① Нека $V = \left\{ \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & a_{33} \end{pmatrix} \middle| a_{11}, a_{13}, a_{22}, a_{31}, a_{33} \in \mathbb{R} \right\}$

$U = \{ A \in V \mid a_{22} = a_{13} \}$; $W = \{ A \in V \mid a_{13} + a_{31} + a_{33} = 0 \}$.

а) Да се док-те V е ЛП над $\mathbb{R}$, $U \leq V$, $W \leq V$

Решение: $\underline{V \subseteq M_3(\mathbb{R})}$.

0) $\overline{O} \in V$, защото можем да фиксираме ✓ $a_{11} = a_{13} = a_{22} = a_{31} = a_{33} = 0$

1) Нека $A, B \in V$. Дали $A + B \in V$.

$$\begin{pmatrix} a'_{11} & 0 & a'_{12} \\ 0 & a'_{22} & 0 \\ a'_{31} & 0 & a'_{33} \end{pmatrix} + \begin{pmatrix} a''_{11} & 0 & a''_{12} \\ 0 & a''_{22} & 0 \\ a''_{31} & 0 & a''_{33} \end{pmatrix} = \begin{pmatrix} a'_{11} + a''_{11} & 0 & a'_{12} + a''_{12} \\ 0 & a'_{22} + a''_{22} & 0 \\ a'_{31} + a''_{31} & 0 & a'_{33} + a''_{33} \end{pmatrix} \in V$$

2) Нека $\lambda \in \mathbb{R}$ и $A \in V$

$$\lambda . A = \begin{pmatrix} \lambda a_{11} & 0 & \lambda a_{12} \\ 0 & \lambda a_{22} & 0 \\ \lambda a_{31} & 0 & \lambda a_{33} \end{pmatrix} \in V$$

$\xRightarrow{0, 1, 2} V \leq M_3(\mathbb{R})$

$\Rightarrow V$ е ЛП над $\mathbb{R}$

б) Да се намерят базиси на V, U, W и да се намерят размерностите им

$$\begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & a_{33} \end{pmatrix} = a_{11}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + a_{13}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} +$$

$$+ a_{22}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + a_{31}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + a_{33}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\dim V = 5$

$$\begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{13} & 0 \\ a_{31} & 0 & a_{33} \end{pmatrix} = a_{11}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + a_{13}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + a_{31}E_{31} +$$

$$+ a_{33}E_{33}, \quad \dim U = 4$$

$W = \{ A \in V \mid a_{33} = -a_{13} - a_{31} \}$

$$\begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & -a_{13}-a_{31} \end{pmatrix} = a_{11} E_{11} + a_{13} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} +$$

$$+ a_{22} E_{22} + a_{31} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}; \quad \dim W = 4$$

в) Да се намерят базиси на $U+W$, $U \cap W$ и да се опр. размерностите им.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \cancel{1} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{bmatrix}$$

$E_{11}, E_{13}, E_{31}, E_{33}, E_{22}$ – базис на $U+W$

г) Да се допълнят U и W до базис на M

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Допълваме базиса на U с E_{22}.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \to$$

Допълваме базиса на W с E_{33}

Нека $A \in U \cap W \Rightarrow A \in U$ и $A \in W$

$$A = \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{13} & 0 \\ a_{31} & 0 & a_{33} \end{pmatrix}, \quad 2 \; \underline{a_{13}+a_{31}}+a_{33}=0$$

$$a_{33} = -a_{13} - a_{31}$$

$$A = \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{13} & 0 \\ a_{31} & 0 & -a_{13}-a_{31} \end{pmatrix} =$$

$$= a_{11} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + a_{13} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} + a_{31} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

$\dim U \cap W = 3$

② Нека $V = M_n(\mathbb{F})$; $S = \{A \in V \mid A^T = A\}$;
$T = \{A \in V \mid A^T = -A\}$. Да се док.

а) $S < V$, $T < V$

1) $S < V$

0) $\begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & 0 \end{pmatrix}^T = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \hline 0 & 0 & \dots & 0 \end{pmatrix}$

1) $A, B \in S \Rightarrow \underline{A^T = A}, \underline{B^T = B}$

$\underline{(A+B)}^T = A^T + B^T = \underline{A+B} \in S$

2) $\lambda \in \mathbb{F}$, $A \in S \Rightarrow \underline{A^T = A}$

$(\lambda A)^T = \lambda A^T = \lambda A \in S$

$\xRightarrow{0,1,2} S < V$

$T < V$

0) $\begin{pmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} -0 & \cdots & -0 \\ -0 & \cdots & -0 \\ \vdots & & \vdots \\ -0 & \cdots & -0 \end{pmatrix}$ ✓

1) $A, B \in T \Rightarrow \underline{A^T = -A}, \ \underline{B^T = -B}$

$(A+B)^T = A^T + B^T = -A - B = -(A+B) \in T$

2) $\lambda \in \mathbb{F}, \ A \in T$

$(\lambda A)^T = \lambda A^T = \lambda(-A) = -\lambda A \in T$

$\xRightarrow{0,1,2} T < V$

б) Нека $A \in S$

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{12} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \\ a_{1n} & a_{2n} & & \underline{a_{nn}} \end{pmatrix} \quad \cancel{n^2}$$

; За първи ред имаме n коеф., за втория $n-1$ и т.н. за k-тия $n-k+1$

$$n + n-1 + \dots + 1 = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

обр. базис на S

$$\Rightarrow \dim S = \frac{n(n+1)}{2} \; ; \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ -a_{12} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \\ -a_{1n} & -a_{2n} & & a_{nn} \end{pmatrix} = \begin{pmatrix} -a_{11} & & & \\ & -a_{22} & & \\ & & \ddots & \\ & & & -a_{nn} \end{pmatrix}$$

$$\left| \begin{array}{l} a_{11} = -a_{11} = 0 \\ a_{22} = -a_{22} = 0 \\ \dots \\ a_{nn} = -a_{nn} = 0 \end{array} \right.$$

Всички са n^2

n фиксирани нули

$$\frac{n^2 - n}{2} = \frac{n(n-1)}{2}$$

$$n-1 + n-2 + \dots + 0$$

$$\sum_{i=1}^{n-1} i = \frac{(n-1)(n)}{2}$$

$$\left\{ \begin{pmatrix} 0 & 1 & \dots & 0 \\ -1 & 0 & \dots & 0 \\ \hline 0 & 0 & \dots & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & \dots & 0 \\ 0 & \dots & 0 & \dots & 0 \\ -1 & & & & \\ & & & & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \dots & 1 \\ 0 & & & \\ \vdots & & & \\ -1 & & & \end{pmatrix} \right.$$

базис от $\frac{n(n-1)}{2}$ ел.

в) Да се докаже $V = S \oplus T$

$$\dim S + \dim T = \frac{n(n+1)}{2} + \frac{n(n-1)}{2} =$$

$$= \frac{n^2 + \cancel{n} + n^2 - \cancel{n}}{2} = \frac{\cancel{2}n^2}{\cancel{2}} = n^2$$

k

$n-k$

$n = \{0\}$

$$A \in S \cap T \Rightarrow A^T = A \text{ и } A^T = -A$$

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{12} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{pmatrix}}_{\in S} = \begin{pmatrix} -a_{11} & +a_{12} & \dots & +a_{1n} \\ -a_{12} & 0 & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \\ -a_{1n} & -a_{2n} & \dots & 0 \end{pmatrix} \Rightarrow A = 0$$

$\Rightarrow S \cap T = \{0\}$

$\Rightarrow S \oplus T = V$

$a_{11} = 0;\ a_{12} = -a_{12}$

$a_{nn} = -a_{nn}$ $\Rightarrow$ всички са 0

Лема на Фитинг

Нека V е кмпп и $\varphi \in \operatorname{Hom} V$. Да се докже

а) $(\forall k \in \mathbb{N})\ [\operatorname{Ker}\varphi^k \le \operatorname{Ker}\varphi^{k+1}$ и $\operatorname{Im}\varphi^{k+1} \le \operatorname{Im}\varphi^k]$

$$\varphi^k(v) = \underbrace{\varphi(\varphi(\varphi(\dots}_{k \text{ пъти}}(v)\dots)))$$

Нека $v \in \operatorname{Ker}\varphi^k \Rightarrow \varphi^k(v) = 0$

$$\varphi^{k+1}(v) = \varphi(\varphi^k(v)) = \varphi(0) = 0$$

$\Rightarrow\ v \in \operatorname{Ker}\varphi^{k+1} \Rightarrow \operatorname{Ker}\varphi^k \subseteq \operatorname{Ker}\varphi^{k+1}$

Нека $w \in \operatorname{Im}\varphi^{k+1} \Rightarrow \exists v \in V : \varphi^{k+1}(v) = w$.

Питаме се дали $w \in \operatorname{Im}\varphi^k$, т.е дали $\exists u \in V$, $w = \varphi^k(u)$. $\quad w = \varphi^{k+1}(v) = \varphi^k(\underbrace{\varphi(v)}_{\in V})$; $u = \varphi(v)$

$\Rightarrow \operatorname{Im}\varphi^{k+1} \subseteq \operatorname{Im}\varphi^k$

б) $\operatorname{Ker}\varphi^{k+1} = \operatorname{Ker}\varphi^{k} \Rightarrow \operatorname{Ker}\varphi^{k+1} = \operatorname{Ker}\varphi^{k+2}$

и $\operatorname{Im}\varphi^{k+1} = \operatorname{Im}\varphi^{k+2}$

Д-во: 1) от а) $\operatorname{Ker}\varphi^{k+1} \subseteq \operatorname{Ker}\varphi^{k+2}$

2) $\operatorname{Ker}\varphi^{k+2} \subseteq \operatorname{Ker}\varphi^{k+1}$

Нека $v \in \operatorname{Ker}\varphi^{k+2} \Rightarrow \varphi^{k+2}(v) = 0 = \varphi^{k+1}(\underline{\varphi(v)}) = 0$

$\Rightarrow \varphi(v) \in \operatorname{Ker}\varphi^{k+1} \Rightarrow \varphi(v) \in \operatorname{Ker}\varphi^{k}$

$\Rightarrow \varphi^{k}(\varphi(v)) = 0 = \varphi^{k+1}(v) \Rightarrow v \in \operatorname{Ker}\varphi^{k+1}$

$\Rightarrow \operatorname{Ker}\varphi^{k+2} \subseteq \operatorname{Ker}\varphi^{k+1} \Rightarrow \operatorname{Ker}\varphi^{k+1} = \operatorname{Ker}\varphi^{k+2}$

От а) $\operatorname{Im}\varphi^{k+2} \leq \operatorname{Im}\varphi^{k+1}$ (Ако едно пр-во е под-пр-во на друго и имат равни разм., то двете са равни)

Тв ранга и дефекта

$$d(\varphi^{k+1}) + r(\varphi^{k+1}) = \underline{\dim V}$$

$$d(\varphi^{k+2}) + r(\varphi^{k+2}) = \underline{\dim V}$$

$$\underline{\cancel{d(\varphi^{k+1})}} + r(\varphi^{k+1}) = \underline{\cancel{d(\varphi^{k+2})}} + r(\varphi^{k+2})$$

$$\Rightarrow \operatorname{Im}\varphi^{k+1} = \operatorname{Im}\varphi^{k+2}$$

в) $\exists k \in \mathbb{N}$:

$$\underline{\operatorname{Ker}\varphi} \subsetneq \underline{\operatorname{Ker}\varphi^2} \subsetneq \dots \subsetneq \underline{\operatorname{Ker}\varphi^{k-1}} \subsetneq \underline{\operatorname{Ker}\varphi^k} = \operatorname{Ker}\varphi^{k+1}$$

$$\underline{\operatorname{Im}\varphi} \supsetneq \underline{\operatorname{Im}\varphi^2} \supsetneq \dots \supsetneq \underline{\operatorname{Im}\varphi^{k-1}} \supsetneq \underline{\operatorname{Im}\varphi^k} = \operatorname{Im}\varphi^{k+1}$$

Понеже V е крайн., то няма как всички включвания да са строги и в някой момент ще е изп.

г) Ако $\psi \in \operatorname{Hom} V : \operatorname{Ker}\psi^2 = \operatorname{Ker}\psi$, то

$\operatorname{Ker}\psi \cap \operatorname{Im}\psi = \{0\}$ и $\operatorname{Ker}\psi \oplus \operatorname{Im}\psi = V$

В частност $\exists k \in \mathbb{N} : \operatorname{Ker}\varphi^k + \operatorname{Im}\varphi^k = V$

Нека $v \in \operatorname{Ker}\psi \cap \operatorname{Im}\psi$ $\left| \begin{array}{l} \psi(v) = 0 \\ v = \psi(u) \end{array} \right.$

$\psi(\underbrace{\psi(u)}_{v}) = 0$, но $\operatorname{Ker}\psi^2 = \operatorname{Ker}\psi$

$u \in \operatorname{Ker}\psi^2 \Rightarrow u \in \operatorname{Ker}\psi$

$\Rightarrow \psi(u) = 0 \Rightarrow 0$

$\left| \begin{array}{l} \dim \operatorname{Ker}\varphi^{2k} + \\ \dim \operatorname{Im}\varphi^{2k} = \\ \dim V \end{array} \right.$

$\Rightarrow \operatorname{Ker}\psi \cap \operatorname{Im}\psi = \{0\}$.

от б) $\operatorname{Ker}\varphi^k = \operatorname{Ker}\varphi^{k+1} = \ldots = \operatorname{Ker}\varphi^{2k} = \ldots$

$\operatorname{Im}\varphi^k = \operatorname{Im}\varphi^{k+1} = \ldots = \operatorname{Im}\varphi^{2k} = \ldots$

φ^{2k}

Нека $m = 2k$

$\operatorname{Ker} \varphi^m \oplus \operatorname{Im} \varphi^m = V$

(Лема на Фитинг)