Детерминантни функции

F - (числово) поле

$$f : \underbrace{F^n \times F^n \times \dots \times F^n}_{n} \longrightarrow F$$

$v_1, \dots, v_n \in F^n$; $f(v_1, \dots, v_n) \in F$

Опр. f е полилинейна (ПЛФ), ако $\forall i = 1, \dots, n$ и $\forall v_1, \dots, v_{i-1}, v_i', v_i'', v_{i+1}, \dots, v_n$, $\forall \alpha, \beta \in F$

$$f(v_1, \dots, v_{i-1}, \alpha v_i' + \beta v_i'', v_{i+1}, \dots, v_n) =$$

$$= \alpha f(v_1, \dots, v_{i-1}, v_i', v_{i+1}, \dots, v_n) + \beta f(v_1, \dots, v_{i-1}, v_i'', v_{i+1}, \dots, v_n)$$

Заб. $v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n \in F^n$

$g_i(v) := f(v_1, \dots, v_{i-1}, v, v_{i+1}, \dots, v_n)$ за $v \in F^n$

$f \in$ ПЛФ $\iff \forall i = 1, \dots, n$ $\quad g_i : F^n \to F$ са ЛФ (ЛИ)

$(\forall \alpha, \beta \in F$ и $\forall u, v \in F^n$

$$g_i(\alpha u + \beta v) = \alpha g_i(u) + \beta g_i(v))$$

Често казваме, че f е линейно по всеки аргумент

Заб. Искаме ПЛФ и ако заменим F^n с разл. ЛП $V_1, \dots, V_n$ --- форм. $V_1, \dots, V_n$

Означение $f : F^n \times \ldots \times F^n \to F$, $\sigma \in S_n$

$\sigma f : F^n \times \ldots \times F^n \to F$:

$\forall\, v_1, \ldots, v_n \in F^n \quad (\sigma f)(v_1, \ldots, v_n) = f(v_{\sigma(1)}, v_{\sigma(2)}, \ldots, v_{\sigma(n)})$

Пр. $n = 2$, $\sigma = (1\,2) \Rightarrow (\sigma f)(v_1, v_2) = f(v_2, v_1)$

Опр. f е антисиметрична ф-я (АСФ), ако

$\forall\, i, j : 1 \leq i < j \leq n \quad (i\,j) f = -f$

Заб. $\Leftrightarrow$ $\forall\, \sigma$ - транспоз. $\sigma f = -f$

Зад. $\iff \forall i,j : 1 \le i < j \le n$

$$f(x_1, \dots, x_{i-1}, \underline{x_i}, x_{i+1}, \dots, x_{j-1}, \underline{x_j}, x_{j+1}, \dots, x_n) =$$

$$= -f(x_1, \dots, x_{i-1}, \underline{x_j}, x_{i+1}, \dots, x_{j-1}, \underline{x_i}, x_{j+1}, \dots, x_n)$$

Тв. $f \in AC\mathcal{F} \iff \forall \sigma \in S_n \quad \sigma f = (\operatorname{sign} \sigma) . f$

Д-во $(\Leftarrow)$ очев. (Ако σ е транспоз., то $\operatorname{sign}\sigma = -1$)

$(\Rightarrow)$ $\sigma = \tau_1 \dots \tau_s$, τ_i – транспоз.

$$\sigma f = (\tau_1 \dots \tau_s) f = \tau_1 [(\tau_2 \dots \tau_s) f] = -(\tau_2 \dots \tau_s) f \overset{\text{инд.}}{=} -\overbrace{(-1)^{s-1}}^{= \operatorname{sign}(\tau_2 \dots \tau_s)} f =$$

$$= (-1)^s f \overset{!}{=} \operatorname{sign}\sigma . f$$

Зад. $\sigma, \tau \in S_n \qquad (\sigma\tau)f = \sigma(\tau f)$

$$\left(\forall i = 1, \ldots, n \quad (\sigma\tau)(i) = \sigma(\tau(i)) \right)$$

$$((\sigma\tau)f)(v_1, \ldots, v_n) = f(v_{\sigma\tau(1)}, \ldots, v_{\sigma\tau(n)}) =$$

$$= \sigma\left[f(v_{\tau(1)}, \ldots, v_{\tau(n)})\right] = \sigma(\tau f)$$

Зад. $(i<j)$ $(i\ j) = (j{-}1\ \ j)(j{-}2\ \ j{-}1)\ldots(i{+}1\ \ i{+}2)(i\ \ i{+}1)(i{+}1\ \ i{+}2)\ldots(j{-}1\ \ j)$, т.е.

$\forall$ трансп. (и $\forall$ перм.) е произведение на трансп. от вида $(k\ \ k{+}1)$

__Тв__ f - АСФ $\iff$ $\forall i$ $\quad 1 \leq i < n$

$$(i \;\; i+1) f = -f$$

__Забел.__ т.е. $f(v_1, \dots, \underline{v_i, v_{i+1}}, \dots, v_n) = -f(v_1, \dots, \underline{v_{i+1}, v_i}, \dots, v_n)$

__Тв.__ Нека f е ПЛФ. Тогава f е АСФ

$\iff \forall\, 1 \leq i < j \leq n$ и $\forall\, v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_{j-1}, v_{j+1}, \dots, v_n, v \in F^n$

$$f(v_1, \dots, v_{i-1}, \underline{v}, v_{i+1}, \dots, v_{j-1}, \underline{v}, v_{j+1}, \dots, v_n) = 0$$

__Д-во__ Нека $1 \leq i < j \leq n$; $v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_{j-1}, v_{j+1}, \dots, v_n \in F^n$

$u, v \in F^n$ $\quad g(u, v) := f(v_1, \dots, v_{i-1}, u, v_{i+1}, \dots, v_{j-1}, v, v_{j+1}, \dots, v_n)$

$(\Rightarrow)$ $\forall u, v \in F^n \quad g(u,v) = -g(v,u)$

$\overset{u=v}{\Rightarrow} g(v,v) = -g(v,v) \Rightarrow 2g(v,v) = 0 \Rightarrow g(v,v) = 0$

<u>Забел.</u> Вярно за F: $\operatorname{char} F \neq 2$. Числовите полета са такива

характеристика на полето

$(\Leftarrow)$ $u, v \in F^n$

$$0 = g(u+v, u+v) \overset{\text{ПЛФ}}{=} g(u+v, u) + g(u+v, v) =$$

$$= \underbrace{g(u,u)}_{=0} + g(v,u) + g(u,v) + \underbrace{g(v,v)}_{=0}$$

$$\Rightarrow g(v,u) = -g(u,v)$$

Тв. $f \in$ ПЛАФ (ПЛФ и АСФ); $e_1, \dots, e_n$ - базис на F^n

$v_1, \dots, v_n \in F^n$; $\forall i = 1, \dots, n$ $\quad v_i = \sum_{j=1}^{n} \lambda_{ij} e_j$. Тогава

$$f(v_1, \dots, v_n) = \left[\sum_{\sigma \in S_n} (\operatorname{sign} \sigma)\, \lambda_{1\sigma(1)} \lambda_{2\sigma(2)} \cdots \lambda_{n\sigma(n)} \right] . f(e_1, \dots, e_n)$$

Д-во $\quad f(v_1, \dots, v_n) = f\left(\sum_{i_1=1}^{n} \lambda_{1 i_1} e_{i_1}, \sum_{i_2=1}^{n} \lambda_{2 i_2} e_{i_2}, \dots, \sum_{i_n=1}^{n} \lambda_{n i_n} e_{i_n} \right)$

$\overset{\text{ПЛФ}}{=} \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_n=1}^{n} \lambda_{1 i_1} \lambda_{2 i_2} \cdots \lambda_{n i_n} . f(e_{i_1}, e_{i_2}, \dots, e_{i_n})$

$\overset{\text{АСФ}}{=} \sum_{\sigma = \begin{pmatrix} 1 & \dots & n \\ i_1 & \dots & i_n \end{pmatrix} \in S_n} \lambda_{1\sigma(1)} \cdots \lambda_{n\sigma(n)} . \underbrace{f(e_{\sigma(1)}, \dots, e_{\sigma(n)})}_{= \operatorname{sign}\sigma . f(e_1, \dots, e_n)} =$

$$= \left[\sum_{\sigma \in S_n} (\operatorname{sign} \sigma)\, \lambda_{1\sigma(1)} \cdots \lambda_{n\sigma(n)}\right] f(e_1, \dots, e_n)$$

3a $i = 1, \dots, n$ $\qquad v_i = (\lambda_{i1}, \dots, \lambda_{in}) \in F^n$

$$\phi(v_1, \dots, v_n) := \sum_{\sigma \in S_n} (\operatorname{sign} \sigma)\, \lambda_{1\sigma(1)} \lambda_{2\sigma(2)} \cdots \lambda_{n\sigma(n)}$$

<u>Tb</u> $\phi \in \overline{\Pi} \wedge \phi$

<u>D-Co</u> $i = 1, \dots, n \qquad v_i = \alpha v_i' + \beta v_i''$

$$v_i' = (\lambda_{i1}', \dots, \lambda_{in}'), \quad v_i'' = (v_{i1}'', \dots, v_{in}'')$$

$$\phi(v_1, \dots v_n) = \sum_{\sigma \in S_n} (\operatorname{sign} \sigma)\, \lambda_{1\sigma(1)} \dots \underbrace{(\alpha \lambda'_{i\sigma(i)} + \beta \lambda''_{i\sigma(i)})}_{\lambda_{i\sigma(i)}} \dots \lambda_{n\sigma(n)}$$

$$= \alpha \sum_{\sigma \in S_n} (\operatorname{sign} \sigma)\, \lambda_{1\sigma(1)} \dots \lambda'_{i\sigma(i)} \dots \lambda_{n\sigma(n)} +$$

$$+ \beta \sum_{\sigma \in S_n} (\operatorname{sign} \sigma)\, \lambda_{1\sigma(1)} \dots \lambda''_{i\sigma(i)} \dots \lambda_{n\sigma(n)} =$$

$$= \alpha\, \phi(v_1, \dots v'_i, \dots v_n) + \beta\, \phi(v_1, \dots v''_i, \dots v_n)$$

<u>Зад.</u> $\sigma, \tau \in S_n$

$$\lambda_{\sigma(1)\tau(1)} \dots \lambda_{\sigma(n)\tau(n)} = \lambda_{1(\tau\sigma^{-1})(1)} \dots \lambda_{n(\tau\sigma^{-1})(n)}$$

$\sigma(k)=i$, $k=\sigma^{-1}(i)$ $\qquad \lambda_{ix} = \lambda_{\underbrace{\sigma(k)}_{i}\,\tau(k)} \to x = \tau(k) = \tau(\sigma^{-1}(i)) = (\tau\sigma^{-1})(i)$

<u>Заув.</u> $\lambda_{\sigma(1)\tau(1)} \dots \lambda_{\sigma(n)\tau(n)} = \lambda_{(\sigma\tau^{-1})(1),1} \dots \lambda_{(\sigma\tau^{-1})(n),n}$

<u>Заув.</u> $S_n = \{\sigma^{-1} \mid \sigma \in S_n\} = \{\sigma\tau \mid \sigma \in S_n\} =$

$\tau \in S_n$ - фікс. $\qquad = \{\tau\sigma \mid \sigma \in S_n\} = \{\sigma^{-1}\tau \mid \sigma \in S_n\} = \{\tau\sigma^{-1} \mid \sigma \in S_n\}$

<u>Тв.</u> $\forall \tau \in S_n \quad \tau\Phi = (\operatorname{sign}\tau)\,\Phi$. В частності

Φ є АСФ

$\underline{D\text{-}bo}$ $(\tau\phi)(v_1, \dots, v_n) = \phi(v_{\tau(1)}, \dots, v_{\tau(n)}) =$

$$= \sum_{\sigma \in S_n} (\operatorname{sign} \sigma) \lambda_{\tau(1)\sigma(1)} \dots \lambda_{\tau(n)\sigma(n)} =$$

$$= \sum_{\sigma \in S_n} (\operatorname{sign} \sigma) \lambda_{1,(\sigma\tau^{-1})(1)} \dots \lambda_{n,(\sigma\tau^{-1})(n)} \cdot \frac{\overline{\operatorname{sign} \tau^{-1}}}{\operatorname{sign} \tau^{-1}} =$$

$$= \frac{1}{\operatorname{sign} \tau^{-1}} \cdot \sum_{\sigma \in S_n} (\operatorname{sign} \underbrace{\sigma\tau^{-1}}_{g}) \lambda_{1,\sigma\tau^{-1}(1)} \dots \lambda_{n,\sigma\tau^{-1}(n)}$$

$$= \operatorname{sign} \tau^{-1} \sum_{\substack{g = \sigma\tau^{-1} \\ \sigma \in S_n}} (\operatorname{sign} g) \lambda_{1 g(1)} \dots \lambda_{n g(n)} =$$

$$= (\operatorname{sign} \tau) \cdot \sum_{g \in S_n} (\operatorname{sign} g) \lambda_{1 g(1)} \dots \lambda_{n g(n)}$$

$= (\operatorname{sign} \tau) \cdot \Phi(v_1 \ldots v_n)$

Зад. $\forall \sigma \in S_n \quad \operatorname{sign} \sigma = \operatorname{sign} \sigma^{-1}$

$\sigma = \tau_1 \ldots \tau_k$ - транспоз. $\Rightarrow \sigma^{-1} = \tau_k^{-1} \ldots \tau_1^{-1} = \tau_k \ldots \tau_1$

(τ - транспоз. $\tau^{-1} = \tau$)

$$\operatorname{sign} \sigma \overset{\text{"}}{=} (-1)^k$$

$$\operatorname{sign} \sigma^{-1} = (-1)^k$$

Сл. $\exists$ ПЛАФ $\quad (\Phi)$

Тв Ако $e_1 \ldots e_n$ - станд. базис на $\mathbb{F}^n$ ($e_1 = (1,0,\ldots 0)$, $e_2 = (0,1,\ldots 0)$, $\ldots$ $e_n = (0,0,\ldots 1)$), то

$$\phi(e_1,\dots,e_n)=1$$

Д-во $\quad e_i=(\delta_{i1},\delta_{i2},\dots,\delta_{in}) \quad \left(\delta_{ij}=\begin{cases}1 & i=j\\ 0 & i\neq j\end{cases}\right)$

$$\phi(e_1,\dots,e_n)=\sum_{\sigma\in S_n}(\operatorname{sign}\sigma)\,\delta_{1\sigma(1)}\dots\delta_{n\sigma(n)} =$$

$$\underset{\substack{\forall i\ \sigma(i)=i,\\ \text{т.е. } \sigma=id}}{=}(\underbrace{\operatorname{sign} id}_{1})\,\delta_{11}\dots\delta_{nn}=1$$

Опр. ПЛАФ f се нарича детерминантна (ДФ) функция, ако f е и нормирана, т.е. $f(e_1,\dots,e_n)=1$ за стандартния базис $e_1,\dots,e_n$ на F^n

Заб. f е $D\Phi \iff f$ е $П Л \Phi$, f е $АС\Phi$, f е нормирана

Тв. $\exists$ единствена $D\Phi$ (ϕ)

Заб. $\exists$ - док, че Φ е $ПЛА\Phi$ и Φ е норм.

единственост и формулата:

Ако f е $ПЛА\Phi$ и $e_1, \dots, e_n$ - базис на F^n, то

$$v_i = \sum_{j=1}^{n} \lambda_{ij} e_j \,; \quad \boxed{f(v_1, \dots, v_n) = \Phi(\lambda_1, \dots, \lambda_n) \cdot f(e_1, \dots, e_n)}$$

$$\lambda_i = (\lambda_{i1}, \dots, \lambda_{in}) \in F^n$$

Ако f е нормирана, то $f \equiv \Phi \rightarrow$ единственост

Зад. Φ – единствената ДФ

Опр. $A \in M_n(F)$; $A = (a_{ij})$

$i = 1, \ldots, n \qquad a_i = (a_{i1}, \ldots, a_{in}) \in F^n$

$$\det A = |A| = \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix}_A = \Phi(a_1, \ldots, a_n)$$

детерминанта на матр. A

Зад. Стойността на единствената ДФ Φ в редовете на матр. A

<u>Зад.</u> 1) $\det A = \sum_{\sigma \in S_n} (\operatorname{sign} \sigma)\, a_{1\sigma(1)} \dots a_{n\sigma(n)}$

2) $\det E_n = 1$

<u>Тв.</u> $A \in M_n(F) \Rightarrow \det A = \det A^t$

<u>Д-во</u> $B = A^t$; $b_i = (b_{i1}, \dots, b_{in}) = (a_{1i}, a_{2i}, \dots, a_{ni})$

$\det B = \sum_{\sigma \in S_n} (\operatorname{sign} \sigma)\, b_{1\sigma(1)} \dots b_{n\sigma(n)} =$

$= \sum_{\sigma \in S_n} (\operatorname{sign} \sigma)\, a_{\sigma(1)1} \dots a_{\sigma(n)n} =$

$$= \sum_{\sigma \in S_n} (\operatorname{sign} \sigma)\, a_{1\sigma^{-1}(1)} \cdots a_{n\sigma^{-1}(n)} =$$

$$\underset{\substack{\operatorname{sign}\sigma = \\ \operatorname{sign}\sigma^{-1}}}{=} \sum_{\sigma \in S_n} (\operatorname{sign} \underbrace{\sigma^{-1}}_{g})\, a_{1\sigma^{-1}(1)} \cdots a_{n\sigma^{-1}(n)} =$$

$$= \sum_{g^{-1} \in S_n \Leftrightarrow g \in S_n} (\operatorname{sign} g)\, a_{1g(1)} \cdots a_{ng(n)} =$$

$$= \sum_{g \in S_n} (\operatorname{sign} g)\, a_{1g(1)} \cdots a_{ng(n)} = \det A$$

Вт. след 15^{00}

четв 13-15

петък 13-15

неделя – Пезн!