# 21-127 Homework 5

Due: Monday, March 10

**Instructions:** Submit a PDF of your work through Gradescope by 11:59 PM on the due date stated above. All other instructions from previous homeworks and the syllabus apply as usual.

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**1.** For each of the functions below, do the following:

- Find (with proof) its two-sided inverse if it has one.
- If it has a left inverse but not a right inverse, find (with proof) its left inverse and prove that it does not have a right inverse.
- If it has a right inverse but not a left inverse, find (with proof) its right inverse and prove that it does not have a left inverse.
- If it has neither a left nor a right inverse, prove it.

(a) $f : \mathbb{N} \to \mathbb{Z}$, $f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ -\frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases}$;

(b) $g : \mathbb{Z} \to \mathbb{Z}$, $g(n) = \dfrac{n+|n|}{2}$ for all $n \in \mathbb{Z}$.

(c) $h : \mathscr{P}(\mathbb{N}) \to \mathscr{P}(\mathbb{N})$, $h(A) = \mathbb{N} \setminus A$ for all $A \in \mathscr{P}(\mathbb{N})$.

(d) $k : \mathscr{P}(\mathbb{R}) \to \mathscr{P}(\mathbb{R})$, $k(A) = A \cap \mathbb{Q}$ for all $A \in \mathscr{P}(\mathbb{R})$.

**2.** (a) Prove by induction that for all $n \in \mathbb{N}$, the sum of the first $n$ even positive integers is equal to $n(n+1)$.
[In this part, do not use the fact about the sum of the first $n$ integers that we proved in class; instead, prove the required statement directly.]

(b) Prove by induction that for all $n \in \mathbb{N}$, the sum of the first $n$ positive whole cubes is equal to the square of the sum of the first $n$ positive integers.
[A 'whole cube' is the cube of an integer.]

**3.** Prove by induction that $5 \cdot 8^n - 7 \cdot 6^n + 2$ is divisible by 70 for all $n \in \mathbb{N}$.

**4.** The *Fibonacci sequence* is the sequence $f_0, f_1, f_2, \dots$ defined recursively by

$$f_0 = 0, \quad f_1 = 1, \quad f_{n+2} = f_{n+1} + f_n \text{ for all } n \in \mathbb{N}$$

(a) Prove by induction that $\displaystyle\sum_{k=1}^{n} f_{2k-1} = f_{2n}$ for all $n \in \mathbb{N}$.

(b) Prove by induction that $\displaystyle\sum_{k=1}^{n} f_k = f_{n+2} - 1$ for all $n \in \mathbb{N}$.

**5.** The *Tribonacci sequence* is the sequence $t_0, t_1, t_2, \dots$ defined by

$$t_0 = 0, \quad t_1 = 0, \quad t_2 = 1, \quad t_n = t_{n-1} + t_{n-2} + t_{n-3} \text{ for all } n \geqslant 3$$

Prove that $t_n \leqslant 2^{n-3}$ for all $n \geqslant 3$.

**6.** Let $\varphi_0, \varphi_1, \varphi_2, \ldots$ be propositional formulas. The operators of *indexed conjunction* $\bigwedge_{i=1}^{n}$ and *indexed disjunction* $\bigvee_{i=1}^{n}$ are defined by recursion on $n \in \mathbb{N}$ as follows:

- $\bigwedge_{i=1}^{0} \varphi_i = \top$ and $\bigwedge_{i=1}^{n+1} \varphi_i = \left( \bigwedge_{i=1}^{n} \varphi_i \right) \wedge \varphi_{n+1}$ for all $n \in \mathbb{N}$;
- $\bigvee_{i=1}^{0} \varphi_i = \bot$ and $\bigvee_{i=1}^{n+1} \varphi_i = \left( \bigvee_{i=1}^{n} \varphi_i \right) \vee \varphi_{n+1}$ for all $n \in \mathbb{N}$.

where $\top$ represents the true proposition '$0 = 0$', and $\bot$ represents the false proposition '$0 = 1$'.

Prove by induction that $\left( \bigvee_{i=1}^{n} p_i \right) \Rightarrow q \equiv \bigwedge_{i=1}^{n} (p_i \Rightarrow q)$ for all $n \in \mathbb{N}$, where $p_1, p_2, \ldots$ and $q$ are propositional variables.