University of Waterloo
CS240 - Fall 2016
Assignment 1 Part 1

Due Date: Wednesday September 21 at 5:00pm

Please read `http://www.student.cs.uwaterloo.ca/~cs240/f16/guidelines.pdf` for guidelines on submission. Problems 1 – 4 are written problems; submit your solutions electronically as a PDF with file name `a01p1wp.pdf` using MarkUs. We will also accept individual question files named `a01q1w.pdf`, `a01q2w.pdf`, ... , `a01q4w.pdf` if you wish to submit questions as you complete them.

Note: Assignment 1 has been split into 2 parts both worth 2.5% each. Keep in mind that part 1 is due before part 2. There are 39 marks for part 1 and 34 marks for part 2.

Problem 1 [3+3+3+3+3=15 marks]

Provide a complete proof of the following statements from first principles (i.e., using the original definitions of order notation). All logarithms are natural logarithms: $\log = \ln$.

a) $12n^3 + 11n^2 + 10 \in O(n^3)$
Solution:
We need to show that $\exists c$ and $\exists n_0$ such that $0 \leq 12n^3 + 11n^2 + 10 \leq c \cdot n^3$ for all $n \geq n_0$.
Obviously, $12n^3 + 11n^2 + 10 \leq 33n^3$ when $n \geq 1$
Thus, the given statement is true such that $c = 33$ and $n_0 = 1$

b) $12n^3 + 11n^2 + 10 \in \Omega(n^3)$
Solution:
We need to show that $\exists c$ and $\exists n_0$ such that $0 \leq c \cdot n^3 \leq 12n^3 + 11n^2 + 10$ for all $n \geq n_0$.
Obviously, $12n^3 + 11n^2 + 10 \geq 12n^3$ when $n \geq 1$
Thus, the given statement is true such that $c = 33$ and $n_0 = 1$

c) $12n^3 + 11n^2 + 10 \in \Theta(n^3)$
Solution:
Since (a) and (b) have been proved to be true, $12n^3 + 11n^2 + 10 \in \Theta(n^3)$ is true by the fact.

d) $1000n \in o(n \log n)$
Solution:
We need to show that $\forall c$ and $\exists n_0$ such that $0 \leq 1000n < c \cdot n \log n$ for all $n \geq n_0$.
Because $1000n < c \cdot n \log n \Rightarrow 1000 < c \cdot \log n \Rightarrow \frac{1000}{c} < \log n \Rightarrow e^{\frac{1000}{n}} < n$
Thus, $n_0 = e^{\frac{1000}{n}}$.
To prove that, plug n_0 in the equation:
$c \cdot n \log n > c \cdot n \log n_0 = c \cdot n \log e^{\frac{1000}{c}} = cn(1000/c) = 1000n$
So, the proof is done.

e) $n^n \in \omega(n^{20})$
Solution:
We need to show that $\forall c$ and $\exists n_0$ such that $0 \leq c \cdot n^{20} < n^n$ for all $n \geq n_0$.
$cn^{20} < n^n \Rightarrow \log c + 20 \log n < n \log n = (n-1) \log n + \log n$
First, when $\log n > \log c \Rightarrow n > c$.
Second, when $(n-1) \log n > 20 \log n \Rightarrow n - 1 > 20 \Rightarrow n > 21$.
Thus, $n^n > cn^{20}$ when $n > max(c, 21)$.

Problem 2 [4+4=8 marks]

For each pair of the following functions, fill in the correct asymptotic notation among Θ, o, and ω in the statement $f(n) \in \sqcup(g(n))$. Provide a brief justification of your answers. In your justification you may use any relationship or technique that is described in class.

a) $f(n) = \sqrt{n}$ versus $g(n) = (\log n)^2$
Solution:

$lim_{n \to \infty} \frac{(\log n)^2}{\sqrt{n}}$

Using L'Hopital's rule,

$= lim_{n\to\infty} \frac{2\cdot\log n\cdot\frac{1}{n}}{0.5n^{-0.5}}$

$= lim_{n\to\infty} \frac{2\cdot\log n}{0.5n^{0.5}}$

$= lim_{n\to\infty} \frac{2\cdot\frac{1}{n}}{0.25n^{-0.5}}$

$= lim_{n\to\infty} \frac{2}{0.25n^{0.5}}$

$= 0$
$\therefore f(n) \in o(g(n))$.

b) $f(n) = n^3(5 + 2\cos 2n)$ versus $g(n) = 3n^2 + 4n^3 + 5n$
Solution:
$\because -2 \leq 2\cos 2n \leq 2$, $3 \leq 5 + 2\cos 2n \leq 7$, $3n^3 \leq n^3(5 + 2\cos 2n) \leq 7n^3$.
$\therefore f(n) \in \theta(n^3)$.
$\because \theta(3n^2 + 4n^3 + 5n) = \theta(4n^3) = \theta(n^3)$ by "Maximum Rules", $g(n) \in \theta(n^3)$.
$\therefore f(n) \in \theta(g(n))$.

Problem 3 [6+6=12 marks]

Prove or disprove each of the following statements. To prove a statement, you should provide a formal proof that is based on the definitions of the order notations. To disprove a statement, you can either provide a counter example and explain it or provide a formal proof. All functions are positive functions.

a) $f(n) \notin o(g(n))$ and $f(n) \notin \omega(g(n)) \Rightarrow f(n) \in \Theta(g(n))$

Solution:

$f(n) \notin o(g(n))$

$\Rightarrow$ not $(\forall c > 0, \exists n_0 > 0$ s.t. $0 \leq f(n) < cg(n)$ for all $n \geq n_0$

$\Rightarrow \exists c > 0, \exists n > n_0, \forall n_0 > 0, f(n) < 0$ or $f(n) \geq cg(n)$

$\because g(n) \geq 0$ and $f(n) \geq 0$

$\therefore$ We choose $f(n) \geq cg(n) \geq 0$

$\Rightarrow \exists c > 0, \exists n_0 > 0$ s.t. $f(n) \geq cg(n) \geq 0$, $\forall n \geq n_0$

$\Rightarrow f(n) \in \Omega(g(n))$

$f(n) \notin \omega(g(n))$

$\Rightarrow$ not ($\forall c > 0, \exists n_0 > 0$ s.t. $0 \leq cg(n) < f(n)$ for all $n \geq n_0$

$\Rightarrow \exists c > 0, \exists n > n_0, \forall n_0 > 0, cg(n) < 0$ or $cg(n) \geq f(n)$

$\because g(n) \geq 0$ and $f(n) \geq 0$

$\therefore$ We choose $cg(n) \geq f(n) \geq 0$

$\Rightarrow \exists c > 0, \exists n_0 > 0$ s.t. $cg(n) \geq f(n) \geq 0$, $\forall n \geq n_0$

$\Rightarrow f(n) \in O(g(n))$

Therefore, $f(n) \in \Theta(g(n))$.

b) $\min(f(n), g(n)) \in \Theta\left(\frac{f(n)g(n)}{f(n)+g(n)}\right)$

Solution:

Firstly, we need to prove $min(f(n), g(n)) \in O(\frac{f(n)g(n)}{f(n)+g(n)})$.

We need to show that, $\exists c > 0$, $\exists n_0 > 0$, s.t. $0 \leq min(f(n), g(n)) \leq c \cdot \frac{f(n)g(n)}{f(n)+g(n)}$ for all $n \geq n_0$.

If $f(n) > g(n)$, $\Rightarrow g(n) \leq c \cdot \frac{f(n)g(n)}{f(n)+g(n)} \Rightarrow 1 \leq \frac{cf(n)}{f(n)+g(n)} \Rightarrow$

$cf(n) \geq f(n) + g(n) \Rightarrow cf(n) \geq 2f(n) \geq f(n) + g(n) \Rightarrow c \geq 2$ and $n_0 \geq 1$

If $f(n) < g(n)$, $\Rightarrow f(n) \leq c \cdot \frac{f(n)g(n)}{f(n)+g(n)} \Rightarrow 1 \leq \frac{cg(n)}{f(n)+g(n)} \Rightarrow$

$cg(n) \geq f(n) + g(n) \Rightarrow cg(n) \geq 2g(n) \geq f(n) + g(n) \Rightarrow c \geq 2$ and $n_0 \geq 1$

Thus, the first part is proved.

Secondly, we need to prove $min(f(n), g(n)) \in \Omega(\frac{f(n)g(n)}{f(n)+g(n)})$.

We need to show that $\exists c > 0$, $\exists n_0 > 0$, s.t. $0 \leq c \cdot \frac{f(n)g(n)}{f(n)+g(n)} \leq min(f(n), g(n))$ for all $n \geq n_0$.

If $f(n) > g(n)$, $\Rightarrow g(n) \geq c \cdot \frac{f(n)g(n)}{f(n)+g(n)} \Rightarrow 1 \geq \frac{cf(n)}{f(n)+g(n)} \Rightarrow$

$cf(n) \leq f(n) + g(n) \Rightarrow cf(n) \leq f(n) \leq f(n) + g(n) \Rightarrow c \leq 1$ and $n_0 \geq 1$

If $f(n) < g(n)$, $\Rightarrow f(n) \geq c \cdot \frac{f(n)g(n)}{f(n)+g(n)} \Rightarrow 1 \geq \frac{cg(n)}{f(n)+g(n)} \Rightarrow$

$cg(n) \leq f(n) + g(n) \Rightarrow cg(n) \leq g(n) \leq f(n) + g(n) \Rightarrow c \leq 1$ and $n_0 \geq 1$

Thus, the second part is proved.

Therefore, $\min(f(n), g(n)) \in \Theta\left(\frac{f(n)g(n)}{f(n)+g(n)}\right)$ is true.

Problem 4 [4 marks]

Derive a closed form for the following sum:

$$S(n) = \sum_{i=1}^{n} i/2^i.$$

Solution:

$S(n) = 1/2 + 2/4 + 3/8 + 4/16 + ... + n/2^n$

$= (1/2 + 1/4 + 1/8 + 1/16 + ...) + (1/4 + 1/8 + 1/16 + 1/32 + ...) + (1/8 + 1/16 + 1/32 + 1/64 + ...) + ... + (1/2^n)$

$= \frac{1/2}{1/2} + \frac{1/4}{1/2} + \frac{1/8}{1/2} + ... + (1/2^n)$

$= 1 + 1/2 + 1/4 + 1/8 + 1/16 + ... + 1/2^n$

$= \frac{1}{1/2}$

=2