Solvers for the Vlasov Equation

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Overview of The Vlasov Equation

- The fundamental equation of plasma physics. Used to model fusion devices, astrophysical systems, etc. Basis of fluid-theory, MHD.
- Fundamental variable if $f(x, v, t)$, the phase space density of particles
- Describes particles interacting consistently with self-generated fields:

$$\partial_t f(x, v, t) + v \partial_x f + \partial_v (E(x, t) f) = 0$$

$$E'(x, t) = -\int dv\, f(x, v, t) dx$$

- E can also be prescribed for testing. E.g., $E = 0 \implies f(x, v, t) = f(x - vt, v, 0)$.
- Vlasov equation is a hyperbolic conservation law. Simulate using finite volume.

Finite Volumes

- Let (x_i, v_j) be equispaced mesh of domain. Define

$$f(x,v,t) = \frac{1}{\Delta x \Delta v} \int_{x_i}^{x_{i+1}} \int_{v_i}^{v_i} f(x,v,t) dv\, dx$$

- Vlasov equation becomes

$$\frac{df_{i+1/2,j+1/2}}{dt} + \frac{F^x_{i+1,j+1/2} - F^x_{i,j+1/2}}{\Delta x} + \frac{F^v_{i+1/2,j+1} - F^v_{i+1/2,j}}{\Delta v} = 0$$

  with $F^v_{i+1/2,j} = \frac{1}{\Delta x} \int_{x_i}^{x_{i+1}} E(x,t) f(x, v_j, t) dx$, $F^x_{i,j+1/2} = \frac{1}{\Delta v} \int_{v_j}^{v_{j+1}} v f(x_i, v, t) dv$.
- To second order, approximate numerical fluxes by $F^v_{i+1/2,j} \approx v_{j+1/2} f(x_{i+1/2}, v_j)$.
- $f(x_i, v_{j+1/2})$ can be reconstructed from right ($f^+_{i+1/2,j+1/2}$) or left ($f^-_{i,j+1/2}$) of edge.
- Upwinding! Use left value $f^-_{i,j+1/2}$ when $v_{j+1/2} > 0$. Otherwise, use $f^+_{i,j+1/2}$.
- Similar for $f^\pm_{i+1/2,j}$. Upwinding determined by $E(x_{i+1/2})$.

First Order Scheme

- To first order accuracy, set $f^{\pm}_{i,j+1/2} = f_{i+\pm 1/2, j+1/2}$. Use to get $F^x[f], F^v[f]$.
- Use Forward Euler to integrate semi-discrete form. Discrete Vlasov equation reads

$$\frac{f^{n+1}_{i+1/2,j+1/2} - f^{n}_{i+1/2,j+1/2}}{dt} + \frac{F^x_{i+1,j+1/2} - F^x_{i,j+1/2}}{\Delta x} + \frac{F^v_{i+1/2,j+1} - F^v_{i+1/2,j}}{\Delta v} = 0$$

- Scheme is stable and positivity-preserving if $C^n = (\frac{\Delta t}{\Delta x} + \frac{\Delta t \max(|E^n|)}{\Delta v}) < 1$.
- Time step must be adapted at each step to ensure $C^n < 1$.
- E^n solved for at each step using

$$E^n(x_{i+1}) - E^n(x_i) = C + \Delta x \Delta v \sum_{i,j} f^n_{i+1/2,j+1/2}.$$

- C such that the system is solvable. Choose $\sum_i E(x_i) = 0$.
- To second order accuracy, $E(x_{i+1/2})$ is set to $\frac{1}{2}(E(x_i) + E(x_{i+}))$.
- The first-order scheme is very diffusive. Better solvers are needed.

Second Order Scheme

- To second-order accuracy, use $f^{\pm}_{i,j+1/2} = f_{i+\pm 1/2,j+1/2} \mp \frac{1}{2}(\Delta^x f_{i+\pm 1/2,j+1/2})$.
- $(\Delta^x f_{i+1/2,j+1/2})$ are the reconstructed slopes. We use the central slope

$$(\Delta^x f_{i+1/2,j+1/2})_C = \frac{f_{i+3/2,j} - f_{i-1/2,j}}{2}$$

- Other choices are the left and right slopes

$$(\Delta^x f_{i+1/2,j+1/2})_R = f_{i+3/2,j} - f_{i+1/2,j}, \;\; (\Delta^x f_{i+1/2,j+1/2})_L = f_{i+1/2,j} - f_{i=1/2,j}$$

- Numerical fluxes obtained from $f^{\pm}$ using upwinding (Fromm scheme).
- Time integration of semi-discrete performed using RK(2) and the $E(x_{i+1/2})$ solver.
- Scheme is stable for $C < 1$. A timestep must be repeated if $C^{n+1/2} > 1$.
- Scheme not positivity preserving. Spurious oscillations from numerical dispersion.
- Scheme is second-order in space and time.

Slope Limited Second Order Scheme

- Slope limiter prevents oscillations, better preserves positivity.
- Assume, temporarily, that f is monotonically increasing in x.
- The moncen limiter ensures $f_{i+1/2,j+1/2} < f^{+}_{i,j+1/2} < f^{-}_{i+1,j+1/2} < f_{i+3/2,j+1/2}$.
- Moncen takes $(\Delta^x f_{i+1/2,j+1/2})_M = 2\min((\Delta^x f_{i+1/2,j+1/2})_C, 2(\Delta^x f_{i+1/2,j+1/2})_L, (\Delta^x f_{i+1/2,j+1/2})_R)$.
- $(\Delta^x f_{i+1/2,j+1/2})_M = 0$ at extremum.
- $(\Delta^x f_{i+1/2,j+1/2})_M$ reduces to $(\Delta^x f_{i+1/2,j+1/2})_C$ in smooth regions.
- We replace $(\Delta^x f_{i+1/2,j+1/2})_C$ in last scheme by $(\Delta^x f_{i+1/2,j+1/2})_M$.
- New scheme second-order and stable for $C < 1$. Positivity preserving for $C < 1/2$.
- Vlasov solver with moncen limiting tends to be positive and non-oscillatory.
- A very good method overall.

- We compare the first and slope-limited second-order scheme for simulating a two-stream instability. We show the output at a fixed time.

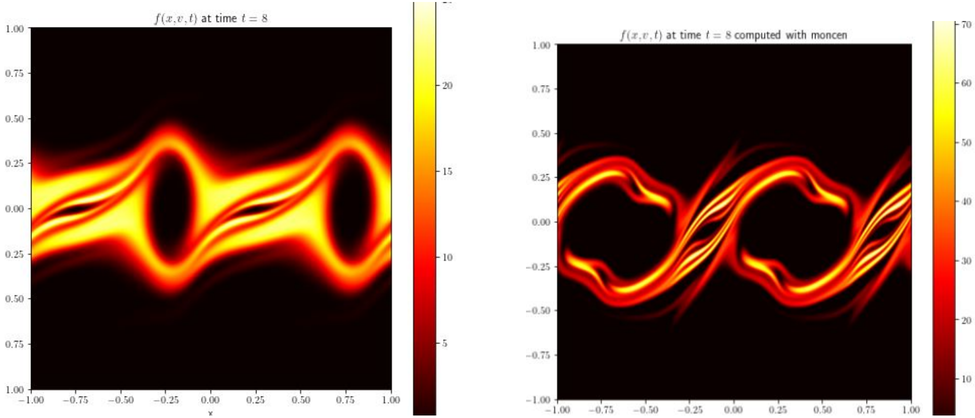

Figure: Left: First order method. Right: Second order w/ limiter.