CSE 6140: Homework 4

Zixuan Wang

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Question 1

(1) Diverse Subset is in NP.
Certificate: A set S of k customers.
Certifier: Check that the set S of k customers is diverse. We can verify it in polynomial time. For each product, we can check if at most 1 customer in S has bought that product. This can be done in $O(km)$, so Diverse Subset is in NP.

(2) Independent Set is reducible to Diverse Subset.

In the Independent Set problem, given a graph G and a number k, we ask whether G contains a set of vertices of at least size k in which no two vertices has an edge.

For the Diverse Subset problem, we can construct an instance as follows: we can set up a graph $G = (V, E)$ and a number k, where each vertex v in V is a customer, and each edge e in E is a product. Each customer v purchases the product e for which the product edge e touches the customer vertex v.

The graph has an Independent Set of size k if and only if this graph has a Diverse Subset of size k. That means if this graph has a Diverse Subset of size k, Therefore, Independent Set can be solved with a black box that solves Diverse Subset.

Since Independent Set is NP-complete and Independent Set $\leq_p$ Diverse Subset, we can show that Diverse Subset is NP-complete.

Question 2

(1) Efficient Recruiting is in NP.
Certificate: A set S of k counselors.
Certifier: Check whether the set S of k counselors is Efficient Recruiting. Given the set of k counselors, we can check if for every sport we have at least one counselor in polynomial time. Therefore, Efficient Recruiting is in NP.

(2) Vertex Cover is reducible to Efficient Recruiting.

In the Vertex Cover problem, given a graph G, a set of vertices S is a vertex cover if every edge in the graph has at least one end in S. We want to ask whether G contains a vertex cover of size at most k.

For the Efficient Recruiting problem, we can construct an instance as follows: we can set up a graph $G = (V, E)$ and a number k, where each vertex v in V is a sport, and each edge e in E is a counselor. For each sport, the counselor is qualified in that sport only when the edge touches the sport vertex.

Clearly, the graph has a Vertex Cover of at most size k if and only if the graph has an Efficient Recruiting of size k. Thus, Vertex Cover can be solved with a black box that solves Efficient Recruiting.

Since Vertex Cover is NP-complete and Vertex Cover $\leq_p$ Efficient Recruiting, we can show that Efficient Recruiting is NP-complete.

Question 3

(1) Truck Loading is in NP.
Certificate: The way of loading n containers into the m trucks.
Certifier: Check if the way is valid. We can easily check that each truck loads only the containers that can be brought together in polynomial time. Therefore, Truck Loading is in NP.

(2) 3-Coloring is reducible to Truck Loading.

The m-coloring problem ($m \geq 3$) is: given a graph G, whether we can color the graph with m colors and no incident nodes have the same color.

For the Truck Loading problem, in the graph, we can create a vertex for every container. There is an edge between two containers if they cannot be placed in the same truck. Also, each color denotes a truck with capacity k.

Clearly, if there is a solution that solves Truck Loading problem with m trucks (without capacity limitation), there would be a solution for m-coloring problem. We can color the nodes in the same color if they are in the same truck. Conversely, if there is a solution to the m-coloring problem, we can assign all containers of the same color to one truck. Therefore, this problem is at least as difficult as the 3-coloring problem and maybe even more difficult due to capacity limitation. Since 3-coloring problem is NP-complete, Truck Loading is NP-complete.

Question 4

(1) Perfect Assembly is in NP.
Certificate: Ordered string of S.
Certifier: Check if the ordered string is a perfect assembly with respect to T. We can verify it by checking if $(s_{i_j}, s_{i_{j+1}})$ is collaborated by a string in the library T. This can be done in polynomial time, so Perfect Assembly is in NP.

(2) Hamiltonian Path is reducible to Perfect Assembly.

The Hamiltonian Path problem is that whether a graph has a cycle that visits each node exactly once.

For the Perfect Assembly problem, we can create a vertex for every string in S. For every string in T, we can create a directional edge $e = (v_i, v_j)$ which represents t_k, and the first l symbols in the edge are equal to the last l symbols in v_i, and the last l symbols in the edge are equal to the first l symbols in v_j. In other words, each edge in the Hamiltonian Path problem is associated with a string that corroborates the pair of vertices connected by that edge.

Clearly, the graph has a Hamiltonian Path if and only if S has a Perfect Assembly with respect to T. Thus, Hamiltonian Path can be solved with a black box that solves Perfect Assembly.

Since Hamiltonian Path is NP-complete, Perfect Assembly is also NP-complete.