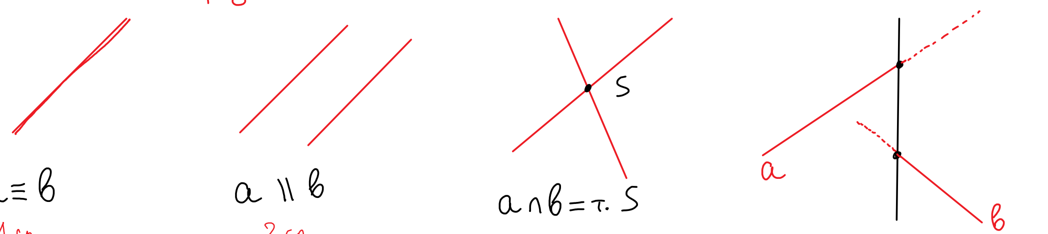
# Трансверзали на 2 кръстосани прави

1 зад. ОКС $Oxyz$

$$a: \begin{cases} x = 5 + 1s \\ y = -1 + 2s \\ z = 11 - 1s \end{cases}, s \in \mathbb{R}, \quad b: \begin{cases} x = -4 - 7p \\ y = 3 + 2p \\ z = 4 + 3p \end{cases}, p \in \mathbb{R}$$

а) Да се определи взаимното положение на $a$ и $b$




$a \equiv b$ — 1 сл.
$a \parallel b$ — 2 сл.
$a \cap b =$ т. $S$ — 3 сл.
$a$ и $b$ са кръстосани — 4 сл.

1) $a \parallel \vec{a}(1, 2, -1)$
$b \parallel \vec{b}(-7, 2, 3)$

$\vec{a} \times \vec{b} \neq \vec{o} \Rightarrow \vec{a}, \vec{b}$ са ЛНЗ $\Rightarrow a \not\equiv b,\ a \not\parallel b$

2) $a \cap b = ? \Rightarrow$
$$\begin{array}{l|l} (1) & x = 5 + 1s = -4 - 7p \\ (2) & y = -1 + 2s = 3 + 2p \\ (3) & z = 11 - 1.s = 4 + 3p \end{array}$$
$s = ?,\ p = ?$ (1) и (2), и (3)

(1)+(3) $\quad 16 = -4p \Rightarrow p = -4 \rightarrow (1) \quad 5 + s = -4 - 7.(-4)$

$s = 19$

$\boxed{\begin{array}{l} p = -4 \\ s = 19 \end{array}}$ $\rightarrow$ (1) да, $\rightarrow$ (2) не, $\rightarrow$ (3) да $\Rightarrow$

$\Rightarrow a \cap b = \varnothing$

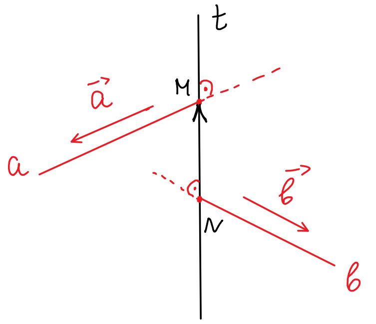
Извод: $a$ и $b$ са кръстосани

б) Да се намерят уравнения на оста на кръстосаните прави $a$ и $b$

$t \cap a = M$
$t \cap b = N$
$t \perp a$
$t \perp b$

$t$ е ос на $a$ и $b$
$MN$ е ос-отсечка на $a$ и $b$

$|MN| = d(a, b)$
разстояние между $a$ и $b$




$M \in a \Rightarrow M(5 + s, -1 + 2s, 11 - s)$

$N \in b \Rightarrow N(-4 - 7p, 3 + 2p, 4 + 3p)$ $(-)$

$\overrightarrow{NM}(9 + 7p + s, -4 - 2p + 2s, 7 - 3p - s)$

$a \parallel \vec{a}(1, 2, -1)$

$$\boxed{\begin{cases} \overrightarrow{NM} \cdot \vec{a} = 0 \\ \overrightarrow{NM} \cdot \vec{b} = 0 \end{cases}}$$

$b \parallel \vec{b}(-7, \; 2, \; 3)$ ос на кръстосани прави

$$\begin{cases} 9+7p+s-8-4p+4s-7+3p+s=0 \\ -63-49p-7s-8-4p+4s+21-9p-3s=0 \end{cases}$$

$$\begin{cases} 6p+6s=6 \\ -62p-6s=+50 \end{cases} \quad (+) \qquad -56p=56$$

$$\boxed{\begin{matrix} p=-1 \\ s=2 \end{matrix}} \begin{matrix} \Rightarrow N(3,1,1) \\ \Rightarrow M(7,3,9) \end{matrix} \qquad \vec{NM}(4,2,8)$$

$$d(a,b) = |\vec{NM}| = \sqrt{16+4+64} = \sqrt{84} = 2.\sqrt{21} \; !$$

$$t: \begin{cases} x = 3 + 2.\lambda \\ y = 1 + \;\; .\lambda \\ z = 1 + \;\; .\lambda \end{cases}, \; \lambda \in \mathbb{R}$$

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2 зад. (Упр.) ОКС $K = Oxyz$

$$a: \begin{cases} x = 7+s \\ y = 2s \\ z = 1+2.s \end{cases}, s \in \mathbb{R} \qquad b: \begin{cases} x = -1+2p \\ y = -4+2p \\ z = 3p \end{cases}, p \in \mathbb{R}$$

? разстоянието между кръстосаните прави $a$ и $b$

? уравнения на оста на $a$ и $b$

3 зад. (Упр.) ОКС $Oxyz$

$$m: \begin{cases} x = 2-s \\ y = s \\ z = 1+2s \end{cases}, s \in \mathbb{R} \qquad g: \begin{cases} x+2y+2z-1=0 \\ y+2z-1=0 \end{cases}$$

? разстоянието между кръстосаните прави $a$ и $b$

? уравнения на оста на $a$ и $b$

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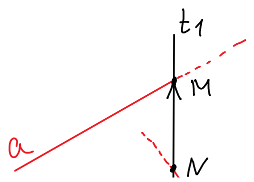
4 зад. (Трансверзали) ОКС $Oxyz$

$$a: \begin{cases} x = 0+p \\ y = -2+p \\ z = -1+2p \end{cases}, p \in \mathbb{R} \; , \quad b: \begin{cases} x+z=0 \\ y+z-2=0 \end{cases} \; , \quad c: \begin{cases} x = 1+2q \\ y = -1+6q \\ z = 2-q \end{cases}, q \in \mathbb{R}$$

а) ? параметрични уравнения на онази трансверзала $t_1$ на кръстосаните прави $a$ и $b$, която е $\parallel c$;

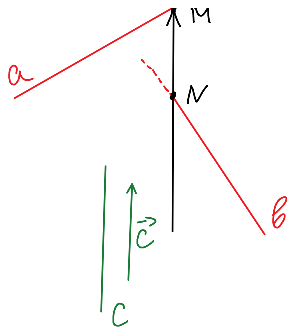
$t_1 \parallel c \Rightarrow t_1 \parallel \vec{c}(2, 6, -1)$




I н. $t_1 \cap a = M(p, -2+p, -1+2p)$ (−)

I н. $t_1 \cap a = M(p, -2+p, -1+2p)$

$t_1 \cap b = N(-\mu, 2-\mu, \mu)$ $(-)$




$b \begin{cases} x+z=0 \\ y+z-2=0 \end{cases}$ търсим параметрични уравнения на $b$

Изб. $z=\mu$ - параметър $\Rightarrow$ $x=-\mu$

$y=2-\mu$

$b \begin{cases} x=-\mu \\ y=2-\mu \\ z=\mu \end{cases}$

$\vec{NM}(p+\mu, -4+p+\mu, -1+2p-\mu)$

$\vec{c}(2, 6, -1)$ колинеарни ($\wedge 3$)

$$\frac{p+\mu}{2} = \frac{-4+p+\mu}{6} = \frac{-1+2p-\mu}{-1}$$

$$\begin{cases} 3(p+\mu) = -4+p+\mu \\ -(p+\mu) = 2(-1+2p-\mu) \end{cases}$$

$$\begin{cases} p+\mu=-2 \\ -5p+\mu=-2 \end{cases} (-) \Rightarrow p=0 \Rightarrow M(0,-2,-1)$$

$\mu=-2 \Rightarrow N(2,4,-2)$

$t_1 \begin{cases} \ni N(2,4,-2) \\ \parallel \vec{c}(2,6,-1) \end{cases} \Rightarrow t_1: \begin{cases} x=2+\eta \cdot 2 \\ y=4+\eta \cdot 6 \\ z=-2+\eta \cdot (-1) \end{cases}, \eta \in \mathbb{R}$

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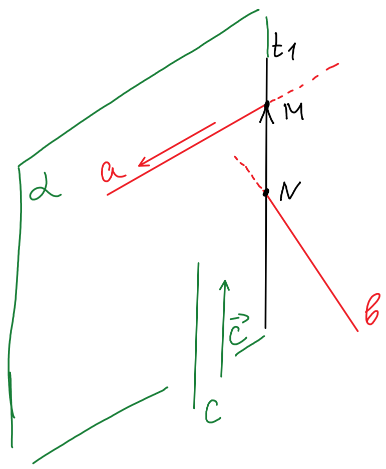
II начин за $t_1 \parallel c$ (Упр.)

1) Намиране уравнение на равнината $\alpha \begin{cases} \supset a \\ \parallel c \end{cases}$

$\alpha \parallel \vec{a}$

$\alpha \parallel \vec{c}$

$\alpha \ni P(0,-2,-1) \in a$




2) $N = b \cap \alpha$

3) $t_1 \begin{cases} \ni N \\ \parallel \vec{c}(2,6,-1) \end{cases}$

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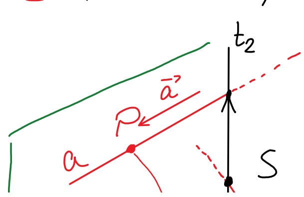
б) ? парам. уравн. на оназч тр. $t_2$ на $a$ и $b$, $t_2 \ni Q(6,0,4)$

1) ? равнина $\beta \begin{cases} \supset a \\ \ni Q \end{cases}$




$Q(6,0,4) \supset$ B

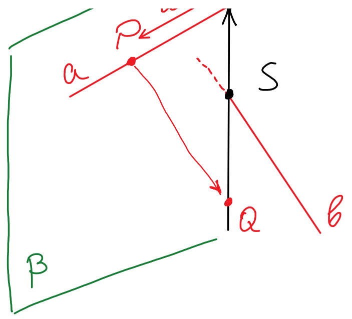
у : равнина $\beta \ni Q$

$Q(6,0,4) \in \beta$

$P(0,-2,-1) \in a \Rightarrow P \in \beta$

$\vec{a}(1,1,2) \parallel \beta$

$\vec{PQ}(6,2,5) \parallel \beta$




$$\beta: \begin{vmatrix} x-6 & y-0 & z-4 \\ 1 & 1 & 2 \\ 6 & 2 & 5 \end{vmatrix} = 0 \qquad \beta: x+7y-4z+10=0 \text{ (Упр.)}$$

2) ?, т. $S = b \cap \beta$

$$b \begin{cases} x = -\mu \\ y = 2-\mu \\ z = \mu \end{cases}$$

$$\begin{cases} x = -\mu \\ y = 2-\mu \\ z = \mu \\ x+7y-4z+10=0 \end{cases}$$

$-\mu+7(2-\mu)-4.\mu+10=0$

$-12\mu+24=0 \Rightarrow \mu=2 \Rightarrow \boxed{S(-2,0,2) \quad Q(6,0,4)}$ $t_2$

$t_2 \parallel \vec{SQ}(8,0,2)$

$$t_2: \begin{cases} x = 6+4.\ell \\ y = 0+0.\ell \\ z = 4+1.\ell \end{cases}, \ \ell \in \mathbb{R}$$

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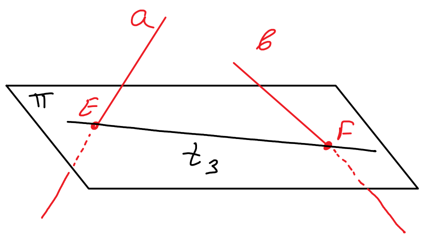
в) (Упр.) $\pi: 2x+2y-z+1=0$

? оназа трансв. $t_3$ на $a$ и $b$, която лежи в $\pi$

?, т. $E = a \cap \alpha$
т. $F = b \cap \alpha$ $\Rightarrow t_3 \begin{cases} \ni E \\ \ni F \end{cases}$

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09.12. Разстояние от точка до равнина

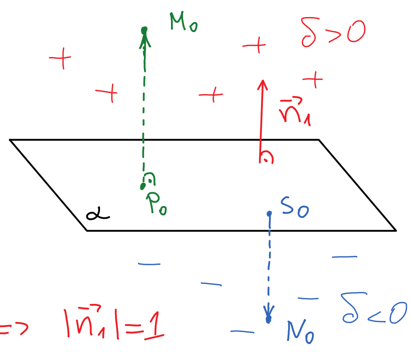
$\boxed{\text{ОКС}}$ $K = Oxyz = O\vec{e_1}\vec{e_2}\vec{e_3}, \ |\vec{e_1}|=|\vec{e_2}|=|\vec{e_3}|=1$

$\alpha: A.x+B.y+C.z+D=0$

$\vec{n_\alpha}(A,B,C) \perp \alpha$

$|\vec{n_\alpha}| = \sqrt{A^2+B^2+C^2}$

$\vec{n_1} = \dfrac{\vec{n_\alpha}}{|\vec{n_\alpha}|} \Rightarrow \boxed{\vec{n_1} = \dfrac{1}{\sqrt{A^2+B^2+C^2}} \cdot (A, B, C)} \Rightarrow |\vec{n_1}| = 1$



$$\alpha: \pm\frac{A.x+B.y+C.z+D}{\sqrt{A^2+B^2+C^2}}=0$$ – нормално уравнение на $\alpha$

$M_0(x_0,y_0,z_0)$ $\quad \delta(M_0,\alpha)$ $\boxed{\pm\ \text{ориентирано}}$ разстояние от $M_0$ до $\alpha$

$$\delta(M_0,\alpha)=\frac{A.x_0+B.y_0+C.z_0+D}{\sqrt{A^2+B^2+C^2}}>0$$ , защото $\overrightarrow{P_0M_0}\uparrow\uparrow\vec{n}_1$

$\delta(N_0,\alpha)<0$ , $\overrightarrow{S_0N_0}\uparrow\downarrow\vec{n}_1$

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1 зад. ОКС $K=Oxyz$

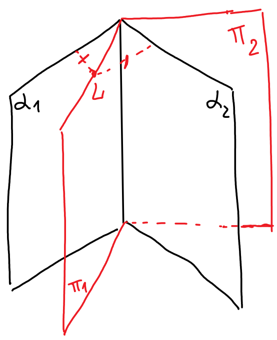
$\alpha_1: 2x-y+2z+3=0$

$\alpha_2: x-2y+2z-3=0$

Да се намерят общи уравнения на ъглополовящите равнини $\pi_1$ и $\pi_2$ на двустенните ъгли, определени от $\alpha_1$ и $\alpha_2$

Решение:

$L\in\pi_1(\in\pi_2)\iff|\delta(L,\alpha_1)|=|\delta(L,\alpha_2)|$




1) Нормални уравнения на $\alpha_1$ и $\alpha_2$

$\alpha_1: 2x-y+2z+3=0$

$\vec{n}_{\alpha_1}(2,-1,2)\Rightarrow|\vec{n}_{\alpha_1}|=\sqrt{4+1+4}=3$

$$\alpha_1: \frac{2x-y+2z+3}{3}=0$$

$\alpha_2: x-2y+2z-3=0$ $\qquad \vec{n}_{\alpha_2}(1,-2,2)\Rightarrow|\vec{n}_{\alpha_2}|=3$

$$\alpha_2: \frac{x-2y+2z-3}{3}=0$$

$$|\delta(L,\alpha_1)|=|\delta(L,\alpha_2)|\iff\left|\frac{2x-y+2z+3}{3}\right|=\left|\frac{x-2y+2z-3}{3}\right| \quad |.3$$

$$2x-y+2z+3=\pm(x-2y+2z-3)$$

$\pi_1: 2x-y+2z+3=+(x-2y+2z-3)$ $\qquad \pi_2: 2x-y+2z+3=-(x-2y+2z+3)$

$\pi_1: x+y+6=0$ $\qquad \pi_2: 3x-3y+4z=0$

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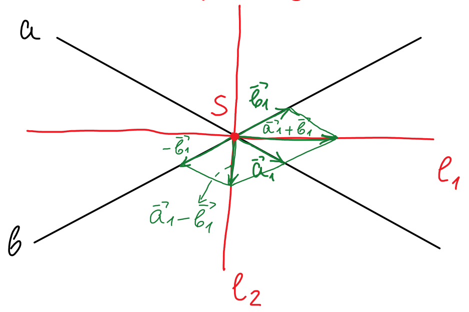
2 зад. ОКС $K=Oxyz$

$$a:\begin{cases}x=-1+2.s\\ y=3-1.s\ ,\ s\in\mathbb{R}\\ z=1+2.s\end{cases} \quad \text{и} \quad b:\begin{cases}x=-1+p\\ y=6-2.p\ ,\ p\in\mathbb{R}\\ z=-1+2.p\end{cases}$$

Да се намерят уравнения на ъглополовящите $\ell_1$ и $\ell_2$ на ъглите между $a$ и $b$.




Решение:

1) $a \parallel \vec{a}(2,-1,2) \Rightarrow |\vec{a}| = 3$

$\vec{a}_1 = \dfrac{\vec{a}}{|\vec{a}|} \Rightarrow \quad \vec{a}_1\left(\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right)$

$b \parallel \vec{b}(1,-2,2) \Rightarrow |\vec{b}| = 3$

$\vec{b}_1 = \dfrac{\vec{b}}{|\vec{b}|} \Rightarrow \quad \vec{b}_1\left(\frac{1}{3}, \frac{-2}{3}, \frac{2}{3}\right)$

2) т. $S = a \cap b \Rightarrow$

$$\begin{array}{ll} (1) & x = -1+2\overset{1}{s} = -1+\overset{2}{p} \ \text{Да} \\ (2) & y = 3-\overset{1}{s} = 6-2\overset{2}{p} \ \text{Да} \\ (3) & z = 1+2\overset{1}{s} = -1+2\overset{2}{p} \ \text{Да} \end{array}$$

$(2)+(3) \Rightarrow \; 4+s=5$

$s=1 \to (2)$

$p=2$

$\Rightarrow S(1,2,3)$

3) $\ell_1 \begin{cases} \ni S(1,2,3) \\ \parallel \vec{a}_1+\vec{b}_1 \left(1, -1, \frac{4}{3}\right) \quad (3,-3,4) \end{cases}$

$\vec{a}_1\left(\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right)$

$\vec{b}_1\left(\frac{1}{3}, \frac{-2}{3}, \frac{2}{3}\right)$

$\ell_1 \begin{cases} x = 1 + \lambda . 3 \\ y = 2 + \lambda . (-3), \ \lambda \in \mathbb{R} \\ z = 3 + \lambda . 4 \end{cases}$

$\ell_2 \begin{cases} \ni S(1,2,3) \\ \parallel \vec{a}_1 - \vec{b}_1 \left(\frac{1}{3}, \frac{1}{3}, 0\right) \quad (1, 1, 0) \end{cases} \Rightarrow \ell_2 : \begin{cases} x = 1 + \mu . 1 \\ y = 2 + \mu . 1 \quad \mu \in \mathbb{R} \\ z = 3 + \mu . 0 \end{cases}$

4) Тъп и остър ъгъл

$\ell_1 \to \measuredangle(\vec{a}_1, \vec{b}_1) \qquad \ell_2 \to \measuredangle(\vec{a}_1, -\vec{b}_1)$

$\vec{a}_1\left(\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right)$

$_1\left(\frac{1}{3}, \frac{-2}{3}, \frac{2}{3}\right)$

$(\vec{a}_1 . \vec{b}_1) = \frac{2}{3}\cdot\frac{1}{3} + \left(-\frac{1}{3}\right)\cdot\left(-\frac{2}{3}\right) + \frac{2}{3}\cdot\frac{2}{3}$

$(\vec{a}_1 . \vec{b}_1) = \frac{8}{9} = \cos\measuredangle(\vec{a}_1, \vec{b}_1) > 0 \Rightarrow$ $\ell_1$ е ъглоп. на острия ъгъл

$\ell_2$ е -ıı- -ıı- тъпия