A Wasserstein-type distance in the space of Wrapped Gaussian Mixtures

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Abstract

We present a closed form expression for the Wasserstein Distance between two Wrapped Gaussian Distributions on the sphere. We then show how to use this distance to extend some results for mixtures of Gaussians in $\mathbb{R}^d$ to mixtures of Wrapped Gaussians on the sphere.

1 Implementation Details

Proofs are left to Section 2.

1.1 Wrapped Gaussian Wasserstein Distance

Let $\mu_i = WN(m_i, \Sigma_i)$, $i = \{0, 1\}$. Let $U_0 S_0 U_0^t$ denote the SVD of Σ_0, with u_j the jth row of U_0. Define $\Sigma_0^* = U_0^* S_0 U_0^{*t}$, where the jth row of U_0^* is the parallel transport of u_j from $T_{m_0}(S^d)$ to $T_{m_1}(S^d)$;

$$u_j^* = u_j - (2(u_j * m_1^t)(|m_0 + m_1|^2))(m_0 + m_1)$$

Then

$$W_2^2(\mu_0, \mu_1) = cos^{-1}(\langle m_0, m_1 \rangle) + tr(\Sigma_0^* + \Sigma_1 - 2(\Sigma_0^{*\frac{1}{2}} \Sigma_1 \Sigma_0^{*\frac{1}{2}})^{\frac{1}{2}})$$

1.2 Wrapped Gaussian Mixture Wasserstein-like Distance

Let $\mu_i = \Sigma_{k=1}^{K_i} \frac{w_{ik}}{\Sigma_k w_{ik}} WN(m_{ik}, \Sigma_{ik})$, $i = \{0, 1\}$, be two wrapped Gaussian mixtures. Let $\Pi(w_0, w_1) = \{W \in \mathbb{R}^{K_0 \times K_1}, \Sigma_i W_{ij} = w_{1i}, \Sigma_j W_{ij} = w_{0j}\}$. Then,

$$WMW_2^2(\mu_0, \mu_1) = \min_{W \in \Pi(w_0, w_1)} \sum_{\ell, k} W_{\ell k} W_2^2(\mu_0^\ell, \mu_1^k)$$

2 Theoretical Details

2.1 Wrapped Gaussian Wasserstein Distance

Let $\mu_i = N(0, \Sigma_i)$, $i = \{0, 1\}$ be Gaussian distributions defined on $\mathbb{R}^{d-1}$, with random variables $X_i \sim \mu_i$. For some $m_i \in S^{d-1}$, let $\tilde{\mu}_i = Exp(m_i)_\# \mu_i$, (and thus $\tilde{\mu}_i = WN(m_i, \Sigma_i)$) and let $\tilde{X}_i \sim \tilde{\mu}_i$. Then,

Proof:

Let $\tilde{X} = \tilde{X}_0$ and $\tilde{Y} = Exp_{m_0}(R_\alpha(Exp^{-1}_{m_1}(\tilde{X}_1)))$. Then,

$$W_2^2(\tilde{\mu}_0, \tilde{\mu}_1) = \inf_{\gamma \in \Pi(\tilde{\mu}_0, \tilde{\mu}_1)} \int_{TS^{d-1}} cos^{-1}(\langle \tilde{x}, \tilde{y} \rangle)\ d\gamma(\tilde{x}, \tilde{y})$$

$$= d^2_{S^{d-1}}(E\tilde{X}_1, E\tilde{X}_2) + \inf_{\gamma \in \Pi(\tilde{\mu}_0, \tilde{\mu}_1)} \int_{T_{m_0}(S^{d-1})} \sqrt{\langle Exp^{-1}_{m_0}(\tilde{x}) - R_\alpha(Exp^{-1}_{m_1}(\tilde{y}), Exp^{-1}_{m_0}(\tilde{x}) - R_\alpha(Exp^{-1}_{m_1}(\tilde{y})) \rangle}\ d\tilde{\gamma}(\tilde{x}, \tilde{y}) \tag{1}$$

$$= cos^{-1}(\langle m_0, m_1 \rangle) + \inf_{\gamma \in \Pi(\mu_0, \mu_1)} \int_{\mathbb{R}^{d-1} \times \mathbb{R}^{d-1}} \sqrt{\langle x - y, x - y \rangle}\ d\gamma(x, y) \tag{2}$$

$$= cos^{-1}(\langle m_0, m_1 \rangle) + tr(\Sigma_0 + \Sigma_1 - 2(\Sigma_0^{\frac{1}{2}} \Sigma_1 \Sigma_0^{\frac{1}{2}})^{\frac{1}{2}})$$

Where the equality in equation 1 follows from Proposition 2 in [5], using m_0 as p_{ref}, and the equality in equation 2 holds since $\forall \gamma \in \Pi(\mu_0, \mu_1)$, if $\tilde{\gamma} = Exp(m_0, m_1)_\# \gamma$ we have that

$$= \int_{TS^{d-1}} cos^{-1}(\langle \tilde{x}, \tilde{y} \rangle)\ d\tilde{\gamma}(\tilde{x}, \tilde{y})$$

$$= \int_{S^{d-1} \times S^{d-1}} \sqrt{\langle Exp^{-1}_{m_0}(\tilde{x}) - R_\alpha(Exp^{-1}_{m_1}(\tilde{y}), Exp^{-1}_{m_0}(\tilde{x}) - R_\alpha(Exp^{-1}_{m_1}(\tilde{y})) \rangle}\ d\tilde{\gamma}(\tilde{x}, \tilde{y})$$

$$= \int_{\mathbb{R}^{d-1} \times \mathbb{R}^{d-1}} \sqrt{\langle x - y, x - y \rangle}\ d\gamma(x, y)$$

$$\implies \inf_{\gamma \in \Pi(\tilde{\mu}_0, \tilde{\mu}_1)} \int_{TS^{d-1}} cos^{-1}(\langle \tilde{x}, \tilde{y} \rangle)\ d\tilde{\gamma}(\tilde{x}, \tilde{y}) = \inf_{\gamma \in \Pi(\mu_0, \mu_1)} \int_{\mathbb{R}^{d-1} \times \mathbb{R}^{d-1}} \sqrt{\langle x - y, x - y \rangle}\ d\gamma(x, y) \tag{3}$$

Furthermore, because we know that the optimal coupling in the Euclidian case is a Normal distribution, we see that the optimal coupling in the spherical case is therefore a wrapped Gaussian distribution.

2.2 Wrapped Gaussian Mixture Wasserstein-like Distance

Our proof is identical to the one presented in section 4.2 of [3], except replacing their W_2^2 with our W_2^2, and their $GMM(*)$ with our $WGMM(*)$.

References

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[3] [6] [4] [1] [2] [5]