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Lab 11

① $\{w \mid w \in \Sigma^*, rev(w) = w\} = L$

Ex/ $w_i = ab$ $w_j = aa$
$w_{ij} = aba$
$ab(aba) \in L$
⊕
$aa(aba) \notin L$

Ex/ $w_i = ab$ $w_j = abbb$
$w_{ij} = aba$
$ab(aba) \in L$ ⊕ $abbb(aba) \notin L$

1. $\Sigma^* = \{\varepsilon, a, b, aa, ab, ba, bb, \ldots\}$
2. $w_i = a^i b^i$ $w_j = a^j b^j$
(any strings where $w_i \neq w_j$)
3. $w_{ij} = (a \mid b) \cdot rev(w_i)$
(a or b appended to the reverse of string i)
$\therefore w_i \cdot w_{ij} = a^i b^i (a|b) rev(a^i b^i) \in L$
⊕
$w_j \cdot w_{ij} = a^j b^j (a|b) rev(a^i b^i) \notin L$

best to use the opposite of the previous character

Ex/ $w_i = abba$ $w_j = bba$
$w_{ij} = babba$
$abba(babba) \in L$
⊕
$bba(babba) \notin L$

① G_1: $S \rightarrow ABS \mid AB$
$A \rightarrow aA \mid a$
$B \rightarrow bA$

1. aabaab → $\boxed{\text{Not in } L(G_1)}$
$S \rightarrow ABS$
$\rightarrow aABS$
$\rightarrow aaBS$
$\rightarrow aabAS$
$\rightarrow aabaS$
$\rightarrow aabaAB$
$\rightarrow aabaaB$
$\rightarrow aabaabA$

A string cannot end on a 'b' because of the third rule, which forces all 'b' characters to be followed by at least one 'a'

2. aaaaba → $\boxed{\text{In } L(G_1)}$
$S \rightarrow AB$
$\rightarrow aAB$
$\rightarrow aaAB$
$\rightarrow aaaAB$
$\rightarrow aaaaB$
$\rightarrow aaaabA$
$\rightarrow aaaaba$

Two 'b' characters cannot be next to each other due to the third rule, so an 'a' would immediately follow

3. aabbaa → $\boxed{\text{Not in } L(G_1)}$

$S \rightarrow ABS$	$S \rightarrow AB$
$\rightarrow aABS$	$\rightarrow aAB$
$\rightarrow aaBS$	$\rightarrow aaB$
$\rightarrow aabAS$	$\rightarrow aabA$

4. abaaba → $\boxed{\text{In } L(G_1)}$
$S \rightarrow ABS$
$\rightarrow aBS$
$\rightarrow abAS$
$\rightarrow abaS$
$\rightarrow abaAB$
$\rightarrow abaaB$
$\rightarrow abaabA$
$\rightarrow abaaba$

② Σ is finite $= \{\varepsilon, a, b\}$

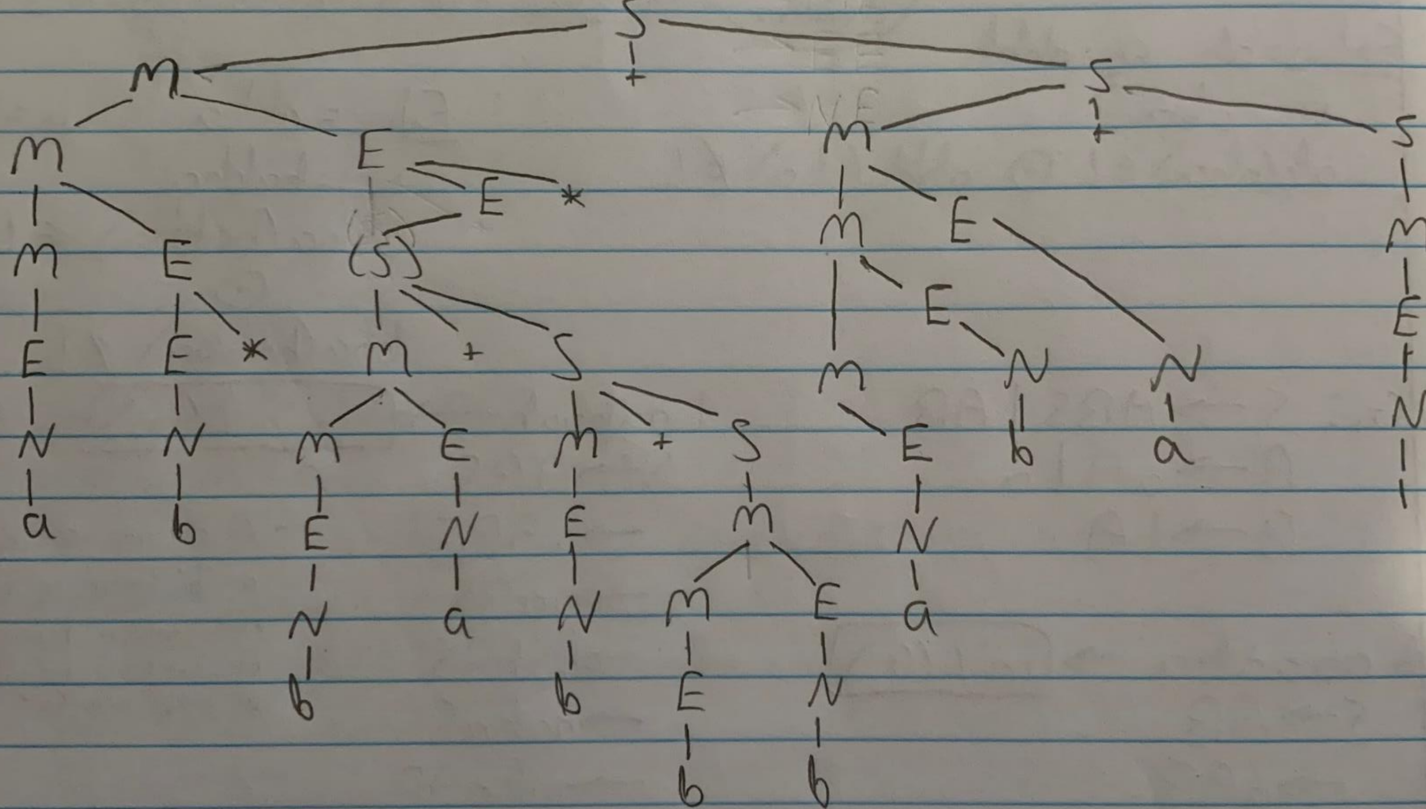
$\Sigma' = \Sigma \cup \{0, 1, +, *, (,)\} = \{\varepsilon, a, b, 0, 1, +, *, (,)\}$

G_2: $S \to M+S \mid M$ $\qquad r = ab*(ba+b+bb)**+aba+1$

$M \to ME \mid E$

$E \to E* \mid N \mid (S)$

$N \to 0 \mid 1 \mid \varepsilon \mid a \mid b$



③ G_3: $S \to a \mid aB \mid \varepsilon$

$B \to bB \mid CB \mid \varepsilon$

$C \to ba \mid \varepsilon$

$\boxed{G_3 = (\{S, B, C\}, S, R)}$

④ $\{a^i b^j a^i \mid i \geq 0,\ j \geq 1\}$

$S \to aSa \mid B$
$B \to b \mid bB$

$\xrightarrow[\text{replace long rules}]{\text{replace terminals}}$

$S \to AX \mid B$
$A \to a$
$B \to b \mid BB$
$X \to SA$

$\xrightarrow{\text{replace short rules}}$

$$\boxed{\begin{array}{l} S \to AX \mid BY \mid b \\ A \to a \\ B \to b \\ X \to SA \\ Y \to BY \mid b \end{array}}$$

$\{a^3 b^5 a^3 \mid i \geq 0,\ j \geq 1\}$

$S \mapsto AX \mapsto aX \mapsto aSA \mapsto aAXA \mapsto aaXA \mapsto aaSAA \mapsto aaAXAA \mapsto$
$aaaXAA \mapsto aaaSAAA \mapsto aaaBYAAA \mapsto aaabYAAA \mapsto aaabBYAAA \mapsto$
$aaabbYAAA \mapsto \ldots \mapsto aaabbbbYAAA \mapsto aaabbbbbAAA \mapsto \ldots \mapsto aaabbbbbaaa$

1 2 3 4

(5) $S \to aSb \mid bSa \mid SS \mid \varepsilon$

Theorem: For all $w \in \Sigma^*$, $S \mapsto^* w$ iff w has an equal amount of as and bs (balanced

$a(w)$ = amount of as in w

$b(w)$ = amount of bs in w

Def: A string $w \in \Sigma^*$ is balanced if.

① $a(w) = b(w)$

② $a(w') \geq b(w')$ or $a(w') \leq b(w')$ for any proper prefix w' of w

$a(\varepsilon) = 0$ $\quad$ $b(\varepsilon) = 0$

$a(aw) = 1 + a(w)$ $\quad$ $b(aw) = b(w)$

$a(bw) = a(w)$ $\quad$ $b(bw) = 1 + b(w)$

$a(Sw) = a(w)$ $\quad$ $b(Sw) = b(w)$

Proof: $L \to R$ ① S is balanced by definition $a(w) = b(w)$

② If $\alpha S \beta$ is balanced, then $\alpha(aSb)\beta$, $\alpha(bSa)\beta$, $\alpha SS\beta$, and $\alpha\beta$ are balanced.

If w is balanced and $w \mapsto w'$ then w' is balanced

$a(\alpha(aSb)\beta) = a(\alpha) + 1 + a(\beta) = b(\alpha) + 1 + b(\beta) = b(\alpha(aSb)\beta)$

$R \to L$ By strong induction on $|w|$

Given a balanced string w

$n=0$ ① $w = \varepsilon$ $\quad S \to \varepsilon$, $\therefore S \mapsto^* \varepsilon$ by def

$n \geq 1$ ② $w = w_1 w_2$, where w_1, w_2 are balanced ($a(w) = b(w)$ for both), then $|w_1| < |w|$ $|w_2| < |w|$

By IH, $S \mapsto^* w_1$, $S \mapsto^* w_2$: $S \overset{3}{\mapsto} SS \mapsto^* w_1 S \mapsto^* w_1 w_2$

$n \geq 1$ ③ $w = a w_1 b$, where w_1 is balanced ($a(w) = b(w)$)

By IH $S \mapsto^* w_1$: $S \overset{1}{\mapsto} aSb \mapsto^* a w_1 b$

$n \geq 1$ ④ Similarly to case 3, $w = b w_2 a$, where w_2 is balanced

By IH $S \mapsto^* w_2$: $S \overset{2}{\mapsto} bSa \mapsto^* b w_2 a$

⑥ $S \to bS \mid Sa \mid aSb \mid \varepsilon$

Generates the set of all strings in Σ^* with any amount and ordering of a's and b's.
$a, b : \Sigma^* \to \mathbb{N}$
$a : \Sigma^* \to \mathbb{N}$
$b : \Sigma^* \to \mathbb{N}$

Theorem: For all $w \in \Sigma^*$, $S \mapsto^* w$ iff w has any amount of a's and b's, including zero of a's and/or b's. Therefore, w can also be the empty string.

Def: A string $w \in \Sigma^*$ has any amount of a's and b's if:
① $a(w) = 0..n$ $\quad b(w) = 0..n$
② $a(w') \leq b(w')$ or $a(w') \geq b(w')$ for any proper prefix w' of w
Rule 1 brings a 'b' to prefix
Rule 3 brings a 'a' to prefix

$a(\varepsilon) = 0$ $\qquad b(\varepsilon) = 0$
$a(aw) = 1 + a(w)$ $\qquad b(aw) = b(w)$
$a(bw) = a(w)$ $\qquad b(bw) = 1 + b(w)$
$a(Sw) = a(w)$ $\qquad b(Sw) = b(w)$

Proof: L→R ① S has any amount of a's and b's by definition
$\{a^i, b^j \mid i \geq 0, j \geq 0\}$ (or $a(w) = 0..n$, $b(w) = 0..n$)
② If $\alpha S \beta$ has any amount/ordering of a's and b's, then $\alpha(bS)\beta$, $\alpha(Sa)\beta$, $\alpha(aSb)\beta$, $\alpha\beta$ have any mixture and count of a's and b's
$a(\alpha(bS)\beta) = a(\alpha) + 0 + a(\beta) \neq b(\alpha) + 1 + b(\beta) = b(\alpha(bS)\beta)$
$a(\alpha(Sa)\beta) = a(\alpha) + 1 + a(\beta) \neq b(\alpha) + 0 + b(\beta) = b(\alpha(Sa)\beta)$
$a(\alpha(aSb)\beta) = a(\alpha) + 1 + a(\beta) = b(\alpha) + 1 + b(\beta) = b(\alpha(aSb)\beta)$
$a(\alpha\beta) = a(\alpha) + a(\beta) \stackrel{?}{=} b(\alpha) + b(\beta) = b(\alpha\beta)$

R$\rightarrow$L By strong induction on $|w|$

Given a string of any amount/ordering of a's and b's

n=0 ① $w = \varepsilon$ $S \xrightarrow{1} \varepsilon$ $\therefore$ $S \mapsto^*$ by def

n≥1 ② $w = aw'b$, where w' has any string of a's and b's

By IH $S \mapsto^* w'$ $\therefore$ $S \xrightarrow{2} aSb \mapsto^* aw'b$

n≥1 ③ $w = bw'$, where w' has any string of a's and b's

By IH $S \mapsto^* w'$ $\therefore$ $S \xrightarrow{1} bS \mapsto^* bw'$

n≥1 ④ $w = w'a$, where w' has any string of a's and b's

By IH $S \mapsto^* w'$ $\therefore$ $S \xrightarrow{2} Sa \mapsto^* w'a$