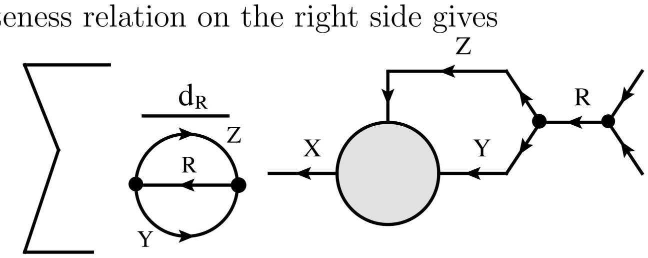
We can perform similar operations on any $3n$-j coefficent to eliminate internal loops until we end up with a sum of products 3-j's and 6-j's. [1]

2.3 Wigner-Eckart Theorem

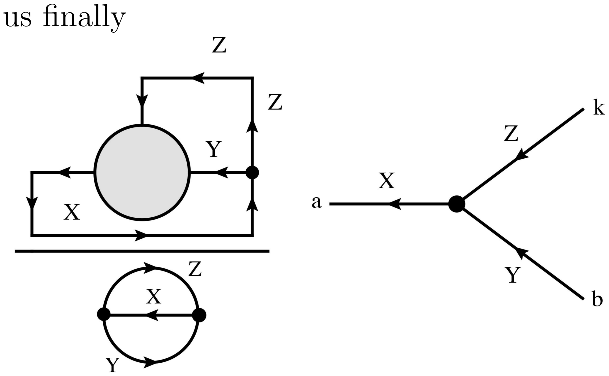
As another example of the need for 3-j and 6-j coefficents consider the Wigner-Eckart theorem. Let $T^{(Z)}$ be an invariant tensor operator with components labelled by k,and let X and Y be irreducible representations with components labelled by a and b respectivley. We depict the matrix element, $\langle X, a| \, T_k^{(Z)} \, |Y, b\rangle$, as



Inserting the completeness relation on the right side gives



And now using Schur's lemma as above we have $X = R$, and the middle section of the diagram must be proportional to the identity on X. To work out the proportion we perform the usual trace, giving us finally



Obviously none of the above processes can be carried out unless we know the Wigner 3-j and 6-j coefficents. For $U(n)$ and $SU(n)$ these can be evaluated using symmeterization operators. It is therefore necessary to give a brief outline of these, and their diagrammatic notation.

2.4 Symmeterizers and Antisymmeterizers

The result of acting on a tensor with a symmeterizer is a sum of all permutations over the chosen indices, multiplied by $\frac{1}{n!}$, where n is the number of indicies being symmeterized. The factor of $\frac{1}{n!}$ ensures that the symmeterizers are idempotent, ie. symmeterizing a tensor which is already symmetric has no effect. As an example consider a three index tensor T. The result of acting on it with a symmeterizer S, is

$$(ST)_{abc} = \frac{1}{6}(T_{abc} + T_{acb} + T_{bac} + T_{bca} + T_{cab} + T_{cba}). \tag{1}$$

An antisymmeterizer has a similar definition, the only difference being that odd index permutations in the sum have a minus sign. The effect of an antisymmeterizer A on T is therefore

$$(AT)_{abc} = \frac{1}{6}(T_{abc} - T_{acb} - T_{bac} + T_{bca} + T_{cab} - T_{cba}). \tag{2}$$

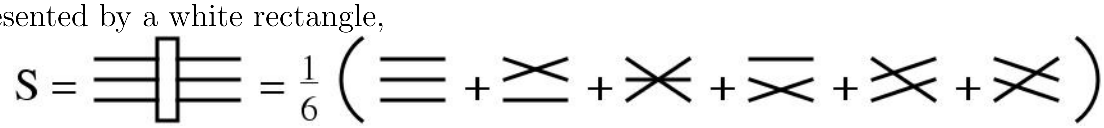
These operators can be written in terms of Kronecker deltas, which is a fact that will be of great importance later. For three indicies we have

$$S_{abc}{}^{def} = \frac{1}{6}(\delta_a^d \delta_b^e \delta_c^f + \delta_a^d \delta_c^e \delta_b^f + ... + \delta_c^d \delta_b^e \delta_a^f). \tag{3}$$

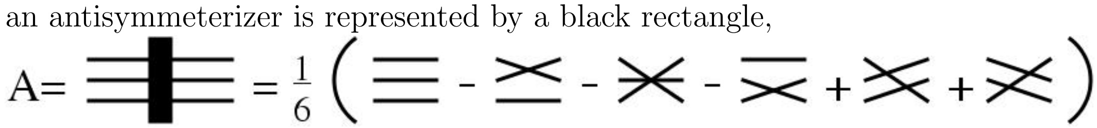
We can also write the antisymmeterizer on n-indices as a product of Levi-Civita tensors. For three indices we have

$$A_{abc}{}^{def} = \epsilon_{abc}\epsilon^{def}. \tag{4}$$

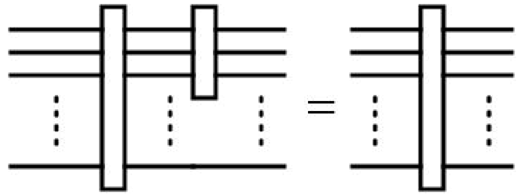
The diagrammatic notation for these operators is particularly simple. A symmeterizer is represented by a white rectangle,



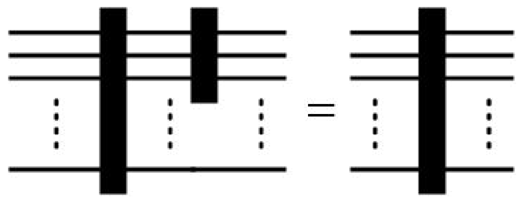
, and an antisymmeterizer is represented by a black rectangle,



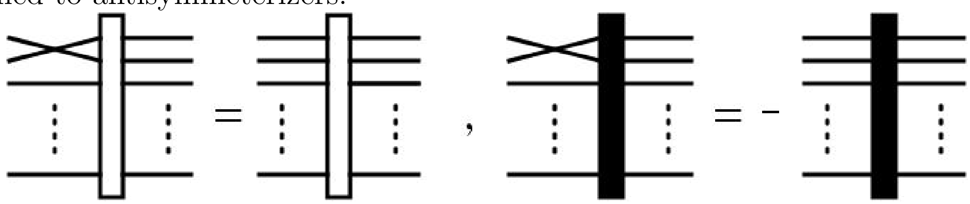
. I have ommitted indicies and arrows here as they are not of importance. The symmeterizers and antisymmeterizers satisfy several intuitive identities. Trying to symmeterized over indices which are already symmetric has no effect. This can be expressed diagrammatically as



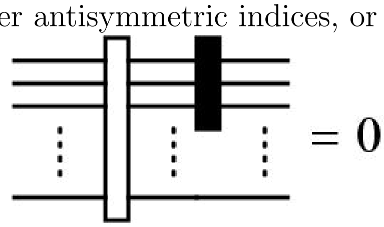
The same applies to antisymmeterizers.



Applying a pairwise index swap has no effect on symmeterizers, but produces a factor of -1 when applied to antisymmeterizers.



As expected, symmeterizing over antisymmetric indices, or vice versa, produces zero.



As you can see birdtrack notation is remarkably simple and intuitive compared with traditional tensor notation in this context. Some of the many benifits will hopefully become clear in the next section, where it is applied in the study the unitary group.

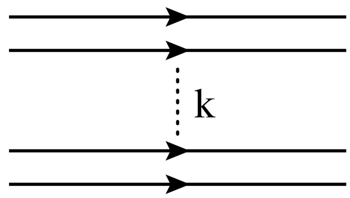
2.5 Representations of $U(n)$

The unitary group $U(n)$, is defined as the set of $n \times n$ unitary matricies, ie. matricies satisfying $G^{\dagger}G = I$, where I is the identity. Since the late 1800's the unitary groups have been an area of intense study. Much of the early work was down by figures such as Frobenius and Schur, with important contributions being made by the english clergyman Alfred Young. [17] Soon after the development of quantum theory the great of importance of the unitary groups was realised by physicists. This stems from the fact that unitary transformations preserve the norm of a vector, which is a requirement imposed on physical trnasformations by the probabilistic nature of quantum mechanics. The unitary group has been well understood for several decades, and most of the theory utilised in this report is several decades old, but as mentioned in the introduction there remains a difficulty in actually calculating numbers of interest.

The fundamental representation of $U(n)$ is simply the set of $n \times n$ unitary matricies. $SU(n)$ is just the subgroup formed by matricies with determinant 1. We denote the underlying $n-$dimensional complex vectorspace by V. A situation often arising in physics is a system of k particles in the fundamental representation. This system also defines a representation of the group defined by,

$$G \circ (v_1 \otimes v_2 \otimes .... \otimes v_k) = (Gv_1 \otimes Gv_2 \otimes .... \otimes Gv_k), \tag{5}$$

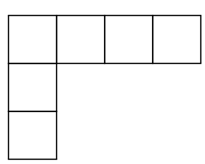
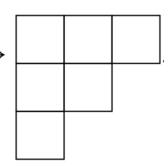
where $v_i \in V$ and $G \in U(n)$. The underlying vectorspace of this representation is $V^{\otimes k}$. If we were only interested in $SU(n)$ then physically this could represent a multiparticle system of quarks for $n = 3$, or a system of spin $\frac{1}{2}$ particles if $n = 2$. We represent the identity element of this representation in birdtrack notation as



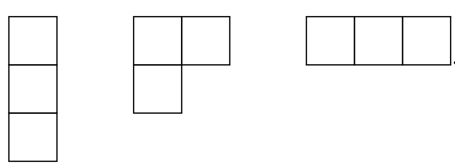
This representation will however be reducible. To understand the physics of this system like spectral decomposition or selection rules, it is necessary to understand how it can be broken into irreducible representations. There are many ways of doing this, but here I will describe the methods first used by Alfred Young in the context of $GL(n)$. [2] These methods are closely intertwined with the representation theory of the symmetric group, so it will first be necessary to give a brief overview of this theory.

2.6 The Symmetric Group

It is a well known fact that the irreducible representations of S_k are labelled by Young diagrams with k boxes. A Young diagram is made up of rows of boxes placed on top of one and other in such a way that the row length is non-increasing as you travel down the diagram. A $k-$box diagram can be seen as labelling an integer partition of k. Take $k = 6$ for example. We have,

$6 = 4 + 1 + 1 \rightarrow$  , or $6 = 3 + 2 + 1 \rightarrow$ . (6)

There are three irreducible representations of S_3, labelled by the following diagrams.

 (7)

Often we denote Young diagrams by the row lengths. The diagrams above would be denoted $(1,1,1)$, $(2,1)$, (3). If we fill the boxes with numbers from $1....k$, then we produce what's called a Young Tableaux. If the numbers are entered in such a way that they are increasing down the columns and non-increasing across rows then this is known as a semi-standard tableau. For example,

$$\begin{array}{|c|c|c|}\hline 2&3&3\\\hline 4&4\\\cline{1-2}\end{array}\qquad\begin{array}{|c|c|c|}\hline 1&3&3\\\hline 2\\\cline{1-1} 4\\\cline{1-1}\end{array}\qquad\begin{array}{|c|c|c|}\hline 1&4&5\\\hline 2&6&6\\\hline\end{array}\tag{8}$$

If every number between 1 and k appears exactly once, with both columns and rows being ordered, then this is known as a standard tableaux.

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array}\qquad\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2\\\cline{1-1} 4\\\cline{1-1}\end{array}\qquad\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array},\tag{9}$$

for instance.

For a $k-$box Young diagram Y, it is possible to label the basis elements of the correseponding irrep of S_k by the standard tableaux which can be obtained from Y. This is made particularly explicit when constructing the representations directly from the action of permutations on tableaux. These constructions are known as Specht modules, and will play a large part later in the report. Take for example the $(2,1)$ representation of S_3. The basis elements of this rep correspond to the tableaux

$$\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\qquad\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\tag{10}$$

This means that the dimension of the irreps is simply the number of standard tableaux obtainable from the corresponding Young Diagram. Although a seemingly simple task, the general formula for this number was not found until the 1954. [18] To determine the dimension we simply enter into every box in the diagram the number of boxes to the right and below it, including itself. This is known as the hook length of a box. The product of all hook lengths for a diagram Y, is denoted $|Y|$. If Y has k boxes then the dimension of the corresponding irrep of S_k is given by

$$d_Y = \frac{k!}{|Y|}.\tag{11}$$

This is known as the hook length formula. As an explicit example consider the Young diagram $(3,3,2,1)$.

$$\begin{array}{|c|c|c|}\hline &&\\\hline &&\\\hline &\\\cline{1-2} \\\cline{1-1}\end{array}\rightarrow\begin{array}{|c|c|c|}\hline 6&4&2\\\hline 5&3&1\\\hline 3&1\\\cline{1-2} 1\\\cline{1-1}\end{array}\implies |Y| = 6\times 4\times 2\times 5\times 3\times 3 = 2160 \implies d_Y = \frac{9!}{2160} = 168.\tag{12}$$

Most of the proofs of this formula are rather advanced but it can be partially justified by a heuristic argument first given by Knuth. [19] For a tableau to be standard the top left entry in every hook must be the smallest. If the entries are random, then the probability of this happening for a hook of length h is simply $\frac{1}{h}$. Imagining the events are independent then the probability of this happening for every hook is then $\frac{1}{|Y|} = p_{stan}$. The number of possible tableaux is $k!$, so the number of these which are standard is then $k! \times p_{stan} = \frac{k!}{|Y|}$. Although this is obviously fallacious, as the events are not independent, the argument can be modified to produce a rigorous proof. [20]

We can induce a representation of the symmetric group by considering the complex vector space made up of linear combinations of permutations. The action of the group is simply the composition of permutations. This is known as the regular representation. It is well known that every irreducible representation of S_k, labelled by a k-box diagram Y, appears d_Y times in the regular representation. This is just an example of the Dimensionality theorem, a basic result in Character theory. So there are d_Y invariant subspaces corresponding to each k-box diagram.

We can associate to every one of these d_Y invariant subspaces an element from the group algebra constructed from a standard tableau of shape Y. These elements are called tableau units by Littlewood, who studied them in the context of character theory, [4] but are known to most mathematicians as Young symmeterizers. [2] Following Cvitanović I will call them Young projectors. [1] The construction procedure is simple. We first take a product of the symmeterizers corresponding to the row entries of a standard tableau λ. Call this element R_λ. For example we have,

$$\lambda = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & \\ \cline{1-2}\end{array} \rightarrow R_\lambda = S_{123} \cdot S_{45} = \frac{1}{12}(1+(12)+(13)+(23)+(123)+(132))(1+(45)). \quad (13)$$

Here I have used traditional cyclic notation with 1 representing the identity. Next we do the same for columns, except with antisymmeterizers. Call this element C_λ.

$$\lambda = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & \\ \cline{1-2}\end{array} \rightarrow C_\lambda = A_{14} \cdot A_{25} = \frac{1}{4}(1-(14))(1-(25)). \quad (14)$$

An important fact is that if we define $A = \mathbb{C}[S_k]$, ie. the group algebra, then the left ring-module $A(R_\lambda \cdot C_\lambda)$ defines an irreducible representation of S_k, corresponding to the diagram shape of λ. [4]

We define the Young projector for a tableau Z as

$$P_Z = \alpha_Z R \cdot C. \quad (15)$$

α_Z is a normalization constant ensuring that the Young projectors are idempotent. It is defined as

$$\alpha_Z = \frac{(\Pi_i |S_i|!)(\Pi_j |A_j|!)}{|Z|}, \quad (16)$$

where $|S_i|$ is the length of the ith row and $|A_j|$ is the length of the jth column. To give an explicit example consider

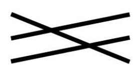
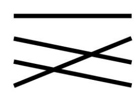
$$Z = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \to P_Z = c_Z S_{12} \cdot A_{13} = \frac{1}{3}(1 + (12))(1 - (13)). \tag{17}$$

We shall now see how these elements can be used to decompose the tensor representation of $U(n)$ constructed above.

2.7 $U(n)$ Tensor Representation

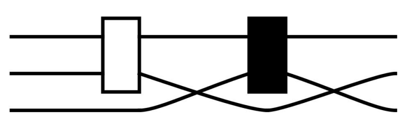
As mentioned earlier we can construct a k-index tensor representation of $U(n)$ by simply taking a direct product of k fundamental representation. The dimension of this new representation is n^k. Like the regular representation of the symmetric group this can be decomposed using Young tableaux. In fact every $U(n)$ irrep in the k-index tensor rep can be labelled by a k-box standard tableaux. Irreps are isomorphic if and only if they're labelled by tableaux of the same shape. It can be shown that all finite dimensional irreps of $U(n)$ can be produced in this way, so that just as with the symmetric group, we can label finite $U(n)$ irreps by Young diagrams. [15] This can be explicitly shown using the Young projectors defined above.

We can induce the regular representation of the symmetric group on the tensor space by considering index permutations of tensors, ie.

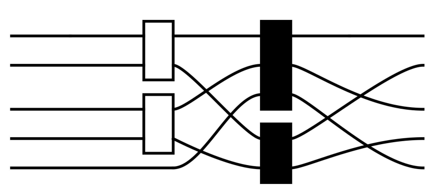
(12) =  , (132) =  , (234) = 

These index swap operators are invariant under unitary transformations as they are simply made up of Kronecker deltas, which are invariant under unitary transformations.

Using the notation for symmeterizers and antisymmeterizers described earlier the Young projectors take on an especially simple form. For example, the P_Z calculated eariler can be written as

$\frac{4}{3}$ 

As another example consider the tableaux $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$. The corresponding Young projector is then written as

2 

Now for $k \leq 4$ the $k-$box Young projectors satisfy the following properties.

1. For any tableau Y, $P_Y^2 = P_Y$. The operators are idempotent.
2. For any two distinct tableaux X and Y, we have that $P_X \cdot P_Y = 0$. This is called mutual orthoganality.
3. $\sum_Y P_Y = 1$, where 1 is the identity element for the $k-$index vector space and the sum is over all $k-$ box standard tableaux. This is known as completness. [1]

The Young projectors are also constructed from Kronecker deltas, which are invariant under the action of unitary transformations. So the Young projectors for $k \leq 4$ are idempotent, mutually orthogonal, complete and invariant under the action of $U(n)$. This therefore implies that they are valid projection operators, and can be used to completley decompose the tensor representation into irreducible pieces.

For $k > 4$ the Young projectors, although idempotent and $U(n)$ invariant, no longer satisfy mutual orhthogonality or completness. For example,

$$X = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array}, \; Y = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2}\end{array}, \quad P_X \cdot P_Y \neq 0. \tag{18}$$

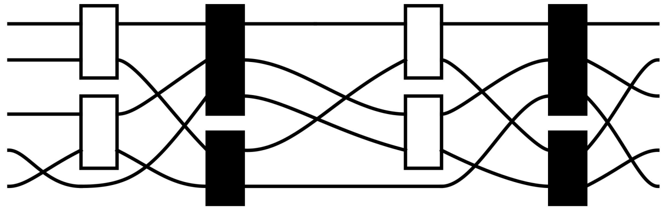
Projectors corresponding to tableaux of different shape are still mutually orhtogonal though.

It is possible to modify the projectors slightly so that they do satisfy these requirments. Littlewood describes a recursive method for constructing orthogonal and complete projection operators for any k from the original Young projectors. [4] Following [11] I will refer to these as Littlewood-Young projectors, and for a tableau Y, I will denote them as L_Y.

To construct these projectors we first need to introduce an ordering on standard Young tableaux of the same shape. For a tableaux X, let X_{ij} be the entry in the ith row and jth column. The row word is defined as $R_X = (R_{11}, R_{12}, ..., R_{21}, ...)$. For two tableaux of the same shape X and Y, we say that X precedes Y if $X_{ij} < Y_{ij}$, for the left most entries in the row words such that $X_{ij} \neq Y_{ij}$. For the diagram $(2, 2, 1)$ the standard tableaux have the following ordering.

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1}\end{array} \prec \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 \\ \cline{1-1}\end{array} \prec \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 \\ \cline{1-1}\end{array} \prec \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 \\ \cline{1-1}\end{array} \prec \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 \\ \cline{1-1}\end{array}. \tag{19}$$

It turns out that this ordering tells that $X \prec Y$ implies that X has two entries in the same row which appear in the same column in Y. [11] This means that $P_Y \cdot P_X = 0$, as an antisymmeterizer will be connected to a symmeterizer by more than two lines. To illustrate this consider the first two tableaux in the ordering above. We can write the product of projectors as



As you can see the third and fourth lines are connected to both a symmeterizer and an antisymmeterizer as 3 and 4 appear in the same row in the first tableau, and the same column in the second tableau. The product therefore vanishes as predicted. In general though it is not true that $X \prec Y \implies P_X \cdot P_Y \neq 0$.

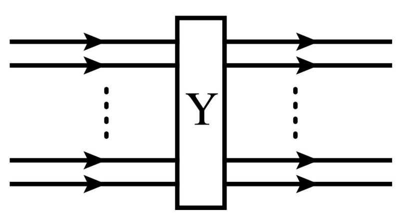
For a Young diagram λ, we can order the standard tableaux produced from it, $\{Y_1, Y_2, ..., Y_n\}$. The Littlewood-Young operators corresponding to the *ith* tableau are defined recursivley as

$$\begin{aligned} L_{Y_1} &= P_{Y_1} && (20)\\ L_{Y_2} &= (1 - L_{Y_1})P_{Y_2} && (21)\\ &\vdots \\ L_{Y_i} &= (1 - L_{Y_1} - L_{Y_2} - .... - L_{Y_{i-1}})P_{Y_i}, && (22) \end{aligned}$$

where P_Y is just the Young projector produced from the tableau Y. It is not hard to prove that these are indeed orthogonal and complete. Furthermore, for a tableau Y, L_Y and P_Y both project into isomorphic irreps of $U(n)$. Often the Littlewood-Young projectors are equal to the regular Young projectors, as most Young projectors are orthogonal even without the modification. For the Young diagram $\lambda = (2, 2, 1)$ above for example, only the first and last Young projectors are not orthogonal, which therefore means that

$$\begin{aligned} L_{Y_i} &= P_{Y_i} \text{ for } \ 1 \leq i < 5, && (23)\\ L_{Y_5} &= (1 - L_{Y_1} - ..... - L_{Y_4})P_{Y_5} = (1 - P_{Y_1})P_{Y_5}. && (24) \end{aligned}$$

The Littlewood-Young projectors are obviously more computationally expensive to construct, but we only need them in scenarios which require completeness and full orhtogonality. The tests performed to check the calculated 3-j's and 6-j's later in the report for example require complete projectors. Often though the usual Young projectors are adequate. If we were just concerned with the properties of one copy of an irrep, regardless of how it was imbedded in a larger space, then the usual Young projectors would be suitable. Through the rest of this report I will often use the following diagram to represent the projector, Young or Littlewood-Young, associated to a tableau Y.



It can be shown that any finite dimensional irrep of $U(n)$ can be obtained by this method of decomposing k copies of the fundamental rep, and since $SU(n)$ is a subgroup of $U(n)$ we can say the same for it. Couple this with the fact that projectors corresponding to tableaux of the same shape produce isomorphic irreps, then we can label every irrep of $U(n)$ and $SU(n)$ by a Young diagram. Non-trivial $U(n)$ irreps are labelled by diagrams with no more than n rows. Otherwise the projectors will contain antisymmeterizers on $n + 1$ or more indices. Since the fundamental rep is n-dimensional we only have n choices for the value of an index.

The antisymmeterized indices will therefore always contain a repeated index, meaning that a tensor is simply sent to zero by these projection operators.

We can also associate to every finite dimensional irrep of $SU(n)$ a Young diagram. Non-trivial irreps of $SU(n)$ are labelled by Young diagrams with no more than $n-1$ rows. This is due to the fact that for $G \in SU(n)$, we have $detG = 1$. This is equivalent to saying that the Levi-Civita tensor on n-indices is left invariant by $SU(n)$, as $G_i^{i'} G_j^{j'} ....G_k^{k'} \epsilon_{i'j'....k'} = det(G)\epsilon_{ij....k} = \epsilon_{ij....k}$. Because of the fact that we can write an antisymmeterizer as a product of Levi-Civita tensors as in 4, this means that $SU(n)$ acts trivially on the n antisymmetric indices. For a projector containing an antisymmeterizer of length n, we can therefore ignore the n antisymmetric indices when considering $SU(n)$. To make this explicit consider the rep of $SU(4)$ produced by the diagram $(4, 4, 2, 1)$. We have

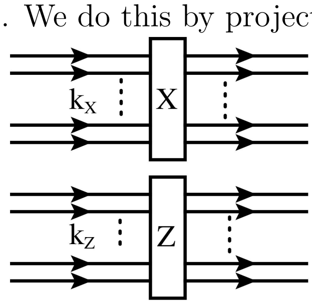
$$\ydiagram{4,4,2,1} \cong \ydiagram{3,3,1} \otimes 1 \cong \ydiagram{3,3,1} \tag{25}$$

where 1 is the trivial rep.

So to summarise, we can label irreps of $U(n)$ by Young diagrams with no more than n rows. $SU(n)$ irreps are labelled by Young diagrams with no more than $n-1$ rows. If we want to explicitly realise these irreps we can use the Young or Littlewood-Young projectors to decompose the tensor representation. Whether we use the Young or Littlewood-Young projectors depends on whether we require them to be mutually orhtogonal and complete for the particular application.

2.8 Product Representations and Projectors

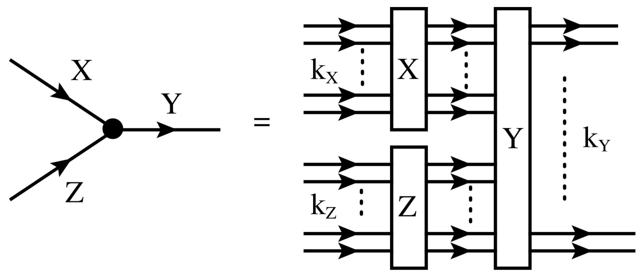
Using the tensor representation we can also produce products of irreducible representations. Consider two irreps of $U(n)$ or $SU(n)$ labelled by Young diagrams λ_1 and λ_2. If the number of boxes in λ_1 and λ_2 is k_X and k_Z respectivley then we can realise the product representation on a $(k_X + k_Z)$-index tensor space. We do this by projecting with the operator



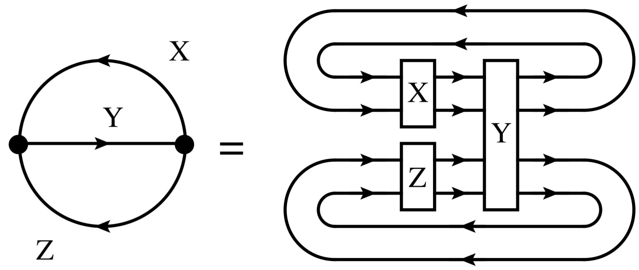
Here X and Z denote some standard tableaux produced from λ_1 and λ_2 respectivley. The values in Z will be shifted upward by k_X to signify the fact that the operator acts on different lines. For an explicit example, imagine we wanted to realise the product rep $\ydiagram{2,1} \otimes \ydiagram{2}$.

We could produce this as above with the projectors associated to the tableaux $X = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$ and $Z = \begin{array}{|c|c|}\hline 4 & 5 \\ \hline\end{array}$.

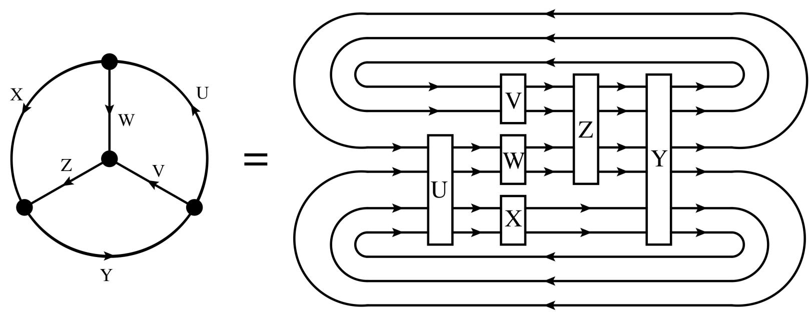
We can then write one of the 3-vertices defined above by



Here Y is some Young tableaux which labels an irrep contained in $X \otimes Z$. The corresponding 3-j coefficent is therefore written as



For tableaux X, Z, Y I will denote the corresponding 3-j by $3j\{X, Z, Y\}$. In terms of projectors we can also represent 6-j's as

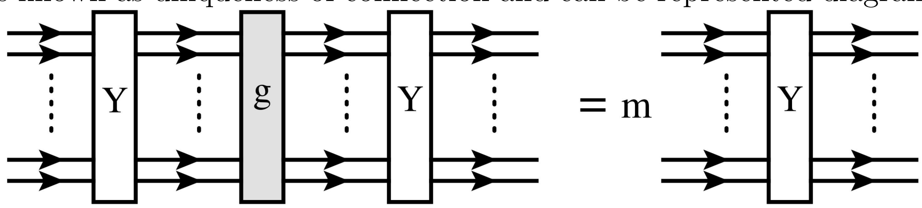


For tableaux U, V, W, X, Z, Y I denote the corresponding 6-j by $6j\{U, V, W, X, Z, Y\}$.

3 Calculation Methods

3.1 Calculation by Hand

The obvious way to determine these coefficents is simply to expand out all the symmeterizers and antisymmeterizers in the projectors and then calculate directly. This method is of course not efficent as the expansion of a symmeterizer on n lines produces $n!$ terms. This becomes computationaly impractical for large n. There are however several identities and relations which greatly simplify the calculations allowing for more efficent computation. First of all we have an important property of the Young projectors, proved in Cvitanović's book. [1] If we insert any permutation g, between two copys of a Young projector Y, then the result is proportional to Y by a factor of $1, 0,$ or -1. This is simply due to the fact there is only one non-zero way to connect the symmeterizers and antisymmeterizers in the projector, up to a sign. If any two lines connect a symmeterizer with an antisymmeterizer then the result is zero. This is known as uniqueness of connection and can be represented diagrammaticaly as



, where $m = 1, 0, -1$.

We can prove a similar property for the Littlewood-Young projectors defined earlier. For a tableau Y_n in an ordering consider the following.

$$L_{Y_n} g L_{Y_n} = (1 - \sum_{i=1}^{n-1} L_{Y_i}) P_{Y_n} g (1 - \sum_{j=1}^{n-1} L_{Y_j}) P_{Y_n} \tag{26}$$

$$= P_{Y_n} g P_{Y_n} - \sum_{i=1}^{n-1} L_{Y_i} P_{Y_n} g P_{Y_n} - P_{Y_n} g \sum_{j=1}^{n-1} L_{Y_j} P_{Y_n} + \sum_{i,j=1}^{n-1} L_{Y_i} P_{Y_n} g L_{Y_j} P_{Y_n}. \tag{27}$$

Using the previous result in conjuction with the fact that $g \sum_{j=1}^{n-1} L_{Y_j}$ is simply a linear combination of permutations, we know that

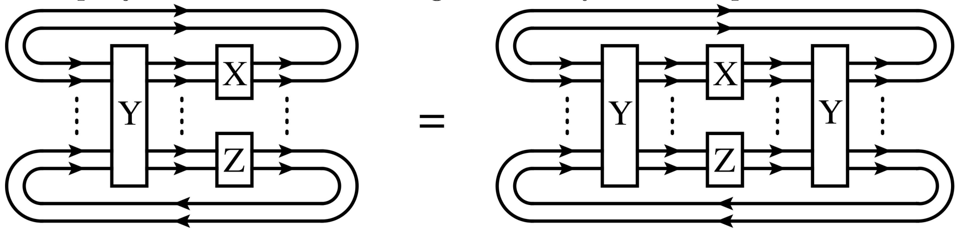
$$P_{Y_n} g P_{Y_n} = m_1 P_{Y_n}, \quad P_{Y_n} g \sum_{j=1}^{n-1} L_{Y_j} P_{Y_n} = m_2 P_{Y_n}, \tag{28}$$

where $m_1 = +1, 0, -1$, and m_2 is a rational number. The above equation now becomes

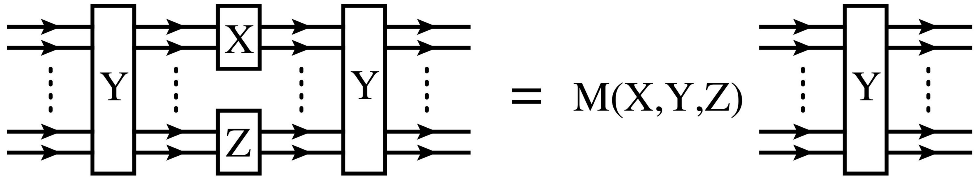
$$L_{Y_n} g L_{Y_n} = m_1 P_{Y_n} - m_1 \sum_{i=1}^{n-1} L_{Y_i} P_{Y_n} - m_2 P_{Y_n} + m_2 \sum_{i=1}^{n-1} L_{Y_i} P_{Y_n} \tag{29}$$

$$= (m_1 - m_2) L_{Y_n}. \tag{30}$$

We can exploit this for $3-j$ coefficents by inserting an additonal copy of the largest projector into the trace, which can always be done as the projectors are idempotent, and sandwiching the two smaller projectors inbetween. Diagramatically this is expressed as



Since the smaller projectors are equal to linear combinations of permutations the earlier result implies that



It is important to note that $M(X,Y,Z)$ only depends on the symmetric group, and is in no way influenced by which unitary or special unitary group we are working with. Now all that is left to do is to trace Y, which is equal to the dimension of the $U(n)$ irrep labelled by Y's Young diagram. We therefore have

$$3j\{X,Z,Y\} = M(X,Y,Z)d_Y. \tag{31}$$

There is a very simple formula for calculating d_Y given by Cvitanović. [1] We take the Young diagram, and in the top left box we insert n, and then in the boxes to the left we insert $n+1, n+2, ...etc.$ We then do the same for the second row, only this time we start from $n-1$ instead. We do this for all rows, starting from $n-i+1$ in the ith row, and finally we take a product of all these numbers to produce a polynomial in n, $f_Y(n)$. The dimension is then given by

$$d_Y = \frac{f_Y(n)}{|Y|}. \tag{32}$$

Let's take an explicit example.

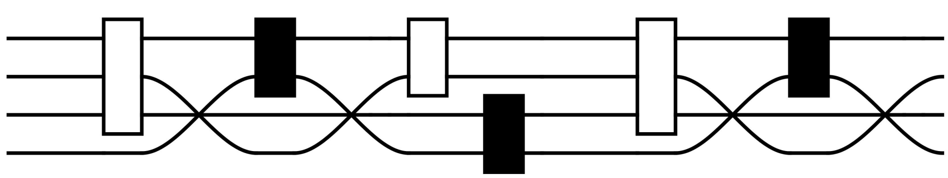
$$Y = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 7 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array} \rightarrow f_Y(n) = \begin{array}{|c|c|c|c|}\hline n & n{+}1 & n{+}2 & n{+}3 \\ \hline n{-}1 & n \\ \cline{1-2} n{-}2 \\ \cline{1-1}\end{array} \rightarrow d_Y = \frac{f_Y(n)}{|Y|} = \frac{n^2(n^2-1)(n^2-4)(n+3)}{144}. \tag{33}$$

This formula is also valid for $SU(n)$ irreps as well.

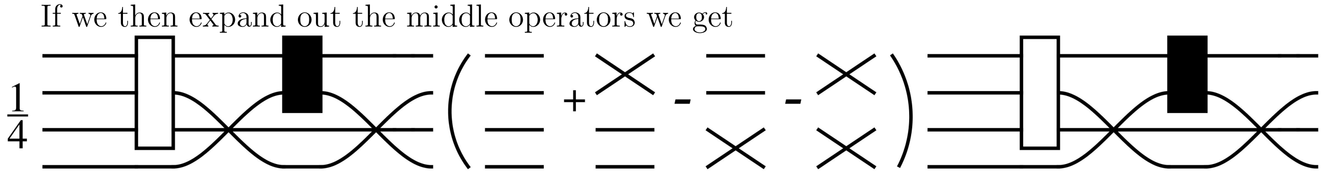
Consider the tableaux,

$$X = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}\,, \ Z = \begin{array}{|c|}\hline 3 \\ \hline 4 \\ \hline\end{array}\,, \ Y = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}\,. \tag{34}$$

We can use the method of snadwiching X and Z between two Y's and then expanding them to calculate the corresponding $3-j$ as follows.



If we then expand out the middle operators we get



We can use this to calculate the $M(X, Y, Z)$ defined earlier. The identity element gives a factor of $\frac{1}{4}$, and because of the symmeterizer on the right we can swap round the first two lines in the second term to get the identity again, giving another $\frac{1}{4}$. The last two terms connect an antisymmeterizer with a symmeterizer by the first and fourth lines. These terms therefore vanish. So $M(X, Y, Z) = \frac{1}{2}$. The corresponding $3 - j$ is therefore given by

$$3j\{X, Z, Y\} = \frac{1}{2} d_Y = \frac{n(n^2 - 1)(n + 2)}{16}. \tag{35}$$

We can use exactly the same technique of sandwiching smaller projectors between the largest one when calculating 6-j coefficents. But these methods are still impractical for larger projectors, and if trying to implement an efficent program to calculate these we must modify our approach.

An important observation is that instead of working in the regular representation of the symmetric group when calculating $M(X, Y, Z)$ we can work in the irreducible representation corresponding to the Young diagram of Y. This is due to the fact mentioned earlier in section 2.6, that the left ring module formed by multiplying by a Young projector forms an irrep of the symmetric group. The same also holds for Littlewood-Young projectors. [4] So instead of working in the $n!$ dimension regular rep we can instead work in the much smaller dimension matrix rep produced by this left ring module. If Y had shape $(5, 3, 1)$, then instead of working with $9! \times 9!$ matrices in the regular representation, we can work with the 162×162 matrices of the $(5, 3, 1)$ irrep of S_9, 162 being the number of standard tableaux of shape $(5, 3, 1)$. In order to generate these irreps we make use of what are known as Specht modules.

3.2 Specht Modules and the Garnir Relations

For a given partition of n labelled λ, we can construct the corresponding irrep of S_n using what are known as Specht modules. Their construction was first outlined by german mathematician Wilhelm Specht in the 1930's. [21] They are based on what are known as tabloids. These are equivalence classes containing Young tableaux of the same shape and containing the same entries in each row, althought not necessarily in the same order. They can be thought of a tableaux with unordered rows. For a tableau Y we denote the corresponding tabloid as $\{Y\}$. For example,

$$Y = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array} \to \{Y\} = \begin{array}{ccc}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array} = \begin{array}{ccc}\hline 1 & 4 & 2 \\ \hline 3 & 5 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array} = \begin{array}{ccc}\hline 2 & 4 & 1 \\ \hline 5 & 3 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array} \tag{36}$$

We define the action of permutation g on a tableaux in the obvious way of simply permuting the entries. This allows us therefore to define an action on tabloids by $g \cdot \{Y\} = \{g \cdot Y\}$. It is simple to check that this is well defined.

Next we define $C(Y) \leq S_n$ as the set of permutations which keep the columns of Y fixed. For the tableaux earlier we have $C(Y) = S_{1,3,6} \times S_{2,5}$, where $S_{\alpha,\beta,\ldots,\gamma}$ is the group of permutations on the symbols $\alpha, \beta.....\gamma$. To every λ tableau we associate an element known as a polytabloid. For a tableau Y this is defined as

$$e_Y = \sum_{g \in C(Y)} \mathrm{sgn}(g) g \cdot \{Y\}. \tag{37}$$

It is simple to show that for any permutation g, and any tableau Y, we have $g \cdot e_Y = e_{g \cdot Y}$. The space spanned by these elements is the Specht module corresponding to λ. We label this S^λ. It can be shown that this vectorspace forms an irreducible representation of S_n, and that furthermore the polytabloids associated to the standard λ tableaux form a basis of it. So the Specht module corresponding to a partition of n, λ, gives a realisation of the irrep of S_n labelled by the Young diagram produced by λ. [3] Take $\lambda = (2,1)$ for example. S^λ is therefore the 2 dimsensional irrep of S_3, with basis elements corresponding to the tableaux $\begin{ytableau} 1 & 2 \\ 3 \end{ytableau}$ and $\begin{ytableau} 1 & 3 \\ 2 \end{ytableau}$.

Applying a permutation to a polytabloid produces another element in S^λ, so if we could find a way of decomposing any element in S^λ into a linear combination of standard polytabloids then this would allow us to construct explicit matrix representations of permutations. This is precisley what the Garnir algorithm performs. It is a simple iterative procedure, first described by Belgian mathematician Henri Garnir. [22] From now on in this section I will identify a polytabloid with the tableau it's produced from. If we are given a non-standard tableau Y, we may decompose it into a sum of standard tableaux by iterativley doing the following. First we order the columns, multiplying by the sign of the permutation needed to produce this ordering. For example

$$Y = \begin{ytableau} 1 & 2 & 3 \\ 6 & 4 \\ 5 \end{ytableau} = -\begin{ytableau} 1 & 2 & 3 \\ 5 & 4 \\ 6 \end{ytableau}. \tag{38}$$

Next we travel along the rows until we come to a point where two elements in a row are out of order, ie. non-increasing. We isolate two strips in the new tableau. One comprised of all the entries in the column below the left element which is out of order including itself, the other composed of the other element which is out of order and all the entries in the column above it. In the example above these strips are $(\{5,6\}\{2,4\})$. We next define an element of the group algebra known as the Garnir element, G. This is the signed sum of all permutations on the elements of these two strips which keep each individual strip vertically ordered. In the example above we have $G = (1 - (45) + (465) + (245) + (46)(25) - (2465))$. Now for any polytabloid, e_T, it can be shown that $G \cdot e_T = 0$. For our example we therefore have

$$G \cdot \begin{ytableau} 1 & 2 & 3 \\ 5 & 4 \\ 6 \end{ytableau} = \begin{ytableau} 1 & 2 & 3 \\ 5 & 4 \\ 6 \end{ytableau} - \begin{ytableau} 1 & 2 & 3 \\ 4 & 5 \\ 6 \end{ytableau} + \begin{ytableau} 1 & 2 & 3 \\ 4 & 6 \\ 5 \end{ytableau} + \begin{ytableau} 1 & 4 & 3 \\ 2 & 5 \\ 6 \end{ytableau} - \begin{ytableau} 1 & 4 & 3 \\ 2 & 6 \\ 5 \end{ytableau} + \begin{ytableau} 1 & 5 & 3 \\ 2 & 6 \\ 4 \end{ytableau} = 0. \tag{39}$$

Since G always contains the identity element this relation allows us to rexpress our original tableaux by taking all other terms to the other side of the equaility. Using this we can now write our original tableau as

$$Y = \ytableaushort{123,64,5} = -\ytableaushort{123,54,6} = -\ytableaushort{123,45,6} + \ytableaushort{123,46,5} + \ytableaushort{143,25,6} - \ytableaushort{143,26,5} + \ytableaushort{153,26,4}. \tag{40}$$

Although not all of the tableaux on the right are standard we can apply this process iterativley to any non-standard tableaux in the sum. Eventually we end up with a linear combination of standard tableaux. [2] After several iterations we get

$$Y = \ytableaushort{123,45,6} - \ytableaushort{123,46,5} - \ytableaushort{134,25,6} + \ytableaushort{135,24,6} + \ytableaushort{134,26,5} - \ytableaushort{136,24,5} - \ytableaushort{135,26,4} + \ytableaushort{136,25,4}. \tag{41}$$

To see how we can construct matrix representations of permutations from these relations consider the irrep of S_4 labelled by the partition $\lambda = (2,1,1)$. In the Specht module construction of this irrep the basis elements are labelled by the three standard tableaux produced λ.

$$e_1 = \ytableaushort{12,3,4}\ , \ e_2 = \ytableaushort{13,2,4}\ , \ e_3 = \ytableaushort{14,2,3}. \tag{42}$$

To construct a permutation matrix we merely have to work out the action of the permutation on these elements. Take the permutation (123) for example.

$$(123) \cdot e_1 = \ytableaushort{23,1,4} = -\ytableaushort{13,2,4} = -e_2, \tag{43}$$

$$\tag{44}$$

$$(123) \cdot e_2 = \ytableaushort{23,1,4} = \ytableaushort{12,3,4} - \ytableaushort{13,2,4} + \ytableaushort{14,2,3} = e_1 - e_2 + e_3, \tag{45}$$

$$\tag{46}$$

$$(123) \cdot e_3 = \ytableaushort{24,3,1} = \ytableaushort{14,2,3} = e_3. \tag{47}$$

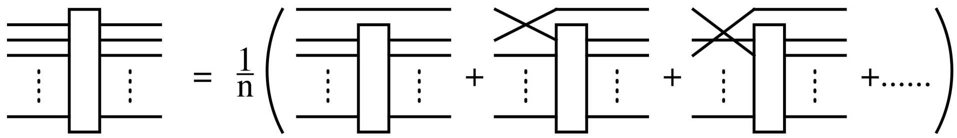
The matrix representation of (123) is therefore

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \tag{48}$$

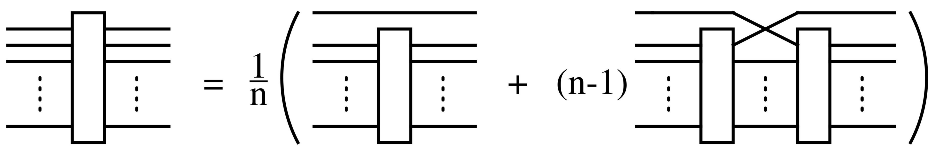
Although this algorithm may seem rather complicated it is relativley simple to implement on a computer. In the program I have written I have made use of the above method in order to calculate $M(X, Y, Z)$ in a rep of the symmetric group with a much smaller dimension than the regular representation. This becomes increasingly more efficent for larger projectors. Computation is made even simpler by another important relation satisfied by symmeterizers and antisymmeterizers.

3.3 Recursive definition of Symmeterizers

We can think of the action of a symmeterizer on n indices as first symmeterizing $n-1$ indices and then summing over the n possible permutations of the unsymmeterized index. Diagrammatically we can write this as



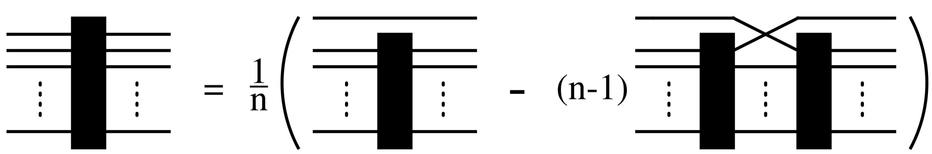
The $\frac{1}{n}$ factor arises from the idempotence constraint. We can insert a symmeterizer on the lower $n-1$ indicies on both sides of this equation. This has no effect on the left side, and since we can swap lines connected to symmeterizers we can collect together the rightmost $(n-1)$ terms on the left side to give



This relation is given by Cvitanović in his book. [1] It is particularly useful for computing the matrix representations of these smmeterizers as we only have to explicitly construct the matricies corresponding to pairwise swaps. If $\mathcal{S}_3$ is the matrix corresponding to a symmeterizer on 3 indices, and σ_{ij} is the matrix representing the permutation (ij), then we have that

$$\mathcal{S}_3 = \frac{1}{6}(\mathcal{S}_2 + 2\mathcal{S}_2\sigma_{12}\mathcal{S}_2) = \frac{1}{6}(I + \sigma_{23} + (I + \sigma_{23})\sigma_{12}(I + \sigma_{23})). \tag{49}$$

Our program therefore only has to compute pairwise swap matricies, rather than arbitrary permutation matricies, which is easier to implement. Furthermore for a symmeterizer over n indices we only have to construct $n-1$ pairwise swap matricies, where as if we were to construct symmeterizer matricies using the brute force method we would have to construct $(n!-1)$ matricies. We obviously don't have to construct the identity. The above statements hold for antisymmeterizers as well, except the recursion relation is instead



So to summarise this section, to calculate $3j\{X,Z,Y\}$ we can use the method of sandwiching the smaller projectors, and then working in the representation of the symmetric group corresponding to Y we can compute the factor $M(X,Z,Y)$. The $3-j$ is equal to $M(X,Z,Y)d_Y$, and since we have a simple formula for d_Y we can easily compute the Wigner coefficent. This method can also be used in almost exactly the same way for calculating 6-j coefficents.

4 Results

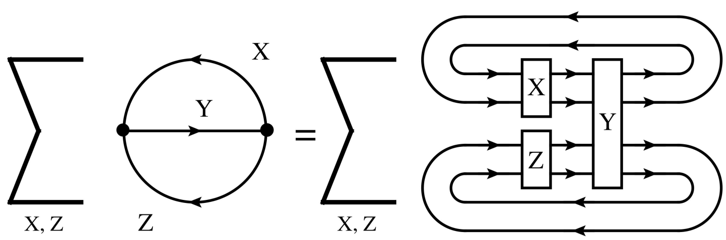
I have written a java program which implements the methods described above to calculate Wigner 3-j and 6-j coefficents. For a 3-j the input is 3 Young tableaux, X, Z and Y, with Y being the largest. The program first calculates the Young or Littlewood-Young projectors in the symmetric group irrep corrseponding to the shape of Y. I will call these P_X, P_Z and P_Y. We then have $P_Y \cdot P_X \cdot P_Z \cdot P_Y = M(X, Z, Y)P_Y$. The 3-j as explained above is simply $M(X, Z, Y)d_Y$. For a 6-j coefficent, we input six tableau, construct the corresponding projectors in the symmeteric group irrep corresponding to the largest tableauxm, snadwich these matrices to find M, and then return Md_Y. There are many different conventions for 3-j's and 6-j's due to the choice of projection operators. This makes checking with existing literature difficult, and in addition to this only the values for $SU(2)$ and $SU(3)$ have been tabulated extensivley. This means that in order ot establish the validity of the program we must perform internal checks. This is made possible by the exsitence of several sum rules, which must be satisfied by these coefficents.

4.1 Sum Rules

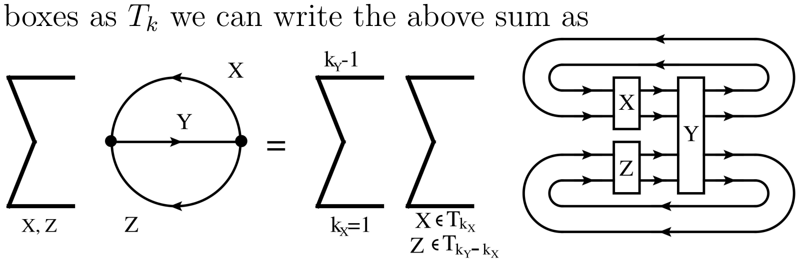
Cvitanović provides some useful sum rules for 3-j's and 6-j's. For example summing over all tableau X and Z we have for any tableau Y,

$$\sum_{X,Z} 3j\{X, Z, Y\} = (k_Y - 1)d_Y, \tag{50}$$

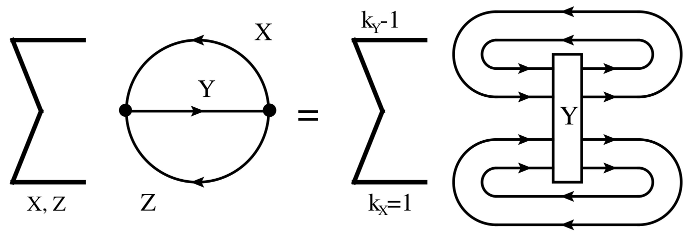
where k_Y is the number of boxes in Y. This can be derived simply using birdtrack notation. First we have



We know that the above 3-j vanishes unless we have $k_X + k_Z = k_Y$. So denoting the set of tableaux with k boxes as T_k we can write the above sum as



If we use the Littlewood-Young projectors then we can utilise completness. The sum of projectors corresponding to tableau with k boxes is simply the identity element on k lines. The sum now becomes



Now since the trace of Y is simply d_Y this proves the result.

This rule provides a convienent test for the validity of calculated results. I have verified my program using this test for all tableaux with up to 9 boxes. It becomes slightly impractical to test for higher. There are 9496 standard tableau with 10 boxes, and a full check of the sum rule for these would require the calculation of over one hundred million 3-j's. However since the algorithm is essentially the same regardless of the projectors size, I would say nine boxes is a sufficent test. As an explicit example consider the tableau $Y = \begin{array}{|c|c|}\hline 1&4\\\hline 2&5\\\hline \multicolumn{1}{|c|}{3}\\\cline{1-1}\end{array}$. Summing $3j\{X,Z,Y\}$ over all tableau X and Z, the zero values produced are

$$3j\left\{\begin{array}{|c|}\hline 1\\\hline\end{array},\ \begin{array}{|c|c|}\hline 2&4\\\hline 3&5\\\hline\end{array},\ \begin{array}{|c|c|}\hline 1&4\\\hline 2&5\\\hline \multicolumn{1}{|c|}{3}\\\cline{1-1}\end{array}\right\} = \frac{n^2(n^2-1)(n-2)}{48},$$

$$3j\left\{\begin{array}{|c|}\hline 1\\\hline\end{array},\ \begin{array}{|c|c|}\hline 2&5\\\hline \multicolumn{1}{|c|}{3}\\\cline{1-1} \multicolumn{1}{|c|}{4}\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 1&4\\\hline 2&5\\\hline \multicolumn{1}{|c|}{3}\\\cline{1-1}\end{array}\right\} = \frac{n^2(n^2-1)(n-2)}{48},$$

$$3j\left\{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array},\ \begin{array}{|c|c|}\hline 3&5\\\hline \multicolumn{1}{|c|}{4}\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 1&4\\\hline 2&5\\\hline \multicolumn{1}{|c|}{3}\\\cline{1-1}\end{array}\right\} = \frac{n^2(n^2-1)(n-2)}{72},$$

$$3j\left\{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array},\ \begin{array}{|c|c|}\hline 3&4\\\hline \multicolumn{1}{|c|}{5}\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 1&4\\\hline 2&5\\\hline \multicolumn{1}{|c|}{3}\\\cline{1-1}\end{array}\right\} = \frac{n^2(n^2-1)(n-2)}{72},$$

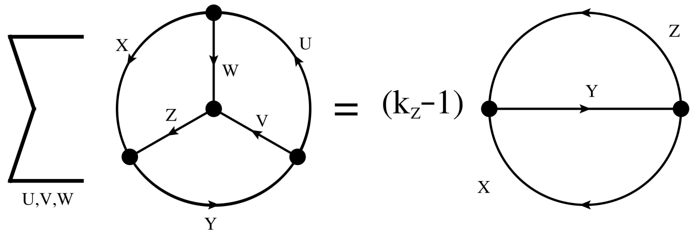
$$3j\left\{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array},\ \begin{array}{|c|}\hline 3\\\hline 4\\\hline 5\\\hline\end{array},\ \begin{array}{|c|c|}\hline 1&4\\\hline 2&5\\\hline \multicolumn{1}{|c|}{3}\\\cline{1-1}\end{array}\right\} = \frac{n^2(n^2-1)(n-2)}{72},$$

$$3j\left\{\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array},\ \begin{array}{|c|}\hline 4\\\hline 5\\\hline\end{array},\ \begin{array}{|c|c|}\hline 1&4\\\hline 2&5\\\hline \multicolumn{1}{|c|}{3}\\\cline{1-1}\end{array}\right\} = \frac{n^2(n^2-1)(n-2)}{24},$$

$$3j\left\{\begin{array}{|c|c|}\hline 1&4\\\hline \multicolumn{1}{|c|}{2}\\\cline{1-1} \multicolumn{1}{|c|}{3}\\\cline{1-1}\end{array},\ \begin{array}{|c|}\hline 5\\\hline\end{array},\ \begin{array}{|c|c|}\hline 1&4\\\hline 2&5\\\hline \multicolumn{1}{|c|}{3}\\\cline{1-1}\end{array}\right\} = \frac{n^2(n^2-1)(n-2)}{24},$$

Adding these up we get $\frac{n^2(n^2-1)(n-2)}{6}$, which is indeed equal to $(k_Y-1)d_Y$.

Once the 3-j values have been checked we can use the following sum rule as a cross check for 6-j coefficents.



This can be proved using the constraints $k_W = k_Z - k_V$, and $k_U = k_Y - k_V$. We can sum k_V from 1 to $k_Z - 1$, and then use the completness relations to prove the result. As an explicit example consider $X = \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}$, $Z = \begin{array}{|c|c|}\hline 3&5\\\hline 4\\\cline{1-1}\end{array}$ and $Y = \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array}$. For the non-vanishing 6-j's my program returns the following values.

$$6j\left\{\begin{array}{|c|c|c|}\hline 2&4&5\\\hline 3\\\cline{1-1}\end{array}, \begin{array}{|c|}\hline 1\\\hline\end{array}, \begin{array}{|c|c|}\hline 2&3\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}, \begin{array}{|c|c|}\hline 3&5\\\hline 4\\\cline{1-1}\end{array}, \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array}\right\} = \frac{5n^2(n^2-1)(n+2)}{576},$$

$$6j\left\{\begin{array}{|c|c|c|}\hline 2&3&5\\\hline 4\\\cline{1-1}\end{array}, \begin{array}{|c|}\hline 1\\\hline\end{array}, \begin{array}{|c|c|}\hline 2&3\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}, \begin{array}{|c|c|}\hline 3&5\\\hline 4\\\cline{1-1}\end{array}, \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array}\right\} = \frac{7n^2(n^2-1)(n+2)}{576},$$

$$6j\left\{\begin{array}{|c|c|}\hline 2&4\\\hline 3&5\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline\end{array}, \begin{array}{|c|c|}\hline 2&3\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}, \begin{array}{|c|c|}\hline 3&5\\\hline 4\\\cline{1-1}\end{array}, \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array}\right\} = \frac{n^2(n^2-1)(n+2)}{192},$$

$$6j\left\{\begin{array}{|c|c|}\hline 2&3\\\hline 4&5\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline\end{array}, \begin{array}{|c|c|}\hline 2&3\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}, \begin{array}{|c|c|}\hline 3&5\\\hline 4\\\cline{1-1}\end{array}, \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array}\right\} = \frac{n^2(n^2-1)(n+2)}{576},$$

$$6j\left\{\begin{array}{|c|c|c|}\hline 2&4&5\\\hline 3\\\cline{1-1}\end{array}, \begin{array}{|c|}\hline 1\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}, \begin{array}{|c|c|}\hline 3&5\\\hline 4\\\cline{1-1}\end{array}, \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array}\right\} = \frac{n^2(n^2-1)(n+2)}{576},$$

$$6j\left\{\begin{array}{|c|c|c|}\hline 2&3&5\\\hline 4\\\cline{1-1}\end{array}, \begin{array}{|c|}\hline 1\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}, \begin{array}{|c|c|}\hline 3&5\\\hline 4\\\cline{1-1}\end{array}, \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array}\right\} = \frac{5n^2(n^2-1)(n+2)}{576},$$

$$6j\left\{\begin{array}{|c|c|}\hline 2&4\\\hline 3&5\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}, \begin{array}{|c|c|}\hline 3&5\\\hline 4\\\cline{1-1}\end{array}, \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array}\right\} = \frac{n^2(n^2-1)(n+2)}{192},$$

$$6j\left\{\begin{array}{|c|c|}\hline 2&3\\\hline 4&5\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}, \begin{array}{|c|c|}\hline 3&5\\\hline 4\\\cline{1-1}\end{array}, \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array}\right\} = \frac{-n^2(n^2-1)(n+2)}{576},$$

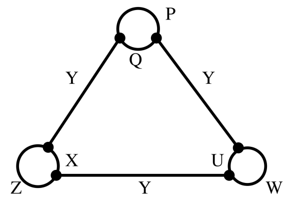
$$6j\left\{\begin{array}{|c|c|}\hline 3&5\\\hline 4\\\cline{1-1}\end{array}, \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}, \begin{array}{|c|}\hline 3\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}, \begin{array}{|c|c|}\hline 3&5\\\hline 4\\\cline{1-1}\end{array}, \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array}\right\} = \frac{n^2(n^2-1)(n+2)}{24},$$

Adding these up we get $\frac{n^2(n^2-1)(n+2)}{12}$. Since we have $3j\{X, Z, Y\} = \frac{n^2(n^2-1)(n+2)}{24}$ in this case, this is consistent with the validity of the 6-j's.

These two checks show the internal consistency of my calculated values.

4.2 Vacuum Bubble Reduction

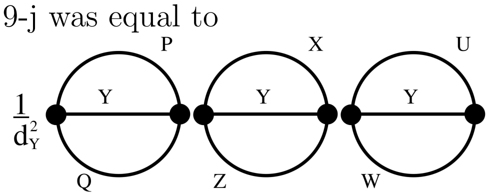
As was mentioned earlier in the report, a two particle scattering process produces a group theory factor in the form of Wigner 3n-j coefficent. [1] By repeated recouplings I showed that any 3n-j can be decomposed into sums of products of 3-j's and 6-j's, by eliminating internal loops. As an example consider the 9-j decomposed earlier.



Now let

$$Y = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array},\; X = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array},\; Z = \begin{array}{|c|c|}\hline 4 & 5 \\ \hline\end{array},\; P = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}, Q = \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 5 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array},\; U = \begin{array}{|c|}\hline 1 \\ \hline\end{array},\; W = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 4 & 5 \\ \hline\end{array}.$$

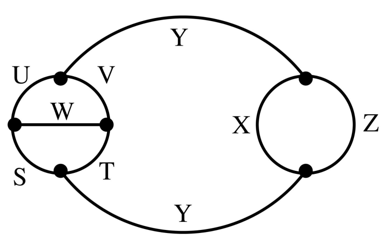
Earlier we showed that the 9-j was equal to



Evalulating this using my program results in

$$\frac{n^2(n^2-1)(n+2)}{144}.$$

As another example consider the following 9-j coefficent.

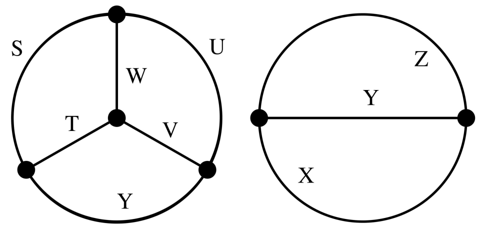


Let

$$Y = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline\end{array}, X = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}, Z = \begin{array}{|c|c|c|}\hline 3 & 4 & 5 \\ \hline 6 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}, S = \begin{array}{|c|c|}\hline 5 & 6 \\ \hline\end{array}, T = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}, W = \begin{array}{|c|}\hline 3 \\ \hline 4 \\ \hline\end{array}, U = \begin{array}{|c|c|c|}\hline 3 & 4 & 5 \\ \hline 6 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}, V = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}$$

Again due to Schur's lemma and the fact that Y is an irrep we know that the right and left sections of the diagram are simply multiples of the identity on Y. As usual we trace to find the particular proportion and end up with

$$\frac{1}{d_Y}$$



Using my program this can then be evaluated to give $\frac{n^2(n^2-1)(n+1)(n+2)}{144}$.

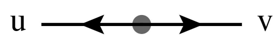
5 Conclusion

The Wigner 3n-j coefficents defined above are of great importance to physicists. A scattering process produces a group theory factor due to symmetries in the system. This factor takes the diagrammatic form of a contraction over three verticies, a 3n-j. In birdtrack notation it is relativley easy to show that any of these coefficents can be expressed in terms of 3-j's and 6-js. The usual method of evaulating these is to simply contract many Clebsch-Gordan coefficents, Lie aglgebra generators etc... together, and calculate explicitly. The methods described above have various advantages over this traditional procedure. First and foremost is the fact that the usual method requires the group dimension to be fixed from the start, which is not ideal for certain applications. In QCD for example, it is often necessary to leave the n in $SU(n)$, as an unkown throughout a calculation. In this case n represents the number of quark colours. [11] The methods above do however allow for n to be left unkown for as long as is needed. The group theory factors are then expressed as polynomials in n.

Another advantage is that many otherwise abtruse facts become relativley clear in birdtrack notation. Take for example the fact that 6-j's and 3-j's are always proportional to the dimension of the largest representation. This would not be clear from a large product of contracted tensors in traditional notation, but in birdtrack notation it takes on an intuitive character. Another example is the reduction of vertex loops in order to show that any Wigner coefficent is equivalent to a sum of products of 6-j's and 3-j's. The methods above also provide us with a way to efficently calculate these numbers, thanks to relations such as the two mentioned previously. The ability to work in some irreducible representation of the symmetric group, rather than the regular representation, also greatly increases computational ease.

The next obvious improvement to be made would be an automation of the reduction process for 3n-j's. This would not be terribly hard to implement, especially since we already have knowlegde of the 3-j's and 6-j's, which provide the numerical weights in the reduction. The problem would therefore reduce to the manipulation of graphs. Another route which could be explored is the use of projection operators other than the Young and Littlewood-Young projectors defined above. Two papers published recently showed that hermitian operators could be contructed in a simple manner from the Young projectors. [11] [16] These projectors satisfy some interseting relations which do not hold for the projectors used in this report. They are however much more costly to construct, and it is questionable if the advantages are worth the computational costs.

Looking farther forward, the biggest open problem in this area is the calculation of invariant scalars for $SO(n)$. For the spinor representations there exist several very efficent calculation methods. [23] The vector representations however continue to cause difficulty. Altough many of the above methods are aplicable the situation is complicated by the existence of a new invarint, namely the metric tensor. Diagrammatically we represnt $g_{uv} = g_{vu}$, as



Take the two index tensor representation for example. For $SU(n)$ we can resolve this simply using the Young projectors into a symmetric and antisymmetric subspace. For $SO(n)$ however things are further complicated by the existence of a new invariant, the trace, $T_{uv,pq} = g_{uv}g_{pq}$. This is signifigant as the symmetric subspace of $V \otimes V$, where V is the fundamental rep of $SO(n)$, can now be further decomposed into a tracless and a singlet subspace. In the singlet subspace elements are simply projected to the identity matrix multiplied by their trace. Altough a simple example it shows that the above methods begin to break down as we can no longer classify states by their index symmetry alone, but must also take in to account the trace in the decompostition.

As far as I am aware there currently exist no general methods for the caluclation of $SO(n)$ invariant scalars, at least not of equal efficency to those above for $U(n)$ and $SU(n)$. The need for such algorithms might soon increase, with many potential grand unified theories containing an $SO(n)$ symmetry. [24] Several variants of String Theory also exhibit $SO(n)$ symmetries, for example, Heterotic String Theory is built around an $SO(32)$ symmetry. [25] Whether or not these theories will bear fruit is impossible to say, but the mathematical interest alone of this area I'm sure will provide ample incentive for it's continued study.

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