$* \le \frac{1}{|\Delta x|}\left|\int_{x_0}^{x_0+\Delta x} \varepsilon \, dt\right| = \frac{1}{|\Delta x|}\cdot \varepsilon \left|\int_{x_0}^{x_0+\Delta x} 1 \, dt\right| = \frac{1}{\cancel{|\Delta x|}} \cdot \varepsilon \, \cancel{|\Delta x|} = \varepsilon$

$\Rightarrow \lim\limits_{\Delta x \to 0} \frac{\Delta F}{\Delta x} = f(x_0) \Rightarrow \exists \; F'(x_0) = f(x_0)$

Следствие: Ако $f(x)$ е непр. в/у $[a,b] \Rightarrow F(x) = \int_a^x f(x)\,dx$ е диф в/у $[a,b]$. При това $F'(x) = f(x)$, $\forall x \in [a,b]$
т.е. $F(x) = \int_a^x f(x)\,dx$ е прим. ф-яна $f(x)$ в/у dx.

Т (Нютон-Лайбниц) Нека $f(x)$ - непр в/у $[a,b]$ и $\Phi(x)$ - прим. ф-я на $f(x)$ в/у $[a,b] \Rightarrow$
$\int_a^b f(x)\,dx = \Phi(b) - \Phi(a) = \Phi(x)\Big|_a^b$

Пример: $\int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$

Д-во:

$F(x) = \int_a^x f(t)\,dt$ е същo е прим. на $f(x)$ в/у $[a,b] \Rightarrow$
$\exists\, c \in \mathbb{R}: \Phi(x) = F(x) + c \; (\forall x \in [a,b]) \Rightarrow$
$\Phi(a) = F(a) + c = 0 + c = c$
$F(a) = \int_a^a f(t)\,dt = 0$
$\Phi(b) = F(b) + \Phi(a) = \int_a^b f(t)\,dt + \Phi a \Rightarrow \int_a^b f(t)\,dt = \Phi(b) - \Phi(a)$

6. Интегриране по части и смяна на пром. в опр. инт.

Т Нека $f(x)$ - непр. в/у $[A,B]$ и $\varphi(t)$ е деф в/у $[\alpha,\beta]$ със ст-сти в $[A,B]$, т.е. $\varphi\,[\alpha,\beta] \to [A,B]$, такава, че:

1) $\varphi(\alpha) = a$, $\varphi(\beta) = b$
2) $\exists\, \varphi'(t)$ - непр. в/у $[\alpha,\beta]$

$\Rightarrow \int_a^b f(x)\,dx = \int_\alpha^\beta f(\varphi(t)) \cdot \underbrace{\varphi'(t)\cdot dt}_{d\varphi(t)} = \int_\alpha^\beta f(\varphi(t))\,d\varphi(t)$

Д-во:

Нека $F(x)$ - прим. ф-я на $f(x)$ в/у $[A,B] \Rightarrow$
$\Rightarrow F(\varphi(t))$ е прим. ф-я на $f(\varphi(t)) \cdot \varphi'(t)$ в/у $[\alpha,\beta]$
$\forall t \in [\alpha,\beta]: [F(\varphi(t))]' = F'(\varphi(t)) \cdot \varphi'(t) = f(\varphi(t)) \cdot \varphi'(t)$
$\overset{\text{Н-Л}}{=} \int_\alpha^\beta f(\varphi(t)) \cdot \varphi'(t)\,dt = F(\varphi(t))\Big|_\alpha^\beta = F(\varphi(\beta)) - F(\varphi(\alpha)) = F(b) - F(a) \overset{\text{Н-Л}}{=}$
$= \int_a^b f(x)\,dx$

Т Нека $f(x)$ и $g(x)$ имат непр. произв. $f'(x)$ и $g'(x)$ в/у $[a,b] \Rightarrow$
$\int_a^b f(x)\,dg(x) = f(x)\cdot g(x)\Big|_a^b - \int_a^b g(x)\,df(x)$

Д-во:

$f(x).g(x)$ – има непр. произв. в/у $[a,b]$, т.к. $(f(x).g(x))' = f'(x)g(x)+f(x)g'(x)$ $\forall x\in[a,b]$

$\overset{н.л.}{\Rightarrow} \int_a^b (f(x)g(x))'dx = \int_a^b [f'(x)g(x)+g'(x)f(x)]dx = \int_a^b f'(x)g(x)dx + \int_a^b g'(x)f(x) =$

$= \int_a^b g(x)\,df(x) + \int_a^b f(x)\,dg(x)$

$\Rightarrow \int_a^b (f(x).g(x))'dx = f(x).g(x)\Big|_a^b - \int_a^b g(x)\,df(x)$

Примери 1) $\int_0^1 \sqrt{1-x^2}\,dx = \int_0^{\frac{\pi}{2}}\sqrt{1-\sin^2 t}\,d\sin t = \int_0^{\frac{\pi}{2}}\sqrt{\cos^2 t}\cos t\,dt = \int_0^{\frac{\pi}{2}}|\cos t|\cos t\,dt =$

$= \int_0^{\frac{\pi}{2}}\cos^2 t\,dt = \int_0^{\frac{\pi}{2}}\frac{1+\cos 2t}{2}dt = \frac12\left[\int_0^{\frac{\pi}{2}}1\,dt + \int_0^{\frac{\pi}{2}}\cos 2t\,dt\right] =$

$= \frac12\left[\frac{\pi}{2}+\frac12\int_0^{\frac{\pi}{2}}\cos 2t\,d2t\right] = \frac12\left[\frac{\pi}{2}+\frac12\sin 2t\Big|_0^{\frac{\pi}{2}}\right] = \frac12\left[\frac{\pi}{2}+\underbrace{\frac12\sin\frac{2\pi}{2}-\frac12\sin\frac{2\cdot 0}{2}}_{0}\right]$

$= \frac12\cdot\frac{\pi}{2} = \frac{\pi}{4}$, където $x=\sin t$

$x=\varphi(t)=\sin t$, $t\in[0,\frac{\pi}{2}]$, $\alpha=0 \Rightarrow \varphi(\alpha)=\varphi(0)=\sin 0=0$

$\beta=\frac{\pi}{2} \Rightarrow \varphi(\beta)=\varphi(\frac{\pi}{2})=\sin\frac{\pi}{2}=1$

$\varphi(t)=\sin t$ има непр. пр. в/у $[0,\frac{\pi}{2}]$

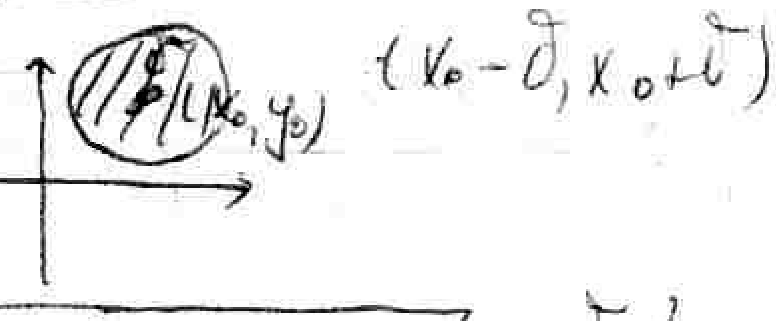
2) $\int_1^2 x\ln x\,dx = \int_1^2 \ln x\,d\left(\frac{x^2}{2}\right) = \frac12\int_1^2 \ln x\,dx^2 = \frac12\left[x^2\ln x\Big|_1^2 - \int_1^2 x^2\ln x\right] =$

$= \frac12\left[(4\ln 2-\underbrace{\ln 1}_{0}) - \int_1^2 x^2\,d\ln x\right] = \frac12\left[(4\ln 2-\ln 1) - \int_1^2 x^2\frac{1}{x}\,dx\right] =$

$= \frac12\left[4\ln 2-\overset{0}{\ln 1} - \frac{x^2}{2}\Big|_1^2\right] = \frac12\left[4\ln 2-\frac32\right] = 2\ln 2-\frac34$

7. Лице на равнинна фигура

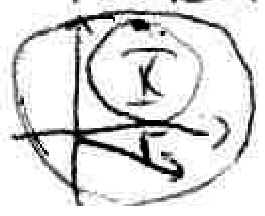
Деф. $R^2 = R\times R = \{(x,y) \mid x,y\in R\}$

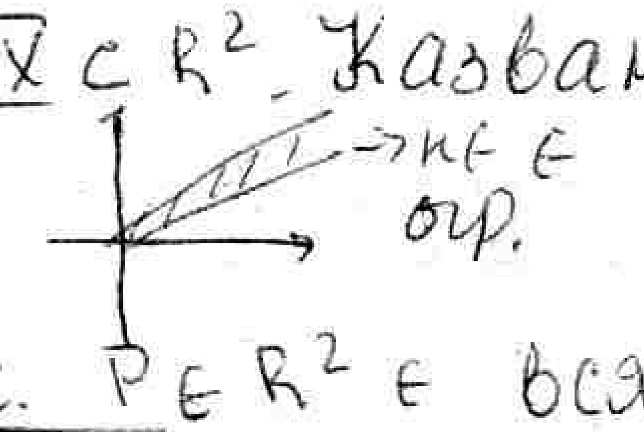


Деф. Нека $M_0(x_0,y_0)\in R^2$, $\delta>0$

$B_\delta(x_0,y_0) = B_\delta(M_0) = \{M\in R^2: |MM_0|<\delta\} = \{(x,y): \sqrt{(x-x_0)^2+(y-y_0)^2}<\delta\}$

Деф. Нека $\overline{X}\subset R^2$. Казваме, че $\overline{X}$ е огр., ако $\exists B_r(0,0): \overline{X}\subset B_r(0,0)$





Деф. Правоъг. $P\in R^2$ е всяко м-во от вида $P=\{(x,y): a\propto x\propto b,\ c\propto y\propto d\}$, където $\propto \leftrightarrow <,\le$

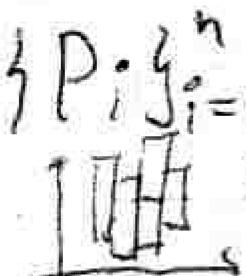
Пример: $P=\{(x,y): 1\le x<2,\ 0\le y\le 3\}$

Деф. Нека $P=\{(x,y): a\propto x\propto b,\ c\propto y\propto d\}$ е прав. Вътрешност на правоъгълник P, наричаме м-вото $P^\circ=\{(x,y): a<x<b,\ c<y<d\}$

Деф. Лице на правоъгълник $P=\{(x,y): a<x<b,\ c<y<d\}$ наричаме числото $S(P)=(b-a)(d-c)$

Деф. М-вото $K\subset R^2$ се нарича клетъчно, ако $\exists \{P_i\}_{i=1}^n$, P_i – правоъг., $P_j^\circ\cap P_i^\circ=\emptyset$, $\forall i,j=1\div n$, $i\neq j$

$K=\bigcup_{i=1}^n P_i$



Деф. Нека K – кл. м-во. $K=\bigcup_{i=1}^n P_i$: $P_i\cap P_j^\circ=\emptyset$ $\forall i,j=1\div n$, $i\neq j$. Лице на K наричаме числото:

$S(K)=\sum_{i=1}^n S(P_i)$