Симетрична група

$\sigma \in S_n$; $\sigma = \begin{pmatrix} 1 & 2 & \dots & n \\ \underbrace{\sigma(1)}_{i_1} & \underbrace{\sigma(2)}_{i_2} & \dots & \underbrace{\sigma(n)}_{i_n} \end{pmatrix}$; $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$

$\sigma(j) = i_j = \sigma(j)$

$\Omega = \{1, 2, \dots, n\}$

$H = \langle \sigma \rangle$

H действа в/у Ω : $h \circ x = h(x)$ $\left(\exists i : h = \sigma^i \right.$

$h \circ x = \underbrace{\sigma(\sigma(\dots(\sigma(x))}_{i} \dots))$

Това наистина е действие

Нека $\Omega = \bigcup_{j=1}^{k} O(i_j)$ $\quad O(i_s) \cap O(i_t) = \emptyset$ за $s \neq t$

<u>Тв.</u> $i \in \Omega \Rightarrow \exists k : O(i) = \{i, \sigma(i), \dots, \sigma^{k-1}(i)\}$ и $i, \sigma(i), \dots, \sigma^{k-1}(i)$ са разл.

$O(i) = \{\sigma^j(i) \mid j \in \mathbb{Z}\} = \{\sigma^j(i) \mid j = 0, 1, \dots, t-1\}$

$|\sigma| = t$ (ред) $\to$ $\langle \sigma \rangle = \{id, \sigma, \sigma^2, \dots, \sigma^{t-1}\}$

k - мин число : $O(i) = \{i, \sigma(i), \dots, \sigma^{k-1}(i)\}$

Ако за $p \neq q$ $\quad \sigma^p(i) = \sigma^q(i) \xRightarrow[p>q]{\text{б.о.о.}} \sigma^{p-q}(i) = i$; $p-q < k$

$O(i) = \{i, \sigma(i), \dots, \sigma^{p-q-1}(i)\}$ $\uparrow\downarrow$ <u>мин k</u>

$(\ s = (p-q)l + r,\ 0 \le r < p-q\ ;\ \sigma^s(i) = \sigma^r(\sigma^{p-q})^l(i) = \sigma^r(i))$

$\Rightarrow \forall p \neq q \quad \sigma^p(i) \neq \sigma^q(i)$

<u>Заб.</u> $\sigma^k(i) = i$

Опр. $\sigma = (i_1\, i_2 \ldots i_k)$ е цикъл, ако

- $\sigma(i_1) = i_2,\ \sigma(i_2) = i_3,\ \ldots,\ \sigma(i_{k-1}) = i_k,\ \sigma(i_k) = i_1$
- $\forall\, j \notin \{i_1 \ldots i_k\}\quad \sigma(j) = j$

Забел. 1) σ е перм. $(\in S_n)$

2) $(i_1\, i_2 \ldots i_k) = (i_2 \ldots i_k\, i_1) = (i_k\, i_1 \ldots i_{k-1})$

Тв. $\sigma \in S_n$; $\Omega = \bigcup_{j=1}^{k} O(i_j)$, $O(i_p) \cap O(i_q) = \emptyset$ за $p \neq q$

за $j = 1 \ldots k$ $\quad O(i_j) = \{\underbrace{i_j, \sigma(i_j), \ldots, \sigma^{n_j - 1}(i_j)}_{\text{разл.}}\}$

$\sigma_j = (i_j\ \sigma(i_j) \ldots \sigma^{n_j-1}(i_j))$ (цикъл); Тогава

$\sigma = \sigma_1 \sigma_2 \dots \sigma_k$

Опр. $\sigma = (i_1 \dots i_k)$, $\tau = (j_1, \dots, j_s)$ – независимые циклы, если $\{i_1, \dots, i_k\} \cap \{j_1, \dots, j_s\} = \varnothing$

Тв. σ, τ – независ. циклы. Тогда

1) $\sigma\tau = \tau\sigma$

2) $|\sigma\tau| = [|\sigma|, |\tau|]$

Д-во 1) $\sigma = (i_1 \dots i_k)$, $j = (j_1 \dots j_s)$; $I = \{i_1, \dots, i_k\}$, $J = \{j_1, \dots, j_s\}$

Зам. $f(x) = x^2 + 1$; $g(x) = \sqrt{x}$ $\quad f(g(x)) = (\sqrt{x})^2 + 1 = x + 1$

$g(f(x)) = \sqrt{x^2+1}$

K: $\Omega = I \cup J \cup K$; $K = \Omega \setminus (I \cup J)$

~~$I \cap J = \varnothing$~~

$I \cap J = I \cap K = J \cap K = \emptyset$

- $i \in I$; $\sigma(i) \in I$; $i, \sigma(i) \notin J \cup K$

$(\sigma\tau)(i) = \sigma(\tau(i)) = \sigma(i)$; $(\tau\sigma)(i) = \tau(\underline{\sigma(i)}) = \sigma(i)$

- $j \in J$ ovanor.

- $k \in K$ $(\sigma\tau)(k) = k = (\tau\sigma)(k)$

2) Ukaz gore, re $\langle\sigma\rangle \cap \langle\tau\rangle = \{id\}$

$\rho \in \langle\sigma\rangle \cap \langle\tau\rangle \Rightarrow \exists p, q : \rho = \sigma^p = \tau^q$

$$\left.\begin{array}{ll} - i \in I & \rho(i) = \tau^q(i) = i \\ - j \in J & \rho(j) = \sigma^p(j) = j \\ - k \in K & \rho(k) = \sigma^p(k) = k \end{array}\right\} \begin{array}{l} \forall i \in \Omega \ \rho(i) = i \\ \Rightarrow \rho = id \end{array}$$

$\Rightarrow |\sigma\tau| = [|\sigma|, |\tau|]$ (тв. от преди)

<u>Д-во на по-предишното тв.:</u>

<u>Заб.</u> $\sigma_1, \dots, \sigma_k$ — $\forall 2$ са независими цикли, б.-е.

твърдим, че $\forall$ перм. се представя като произведение на независими цикли

$$I_j := \{ i_j, \sigma(i_j), \dots, \sigma^{n_j-1}(i_j) \} = O(i_j); \quad \forall p \neq q \ \ I_p \cap I_q = \emptyset$$

$$\cancel{K = \Omega \setminus \left(\bigcup_{j=1}^{k} I_j\right); \ \Omega = I_1 \cup I_2 \cup \dots \cup I_k \cup K}$$

- $i \in I_s \Rightarrow \sigma(i) \in I_s$

$$(\sigma_1 \dots \sigma_k)(i) = (\sigma_1 \dots \sigma_s)\underline{(\sigma_{s+1} \dots \sigma_k)(i)} = (\sigma_1 \dots \sigma_s)(i) =$$

$$= (\sigma_1 \dots \sigma_{s-1})(\sigma_s(i)) = (\sigma_1 \dots \sigma_{s-1})(\sigma(i)) = \sigma(i)$$

Зад. 1) $(i) = id$; обикновено не пишем цикли с дължина 1

2) Ако ги пишем $\longrightarrow$ $\sum_{j=1}^{k} n_j = n$, $n_j = |O(i_j)| = \ell_j$

3) $\tau_1 \dots \tau_k$ са 2x2 независими цикли $\Rightarrow$

$$|\tau_1 \dots \tau_k| = [|\tau_1|, \dots, |\tau_k|]$$

Сл. $|\sigma| = [|\sigma_1|, \dots, |\sigma_k|]$

Тв. $\tau_1 \dots \tau_k = \sigma_1 \dots \sigma_s$; $\tau_1 \dots \tau_k$ и $\sigma_1 \dots \sigma_s$ са независими цикли

Тогава $k = s$ и след преномерация циклите съвпадат

$(\exists \xi \in S_k : \tau_i = \sigma_{\xi(i)})$

Д-во (отчитаме уникалност с дълж. 1)

$i \in \Omega \Rightarrow \exists p, q : \sigma_p = (i \ldots), \; \tau_q = (i \ldots)$

$\sigma_p(i) = \tau_q(i) = (\sigma_1 \ldots \sigma_s)(i) = (\tau_1 \ldots \tau_k)(i)$

Ако $O(i) = \{i, \sigma(i), \ldots, \sigma^{n-1}(i)\}$, то

$(\sigma = \sigma_1 \ldots \sigma_s = \tau_1 \ldots \tau_k)$

$\sigma_p = \tau_q = (i \; \sigma(i) \ldots \sigma^{n-1}(i))$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 1 & 3 & 2 & 5 & 7 & 8 & 11 & 10 & 6 & 9 & 4 \end{pmatrix} = \cancel{(1)}(2\;3)(4\;5\;7\;11)(6\;8\;10\;9)$$

Т.е. $|(i_1 \ldots i_k)| = k$

$|\sigma| = [2, 4, 4] = 4$

Тв. $\forall \rho \in S_n \quad \rho(i_1 i_2 \ldots i_k)\rho^{-1} = (\rho(i_1)\ \rho(i_2) \ldots \rho(i_k))$

Д-ка $\Omega = \{i_1,\ldots,i_k\} \cup J = \{\rho(i_1),\ldots,\rho(i_k)\} \cup \{\rho(j) \mid j \in J\}$

$J = \Omega \setminus \{i_1,\ldots,i_k\}$ не се пресекуваат

- $j = 1,\ldots,k \quad (\rho(i_1 \ldots i_k)\rho^{-1})(\rho(i_j)) = \rho(i_{j+1})$
- $\forall j \in J \quad (\rho(i_1 \ldots i_k)\rho^{-1})(\rho(j)) = \rho(j)$

Сл. $\sigma = (i_1 \ldots i_p)(j_1 \ldots j_q)\ldots(k_1 \ldots k_s)$ (незав. циклуси)

$\Rightarrow \rho\sigma\rho^{-1} = (\rho(i_1) \ldots \rho(i_p))(\rho(j_1) \ldots \rho(j_p))\ldots(\rho(k_1) \ldots \rho(k_s))$

Заб. $\rho(\sigma_1 \ldots \sigma_k)\rho^{-1} = (\rho\sigma_1\rho^{-1})(\rho\sigma_2\rho^{-1})\ldots(\rho\sigma_k\rho^{-1})$

Заб. Две перм. се спрегнати $\Leftrightarrow$ имаат еднаква циклична структура

Опр. $(i\,j)$ — транспозиция

Зab. $(i_1 \dots i_k) = \cancel{(i_k i_1)(i_k i_2)\dots(i_k i_{k-2})(i_k i_{k-1})}$

$$(i_k i_{k-1})(i_k i_{k-2}) \dots (i_k i_2)(i_k i_1)$$

Тв. $\forall$ перм е произведение на транспозиции (не непременно независими)

Тв. $(i_1 j_1)(i_2 j_2) \dots (i_k j_k) = id \Rightarrow k$ — четно

Д-во.

1 $(i\,j)(i\,j) = id$ (намалява бр. с 2)

2 $(i\,j)(k\,l) = (k\,l)(i\,j)$

3 $(j\,k)(i\,j) = (i\,k)(j\,k) = (i\,k\,j)$

4 $(i\,j)(i\,k) = (i\,k)(j\,k) = (i\,k\,j)$

не променят брая

$(i\,j) = (j\,i)$

за $i \in \{i_1, \dots, i_k, j_1, \dots, j_k\}$ през 1, 2, 3

$$\underbrace{(i\,p_1)\dots(i\,p_s)}\underbrace{(q_1\,r_1)\dots(q_t\,r_t)}_{\text{няма } i} = id \qquad \boxed{s+t \equiv k \pmod 2}$$

$\downarrow 4$

$\exists$ няма i или $(i\,x)\underbrace{(\ )\dots(\ )}_{\text{няма } i} = id$ $\quad$ л.стр.$(i) = x$, д.стр.$(i) = i$

$\downarrow$

н.т.к. за друго i

л.стр. $= id$ $\Rightarrow k \equiv 0\,(2) \Rightarrow k$ - четно

Сл. $\sigma_1 \dots \sigma_k = \tau_1 \dots \tau_s$ (σ_i, τ_i - транспозиции) $\Rightarrow k \equiv s \pmod 2$

Д-во $(i\,j)^{-1} = (i\,j)$; $\sigma_1 \dots \sigma_k \cdot \underset{\tau_s^{-1}}{\underset{\shortparallel}{\tau_s}}\, \tau_{s-1} \dots \underset{\tau_1}{\underset{\shortparallel}{\tau_1}} = id$

$\Rightarrow k+s \equiv 0\,(2) \Rightarrow k \equiv s\,(2)$

Опр. $\sigma = \sigma_1 \ldots \sigma_k$, σ_i - транспоз.

$$\operatorname{sign} \sigma = (-1)^k$$

σ - четна, ако k е четно $(\operatorname{sign}\sigma = 1)$

σ - нечетна, — — — нечет. $\quad -1$

Зад. $\operatorname{sign}: S_n \longrightarrow C_2 = \{1, -1\}$ е ХММ

$\sigma = \sigma_1 \ldots \sigma_k$, $\tau = \tau_1 \ldots \tau_s$; σ_i, τ_i - транспоз.

$\sigma\tau = \sigma_1 \ldots \sigma_k \tau_1 \ldots \tau_s$; $\operatorname{sign}\sigma = (-1)^k$, $\operatorname{sign}\tau = (-1)^s$, $\operatorname{sign}\sigma\tau = (-1)^{k+s}$

$\Rightarrow \operatorname{sign}\sigma\tau = \operatorname{sign}\sigma \cdot \operatorname{sign}\tau$

$\operatorname{Ker} \operatorname{sign} = \{\underbrace{\sigma \in S_n \mid \operatorname{sign}\sigma = 1}_{\sigma\text{ - четна}}\} = A_n \trianglelefteq S_n$; $\operatorname{Im}\operatorname{sign} = C_2$

$\Rightarrow S_n / A_n \cong C_2$