Boundary Condition Problem Statement

*** The final question is in bold text at the end of this document***

For this problem statement I am comparing 2 papers which discuss boundary value problems (BVP) for continuity between regions. I have a very good understanding of paper 1 (HAM) and am trying to apply the rules from paper 1 into the rules of paper 2 (HSS). All equations relating to paper 1 will be coloured in red, while all equations relating to paper 2 will be coloured in blue. It is important to note that paper 1 uses +-Z as the normal direction, while paper 2 uses +-Y as the normal direction and both use +-X as the tangential direction.

Let's define a scenario where you want to calculate the unknown coefficients to link two materials (a.k.a regions) through equations. This can be done using boundary conditions. To make sure that the continuity in both the x and y (or z) direction is preserved, we must follow the equations below:

Eqns 1 & 2:

$$\boxed{\begin{aligned} B_z^i(x,z)\big|_{z=h_b} &= B_z^{i+1}(x,z)\big|_{z=h_b} \\ H_x^i(x,z)\big|_{z=h_b} &= H_x^{i+1}(x,z)\big|_{z=h_b}. \end{aligned}}$$

Where $\mathbf{B_z}$ (magnetic flux density in the normal direction), $\mathbf{H_x}$ (magnetic field strength in the tangential), **i** (arbitrary region), **hb** (z coordinate at the boundary between region i and region i+1). Equations 1 and 2 state that Bz from one region must equal Bz of the neighbouring region at the boundary of the regions. Same principle applies to Hx.

Eqns 3 to 5:

$$\boxed{\begin{aligned} \vec{B}^i(x,z) &= B_x^i(x,z)\hat{e}_x + B_z^i(x,z)\hat{e}_z \\ B_x^i(x,z) &= \mu_0 M_{x0}^i + \sum_{n=1}^{N}\left[B_{xsn}^i(z)\sin\left(\omega_n^i x\right)\right. \\ &\qquad\qquad \left. + B_{xcn}^i(z)\cos\left(\omega_n^i x\right)\right] \\ B_z^i(x,z) &= \sum_{n=1}^{N}\left[B_{zsn}^i(z)\sin\left(\omega_n^i x\right) + B_{zcn}^i(z)\cos\left(\omega_n^i x\right)\right] \end{aligned}}$$

Eqn 6:

$$\boxed{\omega_n^i = n\omega_0 = n\frac{2\pi}{x_p}}$$

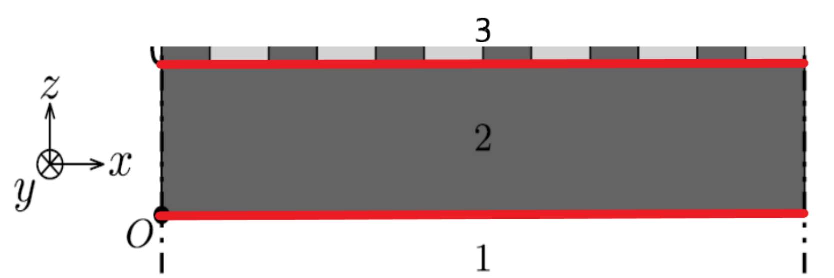
Although only B is defined in equations 3 to 5, H is calculated using **H = B/u**. Where u is the permeability of the material in the region. Equations 7 to 10 expand on the variables in equations 4 and 5 which contain the unknown variables **q_n^i, r_n^i, s_n^i, t_n^i**, that need to be solved in the matrix equation **Ax=B** (will be discussed later). Ignore any terms that include **M_{zcn}, M_{zsn}, J_{ycn}, J_{ysn}** in equations 9 and 10, these are not necessary for the problem.

Eqns 7 to 10:

$$
\begin{aligned}
B^i_{xsn}(z) &= s^i_n\, e^{\omega^i_n z} - t^i_n\, e^{-\omega^i_n z} \\
B^i_{xcn}(z) &= q^i_n\, e^{\omega^i_n z} - r^i_n\, e^{-\omega^i_n z} \\
B^i_{zsn}(z) &= q^i_n\, e^{\omega^i_n z} + r^i_n\, e^{-\omega^i_n z} + \mu_0 M^i_{zsn} + \frac{\mu_0 \mu_r}{\omega^i_n} J^i_{ycn} \\
B^i_{zcn}(z) &= -s^i_n\, e^{\omega^i_n z} - t^i_n\, e^{-\omega^i_n z} + \mu_0 M^i_{zcn} - \frac{\mu_0 \mu_r}{\omega^i_n} J^i_{ysn}.
\end{aligned}
$$

Let's say that we have regions 1, 2, and 3 like in figure 1 below. The red lines highlight the boundary between regions and in this case we want to calculate the equations for the boundary conditions on region 2 and its neighbouring regions. It is important to note that all 3 regions have the same width in the x direction, which results in **$w_n^i = w_n^{i+1}$**. This allows us to simplify equations 4 and 5 (in boundary condition 1 and 2) since the sin(wnx) and cos(wnx) terms cancel each other, leaving only equations 11 to 14.

Fig 1:



Eqns 11 to 14:

$$
\begin{aligned}
B^i_{zsn}(h_b) &= B^{i+1}_{zsn}(h_b) \\
B^i_{zcn}(h_b) &= B^{i+1}_{zcn}(h_b).
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mu_r^{i+1}}{\mu_r^i} B^i_{xsn}(h_b) &= B^{i+1}_{xsn}(h_b) \\
\frac{\mu_r^{i+1}}{\mu_r^i} B^i_{xcn}(h_b) &= B^{i+1}_{xcn}(h_b)
\end{aligned}
$$

Jumping ahead we can discuss the matrix equation Ax=B to solve for the unknown variables q_n^i, r_n^i, s_n^i, t_n^i. The coefficients in front of these variables from the resulting boundary conditions in equations 11 to 14 will be used in matrix A. The unknown variables will be in matrix X. and the result of the equation will be in matrix B. For example, a boundary equation for one value of n is:

$$0q_0^1 + 1r_0^1 + 3s_0^1 - 1t^1 - 3q_0^2 + 0r_0^2 + 0s_0^2 - 5t_0^2 = 4,$$

Then the entries into the equation Ax=B will look like:

$$[0, 1, 3, -1, -3, 0, 0, -5][q_0^1, r_0^1, s_0^1, t_0^1, q_0^2, r_0^2, s_0^2, t_0^2,]^T=[4].$$

If all boundary conditions (continuous and non-continuous) were put into the equation Ax=B then the unknown variables in matrix X are solvable. Going back to figure 1, there are 4N unknown variables to solve for region 2. At each boundary of region 2 there are N sin equations ($\mathbf{B_{Zsn}^i=B_{Zsn}^{i+1}}$ and $\mathbf{u_r^{i+1}/u_r^iB_{xsn}^i=B_{xsn}^{i+1}}$) and N cos equations ($\mathbf{B_{Zcn}^i=B_{Zcn}^{i+1}}$ and $\mathbf{u_r^{i+1}/u_r^iB_{xcn}^i=B_{xcn}^{i+1}}$). This means that there are 4N equations and 4N unknowns, making this a linearly independent set of equations to solve. Having matrix A as a square matrix is important because the solution of matrix X will then be the exact solution to the problem.

The integration of this concept into paper 2 is what I am looking into solving and having issues with. The unknown variables in equation 15 and 16 are $\mathbf{a_n}$ and $\mathbf{b_n}$.

Eqn 15 & 16

$$B_x(x,y,t) = \frac{\partial A_z(x,y,t)}{\partial y} = \sum_{n=-\infty}^{\infty}\left[\lambda_n\left(a_n e^{\lambda_n y} - b_n e^{-\lambda_n y}\right)\right] e^{j\omega_n x} e^{j(2\pi f t + \omega_n v t)},$$

$$B_y(x,y,t) = -\frac{\partial A_z(x,y,t)}{\partial x} = -j\sum_{n=-\infty}^{\infty}\left[\omega_n\left(a_n e^{\lambda_n y} + b_n e^{-\lambda_n y}\right)\right] e^{j\omega_n x} e^{j(2\pi f t + \omega_n v t)}.$$

How do I keep my matrix A square since the number of unknown variables does not match up with the number of equations? Proposed solutions below:

There is one potential solution. This solution maintains that there are 2N equations and 2N variables per region.

Fig 2:

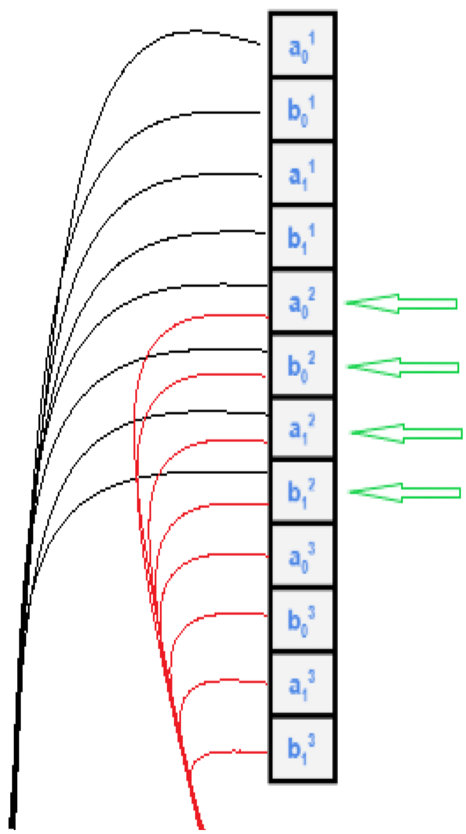
Region i | Region i + 1

$$B_x(x,y,t) = \sum_{n=-\infty}^{\infty}\left[\lambda_n\left(a_n e^{\lambda_n y} - b_n e^{-\lambda_n y}\right)\right] e^{j\omega_n x} e^{j(2\pi f t + \omega_n v t)} = \sum_{n=-\infty}^{\infty}\left[\lambda_n\left(a_n e^{\lambda_n y} - b_n e^{-\lambda_n y}\right)\right] e^{j\omega_n x} e^{j(2\pi f t + \omega_n v t)}$$

$$B_y(x,y,t) = -j\sum_{n=-\infty}^{\infty}\left[\omega_n\left(a_n e^{\lambda_n y} + b_n e^{-\lambda_n y}\right)\right] e^{j\omega_n x} e^{j(2\pi f t + \omega_n v t)} = -j\sum_{n=-\infty}^{\infty}\left[\omega_n\left(a_n e^{\lambda_n y} + b_n e^{-\lambda_n y}\right)\right] e^{j\omega_n x} e^{j(2\pi f t + \omega_n v t)}$$

Then by combining the Bx=Bx condition for both boundaries into one equation and the same approach for Hx=Hx, there are now 2N equations per boundary but the equations overlap in variables. For example:

Fig3:



Region Variables	$Hx_{1\text{-}2}^1=Hx_{1\text{-}2}^2$	$Hx_{2\text{-}3}^2=Hx_{2\text{-}3}^3$	$By_{1\text{-}2}^1=By_{1\text{-}2}^2$	$By_{2\text{-}3}^2=By_{2\text{-}3}^3$
1	X		X	
2	X	X	X	X
3		X		X

Consider this array X where a_n^i and b_n^i are the unknown variables for each harmonic (n) and region (i). This array assumes that there are only 2 harmonics and there are 3 regions. These 3 regions can be seen in the superscript and the 2 harmonics can be seen in the subscript. In figure 3 the unknown variables for all 3 regions can be seen in matrix form (matrix X). The black and red lines highlight the variables used in the respective boundary equation. The 4 green arrows indicate the unknown variables that are used by both equations.

Consider this alternative. Instead of combining the Hx and By equations between regions (like in the figure above), combine the Hx and By equations for each boundary which means that they both use the same variables for each equation.

Now that I think about it, I don't think it is possible to combine the Hx equations for each boundary into 1 and the same goes for By equations and the same goes for the other proposed solution as shown in dark red above. This is because you are changing the original equation by indexing the coefficients from both equations into 1 equation. The only way is to have N equations for one boundary and N equations for the other or changing the unknown variable count to 4N (q,r,s,t).

How did the HAM paper figure this out? That is an important question, because they say N sin and N cos per boundary but there are 4 equations per boundary so maybe I don't understand the HAM paper well enough before continuing onto this