Майкъл Зарков, фн: 45655

Зад. 1 Знам, че $\mathbb{R}^{\leq 4}[x]$ е ЛП относно операциите събиране на полиноми и умножение на полином с число.

$V \subseteq \mathbb{R}^{\leq 4}[x] \implies$ НДУ V да е ЛП е $\forall f,g \in V, \forall \lambda \in \mathbb{R}$

$\lambda f + g \in V.$

$$\lambda f + g \in V \overset{def.}{\iff} (\lambda f + g)(1) + (\lambda f + g)(-1) = 0$$

Нека $f, g \in V$, $\lambda \in \mathbb{R}$:

$(\lambda f + g)(1) + (\lambda f + g)(-1) =$ // деф. сбор на полиноми

$= (\lambda f)(1) + g(1) + (\lambda f)(-1) + g(-1) =$ // деф. умнож. на пол. с число

$= \lambda \cdot f(1) + g(1) + \lambda \cdot f(-1) + g(-1) =$

$= \lambda \cdot (\underbrace{f(1) + f(-1)}_{=0}) + (\underbrace{g(1) + g(-1)}_{=0}) = \lambda \cdot 0 + 0 = 0 \implies$

$f, g \in V$

$\implies \lambda f + g \in V, \forall f,g \in V, \forall \lambda \in \mathbb{R} \overset{НДУ}{\implies} V \leq \mathbb{R}^{\leq 4}[x] \implies V$ е ЛП $\square$

(V е ЛП над $\mathbb{R}$ с операциите $+$ на полиноми и $\cdot$ на пол. с число)

Зад. 2

$\Delta_1 = |p| = p$

$\Delta_2 = \begin{vmatrix} p & -1 \\ p+1 & p \end{vmatrix} = p^2 + p + 1$

Доказваме, че: $(*)\ \forall n \geq 3,\ n \in \mathbb{N},\ \Delta_n = p \cdot \Delta_{n-1} + (p+1)\Delta_{n-2}$

Нека $n \geq 3$. Тогава:

$$\Delta_n = \begin{vmatrix} p & -1 & 0 & \cdots & 0 \\ p+1 & p & -1 & \cdots & 0 \\ 0 & p+1 & p & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & p \end{vmatrix}_{n\times n} = \text{// развиваме по 1-ви стълб.}$$

$$= (-1)^{1+1} \cdot p \cdot \underbrace{\begin{vmatrix} p & -1 & \cdots & 0 \\ p+1 & p & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p \end{vmatrix}_{(n-1)\times(n-1)}}_{=\Delta_{n-1}} + (-1)^{2+1} \cdot (p+1) \cdot \begin{vmatrix} -1 & 0 & \cdots & 0 \\ p+1 & p & \cdots & 0 \\ 0 & p+1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p \end{vmatrix} =$$

развиваме по 1-ви ред.

$$= p \cdot \Delta_{n-1} + (-1) \cdot (p+1) \cdot (-1)^{1+1} \cdot (-1) \cdot \underbrace{\begin{vmatrix} p & -1 & 0 & \cdots & 0 \\ p+1 & p & -1 & \cdots & 0 \\ 0 & p+1 & p & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & p \end{vmatrix}_{(n-2)\times(n-2)}}_{=\Delta_{n-2}} =$$

$= p \cdot \Delta_{n-1} + (p+1) \cdot \Delta_{n-2} \implies (*)$ е вярно.

Финално имаме за $p = 5$:

$\Delta_1 = 5$

$\Delta_2 = 31$

$\Delta_n = 5 \cdot \Delta_{n-1} + 6 \cdot \Delta_{n-2}, \quad n \geq 3,\ n \in \mathbb{N}$

Зад. 3 $\quad A^{-1} = ?$

$$\left(\begin{array}{ccccc|ccccc} 1 & -1 & -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & -1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccccc|ccccc} 1 & -1 & -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & -2 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & -2 & 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & -2 & -2 & 0 & 2 & 1 & 0 & 0 & 1 & 0 \\ 0 & 2 & 2 & 2 & 0 & -1 & 0 & 0 & 0 & 1 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccccc|ccccc} 1 & -1 & -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & -2 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 2 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & -1/2 & 0 & 0 & 0 & 1/2 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 1 & 1/2 & 0 & 0 & 0 & 1/2 \\ 0 & 1 & 1 & 1 & 0 & -1/2 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 2 & 0 & 2 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -2 & 4 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 1 & 1 \end{array}\right) \sim$$

$$\left(\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 1 & 1/2 & 0 & 0 & 0 & 1/2 \\ 0 & 1 & 1 & 1 & 0 & -1/2 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & -1 & 2 & 1/2 & 1/2 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 & 1/6 & 1/6 & 1/6 & 1/6 & 1/3 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 0 & 1/3 & -\frac{1}{6} & -1/6 & -1/6 & 1/6 \\ 0 & 1 & 1 & 1 & 0 & -1/2 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & 0 & -1/6 & -\frac{1}{6} & 1/3 & -\frac{1}{6} & 1/6 \\ 0 & 0 & \cancel{0} & -1 & 0 & 1/6 & 1/6 & 1/6 & -1/3 & -1/6 \\ 0 & 0 & 0 & 0 & 1 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{c|ccccc} & 1/3 & -1/6 & -1/6 & -1/6 & 1/6 \\ & -1/6 & 1/3 & -1/6 & -1/6 & 1/6 \\ E_5 & -1/6 & -1/6 & 1/3 & -1/6 & 1/6 \\ & -1/6 & -1/6 & -1/6 & 1/3 & 1/6 \\ & 1/6 & 1/6 & 1/6 & 1/6 & 1/3 \end{array}\right)$$

$$\underbrace{\qquad\qquad\qquad}_{= A^{-1}}$$

Зад. 4

$$X \cdot \underbrace{\begin{pmatrix} 5 & 3 & 2 \\ 7 & 4 & 3 \\ 3 & 1 & 1 \end{pmatrix}}_{=A} = \underbrace{\begin{pmatrix} 1 & -1 & 0 \\ 14 & 7 & 6 \end{pmatrix}}_{=Y}$$

Намираме A^{-1}:

$$\begin{array}{c} -2 \\ -3 \\ \\ \end{array} \left(\begin{array}{ccc|ccc} 5 & 3 & 2 & 1 & 0 & 0 \\ 7 & 4 & 3 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \sim \begin{array}{c} \\ -1 \\ - \end{array} \left(\begin{array}{ccc|ccc} -1 & 1 & 0 & 1 & 0 & -2 \\ -2 & 1 & 0 & 0 & 1 & -3 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \sim$$

$$\sim \begin{array}{c} - \\ \\ +4 \end{array} \left(\begin{array}{ccc|ccc} -1 & 1 & 0 & 1 & 0 & -2 \\ -1 & 0 & 0 & -1 & 1 & -1 \\ 4 & 0 & 1 & -1 & 0 & 3 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 0 & 1 & 0 & 2 & -1 & -1 \\ 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -5 & 4 & -1 \end{array}\right)$$

$$\Rightarrow A^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ 2 & -1 & -1 \\ -5 & 4 & -1 \end{pmatrix}$$

$$X \cdot A = Y \mid \cdot A^{-1}$$

$$X \cdot A \cdot A^{-1} = Y \cdot A^{-1}$$

$$X = Y \cdot A^{-1} \;;$$

$$X = \begin{pmatrix} 1 & -1 & 0 \\ 14 & 7 & 6 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 & 1 \\ 2 & -1 & -1 \\ -5 & 4 & -1 \end{pmatrix} = \begin{pmatrix} 1-2 & 0 & 2 \\ 28-30 & 3 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 2 \\ -2 & 3 & 1 \end{pmatrix}$$

Зад. 5 $\varphi(x_1, x_2, x_3) = (x_1 + x_2 + 2x_3,\ 2x_1 - x_2 + x_3,\ -2x_1 + x_2 - x_3)$.

Докажете, че $\varphi \in$ ЛО:

$(x_1, x_2, x_3), (y_1, y_2, y_3) \in \mathbb{R}^3$, $\lambda \in \mathbb{R}$

(1) $\varphi(x_1 + y_1, x_2 + y_2, x_3 + y_3) = \big((x_1+y_1) + (x_2+y_2) + 2(x_3+y_3),$

$2(x_1+y_1) - (x_2+y_2) + (x_3+y_3),$

$-2(x_1+y_1) + (x_2+y_2) - (x_3+y_3)\big) =$

$= \big((x_1 + x_2 + 2x_3) + (y_1 + y_2 + 2y_3),$

$(2x_1 - x_2 + x_3) + (2y_1 - y_2 + y_3),$

$(-2x_1 + x_2 - x_3) + (-2y_1 + y_2 - y_3)\big) =$

$= (x_1 + x_2 + 2x_3,\ 2x_1 - x_2 + x_3,\ -2x_1 + x_2 - x_3) +$

$(y_1 + y_2 + 2y_3,\ 2y_1 - y_2 + y_3,\ -2y_1 + y_2 - y_3) =$

$= \varphi(x_1, x_2, x_3) + \varphi(y_1, y_2, y_3)$

(2) $\varphi(\lambda x_1, \lambda x_2, \lambda x_3) = (\lambda x_1 + \lambda x_2 + 2(\lambda x_3),\ 2(\lambda x_1) - \lambda x_2 + \lambda x_3,$

$-2(\lambda x_1) + \lambda x_2 - \lambda x_3) =$

$= (\lambda(x_1 + x_2 + 2x_3),\ \lambda(2x_1 - x_2 + x_3),\ \lambda(-2x_1 + x_2 - x_3)) =$

$= \lambda(x_1 + x_2 + 2x_3,\ 2x_1 - x_2 + x_3,\ -2x_1 + x_2 - x_3) = \lambda \cdot \varphi(x_1, x_2, x_3)$

(1), (2) $\Rightarrow$ $\varphi \in \hom \mathbb{R}^3$

$e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$

$$M_e^e(\varphi) = \begin{pmatrix} | & | & | \\ \varphi(e_1) & \varphi(e_2) & \varphi(e_3) \\ | & | & | \end{pmatrix}$$

$\varphi(e_1) = \varphi(1,0,0) = (1, 2, -2)$

$\varphi(e_2) = \varphi(0,1,0) = (1, -1, 1)$

$\varphi(e_3) = \varphi(0,0,1) = (2, 1, -1)$

$$\Rightarrow M_e^e(\varphi) = \begin{pmatrix} 1 & 1 & 2 \\ 2 & -1 & 1 \\ -2 & 1 & -1 \end{pmatrix}$$

Базис на $\operatorname{Ker}\varphi$:

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 2 & -1 & 1 & 0 \\ -2 & 1 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 3 & 3 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right)$$

$\Rightarrow$ частно решение и базис на горната система е $(1, 1, -1)$
и на $\operatorname{Ker}\varphi$.

Базис на $\operatorname{Im}\varphi$:

$$(M_e^e(\varphi))^T = \begin{pmatrix} 1 & 2 & -2 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 0 & 3 & -3 \\ 1 & -1 & 1 \\ 0 & 3 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix} \Rightarrow \cancel{(1,0,2), (0,1,1) - \text{базис на } \operatorname{Im}\varphi}$$

$\Rightarrow (1,0,0), (0,1,-1)$ – базис на $\operatorname{Im}\varphi$

$\operatorname{Ker}\varphi \subseteq \operatorname{Im}\varphi$:

$(1,0,0) + (0,1,-1) = (1,1,-1) \Rightarrow \forall \lambda \in \mathbb{R},\ \lambda(1,0,0) + \lambda(0,1,-1) = \lambda(1,1,-1)$

но вс. в-ри от $\operatorname{Ker}\varphi = \ell(1,1,-1)$ са във вида $\lambda \cdot (1,1,-1)$, $\lambda \in \mathbb{R}$ и

$\lambda \cdot (1,1,-1)$ е ЛК на $\underbrace{(1,0,0)}_{v_1}, \underbrace{(0,1,-1)}_{v_2} \Rightarrow \lambda \cdot (1,1,-1) \in \operatorname{Im}\varphi = \ell(v_1, v_2)$

$\Rightarrow \underline{\operatorname{Ker}\varphi \subseteq \operatorname{Im}\varphi}$

$\underline{r(\varphi^2):}$

$$M_e^e(\varphi^2) = \begin{pmatrix} 1 & 1 & 2 \\ 2 & -1 & 1 \\ -2 & 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 2 \\ 2 & -1 & 1 \\ -2 & 1 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 2 & 1 \\ -2 & 4 & 2 \\ 2 & -4 & -2 \end{pmatrix} \sim \begin{pmatrix} -1 & 2 & 1 \end{pmatrix}$$

$\Rightarrow r(\varphi^2) = 1$