$$(f*g)(n) = \sum_{ij=n} f(i)\,g(j) = \sum_{i|n} f(i)\,g\left(\frac{n}{i}\right)$$

$$[(f*g)*h](n) = \sum_{ij=n} [(f*g)(i)]\,h(j) =$$

$$= \sum_{ij=n} \left(\sum_{kl=i} f(k)\,g(l) \right) h(j) = \sum_{klj=n} f(k)\,g(l)\,h(j)$$

$$[f*(g*h)](n) = \sum_{ij=n} f(i) \left(\sum_{kl=j} g(k)\,h(l) \right) =$$

$$= \sum_{ikl=n} f(i)\,g(k)\,h(l)$$

$\forall f \quad f * e = f$

$f : \mathbb{N} \longrightarrow \mathbb{C}$

$n \longmapsto \begin{cases} 1 & n = 1 \\ 0 & n > 0 \end{cases} \longleftarrow e$

$$\underset{n>1}{f(n)} = (f * e)(n) = \sum_{ij=n} f(i) \cdot e(j) = \sum_{j \mid n} f\left(\frac{j}{n}\right) \cdot e(j) =$$

$$= \sum_{\substack{j \mid n \\ j \neq n, 1}} f\left(\frac{j}{n}\right) \cdot e(j) + \underline{f(1) \cdot e(n)} + f(n) e(1)$$

Твєрдження

$f(1) = (f * e)(1) = f(1) \cdot e(1)$ $\qquad f : \underline{f(1) \neq 0} \to e(1) = 1$

$\underline{f(2)} = (f * e)(2) = \underbrace{f(1)}_{\neq 0} \cdot e(2) + \underline{f(2)}\underbrace{e(1)}_{=1} \to e(2) = 0$

$e(p) = 0$

$\underline{f(4)} = \underbrace{f(1)}_{=1} \cdot e(4) + f(2)\underbrace{\underline{e(2)}}_{=0} + \underline{f(4)} \cdot \underbrace{e(1)}_{=1}$

$(f * e)(n) = \sum f\left(\frac{n}{d}\right) e(d) + \underline{f(1)} \cdot e(n)$

$d \mid n \quad = $ опр.

$d \neq n \quad 0$ доп.

$d < n$ $\qquad \underset{\neq 0}{} \qquad \underline{\to e(n) = 0}$

$\underline{\text{I}}$ достаточно

II проверка, что $\underline{e}$ свёртка

$f * g = e \qquad g - ?$

$$\underset{\substack{\shortparallel \\ 1}}{e(1)} = \underline{f(1)} \cdot g(1) \qquad g(1) = \frac{1}{f(1)}$$

$$0 = e(n) = (f * g)(n) = \sum_{\substack{d \mid n \\ d < n \\ (d \neq n)}} f\left(\frac{n}{d}\right) \underline{g(d)} + \underset{\substack{\neq \\ 0}}{\underline{f(1)}}\, g(n)$$

$$f \to F(n) = \sum_{d \mid n} f(d).1 = (f * 1)(n) \qquad \underline{1(n) = 1 \ \forall n}$$

$F = f * 1$; $\quad \mu * 1 = M \equiv e \quad$ | ф-ла за обр. $F = f * 1 \to f = F * \mu$

$id : \mathbb{N} \to \mathbb{C}$, $\; n \mapsto n \qquad \varphi * 1 = id \to \varphi = id * \mu$

Трета теорема за ХОМ

$H \lhd G,\ A \lhd G,\ H < A \Rightarrow (G/H)/(A/H) \cong G/A$

<u>Зад.</u> $H \lhd A$

<u>Заб</u> $H \lhd G \overset{\text{деф}}{\Longleftrightarrow} \forall g \in G, \forall h \in H \quad ghg^{-1} \in H$ и $H < G$

$\Longleftrightarrow \forall g \in G \quad gH = Hg$

<u>Д-во</u> $\varphi : G/H \longrightarrow G/A$

$gH \longmapsto gA$

• коректност: $g_1H = g_2H \Leftrightarrow g_1^{-1}g_2 \in H < A \Rightarrow g_1^{-1}g_2 \in A$

$\Rightarrow g_1A = g_2A \Rightarrow \underline{\varphi(g_1H)} = g_1A = g_2A = \underline{\varphi(g_2H)}$

- $\operatorname{Im}\varphi = G/A$
- $\operatorname{Ker}\varphi = \{ gH \in G/H \mid \varphi(gH) = gA = e \cdot A \}$

$$= \{ gH \mid g \in A \} = A/H$$

$$\underset{\text{zu Hom}}{\overset{I}{\Longrightarrow}} \quad \underbrace{(G/H)}_{} / \underbrace{(A/H)}_{\operatorname{Ker}\varphi} \cong \underbrace{G/A}_{\operatorname{Im}\varphi}$$

Действие на група в/у множество

G - гр. ; X - множ.

<u>Опр.</u> Казваме, че G действа в/у X, ако

$$\circ : G \times X \longrightarrow X$$
$$(g, x) \longmapsto g \circ x$$

и

1) $\forall x \in X \qquad e_G \circ x = x$

2) $\forall g_1, g_2 \in G$ и $\forall x \in X \qquad (g_1 g_2) \circ x = g_1 \circ (g_2 \circ x)$

<u>Пр.</u> 1) $X = F^n$ - ЛП над F ; $G = GL_n(F)$
(като стълбове)

$a = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in F^n, \quad g \in G \; ; \quad g \circ a = g \cdot a$

- $e = E_n \quad \forall g \quad e \circ a = a$
- $(g_1 g_2) \circ a = (g_1 g_2) a = g_1 (g_2 a) = g_1 \circ (g_2 \circ a)$

Пр. $X, \quad S(X) = \{ \sigma : X \to X \mid \sigma \text{ - биекция} \}$ с $\circ$

$(\sigma \circ \tau)(x) = \sigma(\tau(x)) \quad \forall x \in X$

е группа

$\sigma \circ x = \sigma(x)$

- $id \circ x = x$
- $(\sigma \tau) \circ x = (\sigma \tau)(x) = \sigma(\tau(x)) = \sigma \circ (\tau \circ x)$

Нека $\Phi : G \times X \longrightarrow X$

$$(g, x) \longmapsto g \circ x$$

$\forall g \in G \qquad \varphi_g : X \longrightarrow X$

$$x \longmapsto g \circ x = \Phi(g, x)$$

Ако е действие

- $\varphi_e = id$
- $\varphi_{g_1 g_2} = \varphi_{g_1} \circ \varphi(g_2)$

композиция на изобр.

$$\Big(\varphi_{g_1 g_2}(x) = (g_1 g_2) \circ x = g_1 \circ (g_2 \circ x) = \varphi_{g_1}(\varphi_{g_2}(x)) =$$

$$= (\varphi_{g_1} \circ \varphi_{g_2})(x)$$

Тв. $\forall g \in G \quad \varphi_g \in S(X)$ (е биекция)

Д-во $(\varphi_g)^{-1} = \varphi_{g^{-1}}$, защото

$$\varphi_g \circ \varphi_{g^{-1}} = \varphi_{gg^{-1}} = \varphi_e = id = \ldots = \varphi_{g^{-1}} \circ \varphi_g$$

$$\varphi : G \longrightarrow S(X)$$
$$g \longmapsto \varphi_g$$

от гр.

Тв φ е хмм

Д-во $\varphi(g_1 g_2) = \varphi_{g_1 g_2} = \varphi_{g_1} \circ \varphi_{g_2} = \varphi(g_1) \circ \varphi(g_2)$

Тв. Ако $\varphi : G \to S(X)$ хом, то

$\phi : G \times X \to X \; : \; \phi(g,x) = (\varphi(g))(x) \quad (= g \circ x)$

е действие на G в/у X

Д-во 1/ $x \in X$

$$\phi(e,x) = (\varphi(e))(x) = id_X(x) = x$$

2/ $x \in X$; $g_1, g_2 \in G$

$$(g_1 g_2) \circ x = (\varphi(g_1 g_2))(x) \underset{\text{хом}}{\overset{\varphi}{=}} (\varphi(g_1) \circ \varphi(g_2))(x) =$$

$$= \varphi(g_1)((\varphi(g_2))(x)) = g_1 \circ (g_2 \circ x)$$

$x, y \in X \qquad x \sim y \overset{\text{def}}{\Longleftrightarrow} \exists g \in G : y = g \circ x$

<u>Тв</u> $\sim$ — РЕ

<u>Д-во</u> 1/ $x \sim x \quad \Leftarrow \quad x = e \circ x$

2/ $(x \sim y \Rightarrow y \sim x) \Leftarrow (y = g \circ x \Rightarrow x = g^{-1} \circ y)$

3/ $(x \sim y, y \sim z \Rightarrow x \sim z) \Leftarrow (y = g_1 \circ x, z = g_2 \circ y \Rightarrow z = (g_2 g_1) \circ x)$

<u>Опр.</u> $O_G(x) = O(x) = [x]$ — орбита на $\underline{\underline{x}}$

<u>Зоб.</u> $O(x) = \{ g \circ x \mid g \in G \}$

$g_1 \circ x = g_2 \circ x \iff (g_1^{-1} g_2) \circ x = x$

Опр. $St_G(x) = St(x) = \{ g \in G \mid g \circ x = x \}$

стабилизатор для x

Тб $St_G(x) < G$

Д-во • $g \circ x = x \Rightarrow x = g^{-1} \circ x$

• $g_1 \circ x = g_2 \circ x = x \Rightarrow (g_1 g_2) \circ x = x$

Зам. $g_1 \circ x = g_2 \circ x \iff g_1^{-1} g_2 \in St(x) \iff g_1 St(x) = g_2 St(x)$

Тб $|G| = |St(x)| \cdot |O(x)|$; $|O(x)| = |G : St(x)|$

Опр. $|O(x)|$ - дължина на орбитата $O(x)$

Заб. $O(x) = [x]$

Тв. $|X| = \sum |O(x_i)| = \sum |G : St(x_i)|$

сумирането е по всички различни представители на непресичащите се орбити

Д-во $X = \bigcup O(x_i)$

$i \neq j \quad O(x_i) \cap O(x_j) = \emptyset$

Заб. $O(x) \to \{gH \mid g \in G\} \qquad H = St(x)$

$g \circ x \longmapsto gH$ биекция

Заб. $\Phi : G \times X \to X$; $\varphi : G \to S(X)$ ХМОМ

$$\ker\varphi = \{g \in G \mid \underbrace{\varphi(g)}_{\varphi_g} = id\} = \{g \in G \mid \forall x \in X \ \underbrace{\varphi_g(x)}_{= g\circ x} = x\}$$

$$\Rightarrow \ker\varphi = \bigcap_{x \in X} St(x)$$

Примери за действие

1) $G = X$ $\qquad g \circ x = gxg^{-1}$ – спрягане

$$(g_1 g_2)\circ x = g_1 g_2 x g_2^{-1} g_1^{-1} = g_1 \circ (g_2 \circ x)$$

2) $G = X$ $\qquad g \circ x = gx$

3) G, $X = \{M \subseteq G\}$ ($\{M \subseteq G \mid |M| = m\}$ m - фикс.)

- $g \circ M = gM$
- $g \circ M = Mg^{-1}$
- $g \circ M = gMg^{-1}$ - спрягане

4) $X = \{M \leq G\}$ $\quad g \circ M = gMg^{-1}$ $\quad |M| = |gMg^{-1}|$

Зад. $H < G$ $\quad gH < G \Leftrightarrow g \in H$

Тв. $y \in O(x)$ (x и y са в една орбита)

$\Leftrightarrow St(x)$ и $St(y)$ са спрегнати

Д-во $\exists g \in G : \ y = g \circ x$

$$t \in St(y) \iff t\circ y = y \iff t\circ(g\circ x) = g\circ x$$

$$\iff (g^{-1}tg)\circ x = x \iff g^{-1}tg \in St(x)$$

$$\iff t \in g\,St(x)\,g^{-1}$$

$$\Rightarrow St(y) = g\cdot St(x)\cdot g^{-1}$$

Приложение

1/ Теорема на Кейли

$\forall G : |G| = n \quad \exists G_1 < S_n$ и $\varphi : G \to G_1$ ИМ

($\forall$ крайна група G е изоморфна на подгрупа на S_n, $n = |G|$)

Д-во $G = X$; $g \circ x = gx$; $|G| = |X| = n$

$$\varphi : G \longrightarrow S(X) \cong S_n$$

$\operatorname{Ker}\varphi = \bigcap\limits_{x \in X} St(x) = \{e\}$ $\Rightarrow$ φ е инективно

$$St(x) = \{ g \in G \mid g \circ x = gx = x \} = \{e\}$$

$$G_1 = \underset{< S_n}{\operatorname{Im}\varphi} \cong G/\{e\} \cong G$$

2) Теорема на Поанкаре

G, $H < G$, $|G:H| = n \Rightarrow \exists N \trianglelefteq G$: $\begin{cases} N < H \\ n \mid |G:N| \mid n! \end{cases}$

Д-во $G; \; X = \{gH \mid g \in G\}$

$\varphi: t \circ (gH) = (tg)H$ ... действие

$\varphi: G \to S(X) \cong S_n$

$St(gH) = \{t \in G \mid (tg)H = gH\} =$

$= \{t \in G \mid (g^{-1}tg)H = H\} = \{t \mid g^{-1}tg \in H\}$

$= gHg^{-1}$

Зад $aH = bH \iff caH = cbH \iff aHc = bHc$

$N = \operatorname{Ker}\varphi = \bigcap_{g \in G} St(gH) = \bigcap_{g \in G} gHg^{-1} < H$

Т-ма ХММ $\Rightarrow N \lhd G$ и $G/N \cong \underset{\operatorname{Im}\varphi}{G_1} < S_n$

$\Rightarrow |G:N| \,/\, n! = |S_n|$

$N < H < G$ и $|G:N| = \underbrace{|G:H|}_{n} \cdot |H:N| \Rightarrow n \,/\, |G:N|$

3) $G = X$ $\qquad g \circ x = g x g^{-1}$

— $St(x) = \{g \in G \mid g x g^{-1} = x\} = \{g \mid gx = xg\}$

централизатор на x; бележим C_x

$O(x) = C(x)$ — клас спрегнати елементи

— $\varphi : G \to S(X)$

$$\ker \varphi = \bigcap_{x \in G} St(x) = \{g \in G \mid \forall x \ gx = xg\} = Z(G) \trianglelefteq G$$

център на G

- формули за класовите

$$|G| = \sum |O(x_i)| + |Z(G)| =$$

$$= |Z(G)| + \sum |G : C_{x_i}|$$

x_i се претставници на $\forall$ разл. неедноелементни орбити

$$|O(x)| = 1 \iff |G : C_x| = 1 \iff G = C_x \iff x \in Z(G)$$