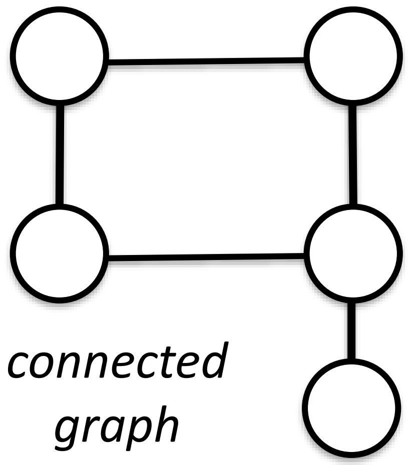
Breadth-first Search & Depth-first Search

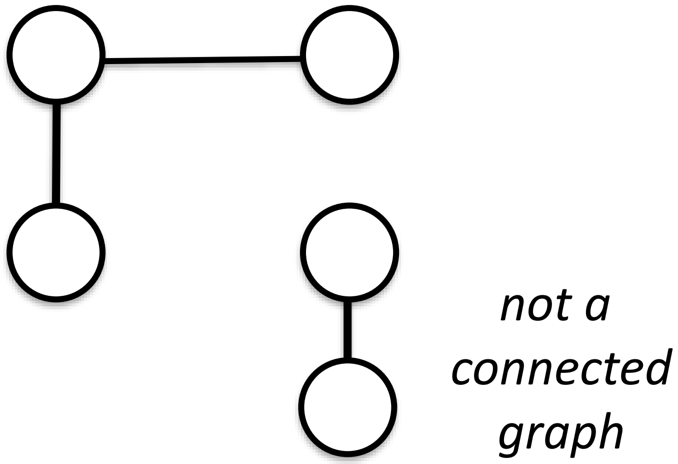
For use in CSE6010 only

Not for distribution

Statement of Motivating Problem

Definition: A *connected* undirected graph is one where there exists a path between every pair of vertices

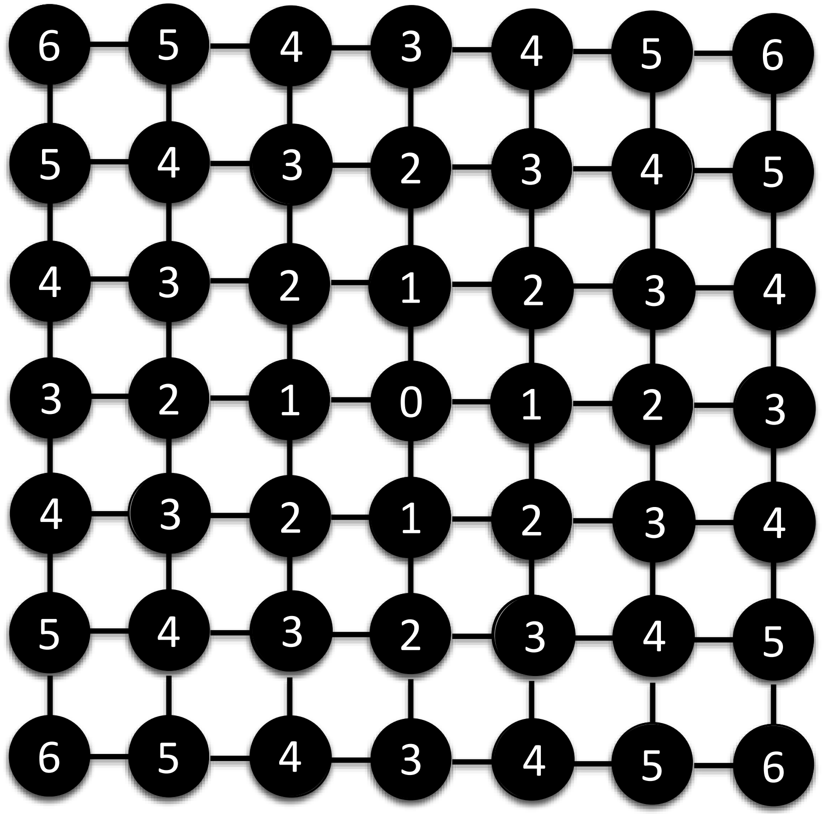




Problem: Given a connected undirected graph G (vertex set V, edge set E) and a source node S

- Visit every node in the graph starting from S
- Find the minimum distance from S to every other vertex assuming edges have unit weight.

Basic Idea



Done!

Note: traditionally, color is used to differentiate vertices we've visited already from those we have not visited yet (alt: condition)

- Initially white (unvis)
- Gray (vis) when a vertex is first visited to "mark" it, but its neighbors have not yet been explored
- Black (done) when done exploring neighbors of vertex

- Starting from S, mark all vertices 1 hop from S (neighbors of S)
- Mark all vertices 2 hops from S
  - The "2 hop" vertices are all unmarked neighbors of the "1 hop" vertices
  - So… mark neighbors of "1 hop vertices" that are not already marked as "2 hop vertices"
- Mark all vertices 3 hops from S by marking those neighbors of "2 hop" vertices that have not been marked yet, …

Breadth-first Algorithm

Vertices numbered 0, 1, 2, …, N-1

D[i] = number of hops (distance) from S to i

C[i] = color of vertex i differentiates three “categories” of nodes:

- White/unvis: vertex has not been visited (marked) yet
- Gray/vis: vertex has been visited and its distance computed, but its neighbor vertices still need to be examined
- Black/done: done with vertex (neighbors examined)

Initially

- Source node S: D[S]=0; C[S]=vis // gray
- C[i] = unvis, D[i] undefined for all i ≠ S

Breadth-first Algorithm (cont.)

Introduce FIFO queue

- Holds all vis vertices (still to be processed)
- Initially holds only source vertex S

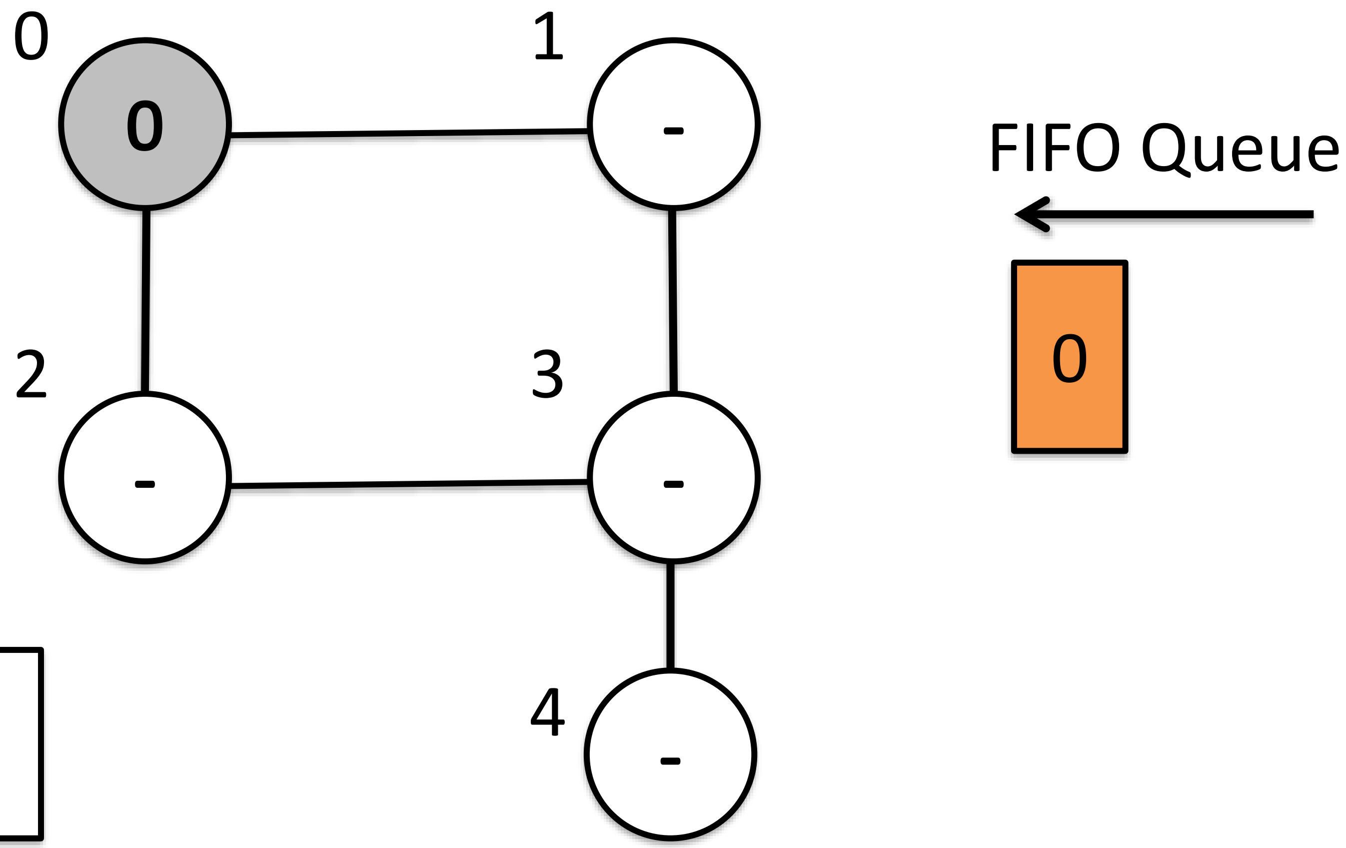
Algorithm

```
While (FIFO queue not empty)
  V = Next vertex from FIFO queue (remove it from queue)
  For each neighbor i of V
    If (C[i]==unvis)
      D[i] = D[V]+1
      C[i] = vis
      Add i to FIFO queue
  C[V] = done
```

Why do we only need to worry about neighbors of V that are unvisited?

Example

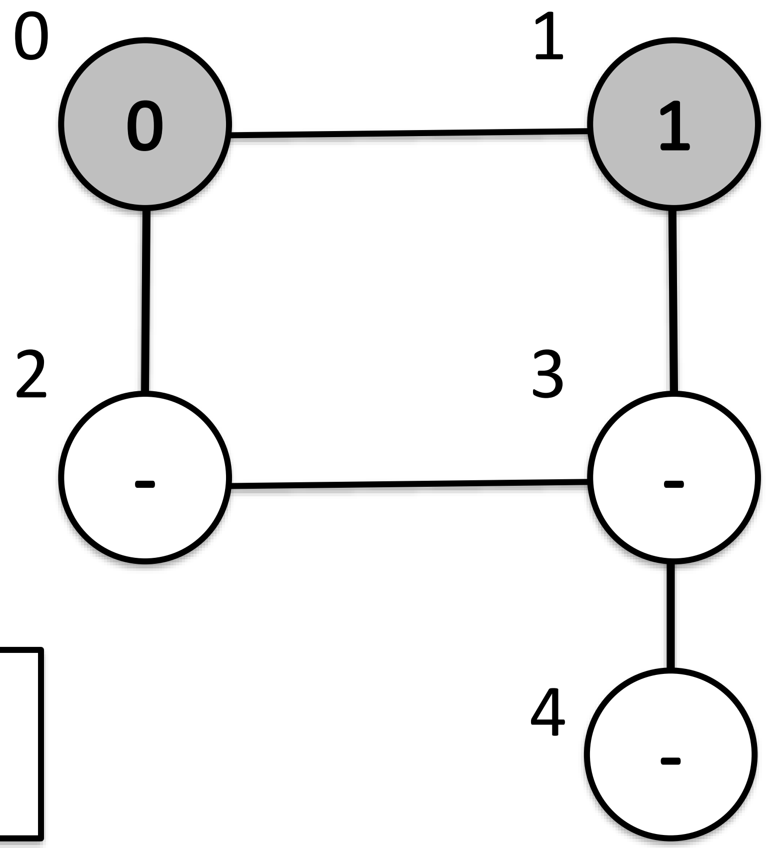
S=0



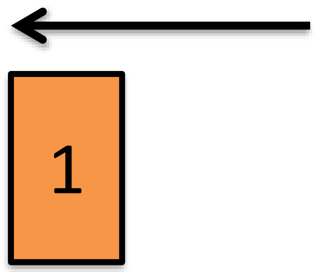
D:	0	-	-	-	-
C:					

Example

S=0



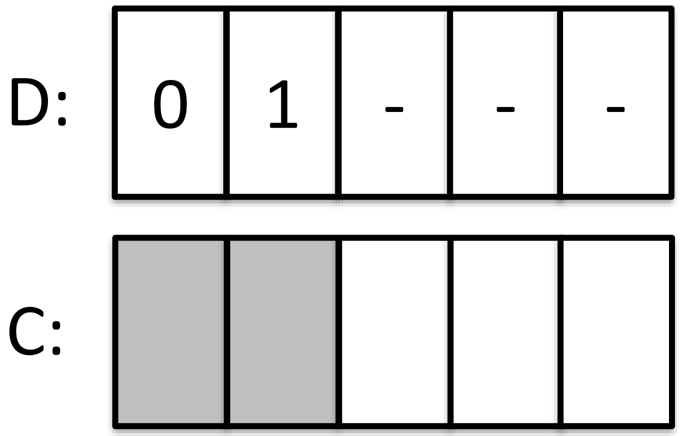
FIFO Queue



Process V=**0** (remove fr/Q):

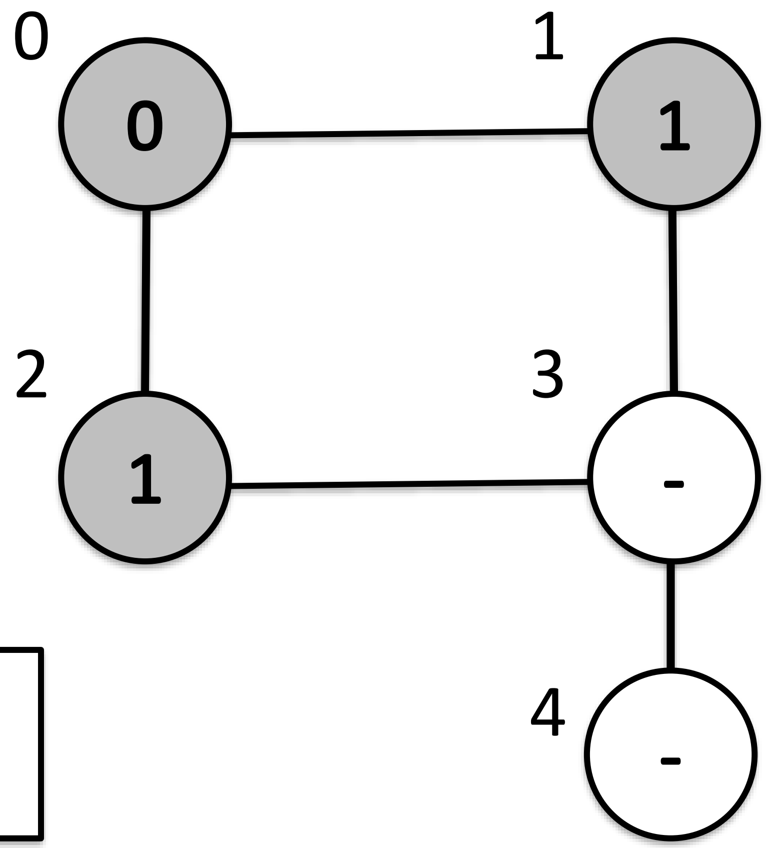
Neighbor i=1

- D[1]=1
- C[1]=vis
- Add 1 to queue

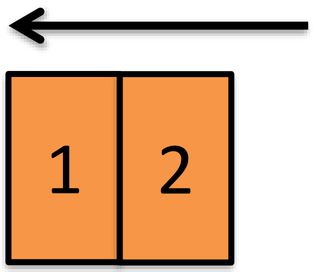


Example

S=0



FIFO Queue



Process V=**0**:

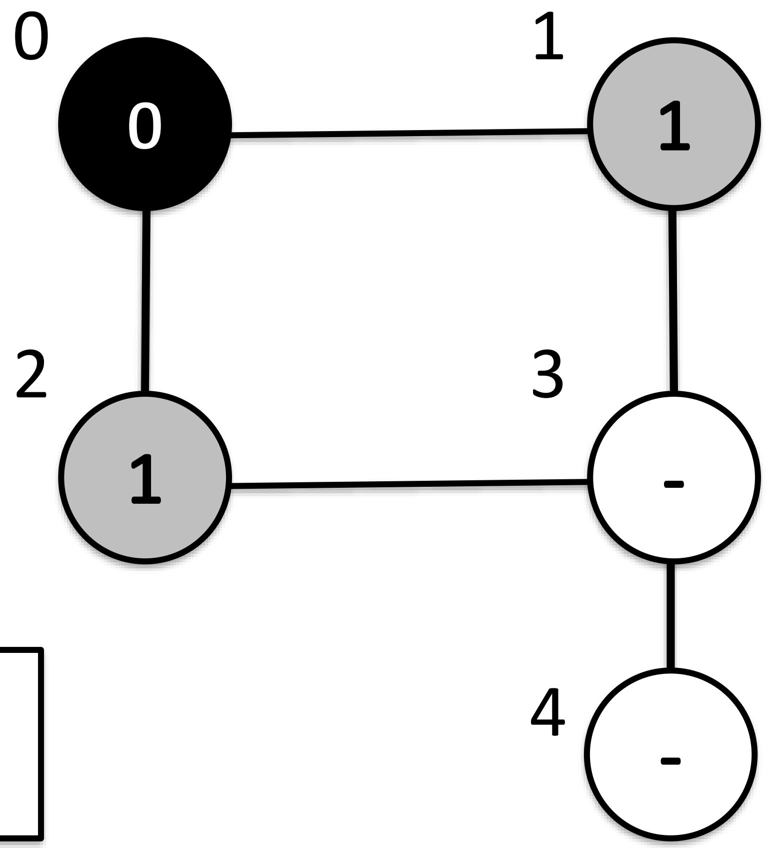
Neighbor i=2

- D[2]=1
- C[2]=vis
- Add 2 to queue

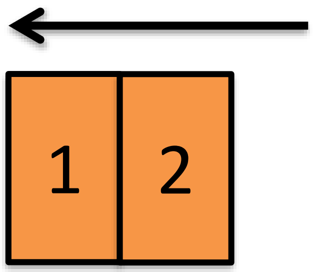
D:	0	1	1	-	-
C:					

Example

S=0



FIFO Queue



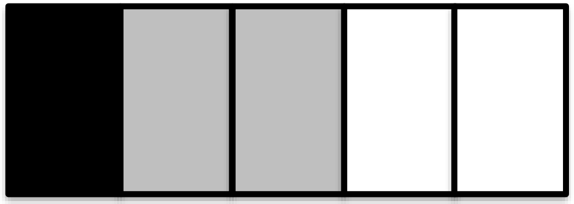
Process V=**0**:

- C[0]=done

D:

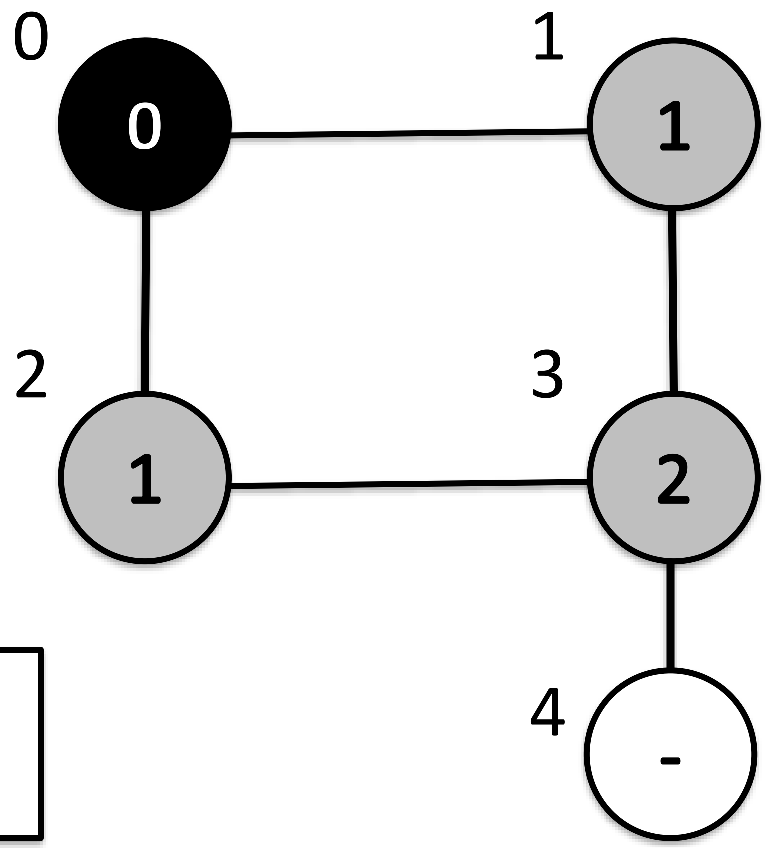
0	1	1	-	-

C:

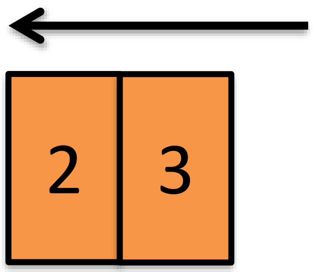


Example

S=0



FIFO Queue



Process V=**1** (remove fr/Q):

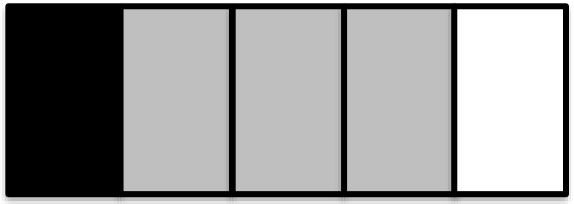
Neighbor i=3

- D[3]=2
- C[3]=vis
- Add 3 to queue

D:

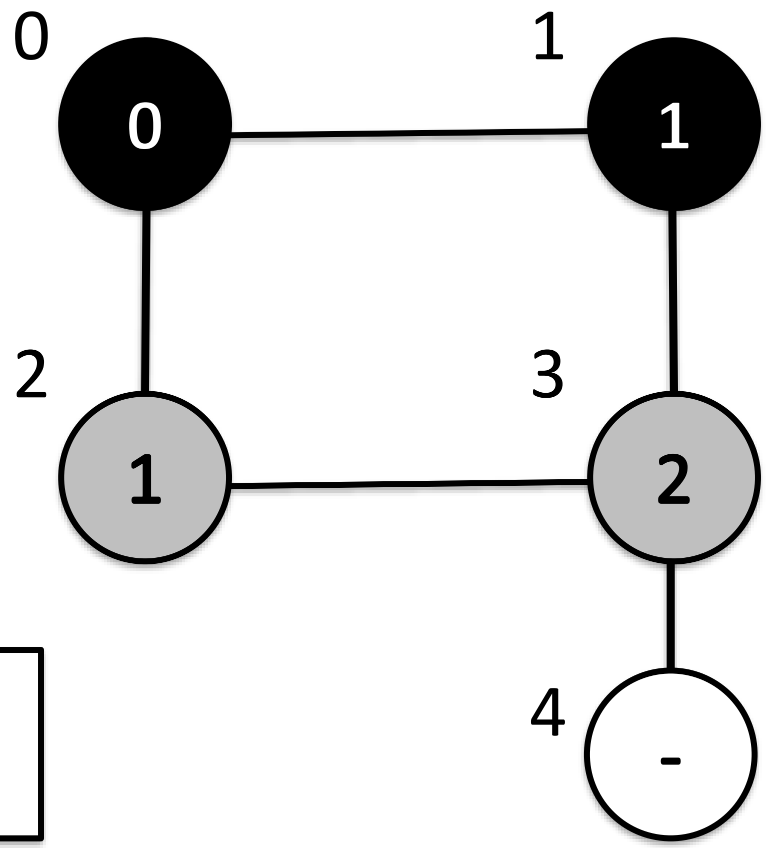
0	1	1	2	-

C:

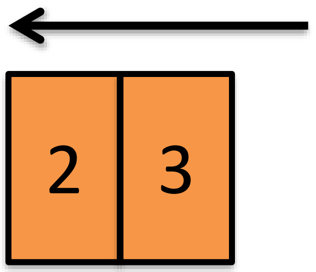


Example

S=0



FIFO Queue



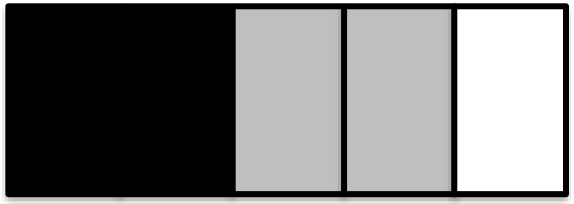
Process V=**1**:

- C[1]=done

D:

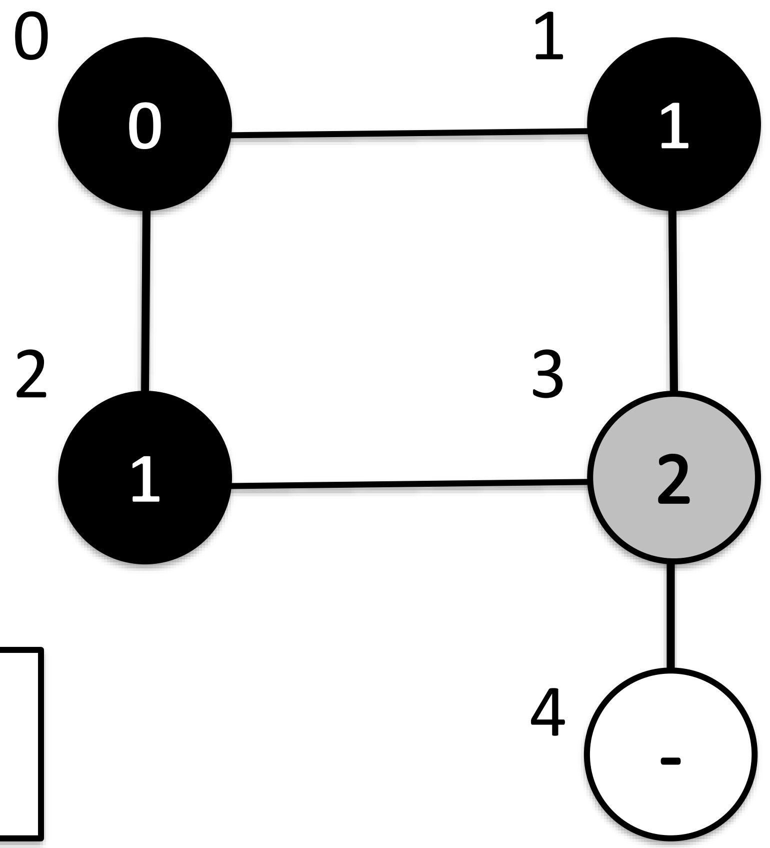
0	1	1	2	-

C:

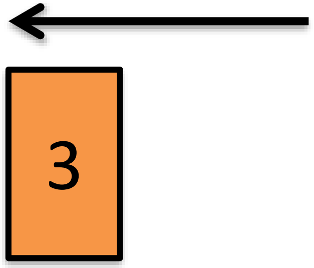


Example

S=0



FIFO Queue



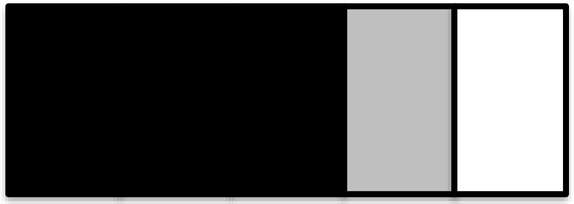
Process V=**2** (remove fr/Q):

- C[2]=done

D:

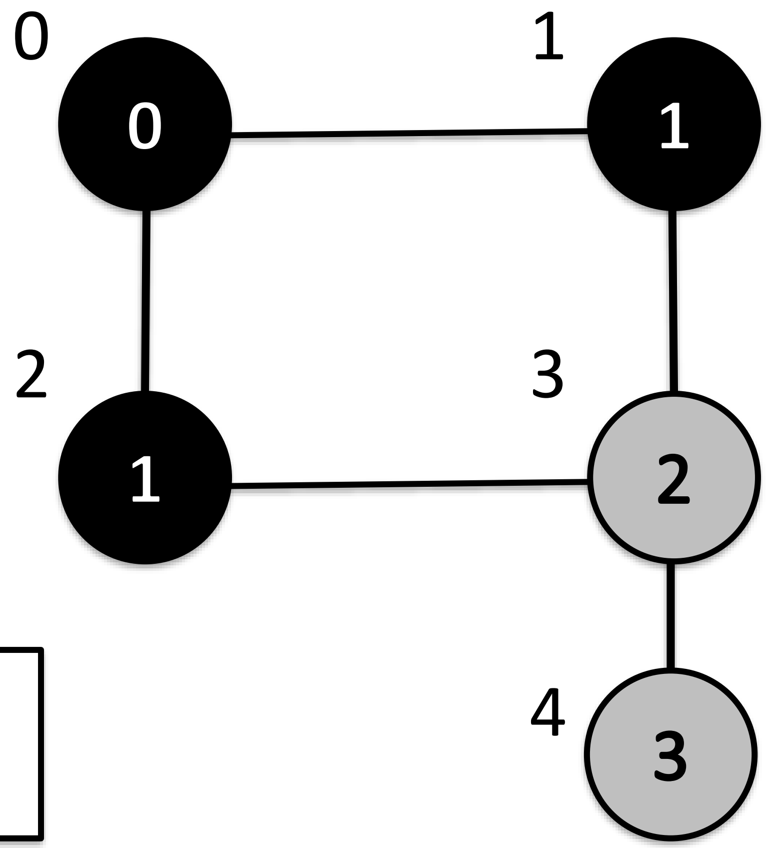
0	1	1	2	-

C:

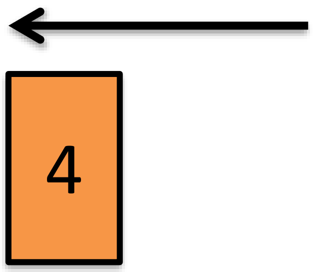


Example

S=0



FIFO Queue



Process V=**3** (remove fr/Q):

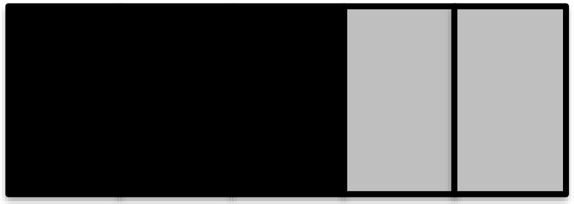
Neighbor i=4

- D[4]=3
- C[4]=vis
- Add 4 to queue

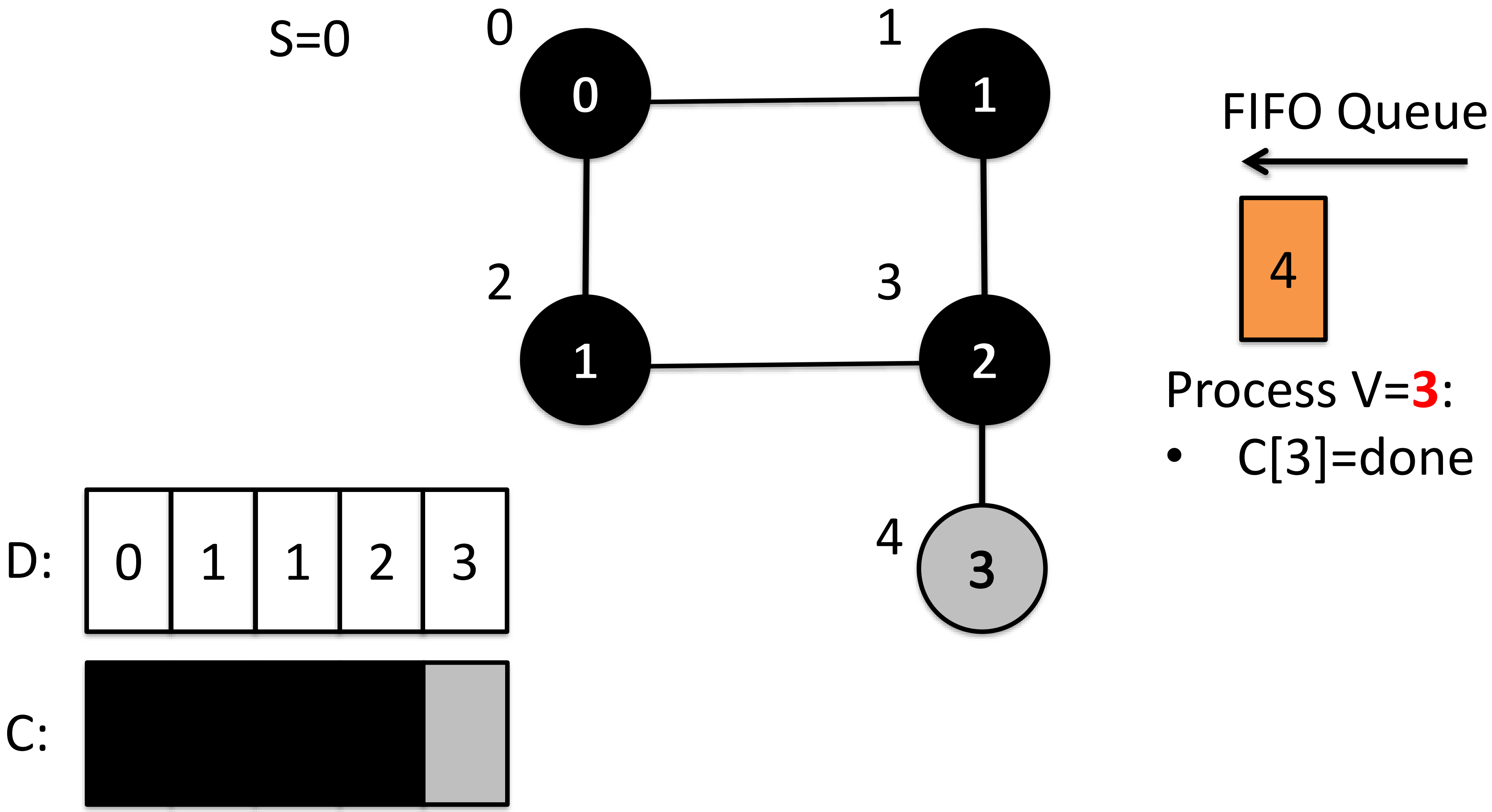
D:

0	1	1	2	3

C:



Example

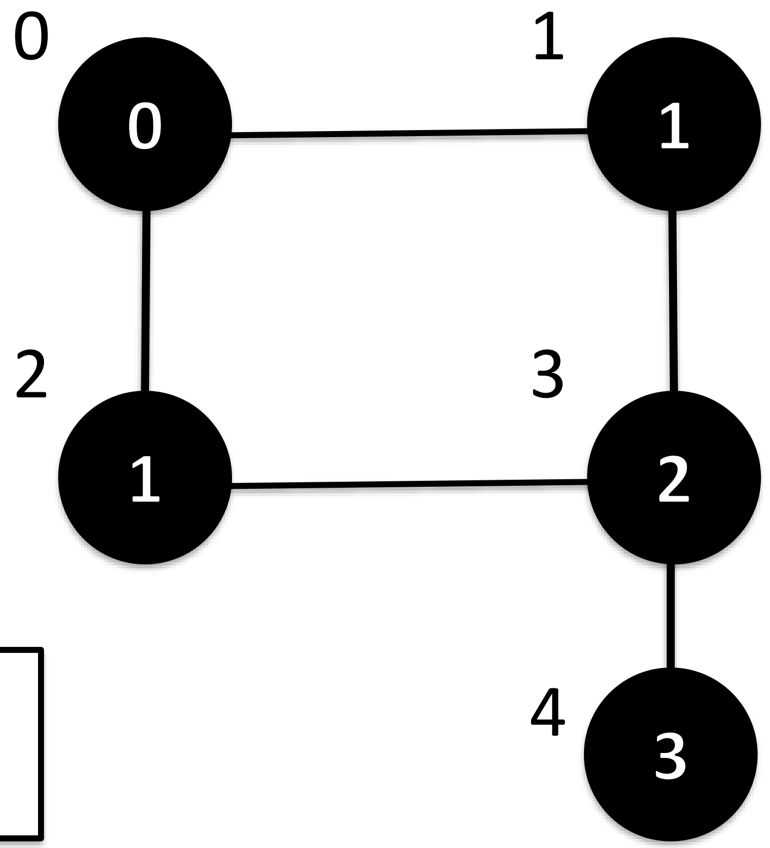


Process V=**3**:

- C[3]=done

Example

S=0



FIFO Queue ←

Process V=**4** (remove fr/Q):

- C[4]=done

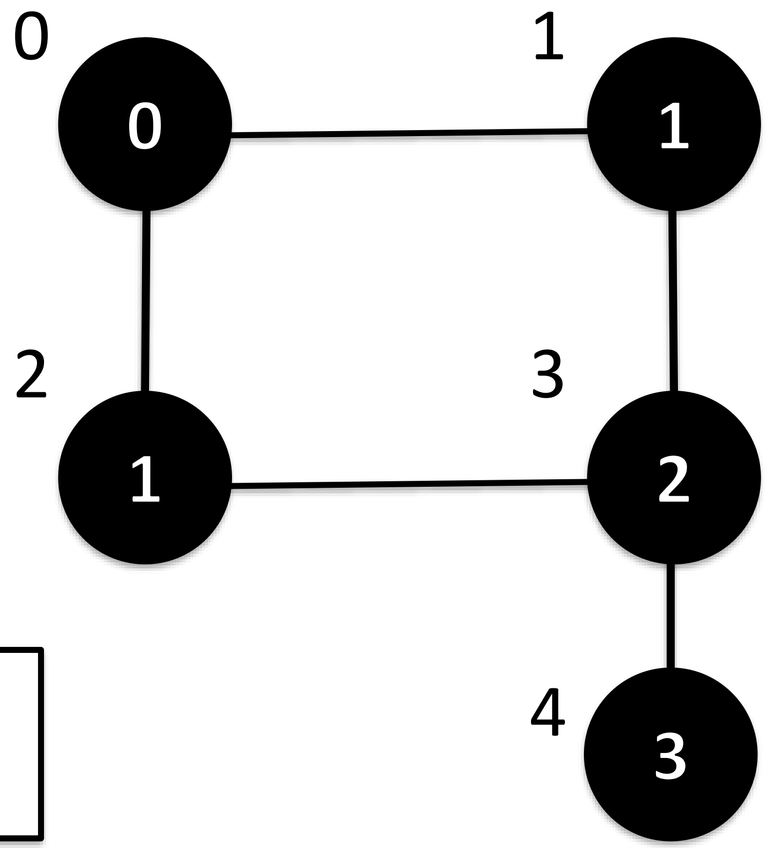
D:

0	1	1	2	3

C:

Example

S=0



FIFO Queue

DONE!

D:

0	1	1	2	3

C:

D[0]=0; D[1]=1; D[2]=1; D[3]=2; D[4]=3

Execution Time

While (FIFO queue not empty)

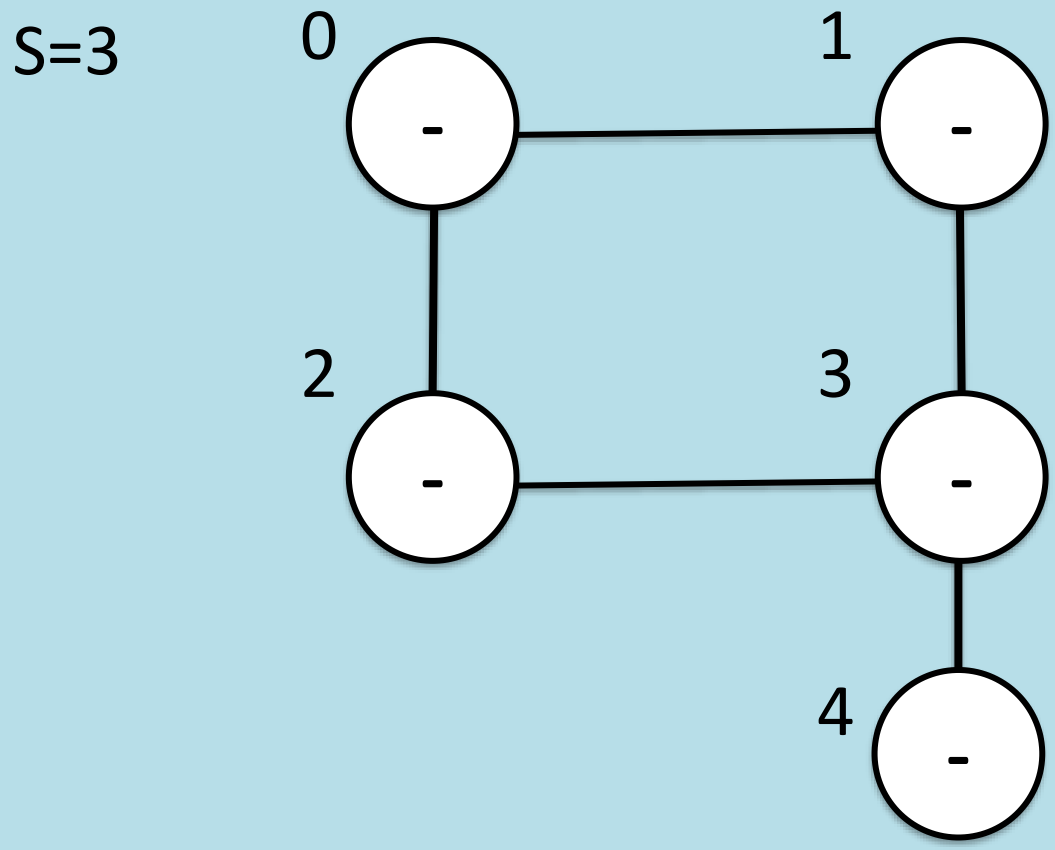
- V = Remove next vertex from FIFO queue
- For each neighbor i of V
  - If (C[i]==unvis)
    - D[i] = D[V]+1
    - C[i] = vis
    - Add i to FIFO queue
- C[V] = done

- Outer loop (While): each vertex added to/removed from FIFO one time: time proportional to the number of nodes
- Inner loop (For): time proportional to number of edges (e.g., all adjacency list entries scanned)

Total time proportional to sum of number of vertices and number of edges in graph: O(V+E)

Breadth-first search example

- Run through BFS on the same graph using S=3 as the starting point! Show how the D and C arrays are updated (can be combined for ease of display) as well as the FIFO queue.

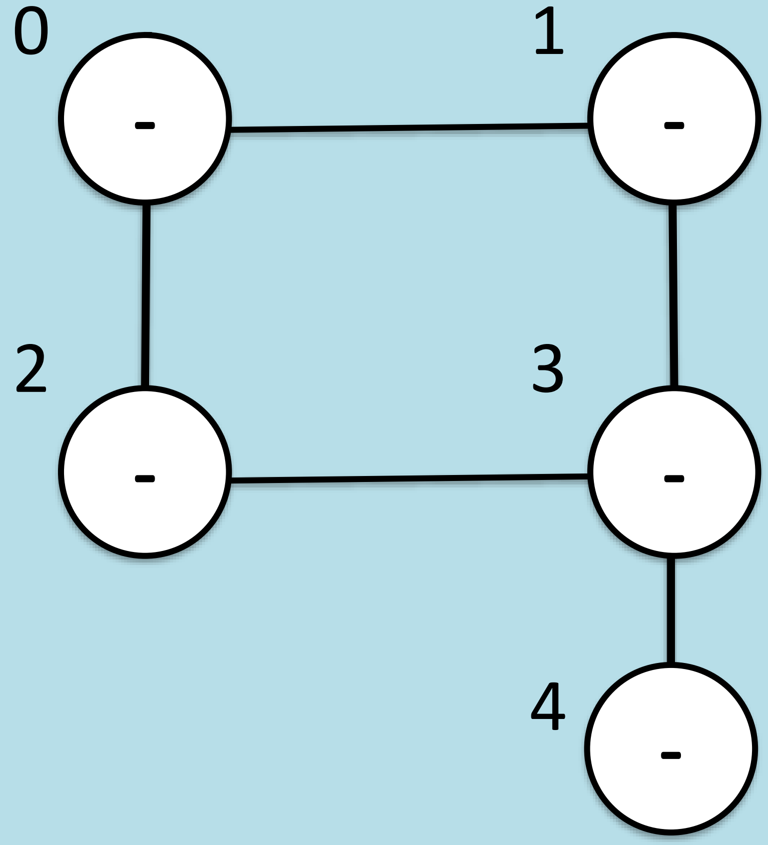


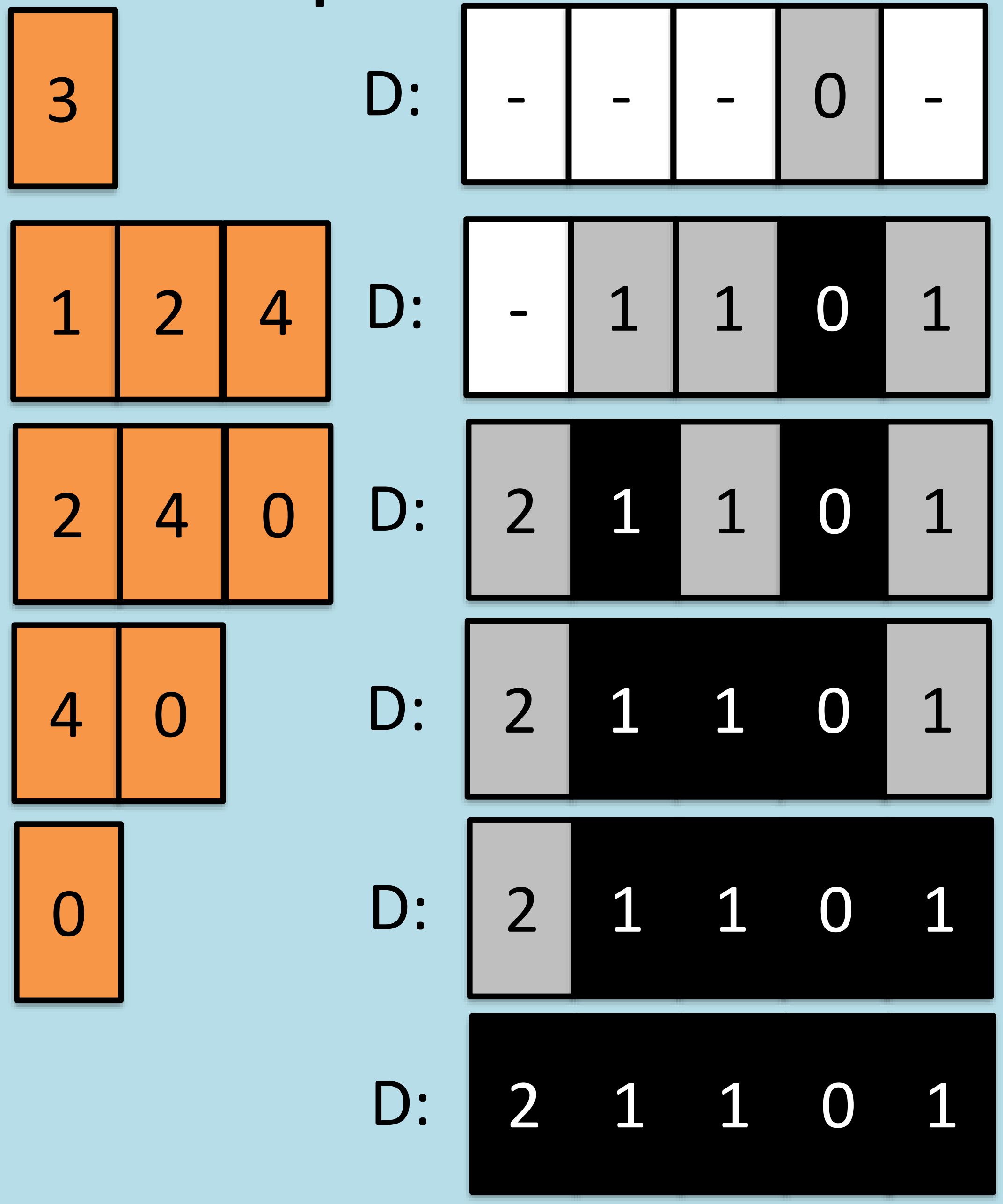
- Thought question: how many different colors/conditions are really needed to execute this algorithm?

Breadth-first search example

- Run through BFS on the same graph using S=3 as the starting point!

S=3



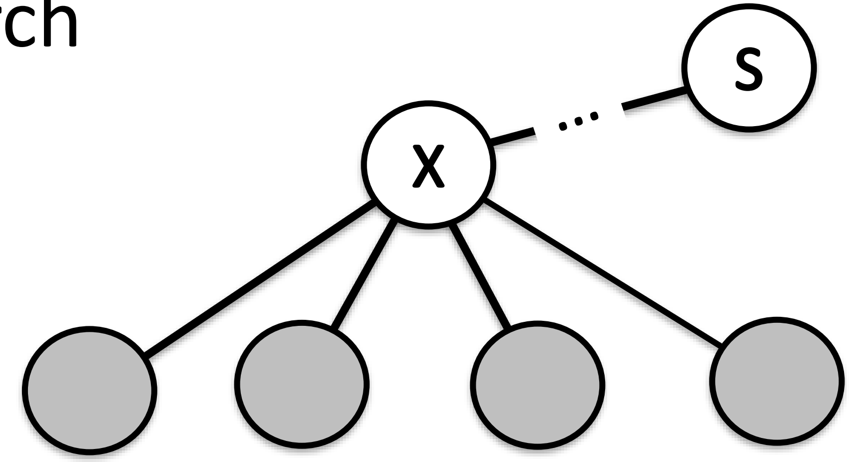


Breadth-first search example

- How many different colors/conditions are really needed to execute this algorithm?
  - Really only two colors/conditions are needed, to differentiate between nodes that have been added to the queue already (at some point) and those that have not. There is no risk of re-adding a node to the queue once it has been processed.

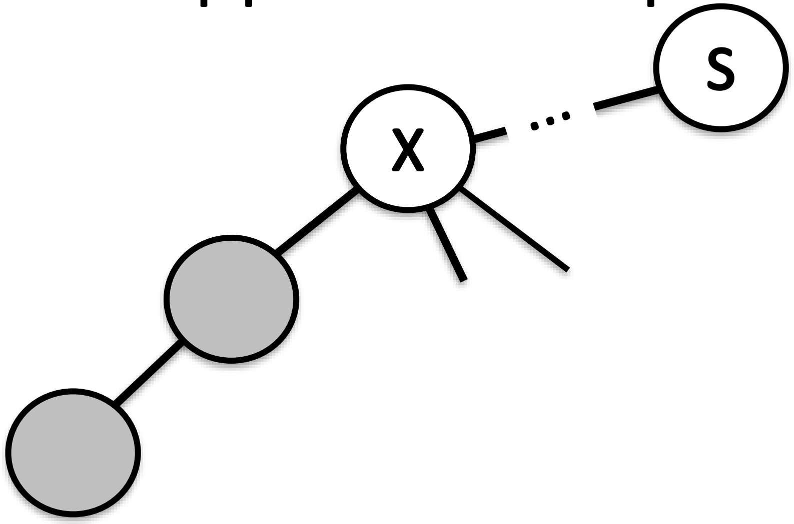
Search Approaches

- Previous algorithm uses an approach called “breadth-first” search



Visit neighbors of X before exploring nodes further from S

- Another approach: “depth-first” search

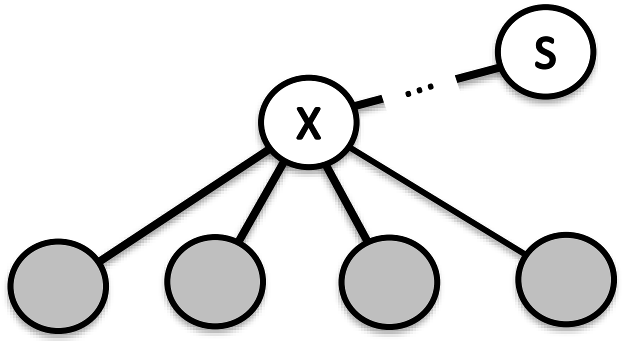


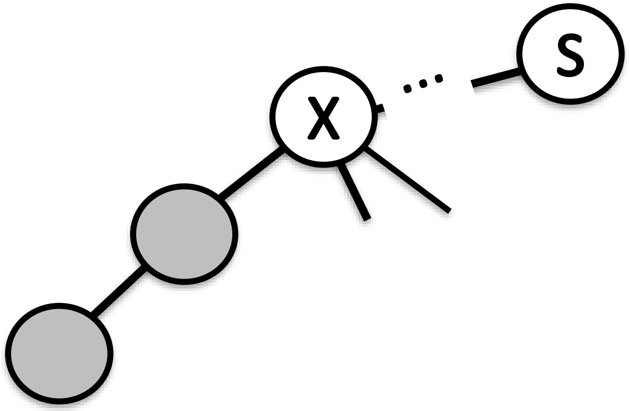
Visit nodes further from S before visiting all of X’s neighbors

Analogy: Literature Survey (or Web Crawler)

Paper citations can be viewed as a graph

- Node: a single paper
- Link: reference another paper in the bibliography





Breadth-first search (BFS)

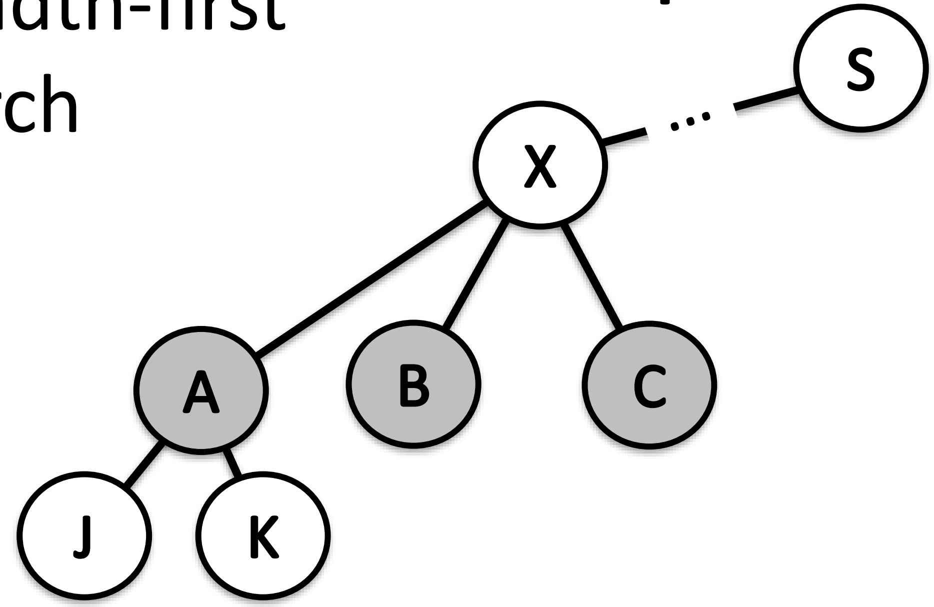
- Read one paper
- Read all papers cited in the bibliography of this paper
- For the set of papers you just read, compile list of the papers they cite that you have not already read; read these
- Repeat previous step until every paper that was cited has been read

Depth-first search (DFS)

- Read one paper
- Read one paper cited by the paper you just read
- Repeat previous step until you reach a paper that only cites papers you've already read
- ***Backtrack*** to the last previously read paper that cites a paper you haven't read, and repeat DFS procedure starting from this paper

Implementation

Breadth-first Search

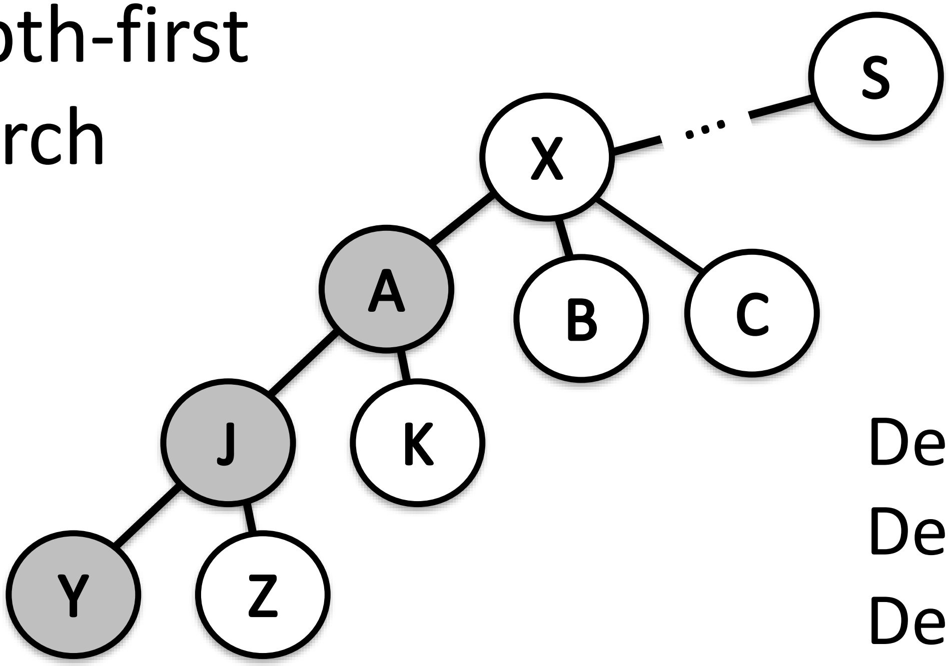


FIFO Queue
(delete left; insert right)

B	C	J	K

Delete (visit) B
Delete (visit) C
Delete (visit) J
Delete (visit) K

Depth-first Search



LIFO Stack
(delete left; insert left)

Y	Z	K	B	C

Delete (visit) A
Add (push)

Delete (visit) Y Delete (visit) B
Delete (visit) Z Delete (visit) C
Delete (visit) K (backtracking)

Depth ~~Breadth~~-first Search

Initially (neglecting distance calculation)

- C[S]=vis; C[i]=unvis, i≠S
- LIFO Stack ~~FIFO Queue~~ holds S

Algorithm

While (LIFO Stack ~~FIFO queue~~ not empty)

V = Remove next vertex from LIFO Stack ~~FIFO queue~~

For each neighbor i of V

If (C[i]==unvis)

C[i] = vis

Add i to LIFO Stack ~~FIFO queue~~

C[V] = done

Depth-first Search (Recursion)

```
// LIFO implemented via recursion
// two colors/conditions: visited, unvisited
for all vertices v
    if (C[v]==unvis) DFS (v);  // perform DFS from vertex v
```

DFS (v)

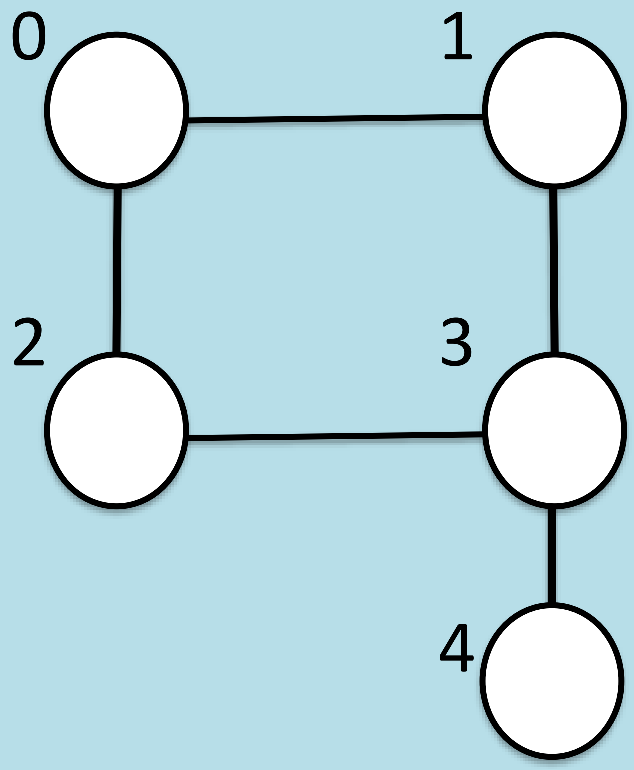
```
C[v] = vis
for each neighbor u of v
        if (C[u]==unvis) DFS (u);
```

Things You Should Know

- Breadth-first search
  - Approach to visit all nodes of a connected graph
  - Visitation, in effect, builds a tree (aka spanning tree; no loops)
  - Visit nodes in level i of tree before exploring level i+1
  - Implemented using a FIFO queue
  - Can be used to compute minimum path from source S
- Depth-first search
  - Also visits all nodes in a connected graph and builds spanning tree
  - Go down tree, level by level, as quickly as possible, then backtrack to visit others in each level
  - Implemented using a LIFO stack
  - Simple, elegant description using recursion
  - Computes minimum path length?

Depth-first search example

- Run through DFS on the same graph as before using S=0 as the starting point using the stack approach. In what order are the nodes visited, assuming that when we add multiple nodes to the stack we push the highest-numbered node first?
- What if the starting node is changed to S=3?
- Analyze the execution time for DFS.



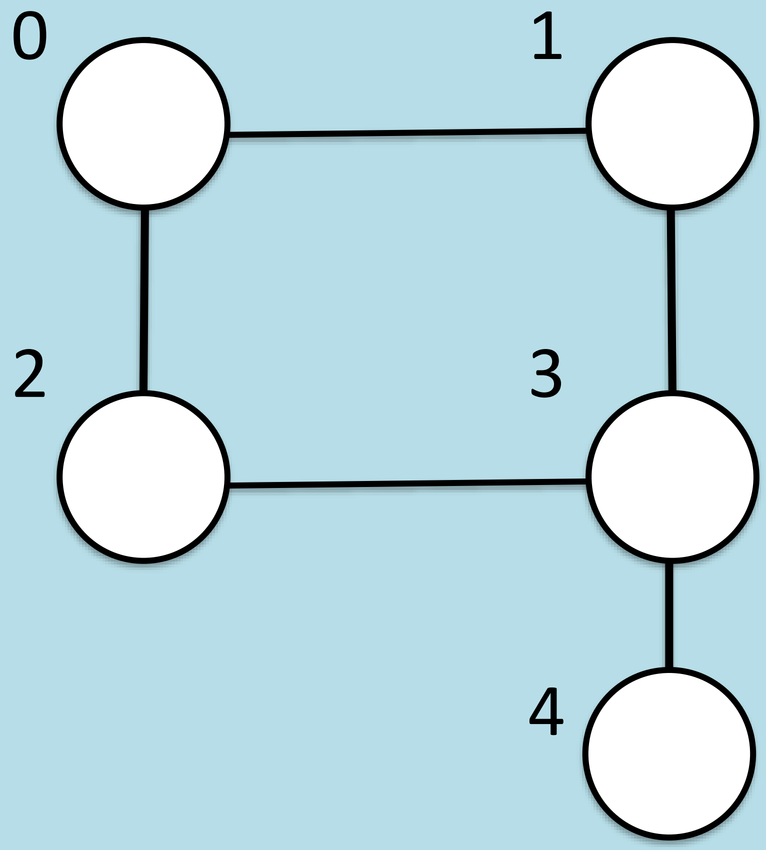
Initialization
- C[S]=vis; C[i]=unvis, i≠S
- Stack holds S

Algorithm

While (stack not empty)
  V = next vertex from stack
  For each neighbor i of V
    If (C[i]==unvis)
      C[i] = vis
      Add i to stack
  C[V] = done

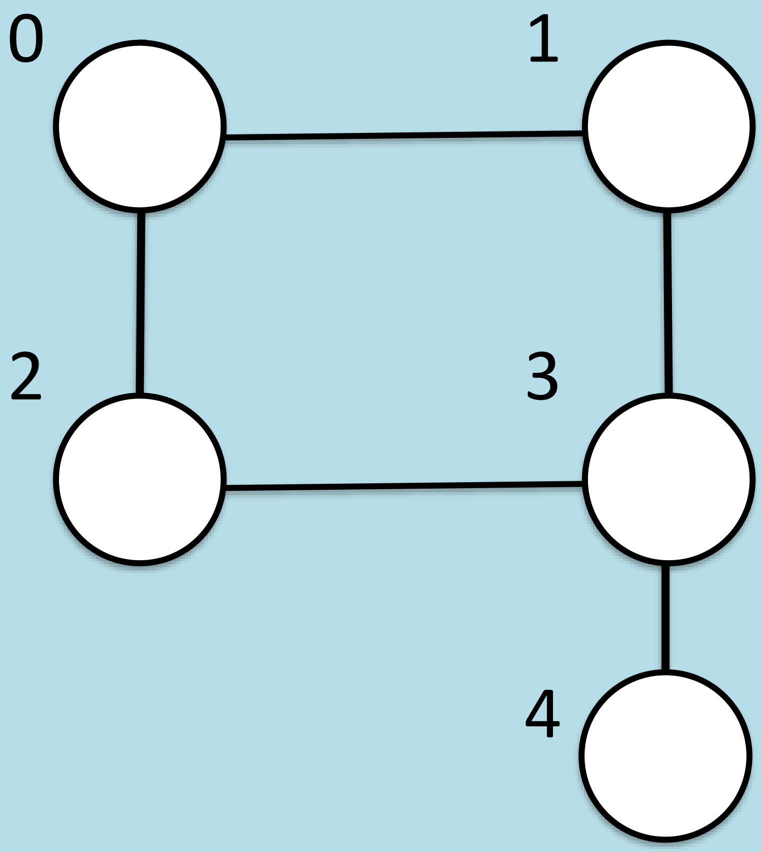
Depth-first search example

- DFS using S=0 as the starting point. In what order are the nodes visited, assuming that when we add multiple nodes to the stack we push the highest-numbered node first?
- Nodes are visited as follows:
  - 0; neighbors are 1 and 2, push to stack: 1 2
  - 1; neighbors are 0 and 3, push 3 to stack: 3 2
  - 3; neighbors are 1, 2, 4, push 4 to stack: 4 2
  - 4; neighbor is 3, time to backtrack
  - 2



Depth-first search example

- DFS using S=3 as the starting point.
- Nodes are visited as follows:
  - 3; neighbors are 1, 2, 4, push to stack: 1 2 4
  - 1; neighbors are 0 and 3, push 0 to stack: 0 2 4
  - 0; neighbors are 1 and 2, time to backtrack: 2 4
  - 2; neighbors are 0 and 3, backtrack again: 4
  - 4



Depth-first search example

- Analyze the execution time for DFS.

  The execution time is the same as for BFS: O(V+E).

  Initially (neglecting distance calculation)

  C[S]=vis; C[i]=unvis, i≠S

  LIFO stack holds S

  Algorithm

  While (LIFO stack not empty) ← *Loop over all vertices*

    V = Remove next vertex from LIFO stack

    For each neighbor i of V ← *Sweep through entries of adjacency list; cumulatively all edges will be traversed*

      If (C[i]==unvis)

        C[i] = vis

        Add i to LIFO stack

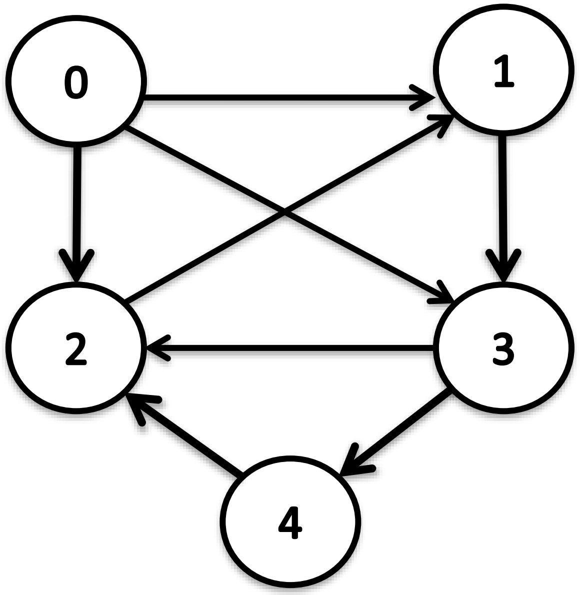
    C[V] = done

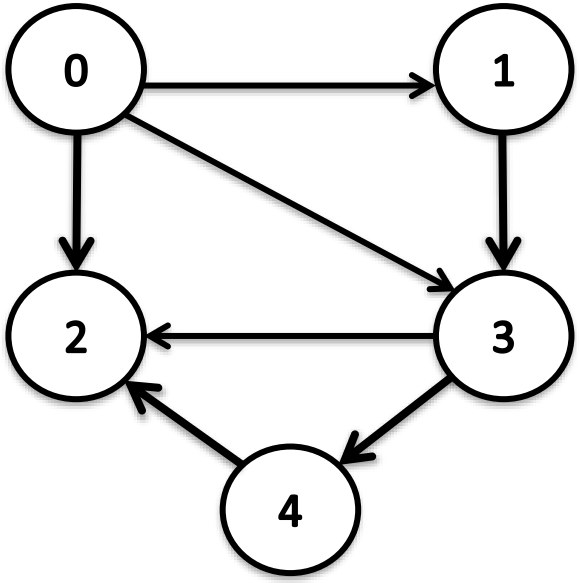
Topological Sorting

- Sometimes it is useful to perform a ***topological sort*** of a graph to indicate precedences among events.
- We will do this only for a directed acyclic graph (“dag”).
  - Edges are directed (arrows).
  - Graph does not contain any cycle: in a series of ordered vertices with each consecutive pair of vertices connected by an edge, no vertices repeat.
- A topological sort is a linear ordering of all the vertices of a dag such that if G contains an edge (u,v), u appears before v in the ordering.
  - Essentially, a listing of vertices along a horizontal line such that all directed edges go from left to right.

Topological Sorting

- Examples:
  - Top graph has cycles (e.g., 1321, 34213)
  - Bottom graph does not and thus is a dag.
- Topological sorts provide a potential order of events.
  - For a curriculum, vertices could be courses and edges could indicate prerequisites.
  - A topological sort could list an ordering for taking those courses.
- Topological sorts are not necessarily unique.





Topological Sorting

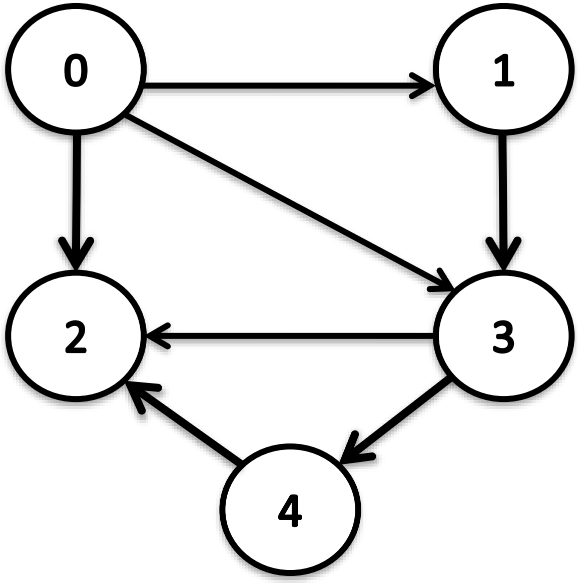
- To do topological sorting, we can use recursive DFS with one modification: add a stack to store each completed vertex (returning to 3 conditions/ colors for explanation).
- Then pop from the stack in order to obtain the topological sort.

```
C[i]=unvis;
for all vertices v
    if (C[v]=unvis)  DFS (v)  // perform DFS from vertex v
```

DFS (v)

```
C[v] = vis
for each neighbor u of v
      if (C[u]==unvis) DFS (u)
C[v] = done, Add v to output stack
```

Topological Sort



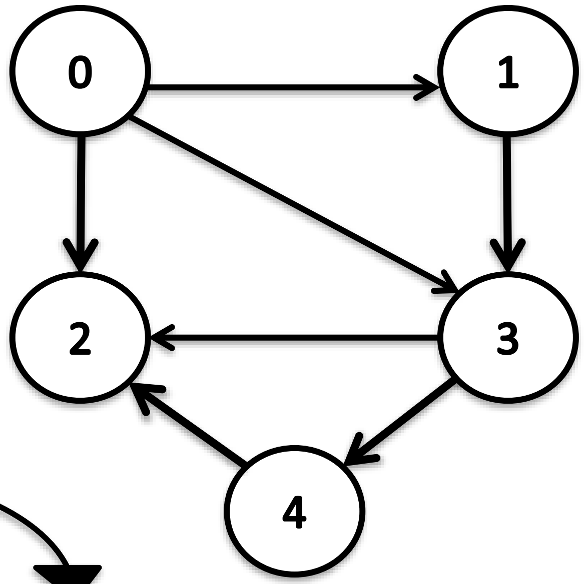
C[i]=unvis
for all vertices v
if (C[v]=unvis) DFS (v)

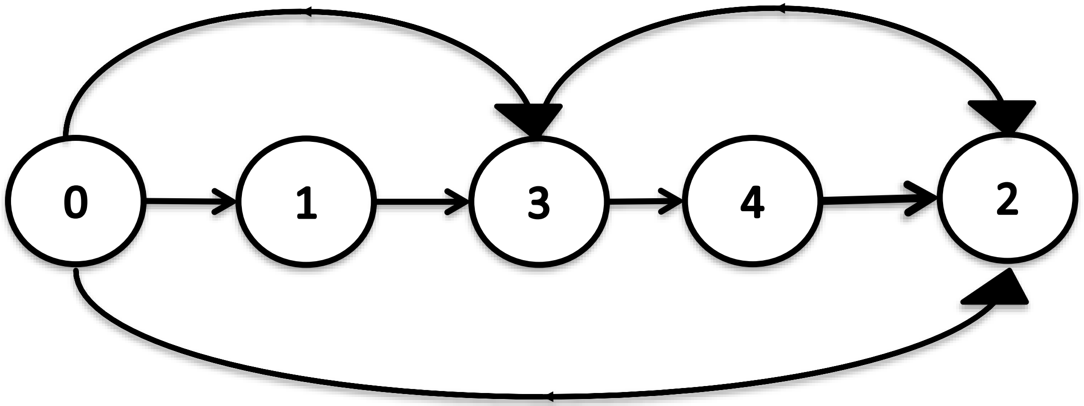
DFS (v)
C[v] = vis
for each neighbor u of v
if (C[u]==unvis) DFS (u)
C[v] = done, Add v to output stack

C = [u u u u u]
DFS[0]
C = [v u u u u]
DFS[1]
C = [v v u u u]
DFS[3]
C = [v v u v u]
DFS[2]
C = [v v v v u]
No unvisited neighbors
C = [v v d v u], output = [2]
DFS[4]
C = [v v d v v]
No unvisited neighbors
C = [v v d v d], output = [4 2]
C = [v v d d d], output = [3 4 2]
C = [v d d d d], output = [1 3 4 2]
C = [d d d d d], output = [0 1 3 4 2]

Confirming the Topological Sort

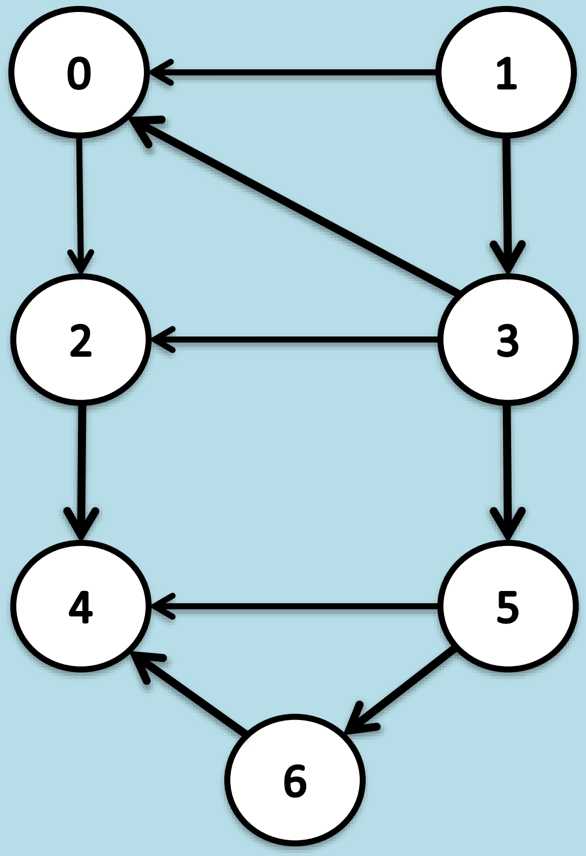
- We can confirm the topological sort result: [0 1 3 4 2]
- Check that for all edges [u, v], u appears before v
- List of edges:
  - (0, 1)
  - (0, 2)
  - (0, 3)
  - (1, 3)
  - (3, 2)
  - (3, 4)
  - (4, 2)





Topological Sort Example

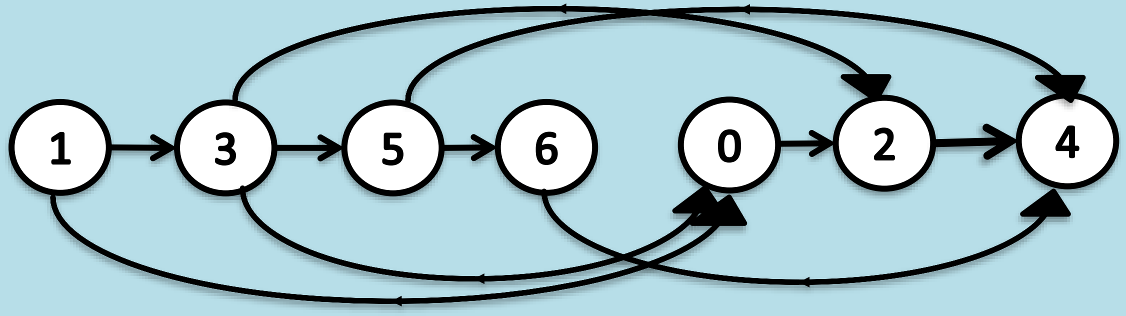
- Use the extended DFS algorithm to perform a topological sort of the graph, starting with vertex 0.
- Repeat, but assume we process vertices in the order 3 4 5 6 0 1 2 (essentially this is a relabeling).



Topological Sort Example

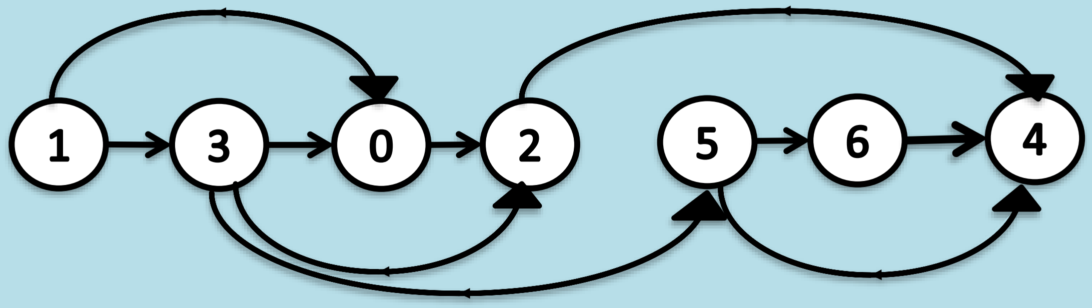
- Use the extended DFS algorithm to perform a topological sort of the graph below, starting with vertex 0.

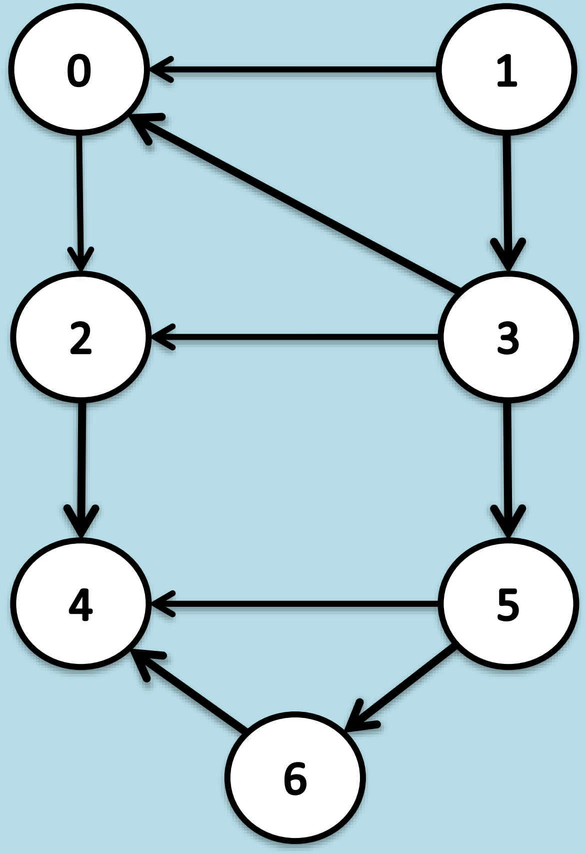
[1 3 5 6 0 2 4]



- Repeat, but assume we process vertices in the order 3 4 5 6 0 1 2 (essentially this is a relabeling).

[1 3 0 2 5 6 4]





Topological Sorting Complexity

- Topological sorting is just DFS with additional storage.
- New storage is $O(V)$.
- Thus, topological sorting remains $O(V + E)$.
- This is one approach for topological sorting; multiple topological sorts can exist, and not all topological sorts can be produced by this algorithm.