$$v = \sum_{i=1}^{n} \lambda_i e_i \qquad \begin{aligned} \varphi_v : F &\longrightarrow V \\ 1 &\longmapsto v \end{aligned} \qquad M^{e}_{1}(\varphi_v) = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} = \lambda$$

$$v = \sum \mu_i f_i \qquad M^{f}_{1}(\varphi_v) = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix} = \mu$$

$$\begin{aligned} \varphi : V &\longrightarrow V \\ e_i &\longmapsto f_i \end{aligned} \qquad M^{e}_{e}(\varphi) = T^{f}_{e} \; ; \; M^{e}_{f}(id_V) = T^{f}_{e}$$

$$\begin{aligned} id_V : V &\longrightarrow V \\ f_i &\longmapsto f_i \end{aligned}$$

$$id_V \circ \varphi_v = \varphi_v$$

$$\begin{tikzcd} F \arrow[r, "\varphi_v"] \arrow[rr, bend right, "\varphi_v"'] & V \arrow[r, "id"] & V \end{tikzcd}$$

$$1 \qquad f \qquad e$$

$$M^{e}_{1}(id_V \circ \varphi_v) = M^{e}_{1}(\varphi_v) = \lambda$$

$$\shortparallel$$

$$M^{e}_{f}(id_V) \cdot M^{f}_{1}(\varphi_v) = T^{f}_{e}\,\mu$$

$$\boxed{\lambda = T^{f}_{e}\,\mu}$$

$$\lambda = T_e^f \mu \quad ; \quad \mu = (T_e^f)^{-1} \lambda = T_f^e \lambda$$

<u>Заб.</u> Горните формули са ф-ли за смяна на координатите на вектор при смяна на базиса

<u>ТВ</u> $\varphi \in Hom(U, V)$

$e_1, \dots, e_n ; e_1', \dots, e_n'$ − базиси на U

$f_1, \dots, f_m ; f_1', \dots, f_m'$ − базиси на V

$$\underline{M_{e'}^{f'}(\varphi)} = M_{e'}^{f'}(id_V \circ \varphi \circ id_U) = M_f^{f'}(id_V)\, M_e^f(\varphi) \cdot M_{e'}^{e}(id_U) = T_{f'}^{f} \cdot M_e^f(\varphi) \cdot T_e^{e'} = \underline{(T_f^{f'})^{-1} M_e^f(\varphi)\, T_e^{e'}}$$

Зад.

$$\begin{array}{lcccccccc}
 & \underline{U} & \xrightarrow{id_U} & U & \xrightarrow{\varphi} & V & \xrightarrow{id_V} & \underline{V} & \qquad U \xrightarrow{\psi} V \\
\text{ge б.} & e'_i & \longmapsto & e'_i & & f_i & \longmapsto & f_i & \\
\text{баз.} & e' & & e & & f & & f' & \qquad e' \quad f'
\end{array}$$

$$\psi = id_V \circ \varphi \circ id_U$$

Зад. $A = M_e^f(\varphi)$; $B = M_{e'}^{f'}(\varphi)$;

$S = T_e^{e'}$; $T = T_f^{f'}$

$$\Rightarrow \boxed{B = T^{-1} A S}$$

Зад (за оператори) $\varphi \in \text{Hom}\, V$; $e_1 \ldots e_n$; $f_1 \ldots f_n$ - бази

$A = M_e^e(\varphi)$; $B = M_f^f(\varphi)$; $T = T_e^f \Rightarrow B = T^{-1} A T$

<u>Тв</u> $\varphi \in \mathrm{Hom}(U,V)$, $u \in U$, $v = \varphi(u) \in V$

$e_1, \dots, e_n$ - базис на U; $f_1, \dots, f_m$ - базис на V

$A = M_e^f(\varphi)$; $M_1^e(\underset{\substack{\cap \\ \mathrm{Hom}(F,U) \\ 1 \longmapsto u}}{\varphi_u}) = \xi$; $M_1^f(\underset{\substack{\cap \\ \mathrm{Hom}(F,V) \\ 1 \longmapsto v}}{v}) = \eta$

$\left(\xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}, \eta = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_m \end{pmatrix}, u = \sum_{i=1}^{n} \xi_i e_i, v = \sum_{i=1}^{m} \eta_i f_i \right)$

Тогава $\eta = A\xi$

<u>Заб.</u> $\xi \in F_{\underline{n} \times 1}$, $\eta \in F_{m \times 1}$, $A \in F_{m \times \underline{n}}$

<u>Д-во</u>

$$F \xrightarrow{\varphi_u} U \xrightarrow{\varphi} V \qquad \Big| \qquad F \xrightarrow{\varphi_v} V$$

$$\underbrace{1 \longmapsto u}_{1 \qquad\qquad e} \qquad\qquad f \qquad\qquad \Big| \qquad \begin{matrix} 1 \longmapsto v \\ 1 \qquad\quad f \end{matrix}$$

б.

$\varphi_v = \varphi \circ \varphi_u$, загалом $\Big|$ $\varphi_v(1) = v$

$(\varphi \circ \varphi_u)(1) = \varphi(\varphi_u(1)) = \varphi(u) = v$

$$M_1^f(\varphi_v) = M_1^f(\varphi \circ \varphi_u) = M_e^f(\varphi) \cdot M_1^e(\varphi_u)$$

$$\eta = A\xi$$

<u>Зав.</u> $v = \varphi(u)$; $v \xrightarrow{f} \eta$, $u \xrightarrow{e} \xi$, $\varphi \xrightarrow{e,f} A$

$$\begin{matrix} v & = & \varphi & (u) \\ \downarrow & & \downarrow & \downarrow \\ \eta & = & A & \xi \end{matrix}$$

Зшб. 1) В нашем базисе (подходящем)

$$\varphi \to A,\ \psi \to B \Rightarrow \left|\begin{array}{l} \varphi + \psi \longrightarrow A + B \\ \lambda \varphi \longrightarrow \lambda A \\ \varphi \psi \longrightarrow AB \end{array}\right.$$

2) $V \begin{array}{l} \xrightarrow{e} \xi \\ \xrightarrow[f]{} \eta \end{array}$ (коорд.) $\quad \xi = T\eta \quad\quad T = T_e^f$

$\underline{\eta = T^{-1}\xi}$

3) $\varphi \to A,\ v = \varphi(u),\ u \to \xi,\ v \to \eta \Rightarrow \eta = A\xi$

4) $\varphi \begin{array}{l} \xrightarrow{e,f} A \\ \xrightarrow[e',f']{} B \end{array} \Rightarrow B = S^{-1} A T,\ S = T_{e'}^{f'},\ T = T_e^f$

4′) (для операторов) $\varphi \begin{array}{l} \xrightarrow{e} A \\ \xrightarrow[f]{} B \end{array} \Rightarrow B = T^{-1} A T,\ T = T_e^f$

Тв $\varphi \in Hom(U, V)$; $U_1 \leq U$, $V_1 \leq V$

$\boxed{\varphi|_{U_1} \in Hom(U_1, V_1)}$ $\left(\forall u \in U_1 \quad \varphi|_{U_1}(u) = \varphi(u) \in V_1\right)$

$e_1, \dots, e_k$ - базис на U_1

$e_1, \dots, e_n$ - базис на U $(k \leq n)$

$f_1, \dots, f_s$ - базис на V_1

$f_1, \dots, f_m$ - базис на V $(s \leq m)$

$A = M_e^f(\varphi) \in F_{m \times n}$, $A_1 = M_e^f(\varphi|_{U_1}) \in F_{s \times k}$

Тогава $A = \begin{array}{cc} \left(\begin{array}{c|c} A_1 & * \\ \hline 0 & * \end{array}\right) & s \\ k & \end{array}$

Док. За $i = 1, \dots, k$ $\quad \varphi(e_i) \in V_1 = \ell(f_1, \dots, f_k)$

$$\varphi(e_i) = \sum_{j=1}^{m} a_{ji} f_j = \sum_{j=1}^{k} a_{ji} f_j \qquad (A = (a_{ij}))$$

Тв $\varphi \in \mathrm{Hom}(U, V)$, $U = U_1 \oplus U_2$, $V = V_1 \oplus V_2$

$\varphi|_{U_1} \in \mathrm{Hom}(U_1, V_1)$, $\varphi|_{U_2} \in \mathrm{Hom}(U_2, V_2)$

$e_1, \dots, e_k$ - б. на U_1; $e_{k+1}, \dots, e_n$ - б. на U_2

$f_1, \dots, f_s$ - б. на V_1, $f_{s+1}, \dots, f_m$ - б. на V_2

$A = M_e^f(\varphi)$, $A_1 = M_e^f(\varphi|_{U_1})$, $A_2 = M_e^f(\varphi|_{U_2})$

Тогава $\quad A = \left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right)$

<u>Сл.</u> $U = U_1 \oplus U_2$, $V = V_1 \oplus V_2$

$\varphi \in \mathrm{Hom}(U_1, V_1)$, $\psi \in \mathrm{Hom}(U_2, V_2)$

Тогава $\exists!\ \theta = \varphi \oplus \psi \in \mathrm{Hom}(U, V)$: $\begin{cases} \theta|_{U_1} = \varphi \\ \theta|_{U_2} = \psi \end{cases}$

<u>Д-во</u> $\varphi \to A_1$, $\psi \to A_2$; $\left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right) \to \theta$

<u>Заб.</u> Горното ще го използваме, но е важно

Тв. $U = U_1 \oplus U_2$; $\varphi \in Hom(U_1, V)$, $\psi \in Hom(U_2, V)$

Тогава $\exists!\, \theta$: $\begin{cases} \theta|_{U_1} = \varphi \\ \theta|_{U_2} = \psi \end{cases}$

Д-во $e_1 \ldots e_k$ – б. на U_1; $e_{k+1} \ldots e_n$ – б. на U_2

($e_1 \ldots e_n$ – б. на U)

Дали ще θ: $\begin{cases} \theta(e_i) = \varphi(e_i) & i = 1, \ldots, k \\ \theta(e_i) = \psi(e_i) & i = k+1, \ldots, n \end{cases}$

$\Rightarrow \exists!$ такова θ

Ядро, образ, ранг и дефект на ЛИ. Теорема за ранга и дефекта

Опр. $\varphi \in Hom(U, V)$

$Ker\, \varphi = \{ u \in U \mid \varphi(u) = \theta_V \}$ ядро на ЛИ φ

$Im\, \varphi = \{ v \in V \mid \exists u \in U : \varphi(u) = v \}$ образ на φ

Забел. $Im\, \varphi = \{ \varphi(u) \mid u \in U \} = \varphi(U)$

Забел. $Ker\, \varphi \subseteq U$, $Im\, \varphi \subseteq V$

Тв (За упр.) $Ker\, \varphi \leq U$, $Im\, \varphi \leq V$

Опр. $r(\varphi) = \dim \operatorname{Im} \varphi$ — ранг на φ

$d(\varphi) = \dim \operatorname{Ker} \varphi$ — дефект на φ

Теорема за ранга и дефекта

$\varphi \in \operatorname{Hom}(U, V)$ $(U, V$ — КМЛП$)$

Тогава $\boxed{r(\varphi) + d(\varphi) = \dim U}$

Д-во $e_1, \dots, e_k$ — базис на $\operatorname{Ker} \varphi < U$ $(\Rightarrow d(\varphi) = k)$
$(\Rightarrow$ за $i = 1, \dots, k$ $\varphi(e_i) = \theta_V)$

Допълваме до базис $e_1, \dots, e_k, e_{k+1}, \dots, e_n$ на U $(\Rightarrow \dim U = n)$

Трябва да док., че $r(\varphi) = n - k$

Ще докажем, че $\underbrace{\varphi(e_{k+1}), \dots, \varphi(e_n)}_{n-k;\ \in \operatorname{Im}\varphi}$ е базис на $\operatorname{Im}\varphi$

1. Пораждащо мн-во.

$$v \in \operatorname{Im}\varphi \Rightarrow \exists u \in U : \varphi(u) = v$$

$$\exists \lambda_i : u = \sum_{i=1}^{n} \lambda_i e_i$$

$$v = \varphi(u) = \varphi\left(\sum_{i=1}^{n} \lambda_i e_i\right) = \sum_{i=1}^{n} \lambda_i \varphi(e_i) = \sum_{i=k+1}^{n} \lambda_i \varphi(e_i)$$

$$\Rightarrow v \in \ell(\varphi(e_{k+1}), \dots, \varphi(e_n))$$

$$\Rightarrow \operatorname{Im}\varphi = \ell(\varphi(e_{k+1}), \dots, \varphi(e_n))$$

Зад. $\varphi \in \mathrm{Hom}(U, V)$; $e_1 \ldots e_n$ – базис в U

$\Rightarrow \mathrm{Im}\,\varphi = \ell(\varphi(e_1), \ldots, \varphi(e_n))$

2. ЛН.

Пусть λ_i : $\sum\limits_{i=k+1}^{n} \lambda_i \varphi(e_i) = \theta_V$

$\Rightarrow \varphi\left(\sum\limits_{i=k+1}^{n} \lambda_i e_i\right) = \theta_V \Rightarrow \sum\limits_{i=k+1}^{n} \lambda_i e_i \in \mathrm{Ker}\,\varphi$

$\Rightarrow \exists \lambda_i : \sum\limits_{i=k+1}^{n} \lambda_i e_i = \sum\limits_{i=1}^{k} \lambda_i e_i$ ($e_1 \ldots e_k$ – базис на $\mathrm{Ker}\,\varphi$)

$$\Rightarrow \sum_{i=1}^{k} \lambda_i e_i + \sum_{i=k+1}^{n} (-\lambda_i) e_i = \theta$$

$$\xRightarrow[\text{базис}]{e_1 \ldots e_n} \forall \lambda_i = 0 \Rightarrow \lambda_{k+1} = \ldots = \lambda_n = 0$$

$\Rightarrow \varphi(e_{k+1}), \ldots, \varphi(e_n)$ – ЛН

1,2 $\Rightarrow \varphi(e_{k+1}), \ldots, \varphi(e_n)$ – базис на $\mathrm{Im}\,\varphi$