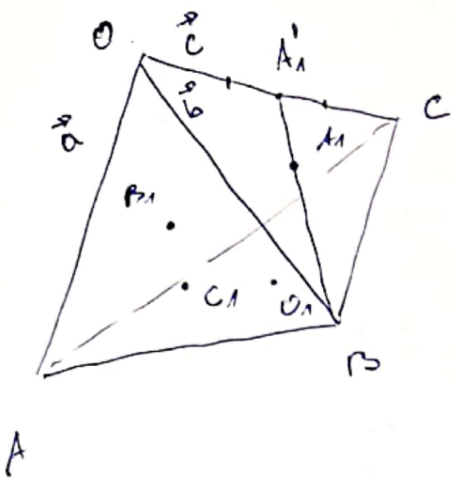
Вариант 1



$\vec{AA_1}$, $\vec{BB_1}$, $\vec{CC_1}$?

а) $\vec{AA_1} = \vec{AB} + \vec{BA_1}$

Нека BA_1' е медиана в $\triangle OBC$

A_1 медицентър в $\triangle OBC$ $\Rightarrow \vec{BA_1} = \frac{2}{3} BA_1'$

BA_1' медиана в $\triangle OBC$

$$\vec{BA_1'} = \frac{1}{2}\left(\vec{BO} + \vec{BC}\right) = \frac{1}{2}\left(-\vec{b} + \vec{OC} - \vec{OB}\right) = \frac{1}{2}\left(-2\vec{b} + \vec{c}\right)$$

$$\vec{BA_1} = \frac{2}{3} \cdot \vec{BA_1'} = \frac{1}{3}\left(\vec{c} - 2\vec{b}\right)$$

$$\vec{AA_1} = \vec{OB} - \vec{OA} + \vec{BA_1} = \vec{b} - \vec{a} + \frac{1}{3}\vec{c} - \frac{2}{3}\vec{b} = -\vec{a} + \frac{1}{3}\left(\vec{c} + \vec{b}\right)$$

$$\vec{BB_1} = \vec{BO} + \vec{OB_1} = -\vec{b} + \frac{1}{3}\left(\vec{a} + \vec{c}\right)$$

$$\vec{CC_1} = \vec{CO} + \vec{OC_1} = -\vec{c} + \frac{1}{3}\left(\vec{a} + \vec{b}\right)$$

$$\vec{OO_1} = \vec{OB} + \vec{BO_1} = \vec{b} + \frac{2}{3} \cdot \frac{1}{2}\left(\vec{BA} + \vec{BC}\right) =$$

$$= \vec{b} + \frac{1}{3}\left(\vec{OA} - \vec{OB} + \vec{OC} - \vec{OB}\right) =$$

$$= \vec{b} + \frac{1}{3}\vec{a} + \frac{1}{3}\vec{c} - \frac{2}{3}\vec{b} = \frac{1}{3}\left(\vec{a} + \vec{b} + \vec{c}\right)$$

б) Док, че $\vec{AA_1} \cap \vec{BB_1} = M$, $AM : MA_1 = BM : MB_1 = 3:1$

$\vec{OM} = \vec{OA} + \vec{AM} = \vec{OB} + \vec{BM}$

т. $M \in \vec{AA_1} \Leftrightarrow \exists! \lambda \in \mathbb{R} : \vec{AM} = \lambda \vec{AA_1}$

т. $M \in \vec{BB_1} \Leftrightarrow \exists! \mu \in \mathbb{R} : \vec{BM} = \mu \vec{BB_1}$

$\vec{OM} = \vec{OA} + \lambda \vec{AA_1} = \vec{a} + \lambda \left(\frac{1}{3}(\vec{b} + \vec{c}) - \vec{a} \right)$

$\vec{OM} = \vec{OB} + \mu \vec{BB_1} = \vec{b} + \mu \left(\frac{1}{3}(\vec{a} + \vec{c}) - \vec{b} \right)$

$\vec{a}(1-\lambda) + \vec{b} \cdot \frac{\lambda}{3} + \vec{c} \cdot \frac{\lambda}{3} = \vec{a} \frac{\mu}{3} + \vec{b}(1-\mu) + \vec{c} \frac{\mu}{3}$

От $\vec{a}, \vec{b}, \vec{c}$ ЛНЗ $\Rightarrow$

$$\begin{cases} 1 - \lambda = \frac{\mu}{3} \\ \frac{\lambda}{3} = 1 - \mu \\ \frac{\lambda}{3} = \frac{\mu}{3} \end{cases} \Rightarrow \lambda = \mu = \frac{3}{4}$$

$\Rightarrow \vec{AA_1} \cap \vec{BB_1} = $ т. M

$\vec{AM} = \frac{3}{4} \vec{AA_1} \Rightarrow \vec{MA_1} = \frac{1}{4} \vec{AA_1}$

$\vec{BM} = \frac{3}{4} \vec{BB_1} \Rightarrow \vec{MB_1} = \frac{1}{4} \vec{BB_1}$

$\Rightarrow \vec{AM} : \vec{MA_1} = 3:1$

$\vec{BM} : \vec{MB_1} = 3:1$

в) O, M, O_1 док, че са колин.

от б) $\vec{OM} = \vec{OA} + \frac{3}{4} \vec{AA_1} = \frac{1}{4}(\vec{a} + \vec{b} + \vec{c})$

$\vec{OM} = \frac{1}{4}(\vec{a} + \vec{b} + \vec{c})$

$\vec{OO_1} = \frac{1}{3}(\vec{a} + \vec{b} + \vec{c})$

$\Rightarrow \vec{OM} = \frac{3}{4} \vec{OO_1} \Rightarrow O, M, O_1$ - колинеарни.

Зад. 2 $\vec{a}, \vec{b}$

$|\vec{a}| = 3$, $|\vec{b}| = 2$, $\measuredangle(\vec{a}, \vec{b}) = \frac{\pi}{3}$

$\vec{OA} = \vec{a} + \vec{b}$, $\vec{OB} = \vec{a} - \vec{b}$, $\vec{OC} = \vec{a} \times \vec{b}$

а) $\exists$ тетраедър $OABC$ $\Leftrightarrow$ $\vec{OA}, \vec{OB}, \vec{OC}$ не са компланарни

$\Leftrightarrow$ смесеното им произведение не е нула

$$(\vec{OA}\,\vec{OB}\,\vec{OC}) = \left((\vec{a}+\vec{b})(\vec{a}-\vec{b})(\vec{a}\times\vec{b})\right) =$$

$$= \left[(\vec{a}+\vec{b}) \times (\vec{a}-\vec{b})\right] \cdot (\vec{a}\times\vec{b}) =$$

$$= \left[\vec{a}\times(\vec{a}-\vec{b}) + \vec{b}\times(\vec{a}-\vec{b})\right] \cdot (\vec{a}\times\vec{b}) =$$

$$= \left[\vec{a}\times\vec{a} - \vec{a}\times\vec{b} + \vec{b}\times\vec{a} - \vec{b}\times\vec{b}\right] \cdot (\vec{a}\times\vec{b})$$

$$= \left[-2(\vec{a}\times\vec{b})\right] \cdot (\vec{a}\times\vec{b}) = \underline{-2(\vec{a}\times\vec{b})^2}$$

$|\vec{a}\times\vec{b}| \neq 0$, защото $\vec{a} \nparallel \vec{b}$, $\vec{a} \neq \vec{0}$, $\vec{b} \neq \vec{0}$ (по усл.)

$\rightarrow$ $\exists$ тетраедър $OABC$

б) $V_{OABC} = \frac{1}{6}\left|(\vec{OA}\,\vec{OB}\,\vec{OC})\right| = \frac{1}{6}\left|(-2)(\vec{a}\times\vec{b})^2\right|$

$|\vec{a}\times\vec{b}| = |\vec{a}| \cdot |\vec{b}| \cdot \sin\measuredangle(\vec{a},\vec{b})_c = 2 \cdot 3 \cdot \frac{\sqrt{3}}{2} = \underline{3\sqrt{3}}$

$V_{OABC} = +\frac{1}{3}(3\sqrt{3})^2 = +\frac{9 \cdot 3}{3} = \underline{9}$

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Зад. 3

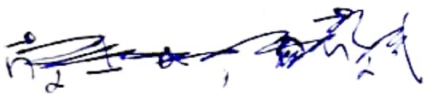
$a: \begin{cases} 2x+y+2z-10=0 \\ 4x-y+z-11=0 \end{cases}$ $\quad b: \begin{cases} x=-5+3q \\ y=5-2q \\ z=3-2q \end{cases}$, $q \in \mathbb{R}$

$a: \begin{cases} 2x+y+2z-10=0 \\ 4x-y+z-11=0 \end{cases} \Rightarrow \begin{array}{l} z=7-2x \\ y=2x-4 \end{array}$

$\Rightarrow$ $a: \begin{cases} x=\lambda \\ y=-4+2\lambda \\ z=7-2\lambda \end{cases}$, $\lambda \in \mathbb{R}$

? Да намерим ~~правата~~ отсечката (AB), $AB \perp a$, $AB \perp b$

$A \in a$, $B \in b$.



Нека $\alpha \perp \vec{a} \Rightarrow \alpha: x+2y-2z+D=0$ $\left(K=O_{xyz}\ \text{ОКС}\right)$

$b \cap \alpha = т.B$

$\begin{cases} x+2y-2z+D=0 \\ x=-5+3q_B \\ y=5-2q_B \\ z=3-2q_B \end{cases} \Rightarrow 3q_B-1+D=0, \boxed{q_B=\frac{-D+1}{3}}$

$\Rightarrow т.B\left(-4-D, 5+\frac{2}{3}(D-1), 3+\frac{2}{3}(D-1)\right)$

$a \cap d = \pi \cdot A$

$$\begin{cases} x + 2y - 2z + D = 0 \\ x = \lambda_A \\ y = -4 + 2\lambda_A \\ z = 7 - 2\lambda_A \end{cases} \Rightarrow 9\lambda_A - 22 + D = 0$$

$$\lambda_A = \frac{-D + 22}{9}$$

$$\Rightarrow A\left(\frac{-D+22}{9}, -4 + \frac{2}{9}(-D+22), 7 - \frac{2}{9}(-D+22) \right)$$

$$\vec{AB} = \vec{OB} - \vec{OA} \Rightarrow$$

$$\Rightarrow \vec{AB}\left(\frac{-8D - 58}{9}, \frac{8D+31}{9}, \frac{4D+2}{9} \right)$$

Условие $\vec{AB} \perp \vec{b} \Rightarrow \vec{AB} \cdot \vec{b} = 0$, $\vec{b}(3, -2, -2)$

$$\frac{-8D-58}{9} \cdot 3 - \frac{8D+31}{9} \cdot 2 - \frac{4D+2}{9} \cdot 2 = 0$$

$\Rightarrow D_2 = -5$ заместване $\Rightarrow \vec{AB}(-2, -1, -2)$

$$|(AB)| = |\vec{AB}| = \sqrt{(-2)^2 + (-1)^2 + (-2)^2} = \sqrt{9} = 3$$

б) $a \cap \gamma = P$

$$\Rightarrow \begin{cases} x + 2y - 2z - 23 = 0 \\ x = \lambda_P \\ y = -4 + 2\lambda_P \\ z = 7 - 2\lambda_P \end{cases} \Rightarrow \lambda_P + 2(-4 + 2\lambda_P) - 2(7 - 2\lambda_P) - 23 = 0$$

$$9\lambda_P - 8 - 14 - 23 = 0$$

$$9\lambda_P = 45$$

$$\lambda_P = -5$$

$P(-5, -14, 17)$

$b \cap \pi = Q$

$\Rightarrow$

$$\begin{cases} 3x - 2y - 2z + 14 = 0 \Rightarrow 17qa - 17 = 0 \\ x = -5 + 3qa \quad qa = 1 \\ y = 5 - 2qa \\ z = 3 - 2qa \end{cases}$$

$Q(-2, 3, 1)$

$S_{\Delta PQR} = \frac{1}{2} |\vec{PQ} \times \vec{PR}|$

$\vec{PQ} (3, 17, -16)$, $\vec{PR} (7, 18, -18)$

$\vec{PR} (7, 18, -18)$

$\vec{PQ} \times \vec{PR} \left(\begin{vmatrix} 17 & -16 \\ 18 & -18 \end{vmatrix}, \begin{vmatrix} -16 & 3 \\ -18 & 7 \end{vmatrix}, \begin{vmatrix} 3 & 17 \\ 7 & 18 \end{vmatrix} \right)$

$\vec{PQ} \times \vec{PR} (-18, -58, -65)$

$S_{\Delta PQR} = \frac{1}{2} \sqrt{(-18)^2 + (-58)^2 + (-65)^2}$