Зад. $e_1, \dots, e_n$ – станд. базис на F^n

$$e_i^*(e_j) = \delta_{ij} \; ; \quad (\lambda_1, \dots, \lambda_n) = \sum_{i=1}^{n} \lambda_i e_i \in F^n$$

$$e_k^*\left(\sum_{i=1}^{n} \lambda_i e_i\right) = \lambda_k \; ; \quad e_k^* : F^n \longrightarrow F$$

$$(\lambda_1, \dots, \lambda_n) \longmapsto \lambda_k$$

Зад. $A \in F_{m \times n}$ $= (a_{ij})$; $\underbrace{\{ x \in F^n \mid Ax = 0 \}}_{F^n}$; $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

$e_1, \dots, e_n$ – станд. базис на F^n

$i = 1, \dots, m$ $\quad \varphi_i : \varphi_i(e_j) = a_{ij}$ $\qquad x = \sum_{i=1}^{n} x_i e_i$

$$\varphi_i(x) = \varphi_i\left(\sum_{j=1}^{n} x_j e_j\right) = \sum_j x_j \cdot a_{ij} \longrightarrow \varphi_i : F^n \longrightarrow F$$

$$\varphi_i \in (F^n)^*$$

$\{x \mid Ax = 0\} = \{x \mid \forall i = 1,\dots,m \quad \varphi_i(x) = 0\}$

$= \{x \mid \forall \varphi \in \ell(\varphi_1, \dots, \varphi_m) \quad \varphi(x) = 0\} = (U^*)_0$

$U^* := \ell(\varphi_1, \dots, \varphi_m) \leq (F^n)^*$; $(U^*)_0 < F^n$

<u>Зад.</u> $\varphi \in \mathrm{Hom}(U, V)$, $U_1 \leq U$

$\varphi|_{U_1} \in \mathrm{Hom}(U_1, V) \qquad \forall u_1 \in U_1 \quad \varphi|_{U_1}(u_1) = \varphi(u_1)$

<u>Зад.</u> $U \leq V \qquad i : U \to V,\ u \mapsto u \qquad i = \mathrm{id}_V|_U$

i - влагане на U във $V \qquad i : U \hookrightarrow V$

$(\forall u \in U \quad i(u) = u)$

$\ker i = \{ \underset{\theta_V}{\overset{\shortparallel}{\theta_U}} \}$; $\operatorname{Im} i = U$

Зад. Ако $\varphi : U \to V$ ЛИ и $\ker \varphi = \{\theta_U\}$

$\Rightarrow \varphi$ - инект. ; $\operatorname{Im} \varphi = U_1 \leq V$; $U \cong U_1 < V$

$i^* \in \operatorname{Hom}(V^*, U^*)$

$\ker i^* = \{ v^* \in V^* \mid i^*(v^*) = \theta_{U^*} \} =$

$= \{v^* \in V^* \mid \forall u \in U \quad \underline{(v^* \circ i)(u)} = 0_{U^*}(u) = \underline{0}\}$

$= \{v^* \in V^* \mid \forall u \in U \quad v^*(\underbrace{i(u)}_{=u}) = 0\} = U^0$

<u>Тв.</u> $U \le V$, $i = id_V|_U \Rightarrow U^0 = \ker i^*$

<u>Зов.</u> $U^* = Hom(U, F)$; $\operatorname{Im} i^* \le U^*$

$u^* \in \operatorname{Im} i^* \Leftrightarrow \exists v^* \in V^* : i^*(v^*) = u^* \Leftrightarrow$

$\Leftrightarrow \exists v^* \in V^* : v^* \circ i = u^* \Leftrightarrow$

$$\Leftrightarrow \exists v^* \in V^* : \quad \forall u \in U \quad (v^* \circ \underbrace{i)(u)}_{u} = u^*(u)$$

$$\Leftrightarrow \exists v^* \in V^* : \quad v^*\big|_U = u^*$$

<u>Зад.</u> $U \leq V$; $v^* \in V^*$

$$\begin{tikzcd} U \arrow[r, hook] \arrow[dr, dashed, "v^*|_U"'] & V \arrow[d, "v^*"] \\ & F \end{tikzcd}$$

<u>Зад.</u> $U \leq V$; $\varphi \in \mathrm{Hom}(U, F) \Rightarrow \exists \psi \in \mathrm{Hom}(V, F)$:

$\psi|_U = \varphi$

Вярно е за произволни ЛП

Реш. КМ ЛП

$e_1 \dots e_k$ - базис на U и го допълваме до

$e_1 \dots e_n$ - базис на V $(k \leq n)$

$\Rightarrow \exists! f \in Hom(V, F) : f(e_i) = \begin{cases} \varphi(e_i) & i = 1 \dots k \\ \underset{=0}{\theta} & i = k+1 \dots n \end{cases}$

__Заб.__ Вместо F може да е произволно ЛП

__Заб.__ т.е. $\forall$ ЛИ $U \to F$ $(U \subseteq V)$ може да бъде продължено до изобр. $V \to F$

__Заб.__ Тогава $\forall u^* \in U^* \; \exists v^* \in V^* : v^*|_U = u^*$

$(U < V)$

ТВ. $\operatorname{Im} i^* = U^*$

<u>ТВ</u> $U < V \Rightarrow \dim U + \dim U^0 = \dim V$ (КМЛЙ)

<u>Д-во</u> теор. за ранга и дефекта за i^*

$\dim \operatorname{Ker} i^* + \dim \operatorname{Im} i^* = \dim V^*$

$(d(i^*))$ $(r(i^*))$

- $\dim V^* = \dim V$
- $\operatorname{Ker} i^* = U^0$
- $\operatorname{Im} i^* = U^* \Rightarrow r(i^*) = \dim U^* = \dim U$

$\Rightarrow$

$\dim U^0 + \dim U = \dim V$

Нека сега $\varphi \in Hom(U, V)$; $\varphi^* \in Hom(V^*, U^*)$

$v^* \in Ker\, \varphi^* < V^* \iff \varphi^*(v^*) = \theta_{U^*} \iff v^* \circ \varphi = \theta_{U^*}$

$\iff \forall u \in U \quad (v^* \circ \varphi)(u) = \theta_{U^*}(u) = 0 = \theta_F$

$\iff \underline{\forall u} \in U \quad v^*(\underline{\varphi(u)}) = 0$

$\iff \forall v \in Im\, \varphi \quad v^*(v) = 0$

$\iff v^* \in (Im\, \varphi)^0$

<u>Т.5.</u> $Ker\, \varphi^* = (Im\, \varphi)^0$

<u>Т.6</u> /Упр./Аналог./ $(Ker\, \varphi^*)_0 = Im\, \varphi$

Сл. $\underline{\underline{d(\varphi^*)}} = \dim \operatorname{Ker}\varphi^* = \dim (\operatorname{Im}\varphi)^0 =$

$= \dim V - \dim \operatorname{Im}\varphi = \underline{\dim V - r(\varphi)} \overset{ТРД}{=}$

$= \dim V - (\dim U + d(\varphi)) = \underline{d(\varphi) + \dim V - \dim U}$

Зад. $d(\varphi^*) + r(\varphi) = \dim V$

Зад. $d(\varphi^*) + r(\varphi^*) = \dim V^* = \dim V$

Сл. $r(\varphi) = r(\varphi^*)$

Сл. $\forall A \in F_{m\times n} \Rightarrow rr(A) = rc(A)$ ← Първа теорема за ранга на матрици

Д-во: $e_1, \dots, e_n$ и $f_1, \dots, f_m$ - стандартните базиси на F^n и F^m

$$\varphi = \Phi_e^f(A) \qquad \left(M_e^f(\varphi) = A; \quad \varphi(e_i) = \sum_{j=1}^{m} a_{ji} f_j, \ i = 1..n \right)$$

e^* и f^* – дуалните бази на e и f

$$M_{f^*}^{e^*}(\varphi^*) = A^t$$

$$r(\varphi) = cr(A) \qquad r(\varphi^*) = cr(A^t) = rr(A)$$

$$\underline{r(\varphi) = r(\varphi^*)} \Rightarrow rr(A) = cr(A)$$

Опр. Ранг на матрица $r(A) = rr(A) = cr(A)$

$u^* \in \operatorname{Im}\varphi^* \le U^* \Leftrightarrow \exists v^* \in V^* : \varphi^*(v^*) = u^*$

$\Leftrightarrow \exists v^* \in V^* ; \ v^* \circ \varphi = u^*$

$\Leftrightarrow \exists v^* \in V^* : \forall u \in U \quad v^*(\varphi(u)) = u^*(u)$

- $u^* \in (\ker\varphi)^0 \to \forall u \in \ker\varphi \quad u^*(u) = 0$

  $\to \varphi(u) = \theta_V \to v^*(\varphi(u)) = 0$

  $\Rightarrow \forall u \in \ker\varphi \quad u^*(u) = v^*(\varphi(u)) = 0$

Тв. $(\ker\varphi)^0 \le \operatorname{Im}\varphi^*$ (вярно)

Тв. $(\ker\varphi)^0 = \operatorname{Im}\varphi^*$ за крайни

Д-во $\dim \operatorname{Im}\varphi^* = r(\varphi^*) = r(\varphi) \xlongequal{\text{ТРД}} \dim U - d(\varphi)$

$= \dim U - \dim \underline{\operatorname{Ker} \varphi} = \dim (\operatorname{Ker} \varphi)^0$

$\Rightarrow \dim \operatorname{Im} \varphi^* = \dim (\operatorname{Ker} \varphi)^0 \qquad (\operatorname{Ker}\varphi)^0 \subseteq \operatorname{Im} \varphi^* \quad \longleftrightarrow$

$\Rightarrow \operatorname{Im} \varphi^* = (\operatorname{Ker} \varphi)^0$

<u>ТВ</u> (Аналог./упр.) $(\operatorname{Im} \varphi^*)_0 = \operatorname{Ker} \varphi$

<u>Опр.</u> $\theta : V \longrightarrow V^{**} = (V^*)^*$

$v \longmapsto \theta(v)$

$\theta(v) : V^* \longrightarrow F$

$v^* \longmapsto v^*(v)$

$\forall v \in V$ и $\forall v^* \in V^*$

$(\theta(v))(v^*) = v^*(v)$

__Зад.__ $V = F^n$; $e_1 \ldots e_n$ – стандарт. б. на F^n __дуальная__

$f_1 \ldots f_n$ – дуальная ; $g_1 \ldots g_n$ – дуальная к ней

$$v = \sum_{i=1}^{n} \lambda_i e_i = (\lambda_1 \ldots \lambda_n) \; ; \; v^* = \sum_{i=1}^{n} \mu_i f_i \; ; \; v^*(v) = \sum_{i=1}^{n} \lambda_i \mu_i$$

$$\left(v^*(x_1 \ldots x_n) = \sum_{i=1}^{n} \mu_i x_i \; ; \; v^*(e_i) = \mu_i \right)$$

$$\theta(v) = \sum_{i=1}^{n} \lambda_i g_i$$ Д-во

$$\left(g_i(f_j) = \delta_{ij} \; ; \; f_i(e_j) = \delta_{ij} \right) \; ; \; \varphi = \sum_{i=1}^{n} \lambda_i g_i$$

$$\varphi(f_j) = \lambda_j \; ; \; \varphi(v^*) = \varphi\left(\sum_{i=1}^{n} \mu_i f_i\right) = \sum_i \lambda_i \mu_i = v^*(v) = (\theta(v))(v^*)$$

$$\Rightarrow \underline{\theta(v) = \varphi}$$

$\Rightarrow V$ и $\theta(V)$ са с еднакви коорд. в

базите $e_1 \ldots e_n$ и $d_1 \ldots d_n$ (дуалната на дуалната)

<u>Зад.</u> V - КМЛП $\Rightarrow \theta$ е ИМ

(каноничен изоморфизъм м/у V и V^{**})