Уравнения на прави и равнини в пространството

ОКС $K = Oxyz = O\vec{e}_1\vec{e}_2\vec{e}_3$

I Равнина

$\boxed{\alpha}: \underset{\smile}{A}.x+\underset{\smile}{B}.y+\underset{\smile}{C}.z+D=0$ – общо уравнение на равнина

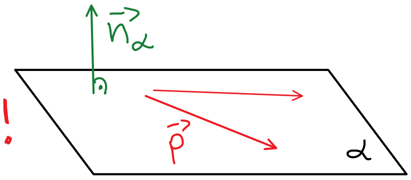
$(A,B,C) \neq (0,0,0)$

1) $\boxed{\alpha}: (k.A).x + (k.B).y + (k.C).z + (k.D) = 0 \qquad k \neq 0$

2) $M_0(x_0,y_0,z_0) \in \alpha \iff \boxed{A.x_0 + B.y_0 + C.z_0 + D = 0}$!

3) Кога $\vec{p}(p_1,p_2,p_3) \parallel \alpha$? $\iff$

$\boxed{A.p_1 + B.p_2 + C.p_3 = 0}$!



4) $\vec{n}_\alpha \perp \alpha \Rightarrow \vec{n}_\alpha(A, B, C)$

Примери:

1) $\alpha_1: x+y+z=0$ $\quad A=1,\ B=1,\ C=1,\ \boxed{D=0}$

т.$O(0,0,0) \in \alpha_1 \iff D=0$

2) $\alpha_2: 2x - y + 3 = 0$ – РАВНИНА $\quad A=2,\ B=-1,\ \boxed{C=0},\ D=3$

$\vec{e}_3(0,0,1) \to A.p_1+B.p_2+C.p_3=0$

$2.0+(-1).0+0.1=0$

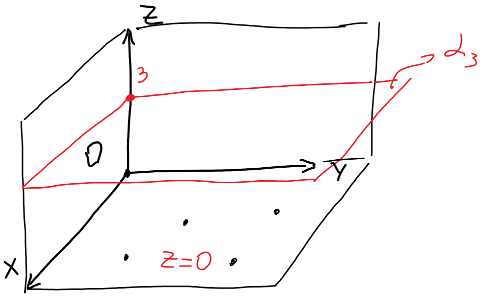
$C=0 \iff \alpha_2 \parallel Oz$

3) $Oxy: \boxed{z=0}$

$Oxz: y=0$

$Oyz: x=0$

$\alpha_3 \parallel Oxy \Rightarrow \alpha_3: z=3$



$\alpha: z=C \iff \alpha \parallel Oxy$

$\alpha_4: x=1 \Rightarrow \alpha_4 \parallel Oyz$

4) $\alpha_5: y+z=0$! $\quad \boxed{A=0},\ B=1,\ C=1,\ \boxed{D=0}$

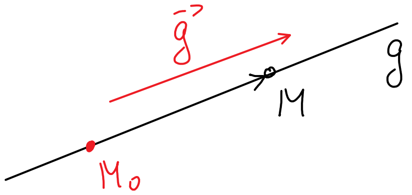
α_5 минава през Ox

II Прави:

1) Координатни параметрични уравнения

$M_0(x_0, y_0, z_0)$

$\vec{g}(a, b, c)$ $\Rightarrow \exists!\, g \begin{cases} \ni M_0 \\ \parallel \vec{g} \end{cases}$



$M(x, y, z)$ - произволна от g

$\overrightarrow{M_0M} = s.\vec{g}$

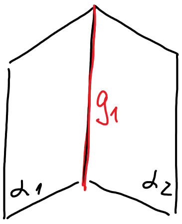
$$g: \boxed{\begin{cases} x(s) = x_0 + s.\,a \\ y(s) = y_0 + s.\,b \\ z(s) = z_0 + s.\,c \end{cases} ,} \quad \underbrace{}_{M_0} \quad \underbrace{}_{\vec{g}}$$

$s \in \mathbb{R}$

!!!

2) Права чрез 2 равнини

$$g_1: \begin{cases} x - 2y + 3 = 0 \rightarrow \alpha_1 \\ y + z + 1 = 0 \rightarrow \alpha_2 \end{cases}$$



$\alpha_1 \nparallel \alpha_2$

$\alpha_1[1, -2, 0, 3]$

$\alpha_2[0, 1\ \ 1\ \ 1]$

$$g_2 \begin{cases} x + z = 0 \rightarrow \alpha_1 , \ B = 0, D = 0 \quad \underline{\alpha_1 \ni O_y} \\ x = 0 \rightarrow \underline{\alpha_2 \rightarrow O_{yz}} \end{cases}$$

$\Rightarrow g_2 \equiv O_y$

$$g_3 \begin{cases} y = 0 \\ x + y = 0 \end{cases} \qquad g_3 \equiv O_z$$

(Защо?)

Взаимни положения на 2 равнини

1) $\alpha_1: 2x + 3y + 1 = 0$

$\alpha_2: 4x + 6.y + 2 = 0$

$$r\begin{pmatrix} 2 & 3 & 0 & 1 \\ 4 & 6 & 0 & 2 \end{pmatrix} = 1 \Leftrightarrow \alpha_1 \equiv \alpha_2$$

2) $\alpha_1: \underline{2}x + \underline{3}y + 1 = 0$

$\alpha_2: \underline{4}x + \underline{6}.y - 12 = 0$

$$r\begin{pmatrix} 2 & 3 & 0 \\ 4 & 6 & 0 \end{pmatrix} = 1$$

$$r\begin{pmatrix} 2 & 3 & 0 & 1 \\ 4 & 6 & 0 & -12 \end{pmatrix} = 2$$

$\alpha_1 \parallel \alpha_2$

3) $\alpha_1 : 2x+3y+1=0$

$\alpha_2 : x+y+z+1=0$

$\alpha_1 \cap \alpha_2 = g \qquad r\begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix} = 2$

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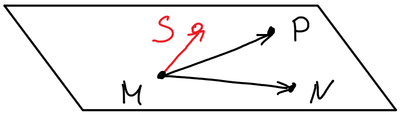
1 зад. ОКС $K = Oxyz$

а) $\underset{\sim}{M}(3,1,4)$, $N(2,1,3)$, $P(1,2,-1)$

Да се намери общо уравнение на равнината

$\alpha \ni M, N, P$

$S(x,y,z)$ - произволна от α



I н. Условие за компланарност на 4 точки

$$\begin{array}{c|cccc|} & x & y & z & 1 \\ M & 3 & 1 & 4 & 1 \\ N & 2 & 1 & 3 & 1 \\ P & 1 & 2 & -1 & 1 \end{array} = 0$$

II н. $$\left.\begin{array}{l} \vec{MS}(x-\underset{\smile}{3},\ y-\underline{1},\ z-\underset{\smile}{4}) \\ \vec{MN}(-1,\ 0,\ -1) \\ \vec{MP}(-2,\ 1,\ -5) \end{array}\right\} \text{компланарни} \Longleftrightarrow \text{ЛНЗ}$$

$$\begin{vmatrix} x-3 & y-1 & z-4 \\ -1 & 0 & -1 \\ -2 & 1 & -5 \end{vmatrix} \begin{matrix} x-3 & y-1 \\ -1 & 0 \\ -2 & 1 \end{matrix} = 0$$

$(y-1).\underset{\sim}{2} + (z-4).(-1) -$

$-[(x-3).(-1) + (y-1).\underset{\sim}{5}] = 0$

$x-3 + (y-1).(-3) - (z-4) = 0$

$$\boxed{\alpha : x-3y-z+4=0}$$

$M \to \alpha \quad 3-3.1-4+4=0$ да

$N \to \alpha \quad 2-3.1-3+4=0$ да

$P \to \alpha \quad 1-3.2-(-1)+4=0$ да

III н. Неопределени коефициенти

$\alpha : \underset{\underset{?}{\shortparallel}}{A}.x + \underset{\underset{?}{\shortparallel}}{B}.y + \underset{\underset{?}{\shortparallel}}{C}.z + \underset{\underset{?}{\shortparallel}}{D} = 0$

$$\begin{array}{l|l} M \to \alpha \Rightarrow & A.3 + B.1 + C.4 + D = 0 \\ N \to \alpha \Rightarrow & A.2 + B.1 + C.3 + D = 0 \\ P \to \alpha \Rightarrow & A.1 + B.2 + C.(-1) + D = 0 \end{array}$$

Отг. $\left| \begin{array}{l} A = -C \\ B = 3.C \\ D = -4C \end{array} \right.$

$(-C, 3C, C, -4C)$

Изб. $C = -1$

$\alpha: x - 3y - z + 4 = 0$

б) $\alpha: x - 3y - z + 4 = 0$

$\gamma: x + y - z + 1 = 0$!

$Q(0, -3, 2)$

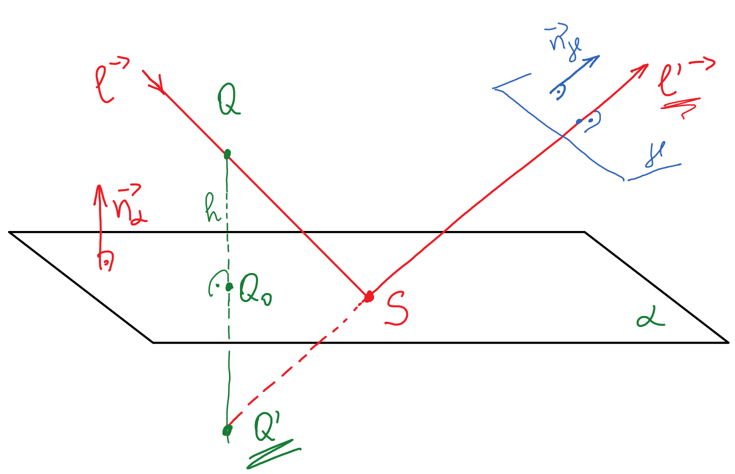
Светлинен лъч $\overrightarrow{l} \ni Q$, отразява се от α и отразеният лъч $\overrightarrow{l'}$ пробожда γ под прав ъгъл.

? уравнения на l и l'.

1) Нека т. $Q \xrightarrow{\sigma_\alpha} Q' \Rightarrow$

$\Rightarrow Q' \ni l'$

$$h \begin{cases} \ni Q(0,-3,2) \\ \parallel \vec{n}_\alpha(1,-3,-1) \end{cases}$$



$$h: \begin{cases} x = 0 + t \cdot 1 \\ y = -3 + t \cdot (-3) \\ z = 2 + t \cdot (-1) \end{cases}, \ t \in \mathbb{R}$$

$$Q_0 = h \cap \alpha \Rightarrow \left| \begin{array}{l} x = t \\ y = -3 - 3t \\ z = 2 - t \\ x - 3y - z + 4 = 0 \end{array} \right. \Rightarrow t - 3.(-3 - 3t) - (2 - t) + 4 = 0$$

$t = -1 \to h \Rightarrow x = -1$

$y = 0$

$z = 3$

$Q_0(-1, 0, 3)$ - средата

$Q\ (0, -3, 2)$

$Q'\ (x', y', z')$

$\Rightarrow \frac{x' + 0}{2} = -1 \quad \frac{y' + (-3)}{2} = 0 \quad \frac{z' + 2}{2} = 3$

$\boxed{Q'(-2, 3, 4)}$

$$2)\ l' \begin{cases} \ni Q'(-2,3,4) \\ \parallel \vec{n}_\gamma(1,1,-1) \end{cases} \Rightarrow l': \begin{cases} x = -2 + \lambda . 1 \\ y = 3 + \lambda . 1 \\ z = 4 + \lambda . (-1) \end{cases}, \lambda \in \mathbb{R}$$

3) ?, т. $S = \ell' \cap \alpha \Rightarrow \left|\begin{array}{l} x = -2+\lambda \\ y = 3+\lambda \\ z = 4-\lambda \\ x-3y-z+4=0 \end{array}\right. \Rightarrow (-2+\lambda)-3.(3+\lambda)-(4-\lambda)+4=0$

$\lambda = -11 \rightarrow \ell'$

$S(-13,-8,15)$

4) $\ell: \begin{cases} \ni Q(0,-3,2) \\ \ni S(-13,-8,15) \end{cases} \Rightarrow \vec{QS}(-13,-5,13)$

$$\ell: \begin{cases} x = 0 + p.(-13) \\ y = -3 + p.(-5) \\ z = 2 + p.13 \end{cases}, \; p \in \mathbb{R}$$

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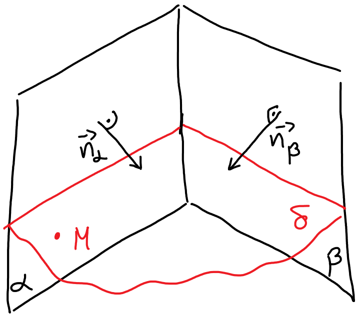
6) ?, общо уравнение на равнина $\delta \begin{cases} \ni M(3,1,4) \\ \perp \alpha: x-3y-z+4=0 \\ \perp \beta: 2x+y+5z-6=0 \end{cases}$

$\delta \perp \alpha \Rightarrow \delta \parallel \vec{n}_\alpha(1,-3,-1)$

$\delta \perp \beta \Rightarrow \delta \parallel \vec{n}_\beta(2,1,5)$

$\delta \ni M(3,1,4)$



$$\delta \; 0 = \left|\begin{array}{ccc|cc} x-3 & y-1 & z-4 & x-3 & y-1 \\ 1 & -3 & -1 & 1 & -3 \\ 2 & 1 & 5 & 2 & 1 \end{array}\right. =$$

$= (x-3).(-15) + (y-1).(-2) + (z-4).1 - [(z-4).(-6) + (x-3).(-1) + (y-1).5] =$

$= (x-3).(-14) + (y-1).(-7) + (z-4).(1+6) = 0 \quad /:(-7)$

$2(x-3) + y-1 - (z-4) = 0$

$\delta: 2x+y-z \boxed{-3} = 0$

$\vec{n}_\alpha(1,-3,-1) \parallel \delta \Leftrightarrow 2.1 + 1.(-3) - 1.(-1) = 0$ Да

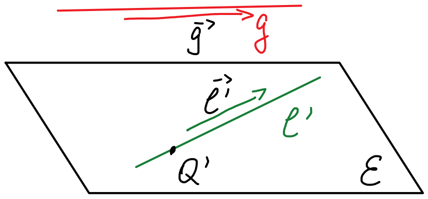
$\vec{n}_\beta(2,1,5) \parallel \delta \Leftrightarrow 2.2 + 1.1 - 1.(5) = 0$ Да

$M(3,1,4) \ni \delta \Leftrightarrow 2.3 + 1 - 4 \boxed{-3} = 0$ Да

$\ni \ell'$ $\quad \begin{cases} x = -2+2.p \end{cases}$

г) ?, общо уравнение на р-на $\varepsilon \begin{cases} \ni \ell' \\ \parallel g \end{cases}$ $\quad g: \begin{cases} x = -2+2\cdot p \\ y = \quad 1p \\ z = 2-1p \end{cases}$

$$\ell': \begin{cases} x = -2 + \lambda\cdot 1 \\ y = 3 + \lambda\cdot 1 \\ z = 4 + \lambda\cdot(-1) \end{cases}, \lambda \in \mathbb{R}$$



$\varepsilon \parallel \vec{e'}(1, 1, -1) \qquad Q'(-2,3,4) \ni \varepsilon$

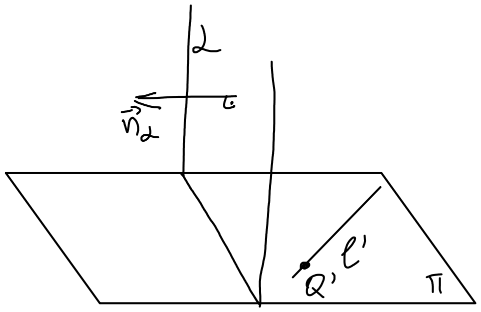
$\varepsilon \parallel \vec{g}(2, 1, -1)$

$$\varepsilon: \begin{vmatrix} x+2 & y-3 & z-4 \\ 1 & 1 & -1 \\ 2 & 1 & -1 \end{vmatrix} = 0$$

Отг: $\varepsilon: y+z-7=0$ (Упр.)

проверка

д) ?, равн. $\pi \begin{cases} \ni \ell' \\ \perp \alpha \end{cases}$



$\pi \parallel \vec{n}_\alpha(1,-3,-1)$

$\pi \parallel \vec{e'}(1, 1, -1)$

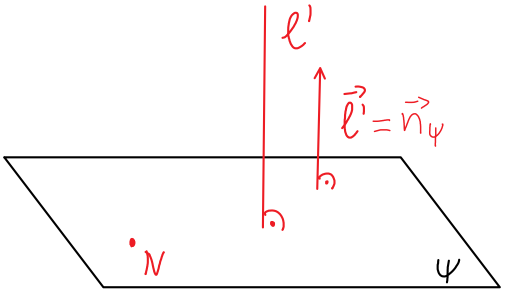
$\pi \ni Q'(-2, 3, 4)$

Отг: $\pi: x+z-2=0$

проверка

е) Да се намери общо уравнение на равнината

$\psi \begin{cases} \perp \ell' \\ \ni N(2,1,3) \end{cases}$



$$\ell': \begin{cases} x = -2 + \lambda\cdot 1 \\ y = 3 + \lambda\cdot 1 \\ z = 4 + \lambda\cdot(-1) \end{cases}, \lambda \in \mathbb{R}$$

$\ell' \parallel \vec{e'}(1, 1, -1) \perp \psi \Rightarrow \vec{\ell'} = \vec{n}_\psi \Rightarrow \vec{n}_\psi(\underset{A}{\overset{\parallel}{1}}, \underset{B}{\overset{\parallel}{1}}, \underset{C}{\overset{\parallel}{-1}}) \Rightarrow$

$\Rightarrow \psi: \underbrace{A}_{1}\cdot x + \underbrace{B}_{1}\cdot y + \underbrace{C}_{-1}\cdot z + \underbrace{D}_{?} = 0$

$\psi: x+y-z+D=0 \qquad \Rightarrow \boxed{\psi: x+y-z=0}$

$\psi: x+y-z+D=0$

$N(2,1,3) \Rightarrow 2+1-3+D=0$ $\Rightarrow$ $\boxed{\psi: x+y-z=0}$

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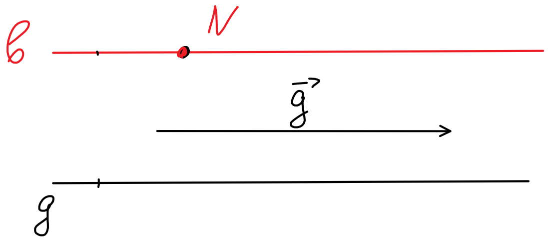
2 зад. ОКС $K=Oxyz$

$g: \begin{cases} 2\overset{5}{x}+\overset{1}{y}-3=0 \\ \underset{5}{x}+\underset{(-1)}{z}+2=0 \end{cases}$ т. $M(1,2,3)$

т. $N(5,-1,+1)$

а) ? коорд. парам. уравнения на правата b:

$b \begin{cases} \ni N \\ \parallel g \end{cases}$



Дали $N \in g$?

$\vec{g} \parallel g \parallel b$

1) ? коорд. парам уравн на g

$g: \begin{cases} 2x+y-3=0 \\ x+z+2=0 \end{cases}$ Избираме $x=s \Rightarrow y=3-2.s$

$z=-2-s$

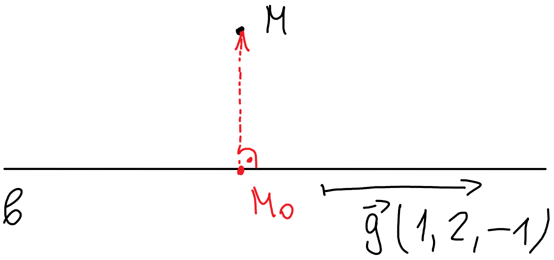
$g: \begin{cases} x = 1s \\ y=3-2.s, \; s\in\mathbb{R} \\ z=-2-1s \end{cases} \Rightarrow g \parallel \vec{g}(1,-2,-1) \Rightarrow$ $\boxed{b: \begin{cases} x=5+p.1 \; =1 \\ y=-1+p.(-2) =2, \; p\in\mathbb{R} \\ z=+1+p.(-1) =3 \end{cases}}$ Не

б) Да се намери разстоянието от т. M до пр. b

$M \notin b$ – проверка

$d(M,b)=|\vec{M_0M}|$

Търсим т. $M_0 =$ орт. пр$_b$ M



$M_0 \in b \Rightarrow M_0(5+p, -1-2p, 1-p)$!!!

$M(\;1\;,\;2,\;3)$

$\vec{M_0M}(1-5-p, 2+1+2p, 3-1+p) \Rightarrow \vec{M_0M}(-4-p, 3+2p, 2+p) \perp \vec{g}(1,-2,-1)$

$\boxed{\vec{M_0M}.\vec{g}=0}$!

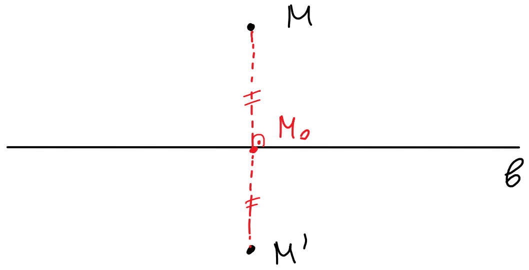
$(-4-p).1+(3+2p).(-2)+(2+p).(-1)=0$

$-4-p-6-4p-2-p=0$

$-6p-12=0 \Rightarrow p=-2 \rightarrow M_0$

$$\boxed{M_0(3,3,3)}$$

$\overrightarrow{M_0M}(-2,-1,0) \Rightarrow |\overrightarrow{M_0M}| = d(M, в) = \sqrt{5}$



$M(1,2,3)$

$M_0(3,3,3) \to$ средата на MM'

$M'(x',y',z')$

Да се намерят коорд. на т. M', ако

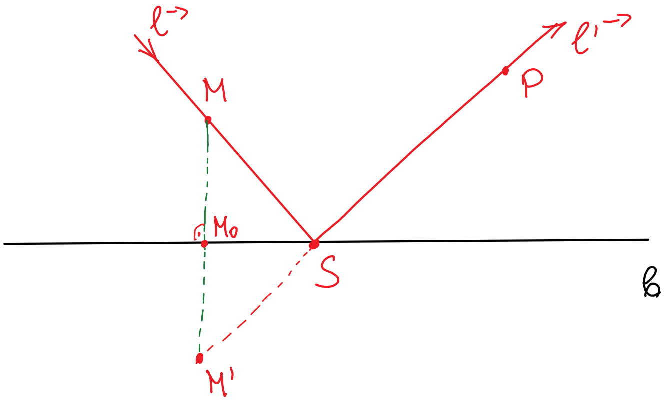
$M \xrightarrow{\sigma_в} M'(x',y',z')$

Симетрия отн. права в пространството.

$\frac{x'+1}{2} = 3 \qquad \frac{y'+2}{2} = 3 \qquad \frac{z'+3}{2} = 3 \Rightarrow M'(5,4,3)$

$M(1,2,3) \xrightarrow{\sigma_в} M'(5,4,3)$

в) (Упр.) Светлинен лъч $\vec{l} \ni M(1,2,3)$, отразява се от в и отразеният лъч $\vec{l'} \ni P(10,-1,0)$. Търсим коорд. парам. уравн. на l и l'



Ако т. $M \xrightarrow{\sigma_в} M'$, то $M' \ni l'$

1) $l' \begin{cases} \ni P(10,-1,0) \\ \ni M'(5,4,3) \text{ от б)} \end{cases}$

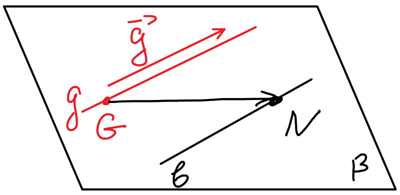
2) ? т. $S = l' \cap в$

3) ?, $l \begin{cases} \ni M(1,2,3) \\ \ni S \end{cases}$

г) ? общо уравнение на р-та $\beta \begin{cases} \ni g \\ \ni в \end{cases}$

$g: \begin{cases} x = 1s \\ y = 3-2.s \\ z = -2-1s \end{cases}, \; s \in \mathbb{R}$

$в: \begin{cases} x = 5 + p.1 \\ y = -1 + p.(-2) \\ z = +1 + p.(-1) \end{cases}$



$g \parallel b \parallel \vec{g}(1,-2,-1) \parallel \beta$ Няма $\vec{n}_b$, няма $\vec{n}_g$

Изб. т. $G \in g$ $\{$за $s=0\}$ $\Rightarrow$ $G(0,3,-2)$

Изб. т. $N \in b$ $\{p=0\}$ $\Rightarrow$ $N(5,-1,1)$ $\Rightarrow$ $\vec{GN}(5,-4,3) \parallel \beta$

$$\left.\begin{array}{l}\beta \parallel \vec{g}(1,-2,-1)\\ \beta \parallel \vec{GN}(5,-4,3)\\ \beta \in G(0,3,-2)\end{array}\right. \Rightarrow \begin{vmatrix} x-0 & y-3 & z+2 \\ 1 & -2 & -1 \\ 5 & -4 & 3 \end{vmatrix} = 0$$ (Упр.)

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3 зад. (Упр.) ОКС $K = Oxyz$

$A(0,2,4)$

$B(1,0,2)$

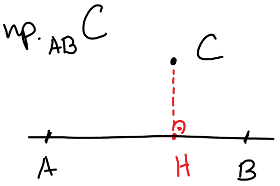
$C(-4,2,1)$

$D(-3,0,-3)$

а) Да се намерят коорд. на т. H = орт. пр.$_{AB} C$

$H \in AB$

$CH \perp AB$



б) $S_{\triangle ABC} = ?$ (| векторно произведение |)

в) $V_{ABCD} = ?$ (смесено произведение)

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4 зад. ОКС $K = Oxyz$

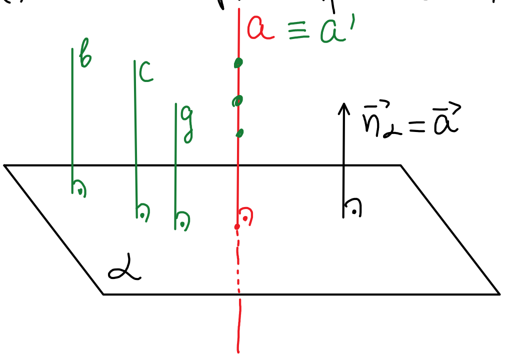
$\alpha: 1x + 2y - 1z - 2 = 0$

а) $a: \begin{cases} x = 1 + \boxed{1}\cdot q \\ y = 2 + \boxed{2}\cdot q \\ z = 3 \boxed{-1}\cdot q \end{cases}, q \in \mathbb{R}$

$a \xrightarrow{\sigma_\alpha} a'$

Да се намерят уравнения на a'.

$a \equiv a'$



$a \parallel \vec{a}(1,2,-1) \perp \alpha$

$\vec{n}_\alpha(1,2,-1) \perp \alpha$

$\sigma_\alpha(a) = a$

б) $a: \begin{cases} x = 2 + 3\cdot p \\ y = 1 - 1\cdot p \\ z = 2 + 1\cdot p \end{cases}, p \in \mathbb{R}$

Да се док., че $a \in \alpha: x+2y-z-2=0$,

$a \xrightarrow{\sigma_\alpha} ?$

Упр.