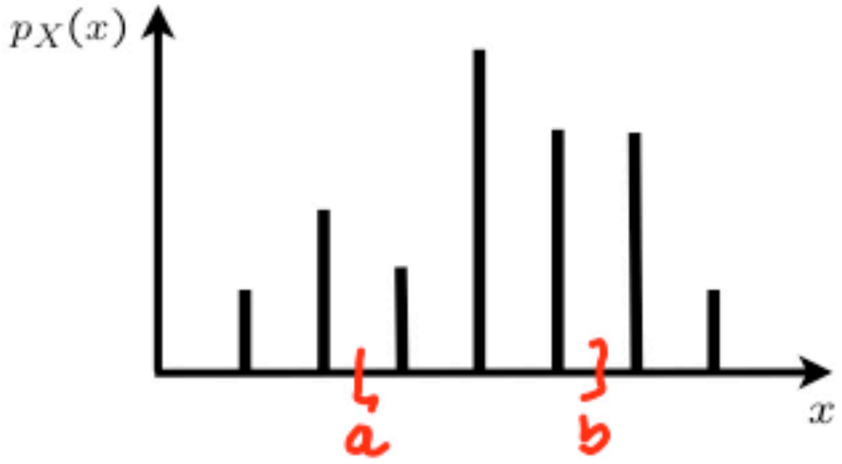
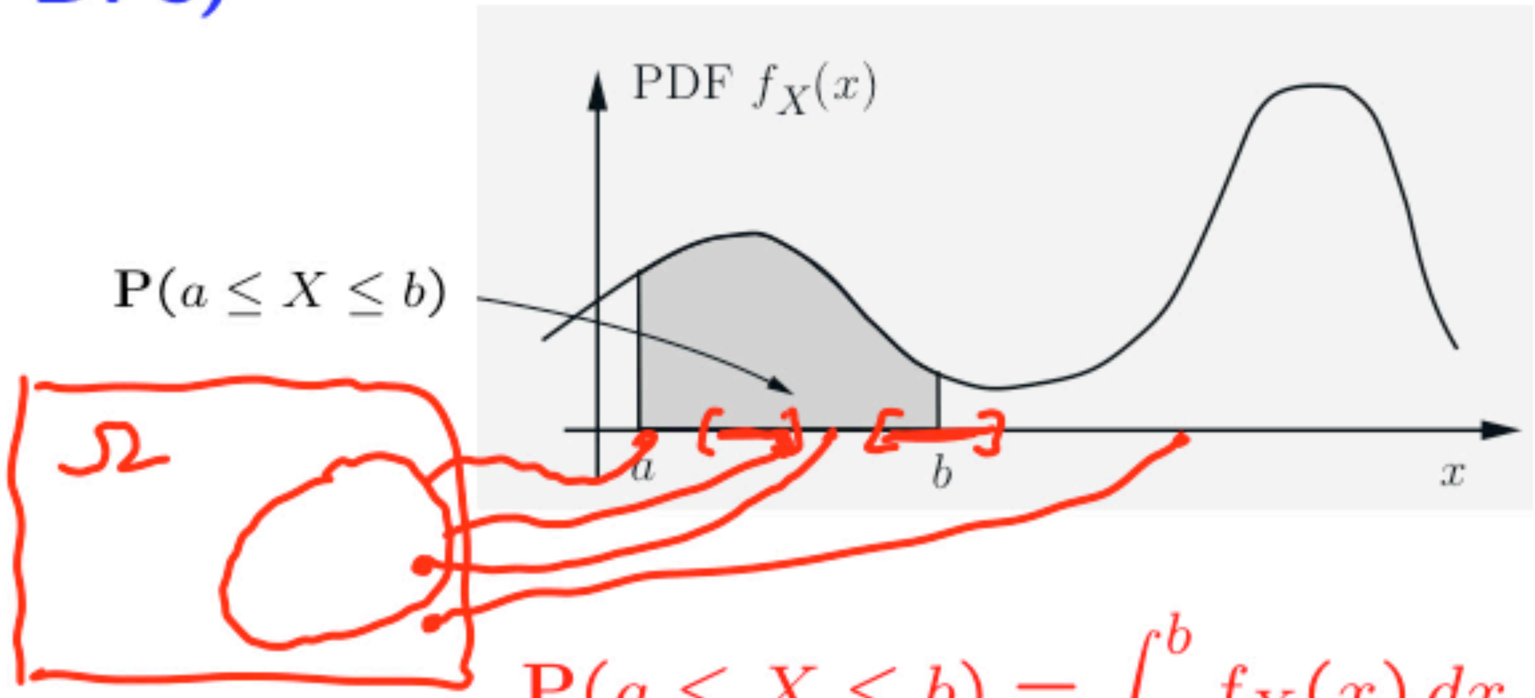
LECTURE 8: Continuous random variables and probability density functions

- Probability density functions
  - Properties
  - Examples
- Expectation and its properties
  - The expected value rule
  - Linearity
- Variance and its properties
- Uniform and exponential random variables
- Cumulative distribution functions
- Normal random variables
  - Expectation and variance
  - Linearity properties
  - Using tables to calculate probabilities

Probability density functions (PDFs)





$$\mathbf{P}(a \le X \le b) = \sum_{x:\, a \le x \le b} p_X(x)$$

$$\mathbf{P}(a \le X \le b) = \int_a^b f_X(x)\, dx$$

$p_X(x) \ge 0 \qquad \sum_x p_X(x) = 1$

- $f_X(x) \ge 0$
- $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$

Definition: A random variable is continuous if it can be described by a PDF

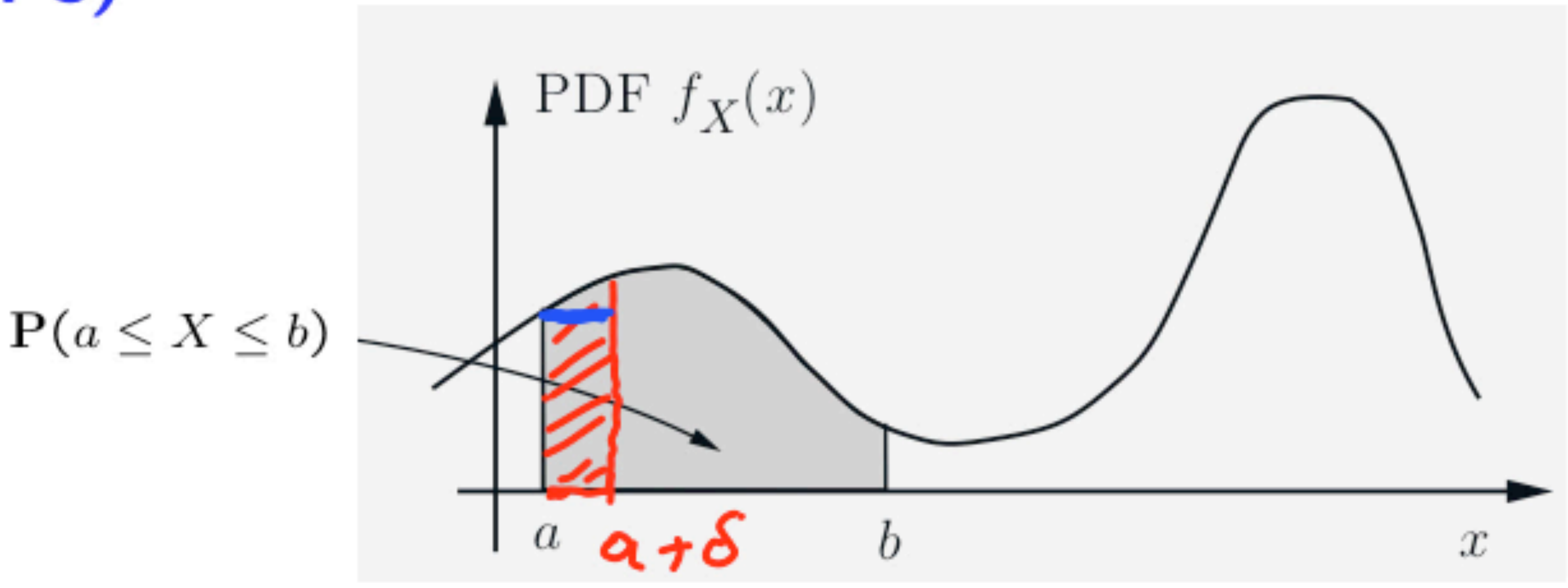
$\mathbf{P}(1 \le X \le 3 \text{ or } 4 \le X \le 5) = \mathbf{P}(1 \le X \le 3) + \mathbf{P}(4 \le X \le 5)$

Probability density functions (PDFs)

$\delta > 0$, small

$\mathbf{P}(a \le X \le a+\delta)$

$\approx f_X(a) \cdot \delta$



$$\boxed{\mathbf{P}(a \le X \le a+\delta) \approx f_X(a) \cdot \delta}$$

$$\boxed{\mathbf{P}(X = a) = 0}$$

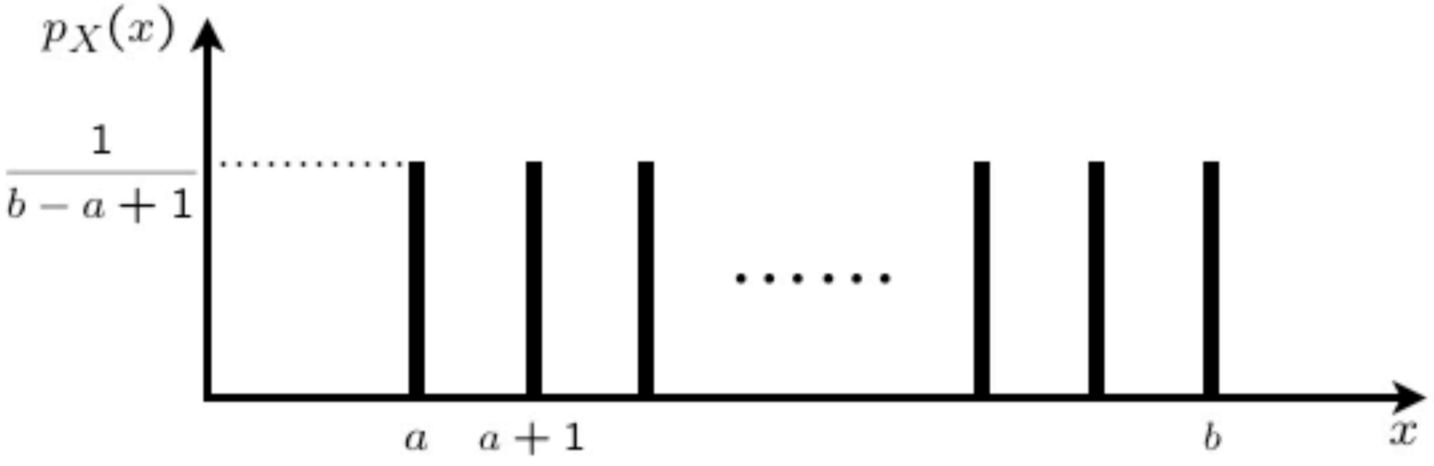
$$\mathbf{P}(a \le X \le b) = \int_a^b f_X(x)\,dx$$

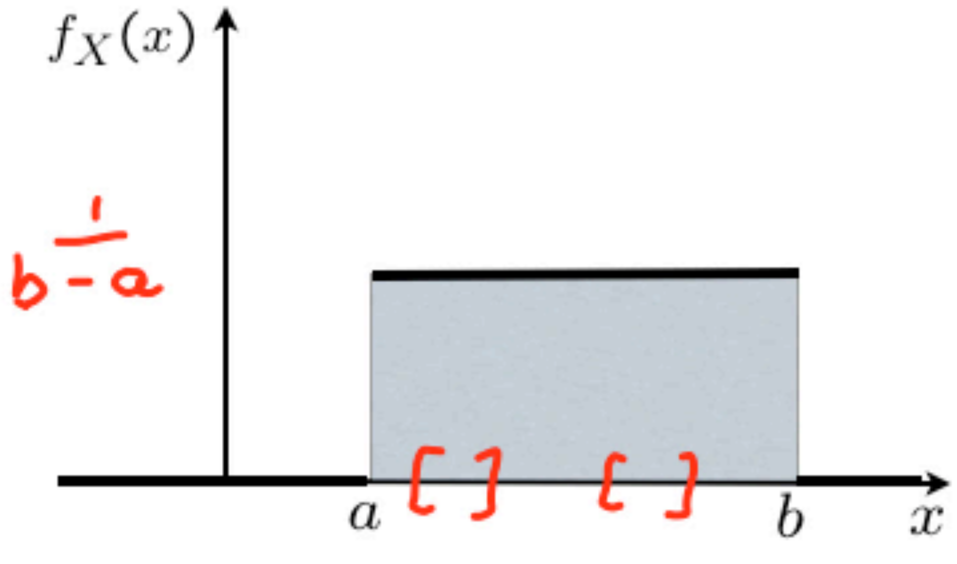
$$f_X(x) \ge 0$$

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

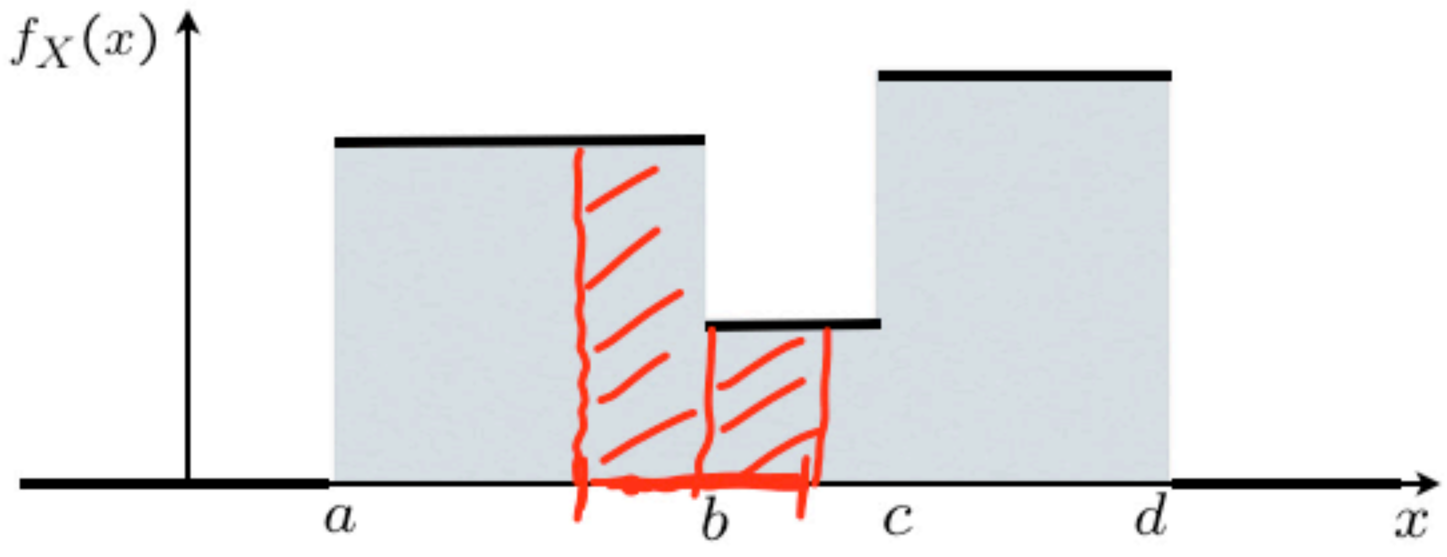
$$\mathbf{P}(a \le X \le b) = \cancel{\mathbf{P}(X = a)} + \cancel{\mathbf{P}(X = b)} + \mathbf{P}(a < X < b)$$

Example: continuous uniform PDF

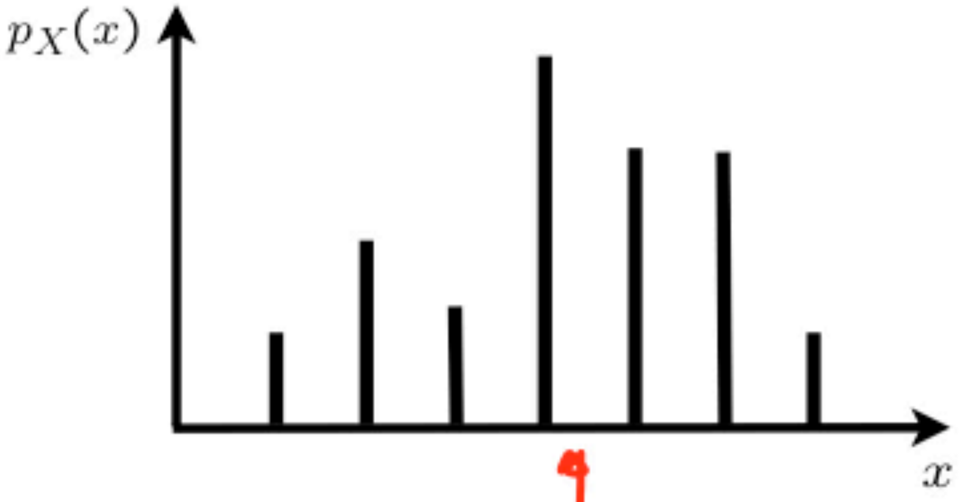




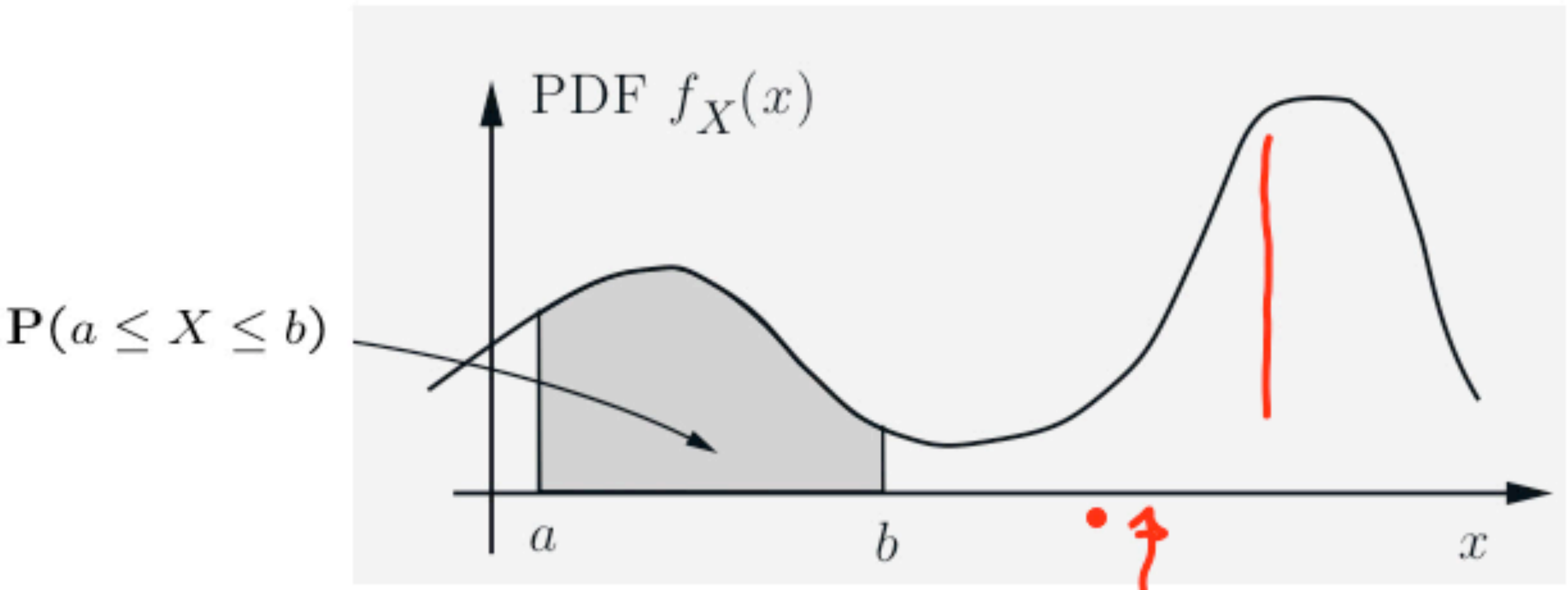
- Generalization: piecewise constant PDF



Expectation/mean of a continuous random variable



$$\mathbf{E}[X] = \sum_x x p_X(x)$$



$$\mathbf{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

- **Interpretation:** Average in large number of independent repetitions of the experiment

Fine print:
Assume $\int_{-\infty}^{\infty} |x|\, f_X(x)\, dx < \infty$

Properties of expectations

- If $X \geq 0$, then $\mathbf{E}[X] \geq 0$
- If $a \leq X \leq b$, then $a \leq \mathbf{E}[X] \leq b$
- Expected value rule:

$$\mathbf{E}[g(X)] = \sum_x g(x) p_X(x)$$

$$\boxed{\mathbf{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx}$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx$$

- Linearity

$$\boxed{\mathbf{E}[aX + b] = a\mathbf{E}[X] + b}$$

Variance and its properties

- **Definition of variance:** $\text{var}(X) = \mathbf{E}[(X-\mu)^2]$

$\mu = E[X]$

- Calculation using the expected value rule, $\mathbf{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$

$$\text{var}(X) = \int_{-\infty}^{\infty} (x-\mu)^2 f_X(x)\, dx$$

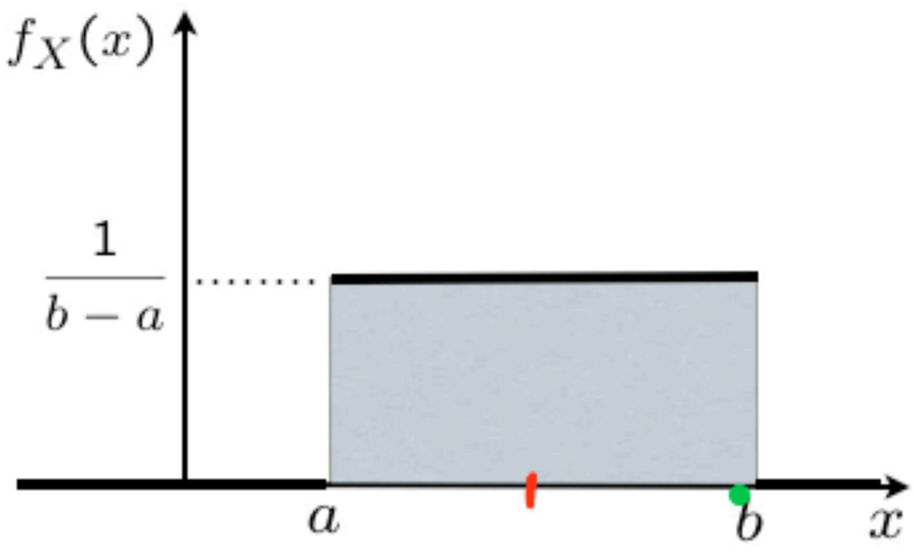
$g(x) = (x-\mu)^2$

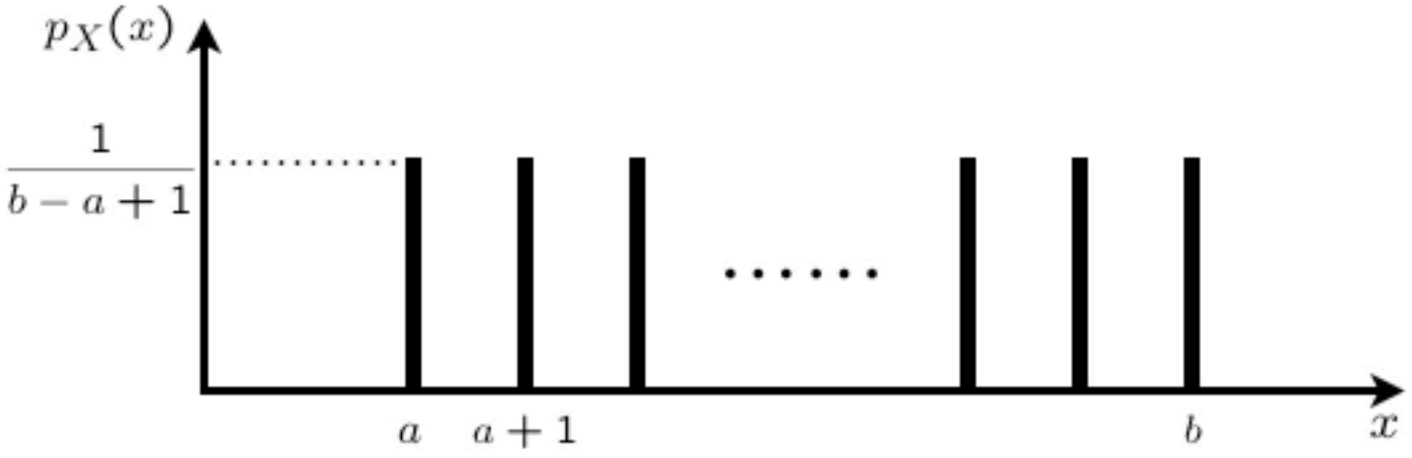
Standard deviation: $\sigma_X = \sqrt{\text{var}(X)}$

✓ $\text{var}(aX+b) = a^2 \text{var}(X)$

✓ A useful formula: $\text{var}(X) = \mathbf{E}[X^2] - \big(\mathbf{E}[X]\big)^2$

Continuous uniform random variable; parameters a, b





$$\mathbf{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

$$= \int_a^b x \cdot \frac{1}{b-a}\, dx = \frac{a+b}{2}$$

$$\mathbf{E}[X^2] = \int_a^b x^2 \cdot \frac{1}{b-a}\, dx = \frac{1}{b-a}\left(\frac{b^3}{3} - \frac{a^3}{3}\right)$$

$$\text{var}(X) = E[X^2] - (E[X])^2 = \boxed{(b-a)^2/12}$$

$$\sigma = \frac{b-a}{\sqrt{12}}$$

$$\mathbf{E}[X] = \frac{a+b}{2}$$

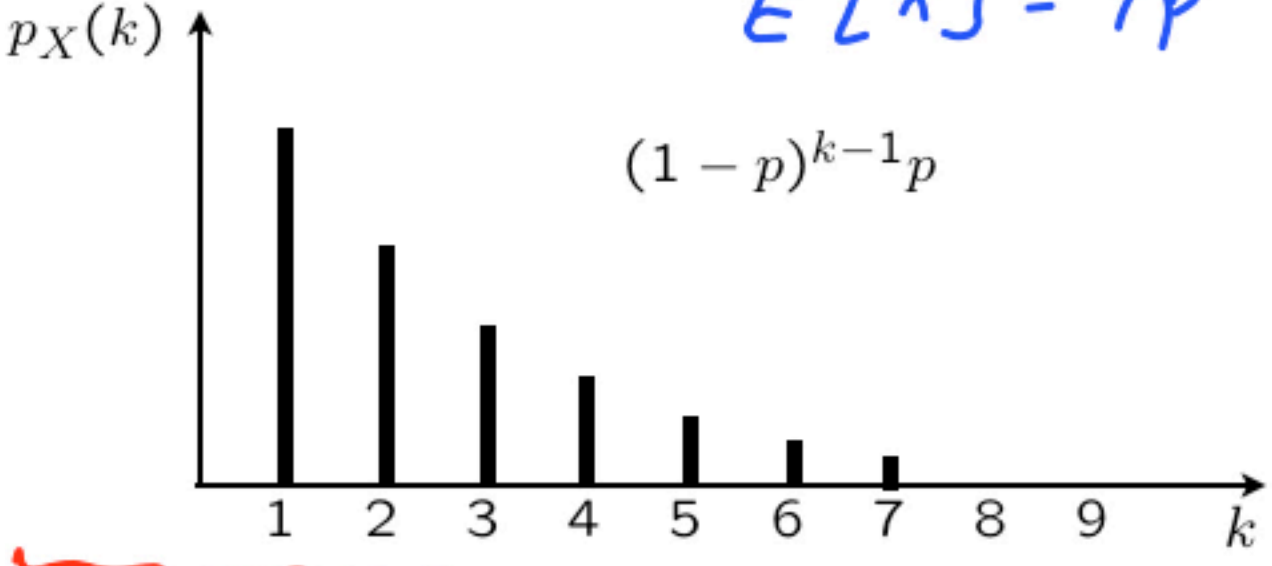
$$\text{var}(X) = \frac{1}{12}(b-a)(b-a+2)$$

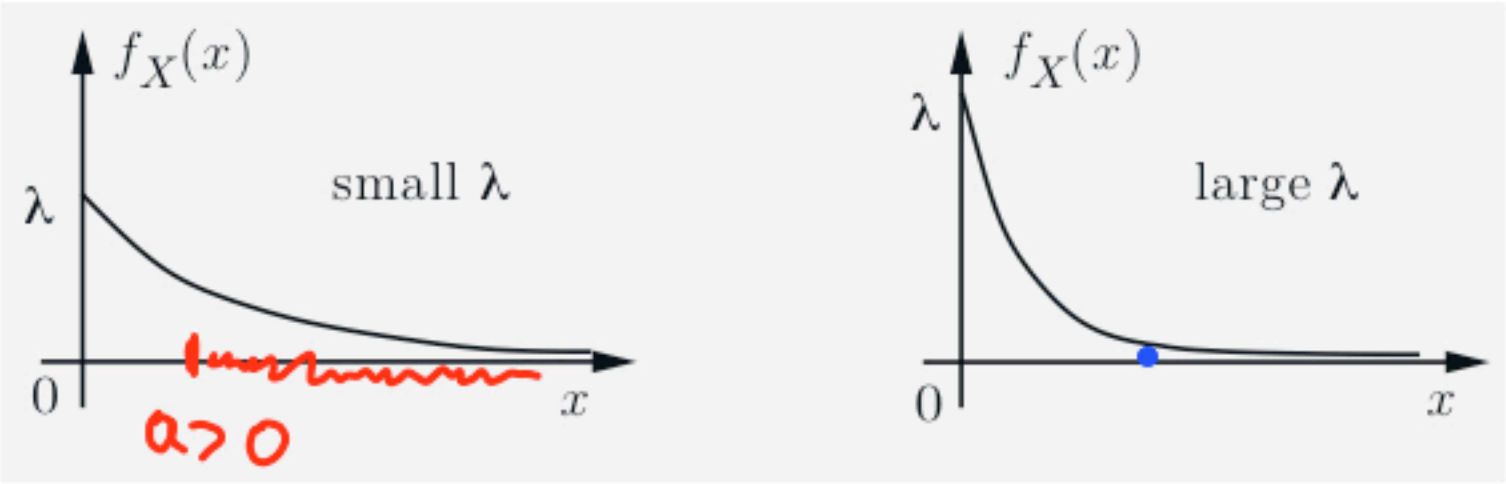
Exponential random variable; parameter $\lambda > 0$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$\int f_x(x)\,dx = 1$

$E[x] = 1/p$





$$\boxed{P(X \geq a)} = \int_a^{\infty} \lambda e^{-\lambda x}\,dx$$

$$\left[\int e^{ax}\,dx = \frac{1}{a} e^{ax} \quad a \leftrightarrow -\lambda\right]$$

$$= \lambda \cdot \left(-\frac{1}{\lambda}\right) e^{-\lambda x}\Big|_a^{\infty}$$

$$= -e^{-\lambda \cdot \infty} + e^{-\lambda a} = \boxed{e^{-\lambda a}}$$

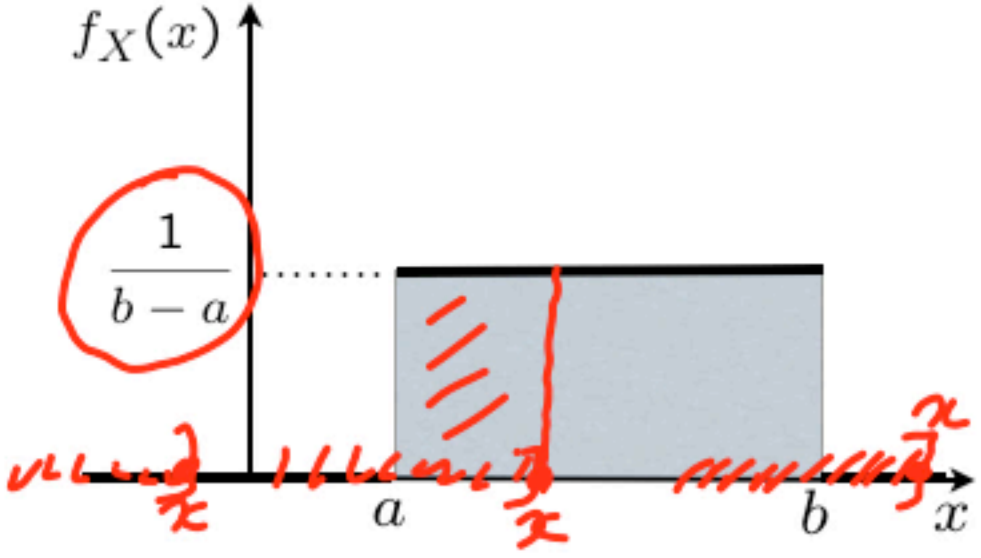
$$\mathbf{E}[X] = \int_0^{\infty} x \cdot \lambda e^{-\lambda x}\,dx = 1/\lambda$$

$$\mathbf{E}[X^2] = \int_0^{\infty} x^2 \lambda e^{-\lambda x}\,dx = 2/\lambda^2$$

$$\text{var}(X) = E[X^2] - (E[X])^2 = 1/\lambda^2$$

Cumulative distribution function (CDF)

CDF definition: $F_X(x) = \mathrm{P}(X \le x)$



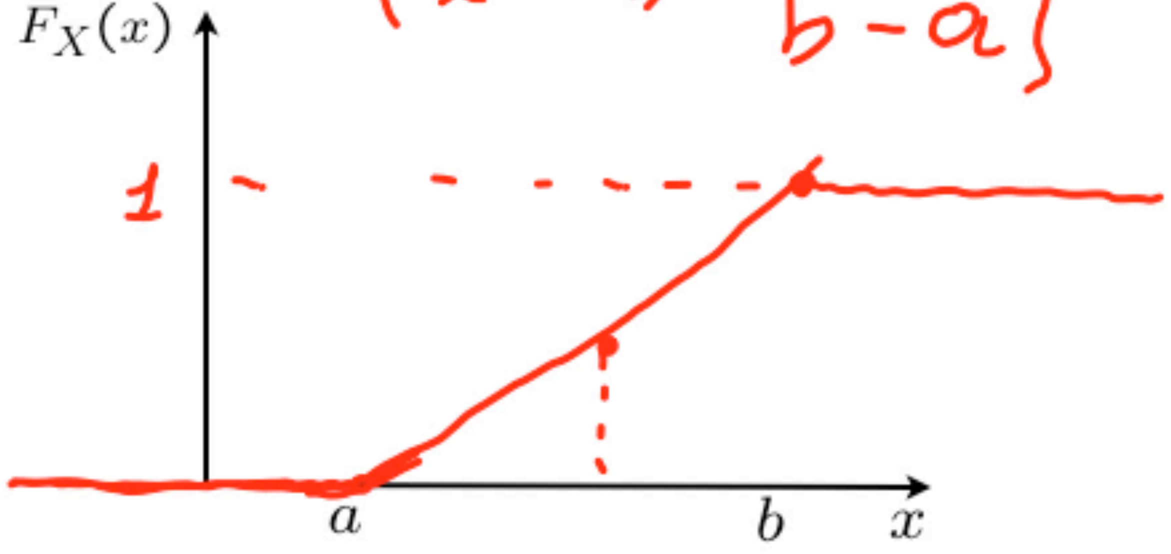
- Continuous random variables:

$$F_X(x) = \mathrm{P}(X \le x) = \int_{-\infty}^{x} f_X(t)\, dt$$

$$\underbrace{P(X \le 4)}\ = \underbrace{P(X \le 3)} + P(3 < X \le 4)$$

$$\boxed{\frac{dF_X}{dx}(x) = f_X(x)}$$

$$(x-a)\cdot\frac{1}{b-a}$$

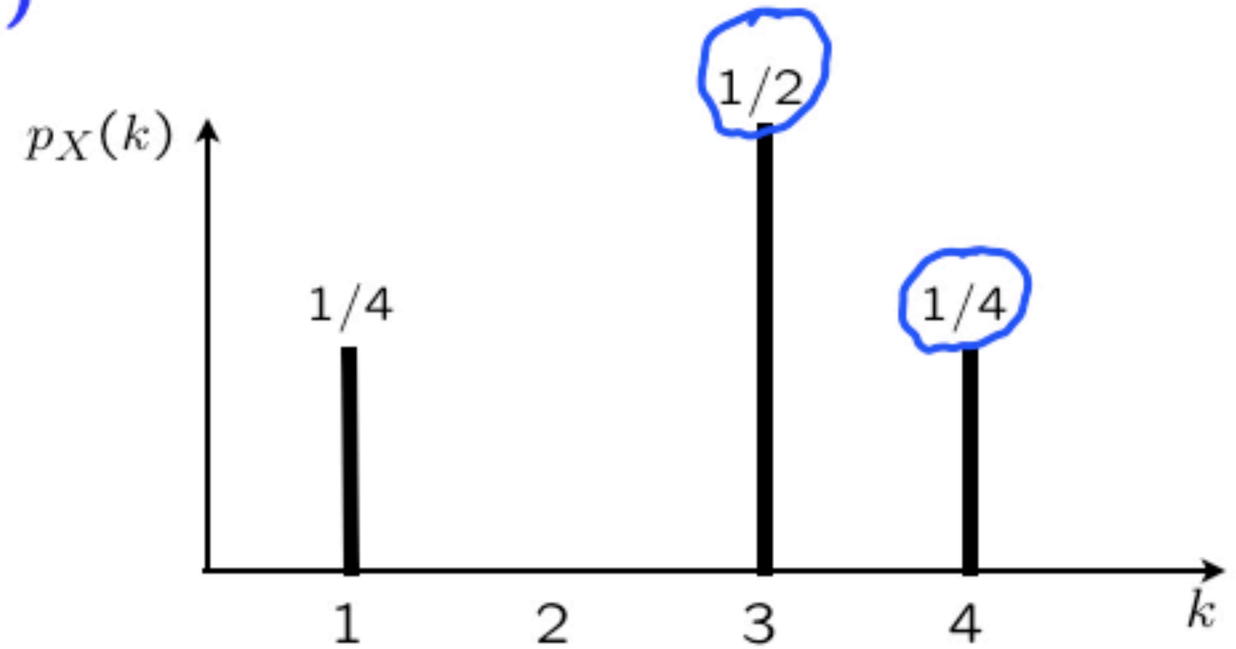


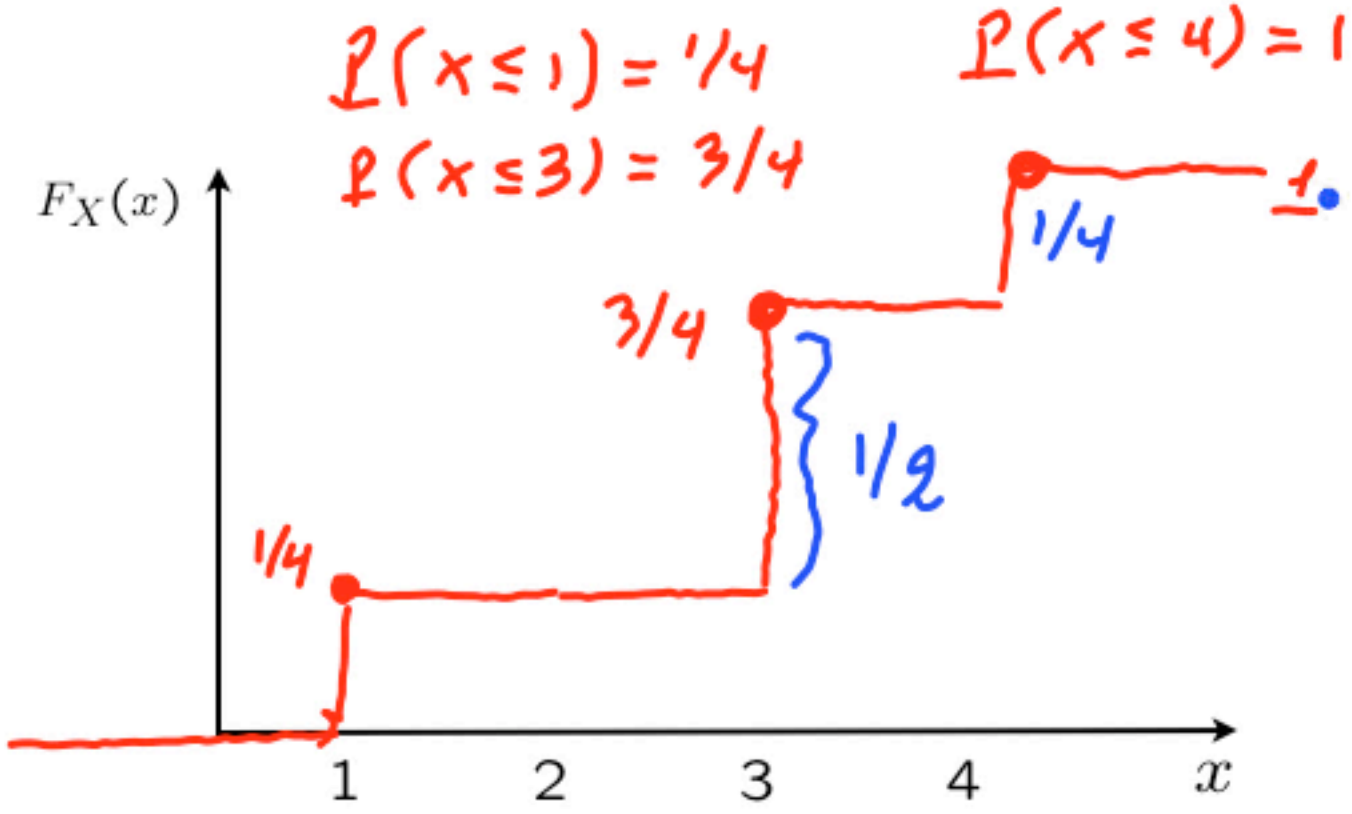
Cumulative distribution function (CDF)

CDF definition: $F_X(x) = \mathrm{P}(X \le x)$

- Discrete random variables:

$$F_X(x) = \mathrm{P}(X \le x) = \sum_{k \le x} p_X(k)$$





General CDF properties

$$F_X(x) = \mathbf{P}(X \le x)$$

- Non-decreasing   If $y \geq x \Rightarrow F_X(y) \geq F_X(x)$

- $F_X(x)$ tends to 1, as $x \to \infty$

- $F_X(x)$ tends to 0, as $x \to -\infty$

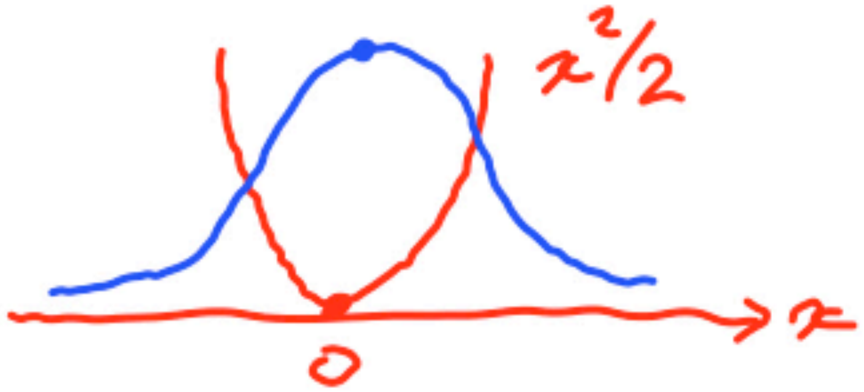
Normal (Gaussian) random variables

- Important in the theory of probability
  - Central limit theorem

- Prevalent in applications
  - Convenient analytical properties
  - Model of noise consisting of many, small independent noise terms

Standard normal (Gaussian) random variables

- Standard normal $N(0,1)$: $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$



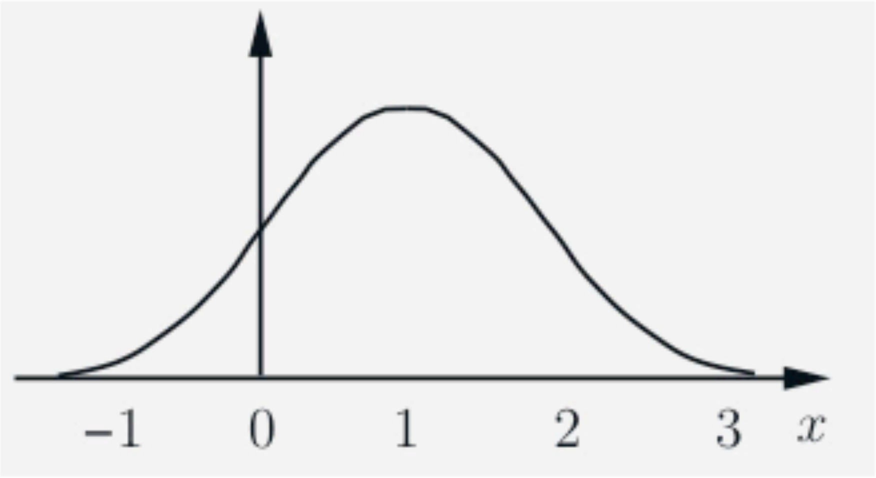
calculus:

$$\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \sqrt{2\pi}$$

- $\mathbf{E}[X] = 0$

- $\mathrm{var}(X) = 1$ integrate by parts

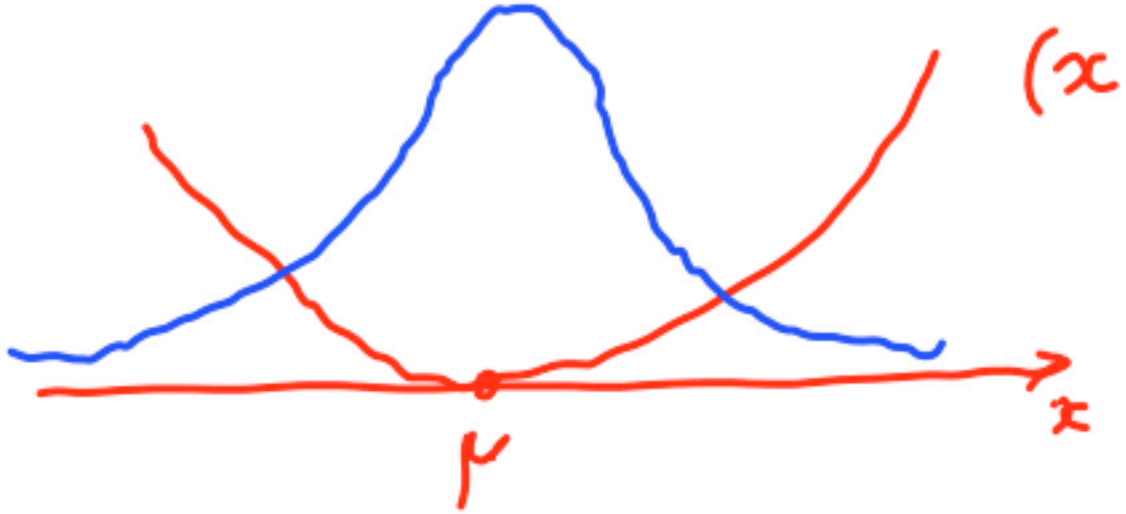
General normal (Gaussian) random variables



- General normal $N(\mu, \sigma^2)$: $f_X(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{\underbrace{-(x-\mu)^2/2\sigma^2}}$

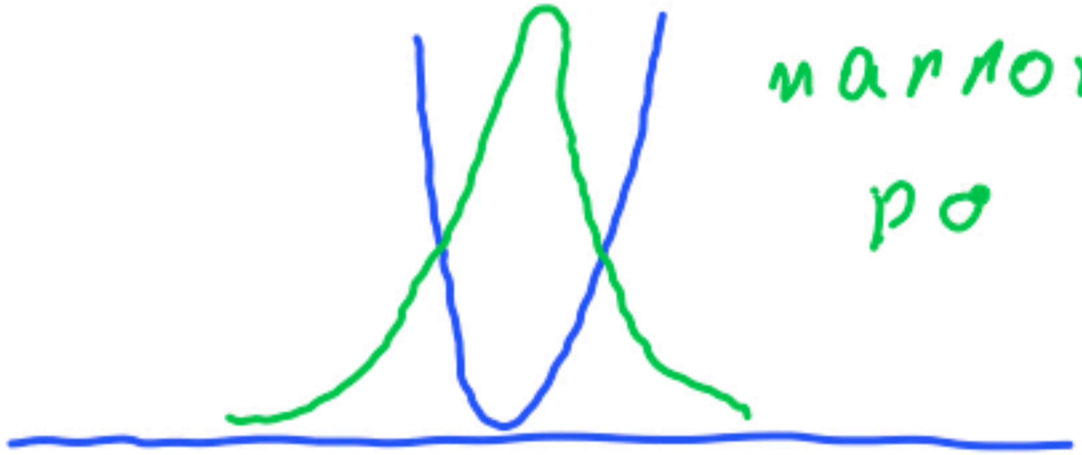
  $\sigma > 0$

$(x-\mu)^2/2\sigma^2$



σ small

narrower pd



- $\mathbf{E}[X] = \mu$

- $\text{var}(X) = \sigma^2$

Linear functions of a normal random variable

- Let $Y = aX + b \quad X \sim N(\mu, \sigma^2)$

$\mathbf{E}[Y] =$ $a\mu + b$

$\mathrm{Var}(Y) =$ $a^2\sigma^2$

- Fact (will prove later in this course):

$$\boxed{Y \sim N(a\mu + b, a^2\sigma^2)}$$

- Special case: $a = 0$?

$Y = b$ discrete

$N(b, 0)$

Standard normal tables

- No closed form available for CDF

but have tables, for the standard normal

$Y \sim N(0,1)$

$\Phi(y) = F_Y(y) = P(Y \le y)$

$\Phi(0) = P(Y \le 0) = 0.5$

$\Phi(1.16) = 0.8770$ $\quad \Phi(2.9) = 0.9981$

$\Phi(-2)$

$=$

$1 - \Phi(2)$

$= 1 - .9772$

	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986

Standardizing a random variable

- Let X have mean μ and variance $\sigma^2 > 0$

- Let $Y = \frac{X - \mu}{\sigma}$ $\quad E[Y] = 0 \quad \mathrm{Var}(Y) = \frac{1}{\sigma^2} \mathrm{Var}(X) = 1$

$$X = \mu + \sigma Y$$

- If also X is normal, then: $Y \sim N(0,1)$

Calculating normal probabilities

- Express an event of interest in terms of standard normal

$X \sim N(6,4) \qquad \sigma = 2$

st. normal

$$\frac{2-6}{2} \le \frac{X-6}{2} \le \frac{8-6}{2}$$

$$P(2 \le X \le 8) = P(-2 \le Y \le 1)$$

$$= P(Y \le 1) - P(Y \le -2)$$

$$= P(Y \le 1) - (1 - P(Y \le 2))$$

	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
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0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
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1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
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1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986