$\varphi: \mathbb{R}[x] \longrightarrow \mathbb{C}$

$f \longmapsto f(i)$

- ХММ
- сюрект. $\rightarrow \operatorname{Im} \varphi = \mathbb{C}$
- $\operatorname{Ker} \varphi = \{ f \mid f(i) = 0 \} =$

$= \{ f = (x-i)q \in \mathbb{R}[x] \mid q \in \mathbb{C}[x] \} =$

$\left(f = \sum a_i x^i, \ \bar{f} = \sum \bar{a}_i x^i \quad f(i) = 0 \Rightarrow \bar{f}(\bar{i}) = 0 \right)$

$= \{ f = \underbrace{(x-i)(x+i)}_{x^2+1} q_1 \mid q_1 \in \mathbb{C}[x] \}$ $\rightarrow \mathbb{R}[x]$

$= \{ f \mid x^2+1 \,/\, f \} = (x^2+1)$

$\Rightarrow \mathbb{C} \cong \mathbb{R}[x] / (x^2+1)$

$\mathbb{R}[x] / (x^2+1) = \mathbb{C}_0$

$\psi: \mathbb{R}[x] \longrightarrow \mathbb{C}_0$

$g \longmapsto g + (x^2+1) = \bar{g}$

$\theta = \psi|_{\mathbb{R}} : \mathbb{R} \longrightarrow \mathbb{C}_0$

$\ker \theta \lhd \mathbb{R} \Rightarrow \boxed{\operatorname{Ker} \theta = \{0\}} \rightsquigarrow \ker \theta = \mathbb{R}$

$\rightarrow \psi = 0$

$\mathbb{R}_0 = \operatorname{Im} \theta \cong \mathbb{R} / \{0\} \cong \mathbb{R}$

$$\begin{array}{ccc} \mathbb{C} & \cong & \mathbb{C}_0 \\ \cup & & \cup \\ \mathbb{R} & \cong & \mathbb{R}_0 \end{array}$$

„отъждествяваме" $\mathbb{R} \subset \mathbb{R}_0$

Корени на полиноми | F - поле

Тв. f - неразл. $\Leftrightarrow$ $F[x]/(f)$ - поле

$$\varphi : F[x] \longrightarrow F[x]/(f) = K \text{ (поле)} \quad f \text{ - неразл.}$$

$$g \longmapsto \bar{g} = g + (f)$$

$$\varphi|_F : F \to K \qquad \operatorname{Im} \varphi|_F \cong F$$

Ще считаме, че $F < K$ („отъждествяване $\operatorname{Im} \varphi|_F$ с F; $\forall a \in F$ $\varphi|_F(a) = a$; $\varphi|_F(a) = \varphi(a) = \bar{a} = a$)

$$g \in F[x] < K[x]$$

$$g = \sum \underset{\in F}{b_i} x^i \qquad \bar{g} = \sum \underset{= b_i}{\bar{b}_i} x^i = \sum b_i x_i = g$$

$$0 = \bar{0} = \bar{f} = \overline{\sum a_i x^i} = \sum \bar{a_i}\, \bar{x}^i = \sum a_i\, \bar{x}^i = f(\bar{x})$$

$$\alpha := \bar{x} = \varphi(x) \in K \quad ; \quad f(\alpha) = 0$$

<u>Тв.</u> $f \in F[x]$ - неразл. $\Rightarrow \exists K > F$ и $\exists \alpha \in K : f(\alpha) = 0$

<u>Зад.</u> $f = (x - \alpha) q$, $q \in K[x]$

<u>Сл.</u> $f \in F[x] \Rightarrow \exists K > F$: f се разлага на линейни множители в K

<u>Опр.</u> $f \in F[x]$, $K > F$, в което f се разлага на линейни и $\alpha_1, \dots, \alpha_n \in K$ са $\forall$ кор. на f

$$K_0 = \bigcap_{\substack{F < P < K \\ \alpha_1, \dots, \alpha_n \in P}} P$$ — поле на разлагане на f над F

<u>Зад.</u> Всички полета на разлагане са изоморфни

Зад. $K_0 = F(\alpha_1 \ldots \alpha_n) = \bigcap\limits_{\substack{F < P < K \\ \alpha_1 \ldots \alpha_n \in P}} P$ $\quad (\alpha_1 \ldots \alpha_n \in K)$

Може да се докаже, че $F(\alpha_1, \ldots, \alpha_n) = \left\{ \frac{f(\alpha_1 \ldots \alpha_n)}{g(\alpha_1 \ldots \alpha_n)} \;\middle|\; \begin{matrix} f, g \in F[x_1 \ldots x_n] \\ g(\alpha_1 \ldots \alpha_n) \neq 0 \end{matrix} \right\}$

$F[\alpha_1 \ldots \alpha_n] = \{ f(\alpha_1, \ldots, \alpha_n) \mid f \in F[x_1, \ldots, x_n] \}$ [полето от частни

Формули на Виет

$f \in F[x]$, $\alpha_1 \ldots \alpha_n \in K > F$ — всички кор. на f (с кратностите)

$f = \sum_{i=0}^{n} a_i x^i$, $a_n \neq 0$, $\deg f = n$

$$f = a_n (x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n) =$$

$$= a_n \left[x^n - \left(\sum_{i=1}^{n} \alpha_i \right) x^{n-1} + \left(\sum_{1 \le i < j \le n} \alpha_i \alpha_j \right) x^{n-2} - \left(\sum_{1 \le i < j < k \le n} \alpha_i \alpha_j \alpha_k \right) x^{n-3} + \ldots + (-1)^n \alpha_1 \ldots \alpha_n \right]$$

$$\sum_{i=1}^{n} \alpha_i = -\frac{a_{n-1}}{a_n}$$

$$\sum_{1 \le i < j \le n} \alpha_i \alpha_j = \frac{a_{n-2}}{a_n}$$

$$\sum_{1 \le i < j < k \le n} \alpha_i \alpha_j \alpha_k = -\frac{a_{n-3}}{a_n}$$

— — — —

$$\alpha_1 \cdots \alpha_n = (-1)^n \frac{a_0}{a_n}$$

Полиноми на повече променливи. Симетрични полиноми

F - поле

Опр. $F[x_1, \dots, x_n] = (F[x_1, \dots, x_{n-1}])[x_n]$

Пръстен от пол. на $\underline{n}$ пром $x_1, \dots, x_n$ с коеф. от F

Заб. $f \in F[x_1, \dots, x_n] \to f = \sum\limits_{\alpha \in I} a_\alpha x^\alpha$ (α - мултииндекс)
$|I| < \infty$

$\alpha = (\alpha_1, \dots, \alpha_n)$ $\quad x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$

Тв. $F[x_1, \dots, x_n]$ е област

Д-во $F[x_1]$ е област [И по инд. по $\underline{n}$

(K - област $\Rightarrow K[x]$ - област)

Лексикографска подредба

$ax^{\alpha} > bx^{\beta}$ $(a, b \in F$; α, β - мултииндекси$)$, ако

$\exists k:$

$\alpha_1 = \beta_1, \alpha_2 = \beta_2, \ldots, \alpha_k = \beta_k, \alpha_{k+1} > \beta_{k+1}$

Заб. $x^{\alpha} = \underbrace{x_1 \ldots x_1}_{\alpha_1} \underbrace{x_2 \ldots x_2}_{\alpha_2} \ldots \underbrace{x_k \ldots x_k}_{\alpha_k} \underbrace{x_{k+1} \ldots x_{k+1}}_{\alpha_{k+1}} x_{k+2}^{\alpha_{k+2}} \ldots x_n^{\alpha_n}$

$x^{\beta} = \underbrace{x_1 \ldots x_1}_{\beta_1} \underbrace{x_2 \ldots x_2}_{\beta_2} \ldots \underbrace{x_k \ldots x_k}_{\beta_k} \underbrace{x_{k+1} \ldots x_{k+1}}_{\beta_{k+1}} x_{k+2}^{\beta_{k+2}} \ldots x_n^{\beta_n}$

Заб. $f = \sum_{\alpha \in I} a_{\alpha} x^{\alpha}$

$\deg_i f = \max\{\alpha_i \mid \alpha \in I\}$

$\deg f = \max\{\sum_{i=1}^{n} \alpha_i \mid \alpha \in I\}$ $\deg fg = \deg f + \deg g$

Зад. $f \in F[x_1 \dots x_n]$ с $M(f)$ - най-големия от едночлените относно ЛН (старши едночлен)

Тв. $M(fg) = M(f)M(g)$

Д-во $M(f) = ax^{\alpha}$, $M(g) = bx^{\beta}$

$\forall$ едночлен cx^{γ} на f $\quad cx^{\gamma} \le ax^{\alpha}$ $\quad$ „=" $\Leftrightarrow \left| \begin{array}{l} c = a \\ \gamma = \alpha \end{array} \right.$

$\forall$ едночлен dx^{δ} на g $\quad dx^{\delta} \le bx^{\beta}$ $\quad$ „=" $\Leftrightarrow \left| \begin{array}{l} d = b \\ \delta = \beta \end{array} \right.$

$\forall$ едночлен на fg е сума на $cdx^{\gamma+\delta}$

Но $cdx^{\gamma+\delta} \le abx^{\alpha+\beta}$, защото

- $\gamma = \alpha$ и $\delta = \beta$ ok $\longrightarrow$ „="
- $x^{\gamma} < x^{\alpha}$ и $\delta = \beta$ $(\delta_i = \beta_i)$

$\exists k: \gamma_1 = \alpha_1 - \gamma_k = \alpha_k$ и $\gamma_{k+1} < \alpha_{k+1}$ $\Big\}$

$\gamma_1 + \delta_1 = \alpha_1 + \beta_1$

$\gamma_k + \delta_k = \alpha_k + \beta_k$

$$\underset{<}{\gamma_{k+1}} + \underset{=}{\delta_{k+1}} < \underset{<}{\alpha_{k+1}} + \underset{=}{\beta_{k+1}} \Rightarrow cd\, x^{\gamma+\delta} < ab\, x^{\alpha+\beta}$$

$$-\; c x^{\gamma} < a x^{\alpha} \wedge d x^{\delta} < b x^{\beta}$$

$$\exists k: \gamma_1 = \alpha_1, \dots, \gamma_k = \alpha_k \wedge \gamma_{k+1} < \alpha_{k+1}$$

$$\exists s: \delta_1 = \beta_1, \dots, \delta_s = \beta_s \wedge \delta_{s+1} < \beta_{s+1}$$

$$t = \min(k, s)$$

$$\gamma_1 = \alpha_1 \dots \gamma_t = \alpha_t \quad \wedge \quad \gamma_{t+1} + \delta_{t+1} < \alpha_{t+1} + \beta_{t+1}$$

$$\delta_1 = \beta_1, \dots, \delta_t = \beta_t$$

$$\downarrow$$

$$\gamma_1 + \delta_1 = \alpha_1 + \beta_1, \dots, \gamma_t + \beta_t = \alpha_t + \beta_t$$

$$\Rightarrow cd\, x^{\gamma+\delta} < ab\, x^{\alpha+\beta}$$

Опр. $f \in F[x_1, \ldots, x_n]$ е симетр., ако $\forall \sigma \in S_n$

$$\sigma \circ f = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = f(x_1, \ldots, x_n)$$

Пр. $2x_1 + x_2$ не е симетричен

Пр/def

$$\left.\begin{aligned}
\sigma_1 &= \sum x_i \\
\sigma_2 &= \sum_{1 \le i < j \le n} x_i x_j \\
\sigma_3 &= \sum_{1 \le i < j < k \le n} x_i x_j x_k \\
&\ldots \\
\sigma_n &= x_1 \ldots x_n
\end{aligned}\right\} \text{ елементарни симетрични полиноми}$$

Заб. f - симетр., ако $\forall \sigma = (i\, j)$ $\sigma \circ f = f_{i \leftrightarrow j}$

$$\left(f(x_1 \ldots x_i \ldots x_j \ldots x_n) = f(x_1 \ldots x_j \ldots x_i \ldots x_n) \right)$$

Основна теорема за симетричните полиноми

$\forall f \in F[x_1 \dots x_n]$ – симетр. $\exists g \in F[x_1 \dots x_n]$:

$$f(x_1 \dots x_n) = g(\sigma_1(x_1 \dots x_n), \sigma_2(x_1 \dots x_n), \dots, \sigma_n(x_1 \dots x_n))$$

Заб. Всеки симетр пол. може да се представи като полином на елементарните симетр. пол.

Заб. Представянето е единствено (без д-во)

Заб. Вярно е и ако F е област на цялост

Заб. $M(\sigma_1^{\alpha_1} \sigma_2^{\alpha_2} \dots \sigma_n^{\alpha_n}) = x_1^{\alpha_1} (x_1 x_2)^{\alpha_2} \dots (x_1 x_2 \dots x_n)^{\alpha_n} =$

$$= x_1^{\underbrace{\alpha_1+\alpha_2+\dots+\alpha_n}_{\beta_1}} x_2^{\underbrace{\alpha_2+\dots+\alpha_n}_{\beta_2}} \dots x_{n-1}^{\underbrace{\alpha_{n-1}+\alpha_n}_{\beta_{n-1}}} x_n^{\underbrace{\alpha_n}_{\beta_n}}$$

$$x_1^{\beta_1} x_2^{\beta_2} \ldots x_n^{\beta_n} = M\left(\sigma_1^{\beta_1-\beta_2}\sigma_2^{\beta_2-\beta_3}\ldots\sigma_{n-1}^{\beta_{n-1}-\beta_n}\sigma_n^{\beta_n}\right)$$

Тв. f - симетр. и $M(f) = a x_1^{\alpha_1}\ldots x_n^{\alpha_n} \Rightarrow \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n$

Д-во Ако $\alpha_1 \geq \alpha_2 \ldots \geq \alpha_k < \alpha_{k+1}$

$$(k\ k+1).f = f \Rightarrow a x_1^{\alpha_1}\ldots x_{k-1}^{\alpha_{k-1}} \underline{x_{k+1}^{\alpha_k} x_k^{\alpha_{k+1}}} x_{k+2}^{\alpha_{k+2}} \ldots x_n^{\alpha_n}$$

същото е едночлен на f,

който е $> a x^{\alpha}$ относно ЛН

Д-во на теор.

Нека $M(f) = a x^{\alpha}$; $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n$

$$M\Big(\underbrace{f - a\sigma_1^{\alpha_1-\alpha_2}\sigma_2^{\alpha_2-\alpha_3}\ldots\sigma_{k-1}^{\alpha_{k-1}-\alpha_k}\sigma_k^{\alpha_k}}_{f_1 \text{ - симетр.}}\Big) < M(f)$$

и индукция по ЛН. Докато процесът е краен?

Има не повече от $(\alpha_1+1)^n$ едночлени

$bx^{\beta} < a x_1^{\alpha_1} \dots x_n^{\alpha_n}$ относно ЛН.;

$\beta_1 \geq \beta_2 \geq \dots \geq \beta_n$

$\Rightarrow$ след краен брой стъпки ще стигнем 0

Пр. $f = x_1^3 + x_2^3 + x_3^3$

$x_1^3 \to 3,0,0 \to \sigma_1^{3-0}\sigma_2^{0-0}\sigma_3^0 \to \sigma_1^3$

$f_1 = f - \sigma_1^3 = -3\sum_{i,j} x_i^2 x_j - 6x_1x_2x_3$

$x_1^2x_2 \to 2,1,0 \to \sigma_1^{2-1}\sigma_2^{1-0}\sigma_3^0 = \sigma_1\sigma_2 = \sum_{i,j} x_i^2 x_j + 3x_1x_2x_3$

$f_1 + 3\sigma_1\sigma_2 = 3x_1x_2x_3 = 3\sigma_3$; $f = f_1 + \sigma_1^3 = 3\sigma_3 - 3\sigma_1\sigma_2 + \sigma_1^3$

Пр $x_1^3 + x_2^3 + x_3^3$

• хорошенький полином

• $x_1^3 \to 3,0,0$ [illegible] $\left| \begin{array}{l} \alpha_1, \alpha_2, \alpha_3 \overset{ЛК}{<} 3,0,0 \\ \alpha_1 + \alpha_2 + \alpha_3 = 3 \\ \alpha_1 \geq \alpha_2 \geq \alpha_3 \end{array} \right.$

α_1	α_2	α_3	
3	0	0	$\to \sigma_1^3$
2	1	0	$\to \sigma_1 \sigma_2$
1	1	1	$\to \sigma_3$

$$f = \overset{}{\underset{1}{\cancel{A}}}\, \sigma_1^3 + \overset{-3}{B}\, \sigma_1 \sigma_2 + C \sigma_3 \qquad \cancel{x_1 = -1,\ x_2 = -1,\ x_3 = 0}$$

$$x_1 = x_2 = 1,\ x_3 = 0 \to \sigma_1 = 2,\ \sigma_2 = 1,\ \sigma_3 = 0,\ f = 2$$

$$2 = B \cdot 2 \cdot 1 + 8 \to B = \cancel{}\ -3$$

$$x_1 = x_2 = x_3 = 1 \qquad \sigma_1 = \sigma_2 = 3 \qquad \sigma_3 = 1 \qquad f = 3$$

$$3 = 3^3 - 3 \cdot 3 \cdot 3 + C \cdot 1 \Rightarrow C = 3 \quad \boxed{f = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3}$$

Зад. $\sum x_1^{\alpha_1}\dots x_k^{\alpha_k}$ – мин. симетр. пол. със старши едночлен $x_1^{\alpha_1}\dots x_k^{\alpha_k}$ $(\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_k)$
$k \leq n$

Пр. $\sum x_1^{\alpha} \cdot \sum x_1^{\beta} = \sum x_1^{\alpha+\beta} + 1 \cdot \sum x_1^{\alpha} x_2^{\beta}$
$\alpha > \beta$

Пр. $\alpha = \beta$ $\quad \sum x_1^{\alpha} \sum x_1^{\alpha} = \sum x_1^{2\alpha} + 2 \sum x_1^{\alpha} x_2^{\alpha}$

Зад. $\sum x_1 x_2$ има $\binom{n}{2}$ събираеми

$\sum x_1^2 x_2$ — " — $n(n-1)$

Опр. $S_k = \sum x_1^k$ – степенни сборове

Зад. $\sum x_1^{\alpha} x_2^{\beta} = S_{\alpha} S_{\beta} - S_{\alpha+\beta}$ за $\alpha > \beta$

$\sum x_1^{\alpha} x_2^{\alpha} = \frac{1}{2}\left(S_{\alpha}^2 - S_{2\alpha}\right)$

Формули на Нютон

$$s_k - \sigma_1 s_{k-1} + \sigma_2 s_{k-2} - \dots + (-1)^k \underline{k}\, \sigma_k = 0$$

$$\sigma_s = 0 \text{ за } s > n$$

<u>Зад.</u> В F с $\operatorname{char} F = 0$ $s_1, \dots, s_n$ се изразяват чрез $\sigma_1, \dots, \sigma_n$ и обратно

Сл. (от осн. теор.) $f \in F[x]$, $\deg f = n$ и $\alpha_1, \dots, \alpha_n \in K \supset F$ са корените ѝ. Тогава $\forall g \in F[x_1, \dots, x_n]$ - симетр. $g(\alpha_1, \dots, \alpha_n) \in F$

<u>Зад.</u> F - област и f е със ст. коеф. 1 същото е вярно