

LECTURE 7: Conditioning on a random variable; Independence of r.v.'s

- Conditional PMFs
 - Conditional expectations
 - Total expectation theorem
- Independence of r.v.'s
 - Expectation properties
 - Variance properties
- The variance of the binomial
- The hat problem: mean and variance

Conditional PMFs

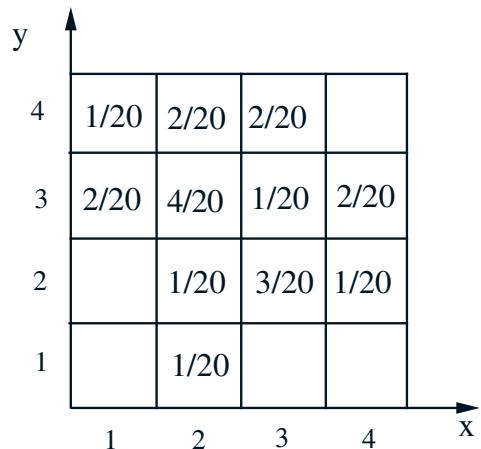
$$p_{X|A}(x | A) = \mathbf{P}(X = x | A)$$

$$p_{X|Y}(x | y) = \mathbf{P}(X = x | Y = y)$$

$$p_{X|Y}(x | y) = \frac{p_{X,Y}(x, y)}{p_Y(y)}$$

defined for y such that $p_Y(y) > 0$

$$\sum_x p_{X|Y}(x | y) = 1$$



$$p_{X,Y}(x, y) = p_Y(y) p_{X|Y}(x | y)$$

$$p_{X,Y}(x, y) = p_X(x) p_{Y|X}(y | x)$$

Conditional PMFs involving more than two r.v.'s

- Self-explanatory notation

$$p_{X|Y,Z}(x | y, z)$$

$$p_{X,Y|Z}(x, y | z)$$

- Multiplication rule

$$\mathbf{P}(A \cap B \cap C) = \mathbf{P}(A) \mathbf{P}(B | A) \mathbf{P}(C | A \cap B)$$

$$p_{X,Y,Z}(x, y, z) = p_X(x) p_{Y|X}(y | x) p_{Z|X,Y}(z | x, y)$$

Conditional expectation

$$\mathbf{E}[X] = \sum_x x p_X(x) \quad \mathbf{E}[X | A] = \sum_x x p_{X|A}(x) \quad \mathbf{E}[X | Y = y] = \sum_x x p_{X|Y}(x | y)$$

- Expected value rule

$$\mathbf{E}[g(X)] = \sum_x g(x) p_X(x) \quad \mathbf{E}[g(X) | A] = \sum_x g(x) p_{X|A}(x) \quad \mathbf{E}[g(X) | Y = y] = \sum_x g(x) p_{X|Y}(x | y)$$

Total probability and expectation theorems

- A_1, \dots, A_n : partition of Ω
- $p_X(x) = P(A_1)p_{X|A_1}(x) + \dots + P(A_n)p_{X|A_n}(x)$

$$p_X(x) = \sum_y p_Y(y) p_{X|Y}(x | y)$$

- $E[X] = P(A_1)E[X | A_1] + \dots + P(A_n)E[X | A_n]$

$$E[X] = \sum_y p_Y(y) E[X | Y = y]$$

- Fine print:
Also valid when Y is a discrete r.v. that ranges over an infinite set,
as long as $E[|X|] < \infty$

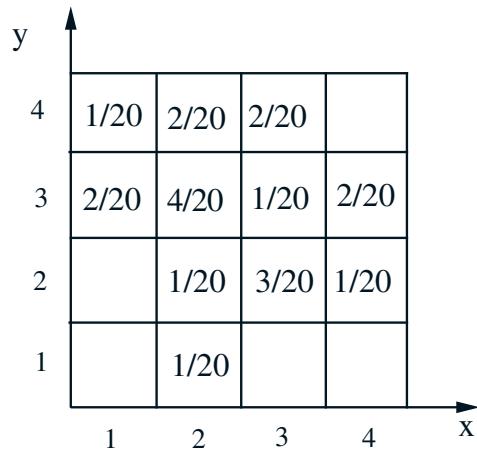
Independence

- of two events: $\mathbf{P}(A \cap B) = \mathbf{P}(A) \cdot \mathbf{P}(B)$ $\mathbf{P}(A | B) = \mathbf{P}(A)$
- of a r.v. and an event: $\mathbf{P}(X = x \text{ and } A) = \mathbf{P}(X = x) \cdot \mathbf{P}(A), \text{ for all } x$
- of two r.v.'s: $\mathbf{P}(X = x \text{ and } Y = y) = \mathbf{P}(X = x) \cdot \mathbf{P}(Y = y), \text{ for all } x, y$
 $p_{X,Y}(x, y) = p_X(x) p_Y(y), \text{ for all } x, y$

X, Y, Z are **independent** if:

$$p_{X,Y,Z}(x, y, z) = p_X(x) p_Y(y) p_Z(z), \text{ for all } x, y, z$$

Example: independence and conditional independence



- Independent?
- What if we condition on $X \leq 2$ and $Y \geq 3$?

Independence and expectations

- In general: $E[g(X, Y)] \neq g(E[X], E[Y])$
- Exceptions: $E[aX + b] = aE[X] + b$ $E[X + Y + Z] = E[X] + E[Y] + E[Z]$

If X, Y are **independent**: $E[XY] = E[X]E[Y]$

$g(X)$ and $h(Y)$ are also independent: $E[g(X)h(Y)] = E[g(X)] \cdot E[h(Y)]$

Independence and variances

- Always true: $\text{var}(aX) = a^2\text{var}(X)$ $\text{var}(X + a) = \text{var}(X)$
- In general: $\text{var}(X + Y) \neq \text{var}(X) + \text{var}(Y)$

If X, Y are independent: $\text{var}(X + Y) = \text{var}(X) + \text{var}(Y)$

- Examples:
 - If $X = Y$: $\text{var}(X + Y) =$
 - If $X = -Y$: $\text{var}(X + Y) =$
 - If X, Y independent: $\text{var}(X - 3Y) =$

Variance of the binomial

- X : binomial with parameters n, p
 - number of successes in n independent trials

$X_i = 1$ if i th trial is a success; (indicator variable)
 $X_i = 0$ otherwise

$$X = X_1 + \cdots + X_n$$

The hat problem

- n people throw their hats in a box and then pick one at random
 - All permutations equally likely
 - Equivalent to picking one hat at a time
- X : number of people who get their own hat
 - Find $E[X]$

$$X_i = \begin{cases} 1, & \text{if } i \text{ selects own hat} \\ 0, & \text{otherwise.} \end{cases}$$

$$X = X_1 + X_2 + \cdots + X_n$$

- $E[X_i] =$

The variance in the hat problem

- X : number of people who get their own hat
 - Find $\text{var}(X)$

$$X_i = \begin{cases} 1, & \text{if } i \text{ selects own hat} \\ 0, & \text{otherwise.} \end{cases}$$

$$X = X_1 + X_2 + \cdots + X_n$$

- $\text{var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

$$X^2 = \sum_i X_i^2 + \sum_{i,j:i \neq j} X_i X_j$$

- $\mathbb{E}[X_i^2] =$

- For $i \neq j$: $\mathbb{E}[X_i X_j] =$

- $\mathbb{E}[X^2] =$