$\boxed{\text{Задача}}$

Докажете, че $\{f_1, f_2\} \models f_3$, където

$f_1 : \forall x \forall y (\exists z (p(x,z) \,\&\, p(z,y)) \Rightarrow p(x,y)) \wedge$

$f_2 : \forall x \forall y \neg \forall z ((p(x,z) \,\&\, p(y,z)) \Rightarrow (p(z,y) \,\&\, \neg p(z,y)))$

$f_3 : \forall x \forall y (\underbrace{\exists x \forall y (p(x,y) \vee \neg p(x,y))}_{\text{И}} \Rightarrow \forall z \exists t (p(y,t) \,\&\, p(x,t) \,\&\, p(z,t))$

Решение:

Трябва да док., че $\{f_1, f_2, \neg f_3\}$ е неизпълнимо

$f_1 \mathrel{\models\!\!\!\dashv} \forall x \forall y (\neg \exists z (p(x,z) \,\&\, p(z,y)) \vee p(x,y)) \mathrel{\models\!\!\!\dashv}$

$\forall x \forall y \forall z (\neg p(x,z) \vee \neg p(z,y) \vee p(x,y))$

$f_2 \mathrel{\models\!\!\!\dashv} \forall x \forall y \neg \forall z (\neg (p(x,z) \,\&\, p(y,z)))$

$\mathrel{\models\!\!\!\dashv} \forall x \forall y \exists z (p(x,z) \,\&\, p(y,z))$

$\underset{z = f(x,y)}{\overset{\text{skolem}}{\sim}} \forall x \forall y (p(x, f(x,y)) \,\&\, p(y, f(x,y)))$

$\neg f_3 \mathrel{\models\!\!\!\dashv} \neg \forall x \forall y \forall z \exists t (p(y,t) \,\&\, p(x,t) \,\&\, p(z,t))$

$\mathrel{\models\!\!\!\dashv} \exists x \exists y \exists z \forall t (\neg p(y,t) \vee \neg p(x,t) \vee \neg p(z,t))$

$\underset{\substack{x = a \\ y = b \\ z = c}}{\overset{\text{skolem}}{\sim}} \forall t (\neg p(b,t) \vee \neg p(a,t) \vee \neg p(c,t))$

Получаваме $D_1 = \{\neg p(x_1, z_1), \neg p(z_1, y_1), p(x_1, y_1)\}$

$D_2 = \{p(x_2, f(x_2, y_2))\}$

$D_3 = \{p(y_3, f(x_3, y_3))\}$

$D_4 = \{\neg p(b, t_4), \neg p(a, t_4), \neg p(c, t_4)\}$

Стратегия:

1. Подреждаме литералите в дизюнктите така, че негативните да са най-отпред
2. Правим резолвенти на дизюнкти без отр. лит. и използваме първи отречен литерал в някой от другите
3. Колапс още в дизюнкта без отречена лит.

$$Res(D_1, D_2) \underset{\substack{x_1 = x_2 \\ z_1 = f(x_2, y_2)}}{=\!=} \{\neg p(f(x_2, y_2), y_1), p(x_2, y_2)\}$$

$$D_5 = \{\neg p(f(x_5, y_5), z_5), p(x_5, z_5)\}$$

2. $$Res(D_1, D_3) \underset{\substack{x_1 = y_3 \\ z_1 = f(x_3, y_3) \\ y_1 = z_6}}{=\!=} \{\neg p(f(x_3, y_3), y_1), p(y_3, y_1)\}$$

$$D_6 = \{\neg p(f(x_6, y_6), z_6), p(y_6, z_6)\}$$

3. $$Res(D_2, D_4) \underset{\substack{x_2 = b \\ f(x_2, y_2) = t_4 \\ y_2 = y_7}}{=\!=} \{\neg p(a, f(b, y_2)), \neg p(c, f(b, y_2))\}$$

$$D_7 = \{\neg p(a, f(b, y_7)), \neg p(c, f(b, y_7))\}$$

4. $$Res(D_3, D_4) \underset{\substack{y_3 = b \\ t_4 = f(x_3, y_3)}}{=\!=} \{\neg p(a, f(y_8, b)), \neg p(c, f(y_8, b))\}$$

$$= D_8$$

5. $Res(D_2, D_5) = \{p(x_5, f(f(x_5,y_5), y_2))\}$

$x_2 = f(x_5, y_5)$

$z_5 = f(x_2, y_2)$

$x_5 = x_9$
$y_5 = y_9$
$y_2 = z_9$

$D_9 = \{p(x_9, f(f(x_9,y_9), z_9))\}$

6. $Res(D_3, D_5) = \{p(x_5, \cancel{f(x_5,y_5)}\, f(x_3, f(x_5,y_5)))\}$

$y_3 = f(x_5, y_5)$

$z_5 = f(x_3, y_3) = f(x_3, f(x_5, y_5))$

$D_{10} = \{p(x_{10}, f(z_{10}, f(x_{10}, y_{10})))\}$

7. $Res(D_3, D_7) = \{\neg p(c, f(b,a))\}$

$y_3 = a$

$x_3 = b$

$y_7 = a$

$D_{11} = \{\neg p(c, f(b,a))\}$

8. $Res(D_3, D_6) = \{p(y_6, f(x_3, f(x_6,y_6)))\}$

$y_3 = f(x_6, y_6)$

$z_6 = f(x_3, y_3) = f(x_3, f(x_6, y_6))$

$D_{12} = \{p(y_{12}, f(z_{12}, f(x_{12}, y_{12})))\}$

9. $Res(D_2, D_6) = \{p(y_6, f(f(x_6,y_6), y_2))\}$

$x_2 = f(x_6, y_6)$

$z_6 = f(x_2, y_2) = f(f(x_6, y_6), y_2)$

$D_{13} = \{p(y_{13}, f(f(x_{13}, y_{13}), z_{13}))\}$

10. $\operatorname{Res}(D_{10}, D_4) = \{\neg p(a, f(z_{10}, f(b, y_{10}))), \neg p(c, f(z_{10}, f(b, y_{10})))\}$

$x_{10} = b$

$t_4 = f(z_{10}, f(\underset{b}{\underset{\shortparallel}{x_{10}}}, y_{10}))$

$D_{14} = \{\neg p(a, f(z_{14}, f(b, y_{14}))), \neg f(c, f(z_{14}, f(b, y_{14})))\}$

11. $D_{15} = \operatorname{Res}(D_{12}, D_{14}) = \{\neg p(c, f(z_{15}, f(b, a)))\}$

$y_{12} = a$

$z_{12} = z_{14}$

$x_{12} = b$

$y_{12} = y_{14}$

12. $D_{16} = \operatorname{Res}(D_{15}, 9) = \blacksquare$

$x_9 = c$

$z_{15} = f(x_9, y_9)$

$z_9 = f(b, a)$