В АГ $|e_1| = |e_2| = 1$, $e_1 \perp e_2$; $\underline{e_i e_j = \delta_{ij}}$

$$(\lambda e_1 + \mu e_2) e_1 = \lambda (e_1 e_1) + \mu (e_2 e_1) = \lambda . 1 + \mu . 0 = \lambda$$

$$e_2 \quad - \quad - \quad - \quad - \quad - \quad - = \mu$$

$\varphi \in \mathrm{Hom}(U, V)$

$v^* \in V^*$ $\qquad v^* \circ \varphi \in U^*$

$$\begin{tikzcd} U \arrow[r, "\varphi"] \arrow[dr, dashed, "v^* \circ \varphi"'] & V \arrow[d, "v^*"] \\ & F \end{tikzcd}$$

<u>Опр.</u> $\varphi \in \mathrm{Hom}(U, V)$ $\quad \varphi^* \in \mathrm{Hom}(V^*, U^*)$:

$\forall v^* \in V^* \quad \varphi^*(v^*) = v^* \circ \varphi$ ($\in U^* \to$ л.ф. е на U, зададено v^* и φ (композиция) са ни)

φ^* – дуално изобр. на φ

Тв (Св-ва на дуалните изобр.) (U, V – КМ ЛП)

1) φ^* е ЛИ (ясно) $(\varphi^* : V^* \to U^*)$

2) $* : Hom(U,V) \to Hom(V^*, U^*)$ е ИМ на ЛП

– $\forall \varphi, \psi \in Hom(U,V) \quad (\varphi+\psi)^* = \varphi^* + \psi^*$ $\Big\}$ ЛИ

– $\forall \lambda \in F$ и $\forall \varphi \in Hom(U,V) \quad (\lambda\varphi)^* = \lambda\varphi^*$

– $*$ е инект. $\Leftrightarrow Ker * = \{0\}$, т.е.

ако $\varphi \in Hom(U,V)$ и $\varphi^* = 0$, то $\varphi = 0$

(Ако $\forall v^* \in V^* \quad \underbrace{\varphi^*(v^*)}_{v^* \circ \varphi} = 0$, то $\varphi = 0$, т.е.

$\forall v^* \in V^*$ и $\forall v \in V$ $\quad (v^* \circ \varphi)(v) = 0$, то $\varphi = 0$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \overset{\shortparallel}{v^*(\varphi(v))}$

$\Big(v^* = e^1, \ldots e^n \Rightarrow \forall v \quad \varphi(v) = 0 \Rightarrow \varphi = 0 \Big)$

$- \ast \in$ ИМ $\big(\operatorname{Ker} \ast = \{0\}$; $\dim \operatorname{Hom}(U, V) =$
$= \dim \operatorname{Hom}(V^*, U^*)\big)$

Зад. $v^* \in V^*$ $\quad (\varphi + \psi)^*(v^*) = v^* \circ (\varphi + \psi) = v^* \circ \varphi + v^* \psi =$

$= \varphi^*(v^*) + \psi^*(v^*) = (\varphi^* + \psi^*)(v^*)$

$\Rightarrow (\varphi + \psi)^* = \varphi^* + \psi^*$

$\varphi \in Hom(U, V)$

$\vec{e}$ $e_1, \dots, e_n$ – базис на U ; $f_1, \dots, f_m$ – базис на V

$\vec{f}$

$A = M_e^f(\varphi)$; $a_{ij} = \underline{f^i(\varphi(e_j))}$

$e^1, \dots, e^n$ – дуален б. на U^* ; $f^1, \dots, f^m$ – дуален на V^*

$\vec{e^*}$ $\vec{f^*}$

$\varphi^* \in Hom(V^*, U^*)$

$B = M_{f^*}^{e^*}(\varphi^*)$

$b_{ij} = (\varphi^*(f^j))(e_i) = (f^j \circ \varphi)(e_i) = f^j(\varphi(e_i)) = a_{ji}$

<u>Опр.</u> $A \in F_{m\times n}$. Матр. $B = A^t \in F_{n\times m}$: $b_{ij} := a_{ji}$

транспонирана матр. на A

<u>Заб.</u> На B редовете са ст. на A

— — — ст. — — — р. — —

<u>Тв.</u> e и f са базиси на U и V;

e^* и f^* са дуалните базиси на U^* и V^*

$\varphi \in \mathrm{Hom}(U, V)$; $A = M_e^f(\varphi)$; $B = M_{f^*}^{e^*}(\varphi^*)$

Тогава $B = A^t$

Th. $\varphi \in \mathrm{Hom}(U,V)$, $\psi \in \mathrm{Hom}(V,W)$

$$\Rightarrow (\psi\circ\varphi)^* = \varphi^*\circ\psi^*$$

Bew.

$$\begin{tikzcd}
U \arrow[r, "\varphi"] \arrow[rr, bend left, "\psi\circ\varphi"] & V \arrow[r, "\psi"] & W
\end{tikzcd}$$

$$\begin{tikzcd}
U^* & V^* \arrow[l, "\varphi^*"'] & W^* \arrow[l, "\psi^*"'] \arrow[ll, bend left, "\varphi^*\circ\psi^*"]
\end{tikzcd}$$

D-Ca $w^* \in W^*$

$$(\psi\circ\varphi)^*(w^*) = w^*\circ(\psi\circ\varphi) = (w^*\circ\psi)\circ\varphi =$$

$$= (\psi^*(w^*))\circ\varphi = \varphi^*(\psi^*(w^*)) = (\varphi^*\circ\psi^*)(w^*)$$

$$\Rightarrow (\psi\circ\varphi)^* = \varphi^*\circ\psi^*$$

Сл. 1) $A, B \in F_{m\times n} \Rightarrow (A+B)^t = A^t + B^t$

(следва от $(\varphi+\psi)^* = \varphi^* + \psi^*$)

2) $\lambda \in F,\ A \in F_{m\times n} \Rightarrow (\lambda A)^t = \lambda A^t$

(следва от $(\lambda\varphi)^* = \lambda\varphi^*$)

3) $A \in F_{n\times m},\ B \in F_{m\times k} \Rightarrow (AB)^t = B^t A^t$

(следва от $(\varphi\circ\psi)^* = \psi^*\circ\varphi^*$)

4) $A \in F_{m\times n} \quad (A^t)^t = A$

Зад. $\varphi \leftrightarrow A$; $A^t \leftrightarrow \varphi^*$; $(A^t)^t \leftrightarrow (\varphi^*)^* = \varphi^{**}$
$\qquad\qquad\qquad\qquad\qquad\qquad\quad \overset{\shortparallel}{A}$

$\Rightarrow \varphi \leftrightarrow \varphi^{**} \in Hom(U^{**}, V^{**})$

$(\varphi \in Hom(U,V)$; $\varphi^* \in Hom(V^*, U^*))$

Пр. $V = F^n$; $e_1 \dots e_n$ – станд. базис $(1,0\dots 0), (0,1\dots 0), \dots (0,0,\dots 1)$

$f \in V^*$; $f : F^n \to F$

$f \leftrightarrow f(e_i) = a_i$ за $i = 1 \dots n$

$v \in V$; $v = \sum_{i=1}^n x_i e_i$; $f(v) = \sum_{i=1}^n x_i f(e_i) = \sum_{i=1}^n x_i a_i = \sum_{i=1}^n a_i x_i$
$\qquad\qquad\shortparallel \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ const
$(x_1 \dots x_n)$

$$f : V \longrightarrow F$$

$$(x_1 \ldots x_n) \longmapsto \sum_{i=1}^{n} a_i x_i \quad ; \quad a_i = f(e_i) = e^i(f)$$

Зад. $e^i : V \to F$

$$(x_1 \ldots x_n) \longmapsto x_i$$

Опр. 1) $U < V$ $\quad U^0 := \{v^* \in V^* \mid \forall u \in U \ \ v^*(u) = 0\}$

анихилатор на U

2) $U^* < V^*$ $\quad U_0 := \{v \in V \mid \forall u^* \in U^* \ \ u^*(v) = 0\}$

анулатор на U^*

Зад. $U^0 < V^*$; $U_0 < V$

Зад. 1) $U < V$; $e_1, \dots, e_k$ – базис на U

$U^0 = \{ v^* \in V^* \mid v^*(e_i) = 0$ за $i = 1, \dots, k \}$

2) $U^* < V^*$; $e^1, \dots, e^k$ – базис на U^*

$U_0 = \{ v \in V \mid e^i(v) = 0$ за $i = 1, \dots, k \}$

1') $U < V$; $e_1, \dots, e_k$ – б. на U ; $e_1, \dots, e_n$ – б. на V $(n \geq k)$

$e^1, \dots, e^n$ – дуален б.

$U^0 = \{ v^* = \sum_{i=1}^{n} x_i^* e^i \mid \underbrace{\left(\sum x_i^* e^i\right)(e_j)}_{= x_j^*} = 0, j = 1, \dots, k \} =$

$= \{ v^* = \sum_{i=k+1}^{n} x_i^* e^i \}$ $\Rightarrow$ $e^{k+1}, \dots, e^n$ – базис на U^0

Вопрос: $e^1, \dots, e^n$ — базис в V^*. Существует ли базис $e_1, \dots, e_n$ в V: $e^1, \dots, e^n$ — двойствен к $e_1, \dots, e_n$