- - Consider $c := 12 + 11 + 10 = 33$ and $n_0 := 1$. Then $0 \le 12n^3 + 11n^2 + 10 \le cn^3$ for all $n \ge n_0$.
  - Consider $c := 1$ and $n_0 := 1$. Then $0 \le cn^3 \le 12n^3 + 11n^2 + 10$ for all $n \ge n_0$.
  - Follows from parts (a) and (b), i.e., let $c_1 := 33$, $c_2 := 1$ and $n_0 := 1$.
  - Let $c > 0$ be given. Set $n_0 > 0$ to be minimal such that $1000 < c \log n_0$ (i.e., $n_0 := 1 + \lfloor e^{1000/c} \rfloor$). Then $0 \le 1000n < cn \log n$ for all $n \ge n_0$.
  - Let $c > 0$ be given. Set $n_0 := 21 + c$. Then, for $n \ge n_0$, we have $n^n = n^{n-20} n^{20} \ge (21 + c)^{1+c} n^{20}$. Since $(21 + c)^{1+c} > c$, this shows that $0 \le cn^{20} < n^n$ for all $n \ge n_0$.

- - $f(n) \in \omega(g(n))$. Either show directly that the definition of ω, or show that $\lim_{n\to\infty} f(n)/g(n) = \infty$.
  - $f(n) \in \Theta(g(n))$. It suffices to show that both $f(n)$ and $g(n)$ are $\Theta(n^3)$.

- - False. Counter example: Consider $f(n) := n$ and $g(n) := \begin{cases} 1 & n \text{ odd} \\ n^2 & n \text{ even} \end{cases}$. To prove the claim false it will be sufficient to show that $f(n) \notin O(g(n))$ and $f(n) \notin \Omega(g(n))$, since then the antecedent of the implication is satisfied while the consequent is not.

    If $f(n) \in O(g(n))$, then there exist constants $n_0 > 0$ and $c > 0$ such that $f(n) \le cg(n)$ for all $n \ge n_0$. But for any odd number $n_1 > c$ we have $f(n_1) = n_1 > c = cg(n_1)$, showing that $f(n) \notin O(g(n))$.

    Similarly, if $f(n) \in \Omega(g(n))$, then there exists constants $n_0 > 0$ and $c > 0$ such that $cg(n) \le f(n)$ for all $n \ge n_0$. But for any even number $n_1 > 1/c$ we have $cg(n_1) = cn_1^2 > n_1 = f(n_1)$, showing that $f(n) \notin \Omega(g(n))$.
  - True. We will show that $f(n)g(n)/(f(n)+g(n)) \le \min(f(n), g(n)) \le 2f(n)g(n)/(f(n)+g(n))$ for all $n \ge 1$. The desired result will then follow from the definition of Θ using $c_1 = 1$, $c_2 = 2$ and $n_0 = 1$.

    For brevity, let f denote $f(n)$ and g denote $g(n)$, $n \ge n_0$. By assumption, f and g are positive, so $fg/(f + g) = \min(f, g) \max(f, g)/(f + g)$, which is less than $\min(f, g)$ since $\max(f, g)/(f + g) < 1$. Similarly, $\min(f, g) = 2fg/(2 \max(f, g)) \le 2fg/(f + g)$.

- We have $S(n) = 2\,S(n) - S(n) = 1 + (\sum_{i=1}^{n-1}(1/2^i)) - n/2^n = 2 - (n + 2)/2^n$.