A Wasserstein-type distance in the space of Wrapped Gaussian Mixtures

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Abstract

We present a closed form expression for the Wasserstein Distance between two Wrapped Normal Distributions on the sphere. We then show how to use this distance to extend some results for mixtures of Gaussians in $\mathbb{R}^d$ to mixtures of Wrapped Normal distributions on the sphere.

1 Theoretical Details

Let $\mu = \mathcal{WN}(m, \Sigma)$ be a wrapped normal distribution defined on the $d-1$ dimensional sphere S^{d-1}. We can define the density of μ in terms of the tangent space $T_m(S_{d-1})$, with orthonormal basis B as;

$$\rho_\mu(x) = det(2\pi\Sigma)exp(-\frac{1}{2}\langle BExp_m^{-1}(x), \Sigma^{-1}BExp_m^{-1}(x)\rangle)$$

Here exp corresponds to the exponential function, while Exp corresponds to the Exponential Map for S^{d-1}.

1.1 Wrapped Normal Wasserstein Distance

Given $\mu_i = WN(m_i, \Sigma_i)$, $i = \{0, 1\}$, where $m_i \in S_{d-1}$, and $\Sigma_i^{d-1\times d-1}$, (and with densities ρ_i), the Wasserstein Distance between μ_0 and μ_1 can be calculated as;

$$WNW_2(\mu_0, \mu_1) = cos^{-1}(\langle m_0, m_1\rangle) + tr(\Sigma_0 + \Sigma_1 - 2(\Sigma_0^{\frac{1}{2}}\Sigma_1\Sigma_0^{\frac{1}{2}})^{\frac{1}{2}}) \quad (1)$$

Proof:

In order to prove this, we must find a continuous map $T : S^{d-1} \times S^{d-1}$ such that;

1) T pushes forward μ_0 to μ_1 ($\iff$ $\rho_0(x) = \rho_1(Tx)|detT'|$)

2) T is optimal

3) The transport cost of T is WNW

2 Implementation Details

[1] [2]

References

[1] Julie Delon and Agnes Desolneux. A wasserstein-type distance in the space of gaussian mixture models, 2019.

[2] Asuka Takatsu. On wasserstein geometry of the space of gaussian measures, 2008.