М - мн-во $\quad * : M \times M \to M$

$M = \{a, b, c\}$

$*$	a	b	c
a	b	a	c
b	c	b	a
c	a	c	b

$a * a = b$

$a * b = a$

$a * c = c$

$b * c = c \ldots$

$\mathbb{V}$ е ЛП над $\mathbb{F}$

$v_1, \dots, v_n \in \mathbb{V}$: $\{v_1, \dots, v_n\}$ обр. базис

на $\mathbb{V}$.

$\underline{\ell(}v_1, \dots, v_n) = \mathbb{V}$, т.е. $\forall v \in \mathbb{V}$

$$v = \lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_n v_n$$

$\lambda_i \in \mathbb{F}$ и трябва $v_1, \dots, v_n$ да са ЛНЗ

① В ЛП $M_2(\mathbb{R})$ над $\mathbb{R}$ са дад.

$A_1 = \begin{pmatrix} 1 & 3 \\ -1 & -3 \end{pmatrix}$, $A_2 = \begin{pmatrix} -1 & -4 \\ 1 & -4 \end{pmatrix}$, $A_3 = \begin{pmatrix} 0 & -1 \\ 0 & -7 \end{pmatrix}$,

$A_4 = \begin{pmatrix} 2 & 7 \\ -2 & 1 \end{pmatrix}$.

а) Да се намери рангът на с-мата вр-ни матрици.

$A = \begin{pmatrix} \underline{a} & \underline{b} \\ \underline{c} & \underline{d} \end{pmatrix} \to (a, b, c, d)$

$$A = a\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{E_{11}} + b\underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{E_{12}} + c\underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{E_{21}} + d\underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{E_{22}}$$

$$\begin{pmatrix} 1 & 3 & -1 & -3 \\ -1 & -4 & 1 & -4 \\ 0 & -1 & 0 & -7 \\ 2 & 7 & -2 & 1 \end{pmatrix} \overset{R_1+3R_3}{\sim} \begin{pmatrix} 1 & 0 & -1 & -24 \\ -1 & 0 & 1 & 24 \\ 0 & -1 & 0 & -7 \\ 2 & 0 & -2 & -48 \end{pmatrix} \underset{\substack{1,2 \\ R_4+7R_3}}{\sim}$$

R_2-4R_3

R_1+3R_3

$$\begin{array}{rrrrr} & 1 & 3 & -1 & -3 \\ + & 0 & -3 & 0 & -21 \\ \hline & 1 & 0 & -1 & -24 \end{array} \qquad \begin{array}{rrrrr} & -1 & -4 & 1 & -4 \\ - & 0 & -4 & 0 & -28 \\ \hline & -1 & 0 & 1 & 24 \end{array}$$

$$\sim \begin{pmatrix} 1 & 0 & -1 & -24 \\ -1 & 0 & 1 & 24 \\ 0 & -1 & 0 & -7 \\ 1 & 0 & -1 & -24 \end{pmatrix} \sim \begin{array}{c} \begin{pmatrix} 1 & 0 & -1 & -24 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ 0 & -1 & 0 & -7 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{pmatrix} \end{array} \begin{array}{l} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array}$$

Повече няма накъде да дтуркаме rank = 2

б) Да се намери една негова МЛНЗП

От а) намерихме, че $\{A_1, A_3\}$ е ЛНЗ

подсистема с мощност ранга

$\Rightarrow$ е максимална.

в) Да се допълни до базис на $M_2(\mathbb{R})$

$$\begin{array}{c} A_1 \\ A_3 \\ E_{21} \\ E_{22} \end{array}\begin{pmatrix} 1 & 0 & -1 & -24 \\ 0 & -1 & 0 & -7 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & -24 \\ 0 & -1 & 0 & -7 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{к-м} \sim E_4 \Rightarrow \{A_1, A_3, E_{21}, E_{22}\}$$

обр. базис на $M_2(\mathbb{R})$

② Да се намери ранга на системата вектори $v_1 = (1-a, 1, 1, 1)$, $v_2 = (1, 1-a, -1, -1)$, $v_3 = (1, -1, 1-a, -1)$, $v_4 = (1, -1, -1, 1-a)$.

Решение:

$$\begin{pmatrix} 1-a & 1 & 1 & 1 \\ 1 & 1-a & -1 & -1 \\ 1 & -1 & 1-a & -1 \\ 1 & -1 & -1 & 1-a \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 1-a & 1 & 1 & 1 \\ 2-a & 2-a & 0 & 0 \\ 2-a & 0 & 2-a & 0 \\ 2-a & 0 & 0 & 2-a \end{pmatrix}$$

1 сл. $a=2$

$$\sim \begin{pmatrix} -1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\operatorname{rank} = 1$

$a \neq 2$

$$\begin{pmatrix} 1-a & 1 & 1 & 1 \\ 2-a & 2-a & 0 & 0 \\ 2-a & 0 & 2-a & 0 \\ 2-a & 0 & 0 & 2-a \end{pmatrix} \begin{matrix} \\ 1:2-a \\ 1:2-a \\ 1:2-a \end{matrix} \sim$$

$$\sim \begin{pmatrix} 1-a & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} -a-2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{rcccc} & 1-a & 1 & 1 & 1 \\ - & 1 & 0 & 0 & 1 \\ \hline & -a & 1 & 1 & 0 \\ - & 1 & 0 & 1 & 0 \end{array} \quad \Bigg| \quad -a-1 \quad 1 \quad 0 \quad 0$$

2 сл. $a = -2, a \neq 2$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \text{rank} = 3$$

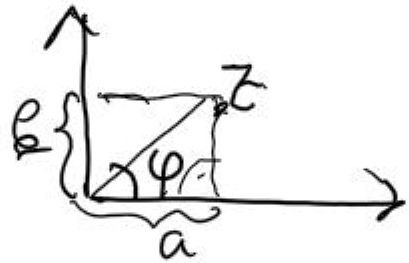
3 сл. $a \neq -2, a \neq 2$ делим 1 ряд $-a-2$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \sim E_4 \quad \text{rank} = 4$$

③ Нека $\omega_0, \dots, \omega_{68}$ са 69-те корени на единицата. Да се реши спрямо k у-нието

$$\omega_k = \omega_{40}\omega_{32}$$

$$\omega^{69} = 1$$



$$\omega_k = \sqrt[69]{|1|}\left(\cos\frac{\arg 1 + 2k\pi}{69} + i\sin\frac{\arg 1 + 2k\pi}{69}\right) =$$

$= 1 \quad \Big| \; \sin\varphi = \frac{b}{z}, \; \cos\varphi = \frac{a}{z} \qquad z = a + b i$

$\varphi = 0\pi + 2k\pi$

$$\omega_k = \cos\frac{2k\pi}{69} + i\sin\frac{2k\pi}{69} \quad k = 0, \dots, 68$$

$$\omega_k = \omega_1^k \quad \forall \; k \in \mathbb{Z}$$

$$\omega_{40} = \omega_1^{40}$$
$$\omega_{32} = \omega_1^{32}$$

$$\omega_1^{40} \cdot \omega_1^{32} = \omega_1^{72} = \omega_1^{69+3} = \omega_1^{69} \cdot \omega_1^{3} =$$

$$= 1 \cdot \omega_1^3 = \omega_1^3 = \omega_3 \Longrightarrow k = 3$$

4) Нека $\omega_0, \ldots, \omega_{68}$ са 69-тте корени на единицата. Да се намери k:

$$\omega_k = \omega_1^{420}$$

Решение: $420 : 69 = 6$

$$\begin{array}{r} 420 \\ -\underline{414} \\ 6 \end{array}$$

$$\frac{6 \cdot 69}{414}$$

$$420 = 6 \cdot 69 + 6$$

$$\omega_1^{420} = \omega_1^{6 \cdot 69 + 6} = \omega_1^{6 \cdot 69} \cdot \omega_1^{6} = \left(\omega_1^{69}\right)^6 \cdot \omega_1^6 = 1^6 \cdot \omega_1^6 =$$

$$= \omega_1^6 = \omega_6 \Rightarrow k = 6$$

⑤ За кои $\lambda \in \mathbb{R}$, в-рът

$b = (2, \lambda, 5, 5)$ е ЛК на $a_1 = (1, 2, 3, 4)$;

$a_2 = (7, 14, 20, 27)$; $a_3 = (5, 10, 16, 19)$.

Решение: Нека $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$, за които

$$\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 = b$$

$$\lambda_1 (1, 2, 3, 4) + \lambda_2 (7, 14, 20, 27) + \lambda_3 (5, 10, 16, 19) = [2, \lambda, 5, 5]$$

$$\begin{cases} \lambda_1 \cdot 1 + 7\lambda_2 + 5\lambda_3 = 2 \\ 2\lambda_1 + 14\lambda_2 + 10\lambda_3 = \lambda \\ 3\lambda_1 + 20\lambda_2 + 16\lambda_3 = 5 \\ 4\lambda_1 + 27\lambda_2 + 19\lambda_3 = 5 \end{cases}$$

$\rightarrow$ кога е съвместима

б) $\mathbb{V} = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \middle| \begin{array}{l} a_{12} = a_{21} \\ a_{31} = a_{13} \\ a_{23} = a_{32} \end{array} \right\}$ над $\mathbb{R}$

Знаеме, че $\underline{\mathbb{V} \subseteq M_3(\mathbb{R})}$.

0) $0 \in \mathbb{V}$ → вземаме нулевата матр.

$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\begin{array}{l} a_{12} = 0 = a_{21} \\ a_{31} = 0 = a_{31} \\ a_{23} = 0 = a_{32} \end{array}$ $\checkmark \Rightarrow 0 \in \mathbb{V}$

1) Нека $A, B \in \mathbb{V} \Rightarrow A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$; $B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$

$\left| \begin{array}{l} a_{12} = a_{21} \\ a_{31} = a_{13} \\ a_{23} = a_{32} \end{array} \right.$ $\left| \begin{array}{l} b_{12} = b_{21} \\ b_{31} = b_{13} \\ b_{23} = b_{32} \end{array} \right.$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$$

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{12} & b_{22} & b_{23} \\ b_{13} & b_{23} & b_{33} \end{pmatrix}$$

$$A+B = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{12}+b_{12} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{13}+b_{13} & a_{23}+b_{23} & a_{33}+b_{33} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}$$

Дали $c_{12} = c_{21}$, $c_{13} = c_{31}$ и $c_{23} = c_{32}$ ✓

$$\Rightarrow A + B \in V$$

2) Дали $\forall \lambda \in \mathbb{R}$ $\lambda \cdot A \in W$?

$A \in W$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ & & \end{pmatrix}$$

$$\lambda A = \begin{pmatrix} \lambda a_{11} & \lambda a_{12} & \lambda a_{13} \\ \lambda a_{21} & \lambda a_{22} & \lambda a_{23} \\ \lambda a_{31} & \lambda a_{32} & \lambda a_{33} \end{pmatrix}$$

При $a_{12} = a_{21} \Rightarrow \lambda a_{12} = \lambda a_{21}$

Аналог. за a_{13} с a_{31} и a_{23} с a_{32}

$\Longrightarrow \lambda A \in W \xrightarrow{\text{отн}} W$ е ЛПП на $M_3\mathbb{R}$

$\Leftrightarrow$ е ЛП ✓

(14) Докажете, че $a_1=(1,2,3)$, $a_2=(2,5,7)$, $a_3=(3,7,11)$ обр. базис на $\mathbb{R}^3$. И намерете коорд. на $a=(1,1,1)$ спрямо този базис.

Решение:

I н. с детерминанта

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 7 & 11 \end{vmatrix}$$

II н. Нареждаме по редове / стълбове и ако рангът е пълен $\Rightarrow$ ЛНЗ

Нека $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$, за които

$\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 = (0,0,0)$

$$\left| \begin{array}{l} \lambda_1 + 2\lambda_2 + 3\lambda_3 = 0 \\ 2\lambda_1 + 5\lambda_2 + 7\lambda_3 = 0 \\ 3\lambda_1 + 7\lambda_2 + 11\lambda_3 = 0 \end{array} \right. \rightarrow (\lambda_1, \lambda_2, \lambda_3) = (0,0,0)$$

$$\lambda_1 \cdot a_1 + \lambda_2 a_2 + \lambda_3 a_3 = (1,1,1)$$

$$\begin{cases} \lambda_1 + 2\lambda_2 + 3\lambda_3 = 1 \\ 2\lambda_1 + 5\lambda_2 + 7\lambda_3 = 1 \\ 3\lambda_1 + 7\lambda_2 + 11\lambda_3 = 1 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 5 & 7 & 1 \\ 3 & 7 & 11 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 2 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{array}\right]$$

$$- \begin{array}{ccc|c} 2 & 5 & 7 & 1 \\ 2 & 4 & 6 & 2 \end{array} \quad - \begin{array}{ccc|c} 3 & 7 & 11 & 1 \\ 3 & 6 & 9 & 3 \end{array} \quad - \begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 2 & 2 & -2 \\ \hline 1 & 0 & 1 & 3 \end{array}$$

$\lambda_3 = 1$

$\lambda_2 = 0$

$\lambda_1 = 4$

1. $\frac{(\sqrt{3}-i)^{16}}{(1+i)^8}$

$z_1 = \sqrt{3} - i$ $\qquad$ $z_1 = r_1(\cos \arg z_1 + i \sin \arg z_1)$

$r_1 = |z_1| = \sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{4} = 2$

$\cos \arg z_1 = \frac{\sqrt{3}}{r_1} = \frac{\sqrt{3}}{2}$ $\qquad$ $\sin \arg z_1 = -\frac{1}{2}$

$\cos > 0,\ \sin < 0 \Rightarrow$ IV кв $\Rightarrow \arg z_1 \in \left(\frac{3\pi}{2}; 2\pi\right)$

$180 \to \pi$ $\qquad$ $180x = 330\pi$ $\qquad$ $300 \to \frac{10\pi}{6}$

$330 \to x$ $\qquad$ $x = \frac{33\cancel{0}\pi}{18\cancel{0}} = \frac{11\pi}{6} \to 330$

$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

$\cos(360 - 60) = \cos 360 \cos 60 + \sin 360 \sin 60 =$

$= \frac{1}{2} \cdot 1 = \frac{1}{2} \Rightarrow \arg z_1 = \ , \ \frac{11\pi}{6}$

$\Rightarrow z_1^{15} = 2^{15}\left(\cos\frac{\cancel{15}\cdot 11\pi}{\cancel{6}_2} + i\sin\frac{\cancel{15}\cdot 11\pi}{\cancel{6}2}\right) =$

$= 2^{15}\left(\cos\frac{55\pi}{2} + i\sin\frac{55\pi}{2}\right)$

$$z_2 = 1+i$$

$$z_2 = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$z_2^8 = 2^4\left(\cos\frac{8\pi}{4} + i\sin\frac{8\pi}{4}\right) = 16(\cos 2\pi + i\sin 2\pi)$$

$$= 16 \cdot 1 = 16$$

$$\frac{z_1^{15}}{z_2^8} = \frac{2^{15}\left(\cos\frac{55\pi}{2} + i\sin\frac{55\pi}{2}\right)}{2^4} =$$

$$= 2^{11}\left(\cos\frac{55\pi}{2} + i\sin\frac{55\pi}{2}\right)$$