Least squares estimation of regression lines

Regression via least squares

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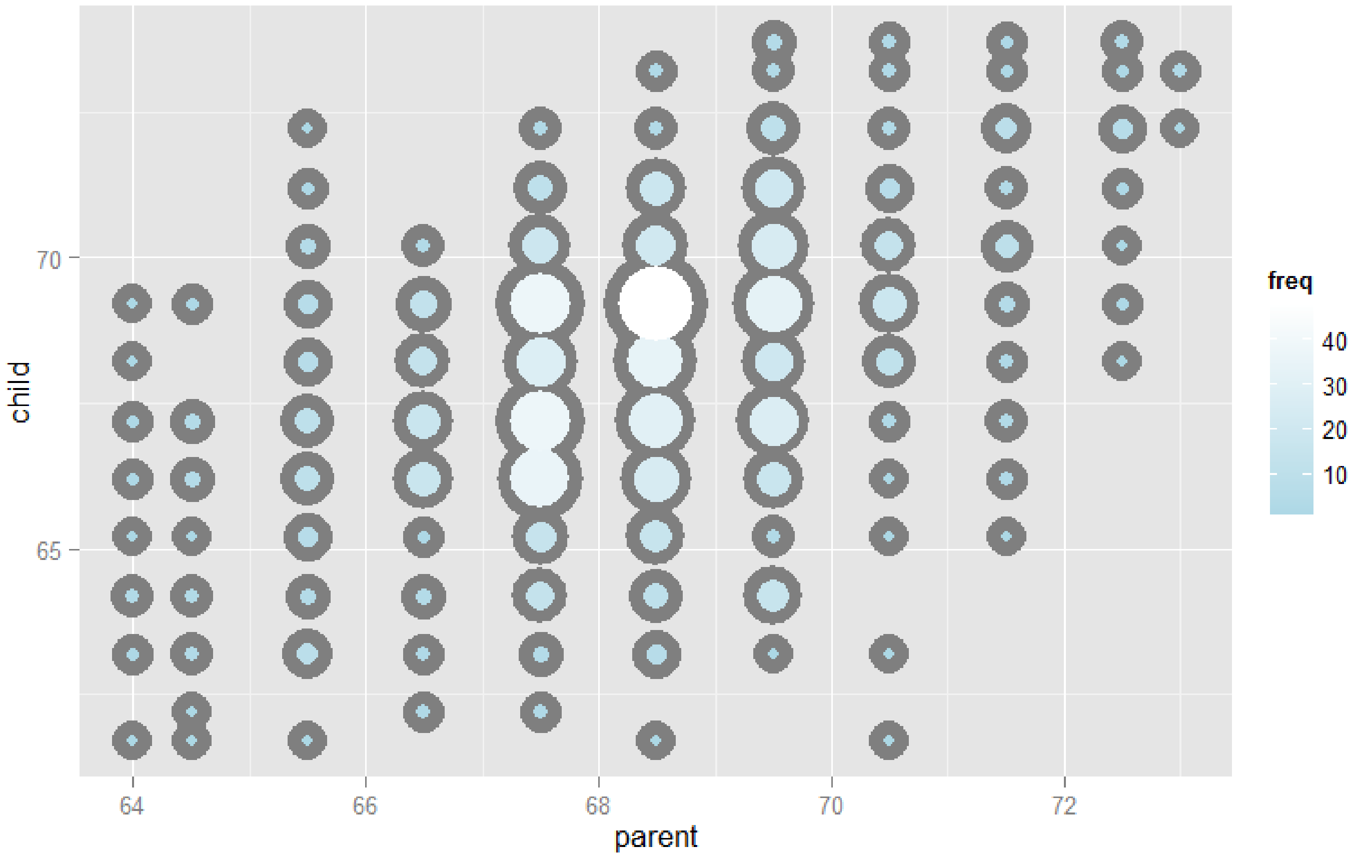
General least squares for linear equations

Consider again the parent and child height data from Galton



Fitting the best line

- Let Y_i be the i^{th} child's height and X_i be the i^{th} (average over the pair of) parents' heights.
- Consider finding the best line
  - Child's Height = β_0 + Parent's Height β_1
- Use least squares

$$\sum_{i=1}^{n} \{Y_i - (\beta_0 + \beta_1 X_i)\}^2$$

Results

- The least squares model fit to the line $Y = \beta_0 + \beta_1 X$ through the data pairs (X_i, Y_i) with Y_i as the outcome obtains the line $Y = \hat \beta_0 + \hat \beta_1 X$ where

$$\hat \beta_1 = Cor(Y, X) \frac{Sd(Y)}{Sd(X)} ~~~ \hat \beta_0 = \bar Y - \hat \beta_1 \bar X$$

- $\hat \beta_1$ has the units of Y / X, $\hat \beta_0$ has the units of Y.
- The line passes through the point $(\bar X, \bar Y)$
- The slope of the regression line with X as the outcome and Y as the predictor is $Cor(Y, X) Sd(X) / Sd(Y)$.
- The slope is the same one you would get if you centered the data, $(X_i - \bar X, Y_i - \bar Y)$, and did regression through the origin.
- If you normalized the data, $\{ \frac{X_i - \bar X}{Sd(X)}, \frac{Y_i - \bar Y}{Sd(Y)}\}$, the slope is $Cor(Y, X)$.

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Revisiting Galton's data

Double check our calculations using R

```
y <- galton$child
x <- galton$parent
beta1 <- cor(y, x) *  sd(y) / sd(x)
beta0 <- mean(y) - beta1 * mean(x)
rbind(c(beta0, beta1), coef(lm(y ~ x)))
```

```
     (Intercept)      x
[1,]       23.94 0.6463
[2,]       23.94 0.6463
```

Revisiting Galton's data

Reversing the outcome/predictor relationship

```
beta1 <- cor(y, x) *  sd(x) / sd(y)
beta0 <- mean(x) - beta1 * mean(y)
rbind(c(beta0, beta1), coef(lm(x ~ y)))
```

```
     (Intercept)      y
[1,]       46.14 0.3256
[2,]       46.14 0.3256
```

Revisiting Galton's data

Regression through the origin yields an equivalent slope if you center the data first

```
yc <- y - mean(y)
xc <- x - mean(x)
beta1 <- sum(yc * xc) / sum(xc ^ 2)
c(beta1, coef(lm(y ~ x))[2])
```

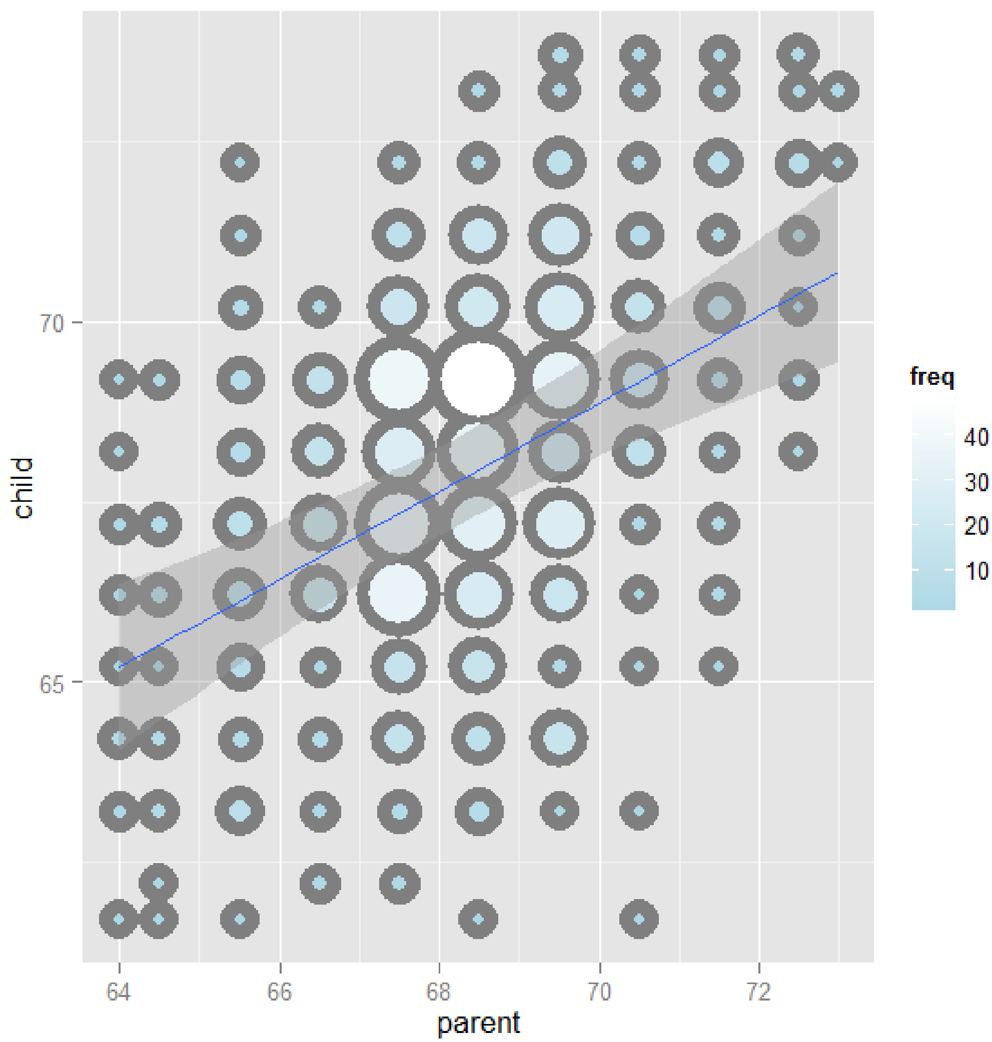
```
            x
0.6463 0.6463
```

Revisiting Galton's data

Normalizing variables results in the slope being the correlation

```
yn <- (y - mean(y))/sd(y)
xn <- (x - mean(x))/sd(x)
c(cor(y, x), cor(yn, xn), coef(lm(yn ~ xn))[2])
```

```
                  xn
0.4588 0.4588 0.4588
```


70
65
child
64
66
68
70
72
parent
freq
40
30
20
10