Формули на Френе

ОКС $K = O\vec{e}_1\vec{e}_2\vec{e}_3$

$c: \vec{x} = \vec{x}(s)$, $\vec{x} \in C^3(J)$

$\vec{t} = \vec{x}'$, $\vec{n} = \dfrac{\vec{x}''}{|\vec{x}''|}$, $\vec{b} = \vec{t} \times \vec{n}$, т. $P \in c$

$P\vec{t}\vec{n}\vec{b}$ - триедър на Френе в точка от c

$$\boxed{\begin{cases} \vec{t}'(s) = \boxed{\varkappa}.\vec{n} \\ \vec{n}'(s) = \boxed{-\varkappa}.\vec{t} + \boxed{\tau}.\vec{b} \\ \vec{b}'(s) = -\boxed{\tau}.\vec{n} \end{cases}}$$

Формули на Френе

I $\varkappa(s)$ - кривина в т. P на линията c

$$\left| \begin{array}{l} \vec{t}' = \vec{x}'' \\ \vec{t}' = \varkappa.\vec{n} \end{array} \right. \Rightarrow \varkappa.\vec{n} = \varkappa.\frac{\vec{x}''}{|\vec{x}''|} = \vec{x}''$$

Извод: $\varkappa(s) = |\vec{x}''(s)| \geq 0$ за $\forall s$

1. $\varkappa(s)$ е скаларна функция, показва изкривяването на линията c спрямо допирателната;

2. Ако в т. $P_0 = \vec{x}(s_0)$ $\varkappa(s_0) = 0$, то P_0 се нарича точка на изправяне;

3. Ако $\varkappa(s) \equiv 0$ за $\forall s \in J$, то линията c е права линия;

4. Ако $\varkappa(s) \neq 0$ за $\forall s \in J$, то c е правилна крива $\Leftrightarrow$ няма точки на изправяне;

5. Ако $\varkappa(s) \equiv const.$, $\varkappa \neq 0$ и c е $\boxed{\text{равнинна,}}$ то линията c е окръжност.

* * *

II $\tau(s)$ – торзия в т. P на линията c

$$\vec{b}' = -\tau . \vec{n} \,/\, . \vec{n} \Rightarrow \tau = -(\vec{b}' . \vec{n})$$

$$\vec{b} = \vec{t} \times \vec{n} \Rightarrow \vec{b}' = \vec{t}' \times \vec{n} + \vec{t} \times \vec{n}'$$

$$\left\{\vec{b}' = \vec{x}'' \times \frac{\vec{x}''}{|\vec{x}''|} + \vec{x}' \times (-\varkappa . \vec{x}' + \right\} \text{ Не}$$

$$\vec{b}' = \underbrace{\vec{x}'' \times \frac{\vec{x}''}{|\vec{x}''|}}_{0} + \vec{t} \times \vec{n}' \to \tau$$

$$\tau(s) = -(\vec{t} \times \vec{n}') . \vec{n} = (\vec{t} \times \vec{n}) . \vec{n}' = (\vec{t} \times \vec{n}) . \left(\frac{\vec{x}''}{\varkappa}\right)'$$

$$\tau(s) = (\vec{t} \times \vec{n}) . \left(\frac{\vec{x}''' . \varkappa - \vec{x}'' . \varkappa'}{\varkappa^2}\right)$$

$$\tau(s) = \left(\vec{x}' \times \frac{\vec{x}''}{\varkappa}\right) . \left(\frac{\vec{x}''' . \varkappa - \vec{x}'' . \varkappa'}{\varkappa^2}\right)$$

$$\boxed{\tau(s) = \frac{(\vec{x}'\,\vec{x}''\,\vec{x}''')}{\varkappa^2}}$$

1. Функцията $\tau(s)$ се дефинира само в точките на правилна крива, т. е. $\boxed{\varkappa \neq 0}$;
2. $\tau(s)$ показва пространственото усукване на c спрямо оскулачната равнина;
3. Ако в т. $P_0 = \vec{x}(s_0)$ $\tau(s_0) = 0$, то P_0 се нарича равнинна точка;
4. Ако $\tau(s) \equiv 0$ за $\forall s \in J$, то c е равнинна крива; $\quad \tau \equiv 0 \Leftrightarrow \vec{b}' = \vec{o} \Leftrightarrow \vec{b}$ - постоянен

*   *   *

III Инвариантност:

$\vec{t}(s), \vec{b}(s), \vec{n}(s)$ - векторни инварианти;

$\varkappa(s), \tau(s)$ - скаларни инварианти;

1. При смяна на параметъра

$c: \vec{x} = \vec{x}(q), q \in J \to t, n, b, \varkappa, \tau$

$c: \bar{x} = \bar{x}(\bar{q}) = x(q(\bar{q})), q \in \bar{J} \to \bar{t}, \bar{n}, \bar{b}, \bar{\varkappa}, \bar{\tau}$

$$\boxed{\bar{t} = \varepsilon . t, \ \bar{n} = n, \ \bar{b} = \varepsilon . t, \ \bar{\varkappa} = \varkappa, \ \bar{\tau} = \tau} \qquad \varepsilon = \operatorname{sign}\left(\frac{dq}{d\bar{q}}\right)$$

2. При смяна на $\overline{OKC}$: $\underline{K = Oe_1e_2e_3 \to \bar{K} = \bar{O}\bar{e}_1\bar{e}_2\bar{e}_3}$

има пълна инвариантност, т.е.

$\underline{\bar{t} = t, \ \bar{n} = n, \ \bar{b} = b, \ \bar{\varkappa} = \varkappa, \ \bar{\tau} = \tau}$

3. При еднаквост в $\mathbb{R}^3$

$\underline{\bar{t} = t, \ \bar{n} = n, \ \bar{b} = \varepsilon . b, \ \bar{\varkappa} = \varkappa, \ \bar{\tau} = \varepsilon . \tau}$

$\underline{\varepsilon = 1}$ при движения

$\underline{\varepsilon = -1}$ при отражения

* * *

IV Формули за пресмятане на $\vec{t}, \vec{n}, \vec{b}, \varkappa$ и τ

1. $c: \vec{x} = \vec{x}(s)$, спрямо естествен параметър

$$\vec{t} = \vec{x}', \quad \vec{n} = \frac{\vec{x}''}{|\vec{x}''|}, \quad \vec{b} = \vec{t} \times \vec{n}$$

$$\varkappa = |\vec{x}''|, \quad \tau = \frac{(\vec{x}' \ \vec{x}'' \ \vec{x}''')}{\varkappa^2}$$

2. $c: \vec{x} = \vec{x}(q)$, спрямо произволен параметър

$$\frac{ds}{dq} = \dot{s} = |\dot{\vec{x}}|$$

$$\vec{t} = \vec{x}' = \frac{d\vec{x}}{ds} = \frac{d\vec{x}}{dq} \cdot \frac{dq}{ds} = \frac{\dot{\vec{x}}}{\dot{s}} = \frac{\dot{\vec{x}}}{|\dot{\vec{x}}|} \Rightarrow \vec{t} = \frac{\dot{\vec{x}}}{|\dot{\vec{x}}|}$$

$$\varkappa(s) = |\vec{x}''|$$

Разгл. $|\vec{x}' \times \vec{x}''| = \underbrace{|\vec{x}'|}_{=1} \cdot |\vec{x}''| \cdot \underbrace{\sin\frac{\pi}{2}}_{=1} = |\vec{x}''| = \varkappa(s)$

$$\vec{x}' = \frac{\dot{x}}{\dot{s}}$$

$$\vec{x}'' = \frac{d}{ds}\left(\frac{\dot{x}}{\dot{s}}\right) = \frac{d}{dq}\left(\frac{\dot{x}}{\dot{s}}\right) \cdot \frac{dq}{ds} = \frac{\ddot{x} \cdot \dot{s} - \dot{x} \cdot \ddot{s}}{\dot{s}^2} \cdot \frac{1}{\dot{s}}$$

$$\vec{x}' \times \vec{x}'' = \frac{\dot{x} \times (\ddot{x} \cdot \dot{s} - \dot{x} \cdot \ddot{s})}{\dot{s}^4} = \frac{\dot{x} \times \ddot{x}}{\dot{s}^3} \Rightarrow$$

$$\varkappa(q) = |\vec{x}' \times \vec{x}''| = \boxed{\frac{|\dot{x} \times \ddot{x}|}{\dot{s}^3}}$$

$$\vec{b} = \vec{t} \times \vec{n} = \vec{x}' \times \frac{\vec{x}''}{|\vec{x}''|} = \frac{\vec{x}' \times \vec{x}''}{|\vec{x}' \times \vec{x}''|} = \frac{\dot{x} \times \ddot{x}}{|\dot{x} \times \ddot{x}|}$$

$$\vec{n} = \vec{b} \times \vec{t}$$

$$\tau(q) = \frac{(\dot{x}\ \ddot{x}\ \dddot{x})}{|\dot{x} \times \ddot{x}|^2}$$

$$\boxed{\vec{t} = \frac{\dot{\vec{x}}}{|\dot{\vec{x}}|}, \quad \vec{b} = \frac{\dot{\vec{x}} \times \ddot{\vec{x}}}{|\dot{x} \times \ddot{x}|}, \quad \vec{n} = \vec{b} \times \vec{t} \qquad \varkappa(q) = \frac{|\dot{\vec{x}} \times \ddot{\vec{x}}|}{\dot{s}^3}, \quad \tau(q) = \frac{(\dot{\vec{x}}\ \ddot{\vec{x}}\ \dddot{\vec{x}})}{|\dot{\vec{x}} \times \ddot{\vec{x}}|^2}}$$

* * *

Пример:

$$\boxed{C: \begin{cases} x^1 = a.\cos q \\ x^2 = a.\sin q \\ x^3 = d.q \end{cases}} \qquad \begin{aligned} &q \in \mathbb{R} \\ &\left.\begin{aligned} a > 0 \\ d > 0 \end{aligned}\right\} - \text{const.} \end{aligned}$$

Пресмятаме $\varkappa(q)$ и $\tau(q)$

1. $\varkappa(q) = \dfrac{|\dot{\vec{x}} \times \ddot{\vec{x}}|}{\dot{s}^3}$

$$\varkappa(q) = \frac{a.\sqrt{a^2+d^2}}{\left(\sqrt{a^2+d^2}\right)^3}$$

$$\boxed{\varkappa(q) = \frac{a}{a^2+d^2}}$$

$$\tau(q) = \frac{(\dot{\vec{x}}\ \ddot{\vec{x}}\ \dddot{\vec{x}})}{|\dot{x} \times \ddot{x}|^2}$$

$$\tau(q) = \frac{a^2.d}{a^2.(a^2+d^2)}$$

$$\boxed{\tau(q) = \frac{d}{a^2+d^2}}$$

$\dot{\vec{x}}(-a.\sin q, a.\cos q, d)$

$\ddot{\vec{x}}(-a.\cos q, -a.\sin q, 0)$

$\dot{x} \times \ddot{x}\,(a.d.\sin q, -a.d.\cos q, a^2)$

$|\dot{\vec{x}}| = \sqrt{(-a.\sin q)^2 + (a.\cos q)^2 + d^2}$

$|\dot{\vec{x}}| = \sqrt{a^2+d^2}$

$|\dot{\vec{x}} \times \ddot{\vec{x}}| = \sqrt{a^2.d^2 + a^4}$

$|\dot{\vec{x}} \times \ddot{\vec{x}}| = a.\sqrt{a^2+d^2}$

$\dddot{\vec{x}}(a.\sin q, -a.\cos q, 0)$

$(\dot{x}\ \ddot{x}\ \dddot{x}) = a^2.d$

Задача за упражнение:

$$c : \begin{cases} x^1 = \operatorname{ch} q \\ x^2 = \operatorname{sh} q \ , q \in \mathbb{R} \\ x^3 = q \end{cases}$$

$$\operatorname{ch} q = \frac{e^q + e^{-q}}{2}$$

$$\operatorname{sh} q = \frac{e^q - e^{-q}}{2}$$

$$\dot{x}(\operatorname{sh} q, \operatorname{ch} q, 1)$$

$$\operatorname{ch}^2 q - \operatorname{sh}^2 q = \underline{1}$$

$$\ddot{x}(\operatorname{ch} q, \operatorname{sh} q, 0)$$

$$\operatorname{ch}^2 q = 1 + \operatorname{sh}^2 q$$

$$\dot{x} \times \ddot{x}(-\operatorname{sh} q, \operatorname{ch} q, -\underline{1})$$

$$(\operatorname{ch} q)^{\cdot} = \operatorname{sh} q$$

$$(\operatorname{sh} q)^{\cdot} = \operatorname{ch} q$$

$$|\dot{x}|^2 = \operatorname{sh}^2 q + \operatorname{ch}^2 q + 1 = $$
$$= 2 . \operatorname{ch}^2 q$$

$$\varkappa(q) = \frac{1}{2\operatorname{ch}^2 q}$$

$$|\dot{\vec{x}}| = \sqrt{2} . \operatorname{ch} q$$

$$\tau(q) = \frac{1}{2\operatorname{ch}^2 q}$$

$$|\dot{x} \times \ddot{x}| = \sqrt{2} . \operatorname{ch} q$$

$$\vec{t} = \frac{\dot{x}}{|\dot{x}|}\left(\frac{\operatorname{sh} q}{\sqrt{2}.\operatorname{ch} q}, \frac{\operatorname{ch} q}{\sqrt{2}.\operatorname{ch} q}, \frac{1}{\sqrt{2}.\operatorname{ch} q}\right)$$

$$\vec{b} = \left(\frac{-\operatorname{sh} q}{\sqrt{2}.\operatorname{ch} q}, \frac{\operatorname{ch} q}{\sqrt{2}.\operatorname{ch} q}, \frac{-1}{\sqrt{2}.\operatorname{ch} q}\right)$$

$$\vec{n} = \vec{b} \times \vec{t}$$

$$\vec{n}\left(\frac{1}{\operatorname{ch} q}, 0, -\frac{\operatorname{sh} q}{\operatorname{ch} q}\right)$$

Проверки: $(\vec{t}.\vec{b}) = 0$, $(\vec{b}.\vec{n}) = 0$, $(\vec{t}.\vec{n}) = 0$, $|\vec{t}| = |\vec{b}| = |\vec{n}| = \underline{1}$

Задача (II начин за пресмятане)

$$C: \begin{cases} x^1 = \cos^3 q \\ x^2 = \sin^3 q \\ x^3 = \cos 2q \end{cases}, \quad q \in \left(0; \frac{\pi}{2}\right)$$

$$\begin{cases} \vec{t}\,' = \varkappa . \vec{n} \\ \vec{n}\,' = -\varkappa . \vec{t} + \tau . \vec{b} \\ \vec{b}\,' = -\tau . \vec{n} \end{cases}$$

$\dot{\vec{x}}(-3\cos^2 q . \sin q,\ 3\sin^2 q . \cos q,\ -2 . \sin 2q)$

$|\dot{\vec{x}}|^2 = 9 . \sin^2 q . \cos^2 q + 16 . \sin^2 q . \cos^2 q$

$|\dot{\vec{x}}|^2 = 25 . \sin^2 q . \cos^2 q$

$\dot{s} = |\dot{\vec{x}}| = 5 . \sin q . \cos q$

1. $\vec{t} = \dfrac{\dot{\vec{x}}}{\dot{s}} \Rightarrow \vec{t}\left(-\dfrac{3}{5} . \cos q,\ \dfrac{3}{5} . \sin q,\ -\dfrac{4}{5}\right)$

2. $\vec{t}\,' = ?$ $\quad \dot{\vec{t}}\left(\dfrac{3}{5} . \sin q,\ \dfrac{3}{5} . \cos q,\ 0\right)$

$\vec{t}\,'\left(\dfrac{3\sin q}{25 \sin q . \cos q},\ \dfrac{3\cos q}{25 \sin q . \cos q},\ 0\right)$

$\varkappa^2 = |\vec{t}\,'|^2 = \dfrac{9}{(25 . \sin q . \cos q)^2}$

$\varkappa = \dfrac{3}{25 . \sin q . \cos q}$

$\vec{n} = \dfrac{\vec{t}\,'}{\varkappa}(\sin q,\ \cos q,\ 0)$

1. $\vec{t} = \vec{x}\,' = \dfrac{\dot{\vec{x}}}{|\dot{\vec{x}}|} = \dfrac{\dot{\vec{x}}}{\dot{s}}$

2. Търсим $\vec{t}\,' = ?$

$\vec{t}\,' = \dfrac{dt}{ds} = \dfrac{d\vec{t}}{dq} . \dfrac{dq}{ds}$

$\vec{t}\,' = \dfrac{\dot{\vec{t}}}{\dot{s}}$

$\varkappa = |\vec{t}\,'|$

$\vec{n} = \dfrac{\vec{t}\,'}{\varkappa}$

3. $\vec{b} = \vec{t} \times \vec{n}$

4. $\vec{b}\,' = \dfrac{\dot{\vec{b}}}{\dot{s}}$

5. $\tau = -(\vec{b}\,' . \vec{n})$

$\vec{b} = \vec{t} \times \vec{n}$ $\quad \vec{t}\left(-\frac{3}{5}\cdot\cos q, \frac{3}{5}\cdot\sin q, -\frac{4}{5}\right)$

$\vec{n}\left(\sin q, \cos q, 0\right)$

$\quad\quad\quad$ I $\quad\quad$ II $\quad\quad$ III

$\vec{b}\left(\frac{4}{5}\cdot\cos q, -\frac{4}{5}\cdot\sin q, -\frac{3}{5}\right)$

проверки: $(t\cdot n)=0$, $(n\cdot b)=0$, $(b\cdot t)=0$, $|\vec{t}|=|\vec{b}|=|\vec{n}|=\underline{1}$

4. $\vec{b}' = \frac{\dot{b}}{\dot{s}} \Rightarrow b'\left(\frac{4}{25\cdot\sin q}, -\frac{4}{25\cdot\cos q},\right.$

$\dot{b}\left(-\frac{4}{5}\cdot\sin q, -\frac{4}{5}\cdot\cos q, 0\right)$, $\dot{s} = 5\cdot\sin q\cdot\cos q$

$\vec{b}' = \frac{\dot{b}}{\dot{s}} = \left(-\frac{4}{25\cdot\cos q}, -\frac{4}{25\cdot\sin q}, 0\right)$

$\vec{n}\quad\left(\sin q, \cos q, 0\right)$

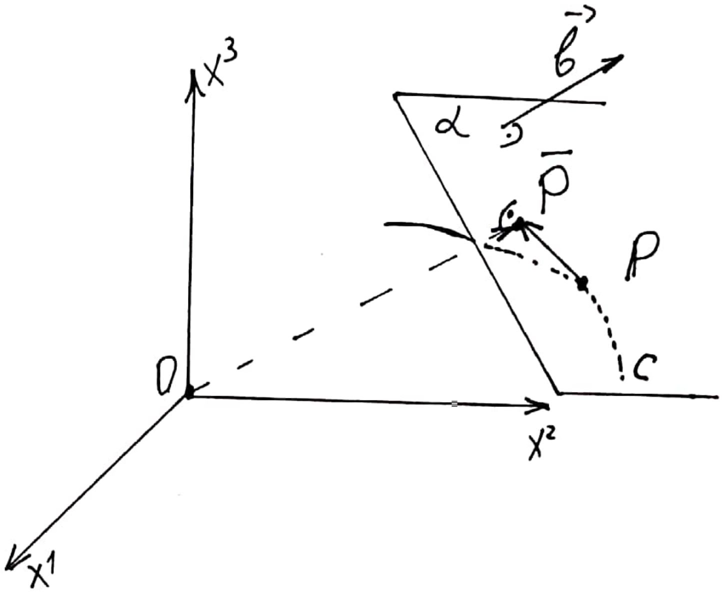
$(\vec{b}'\cdot\vec{n}) = -\frac{4}{25}\cdot\left(\frac{\sin q}{\cos q} + \frac{\cos q}{\sin q}\right) = -\frac{4}{25\cdot\sin q\cdot\cos q} = -\tau$

$\tau = \frac{4}{25\cdot\sin q\cdot\cos q}$

Задача: $c: \begin{cases} x^1 = \cos^3 q \\ x^2 = \sin^3 q \\ x^3 = \cos 2q \end{cases}, \quad q \in \left(0; \frac{\pi}{2}\right)$

Да се намерят уравнения на линия $\bar{c}$ - геометричното място на ортогоналните проекции на т. O върху оскулачните равнини в точките на кривата c.

Решение:



т. $O(0,0,0)$ - начало на ОКС

$P(x^1, x^2, x^3) \in c$

α - оскулачна равнина в т. $P \in c$

$\bar{P}$ = орт. проекция$_\alpha O$

Нека $\bar{P}(\bar{x}^1, \bar{x}^2, \bar{x}^3)$

$\overrightarrow{O\bar{P}} \perp \alpha \Rightarrow \overrightarrow{O\bar{P}} \parallel \vec{b} \Rightarrow \exists!\, \lambda(q): \ \overrightarrow{O\bar{P}} = \lambda . \vec{b}$

Ще намерим $\lambda(q) = ?$ от $\overrightarrow{P\bar{P}} \perp \vec{b}$

$(\overrightarrow{P\bar{P}} . \vec{b}) = 0$

$(\overrightarrow{OP} - \overrightarrow{O\bar{P}}) . \vec{b} = 0$

$(\overrightarrow{OP} - \lambda . \vec{b}) . \vec{b} = 0$

$\lambda(q) = (\overrightarrow{OP} . \vec{b})$

$\overrightarrow{OP}(\cos^3 q, \sin^3 q, \cos 2q)$

$\vec{b}\left(\frac{4}{5} \cdot \cos q, -\frac{4}{5}\sin q, -\frac{3}{5}\right)$

$\lambda(q) = \vec{OP} \cdot \vec{b} = \frac{4}{5} \cdot (\cos^4 q - \sin^4 q) - \frac{3}{5} \cdot \cos 2q$

$\lambda(q) = \frac{4}{5} \cdot (\cos^2 q - \sin^2 q) - \frac{3}{5} \cdot \cos 2q = \frac{1}{5} \cdot \cos 2q$

$\Rightarrow \vec{O\bar{P}} = \frac{\cos 2q}{5} \cdot \vec{b}$

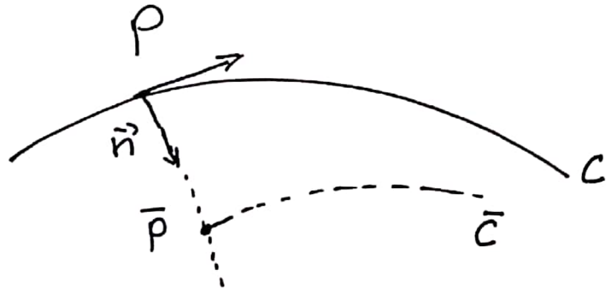
$$\bar{c}: \begin{cases} \bar{x}^1 = \frac{4}{25} \cdot \cos q \cdot \cos 2q \\ \bar{x}^2 = -\frac{4}{25} \cdot \sin q \cdot \cos 2q \quad , q \in (0; \pi) \\ \bar{x}^3 = -\frac{3}{25} \cdot \cos 2q \end{cases}$$

* * *

Задача:

$$c: \begin{cases} x^1 = a.(q - \sin q) \\ x^2 = a.(1 - \cos q), q \in \mathbb{R} \\ x^3 = 4.a.\sin q/2 \quad a > 0 \end{cases}$$

От всяка точка P на линията c по главната нормала и към вдлъбнатата част на c е нанесена отсечка $P\bar{P}$ с дължина $d = 4a^2.\varkappa$. Да се намерят уравнения на линията $\bar{c}$, описана от точките $\bar{P}$. Да се докаже, че $\bar{c}$ е равнинна линия и да се намери уравнение на равнината, която я съдържа.



$P(x^1, x^2, x^3)$

$\bar{P}(\bar{x}^1, \bar{x}^2, \bar{x}^3)$

$$\left.\begin{array}{l}\overrightarrow{P\bar{P}} \uparrow\uparrow \vec{n} \\ |\overrightarrow{P\bar{P}}| = 4.a^2.\varkappa\end{array}\right\} \Rightarrow$$

$$\Rightarrow \overrightarrow{P\bar{P}} = 4.a^2.\varkappa.\vec{n}$$

$$\overrightarrow{O\bar{P}} = \overrightarrow{OP} + 4a^2.\vec{t}' \quad (*)$$

Търсим $\vec{t}' = ?$

$$\dot{\vec{x}}\left(a.(1-\cos q),\ a.\sin q\ ,\ 4.\tfrac{1}{2}.a.\cos q/_2\right)$$

$$|\dot{\vec{x}}|^2 = a^2.\left[(1-\cos q)^2 + \sin^2 q + 4.\cos^2 q/_2\right]$$

$$|\dot{\vec{x}}|^2 = a^2.\left(1 - 2\cos q + \cos^2 q + \sin^2 q + 2.(1+\cos q)\right)$$

$$|\dot{\vec{x}}|^2 = 4.a^2 \Rightarrow |\dot{\vec{x}}| = \dot{s} = 2.a$$

$$\vec{t} = \frac{\dot{\vec{x}}}{\dot{s}}\left(\frac{1-\cos q}{2},\ \frac{\sin q}{2},\ \cos q/_2\right)$$

$$\dot{\vec{t}}\left(\frac{\sin q}{2},\ \frac{\cos q}{2},\ -\frac{\sin q/_2}{2}\right)$$

$$\vec{t}' = \frac{\dot{\vec{t}}}{\dot{s}} \Rightarrow \vec{t}'\left(\frac{\sin q}{4a},\ \frac{\cos q}{4a},\ -\frac{\sin q/_2}{4a}\right) \to (*)$$

$$\overrightarrow{O\bar{P}}: \begin{cases}\bar{x}^1 = a.(q-\sin q) + 4a^2.\dfrac{\sin q}{4a} = a.q \\ \bar{x}^2 = a.(1-\cos q) + 4a^2.\dfrac{\cos q}{4a} = a \\ \bar{x}^3 = 4a.\sin q/_2 - a.\sin q/_2 = 3a.\sin q/_2\end{cases}$$