A Wasserstein-type distance in the space of Wrapped Gaussian Mixtures

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Abstract

We present a closed form expression for the Wasserstein Distance between two Wrapped Gaussian Distributions on the sphere. We then show how to use this distance to extend some results for mixtures of Gaussians in $\mathbb{R}^d$ to mixtures of Wrapped Gaussians on the sphere.

1 Theoretical Details

1.1 Wrapped Gaussian Wasserstein Distance

Let $\mu_i = N(0, \Sigma_i)$, $i = \{0, 1\}$ be Gaussian distributions defined on $\mathbb{R}^{d-1}$, with random variables $X_i \sim \mu_i$. For some $m_i \in S^{d-1}$, let $\tilde{\mu}_i \sim Exp(m_i)_{\#}\mu_i$, and let $\tilde{X}_i \sim \tilde{\mu}_i$. Then the Wasserstein distance between $\tilde{\mu_0}$ and $\tilde{\mu}_1$ can be calculated as;

$$W_2(\tilde{\mu}_0, \tilde{\mu}_1) = cos^{-1}(\langle m_0, m_1 \rangle) + (\Sigma_0 + \Sigma_1 - 2(\Sigma_0^{\frac{1}{2}} \Sigma_1 \Sigma_0^{\frac{1}{2}})^{\frac{1}{2}}) \tag{1}$$

Proof:

Let $\tilde{X} = \tilde{X}_0$ and $\tilde{Y} = Exp_{m_0}(R_\alpha(Exp_{m_1}^{-1}(\tilde{X}_1)))$. Then,

$$W_2(\tilde{\mu}_0, \tilde{\mu}_1) = \inf_{\pi \in \Pi(\tilde{\mu}_0, \tilde{\mu}_1)} E||\tilde{X}_1 - \tilde{X}_2||$$

$$= ||E\tilde{X}_1 - E\tilde{X}_2|| + \inf_{\pi \in \Pi(\tilde{\mu}_0, \tilde{\mu}_1)} E||\tilde{X} - \tilde{Y}|| \tag{2}$$

$$= cos^{-1}(\langle m_0, m_1 \rangle) + \inf_{\pi \in \Pi(\mu_0, \mu_1)} E||X - Y|| \tag{3}$$

$$= cos^{-1}(\langle m_0, m_1 \rangle) + tr(\Sigma_0 + \Sigma_1 - 2(\Sigma_0^{\frac{1}{2}} \Sigma_1 \Sigma_0^{\frac{1}{2}})^{\frac{1}{2}})$$

Where the equality in line 2 holds since;

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(I think for someone who understands geometry, a proof following that of [4] would be trivial. The issue with my approach is that parallel transport on the sphere does not correspond to a simple translation in the tangent space at a given point: A wrapped normal centered at $m_1 \neq m_0$ does not get pulled back to a normal distribution in $T_{m_0}(S^{d-1})$ by $Exp^{-1}(m_1)^{\#}$.)

and the equality in line 3 holds since $\forall \pi \in \Pi(\mu_0, \mu_1)$, if $\tilde{\pi} = Exp(m_0, m_1)_{\#}\pi$ we have that

$$E||\tilde{X} - \tilde{Y}||$$

$$= \int_{S^{d-1} \times S^{d-1}} cos^{-1}(\langle \tilde{x}, \tilde{y} \rangle) \; d\tilde{\pi}(\tilde{x}, \tilde{y})$$

$$= \int_{S^{d-1} \times S^{d-1}} \sqrt{\langle Exp_{m_0}^{-1}(\tilde{x}_0) - R_\alpha(Exp_{m_1}^{-1}(\tilde{x}_1), Exp_{m_0}^{-1}(\tilde{x_0}) - R_\alpha(Exp_{m_1}^{-1}(\tilde{x_1})) \rangle} \; d\tilde{\pi}(\tilde{x_0}, \tilde{x_1})$$

$$= \int_{\mathbb{R}^{d-1} \times \mathbb{R}^{d-1}} \sqrt{\langle x - y, x - y \rangle} \; d\pi(x, y)$$

$$= E||X - Y||$$

$$\implies \inf_{\tilde{\pi} \in \Pi(\tilde{\mu}_0, \tilde{\mu}_1)} E||\tilde{X} - \tilde{Y}|| = \inf_{\pi \in \Pi(\mu_0, \mu_1)} E||X - Y|| \tag{4}$$

1.2 Wrapped Gaussian Mixture Wasserstein-like Distance

Our proof is identical to the one presented in section 4.2 of [2], except replacing their W_2^2 with our W_2, and their $GMM(*)$ with our $WGMM(*)$.

2 Implementation Details

Let $X \in \mathbb{R}^{d \times n}$, $Y \in \mathbb{R}^{d \times m}$, with X_i (resp. Y_i) denoting the the ith column of X (resp. Y).

References

[1] Computational optimal transport. *Foundations and Trends in Machine Learning*, 11(5-6):355–607, 2019.

[2] Julie Delon and Agnes Desolneux. A wasserstein-type distance in the space of gaussian mixture models, 2019.

[3] Clark R. Givens and Rae Michael Shortt. A class of Wasserstein metrics for probability distributions. *Michigan Mathematical Journal*, 31(2):231 – 240, 1984.

[4] Asuka Takatsu. On wasserstein geometry of the space of gaussian measures, 2008.

[2] [4] [3] [1]