Домашна работа 1

Зад. 1

Док., че $n! < n^n$ за $n \geq 2$

Реш:

Д-во по индукция.

1. База: $n = 2$ $\quad 2! < 2^2$ – да
2. Нека е вярно за някое $k = n$, $k \in \mathbb{N}$, $k > 2$
3. Стъпка: за $n = k+1$

$$(k+1)^k = k^k + \underbrace{\binom{k}{1}k^{k-1} + \binom{k}{2}k^{k-2} + \dots + \binom{k}{k-1}k + 1}_{>0}$$

$$\Rightarrow (k+1)^k > k^k \quad (1)$$

$$(k+1)! = k!(k+1) < k^k(k+1) \overset{(1)}{<} (k+1)^k(k+1) = (k+1)^{k+1}$$

$$\Rightarrow (k+1)! < (k+1)^{k+1} \Rightarrow n! < n^n, \ n \geq 2 \quad \checkmark$$

Зад. 2

а) $f(x)=\frac{1}{1+25x^2}$, $x\geq 0$, $f^{-1}(x)=?$

Реш: $f(x)\in(0,1]$

$y=\frac{1}{1+25x^2}$, $1+25x^2=\frac{1}{y}$, $25x^2=\frac{1}{y}-\frac{y}{y}$,

$x^2=\frac{1-y}{25y}$, $x=\frac{\sqrt{1-y}}{5\sqrt{y}}$ $\Rightarrow$ $\boxed{f^{-1}(x)=\frac{\sqrt{1-x}}{5\sqrt{x}},\ x\in(0,1],\quad f^{-1}(x)\in[0,+\infty)}$

Проверка:

$f(f^{-1}(x))=\frac{1}{1+25\left(\frac{\sqrt{1-x}}{5\sqrt{x}}\right)^2}=\frac{1}{\frac{\cancel{x}}{\cancel{x}}+\frac{1-x}{x}}=\frac{x}{x-x+1}=x$ V

$f^{-1}(f(x))=\frac{\sqrt{1-1/1+25x^2}}{5\sqrt{1/1+25x^2}}=\sqrt{\frac{\frac{\cancel{1}+25x^2-\cancel{1}}{\cancel{1+25x^2}}}{\frac{25}{\cancel{1+25x^2}}}}=\sqrt{\frac{\cancel{25}x^2}{\cancel{25}}}=x$ V

б) $f(x)=\frac{5(x-32)}{9}$, $x\in\mathbb{R}$

$y=\frac{5(x-32)}{9}$, $\frac{9}{5}y=x-32$, $\frac{9}{5}y+\frac{160}{5}=x$, $x=\frac{9y+160}{5}$

$\Rightarrow$ $f^{-1}(x)=\frac{9x+160}{5}$, $x\in\mathbb{R}$

Проверка: $f(f^{-1}(x))=\frac{5.\left(\frac{9x+160}{5}-32\right)}{9}=\frac{9x+160-160}{9}=x$ V

$f^{-1}(f(x))=\frac{9\frac{5(x-32)}{9}+160}{5}=\frac{5x-160+160}{5}=x$ V

$\boxed{\text{Отг. } f^{-1}: {}^\circ C\to{}^\circ F;\ f^{-1}(x)=\frac{9x+160}{5}.}$

Зад. 3 $\lim\limits_{n\to\infty}\left(\frac{n^3+5n^2}{n^2+1}-\overset{n^2+1}{\smile}n\right)=\lim\limits_{n\to\infty}\left(\frac{n^3+5n^2-n^3-n}{n^2+1}\right)=$

$$=\lim_{n\to\infty}\frac{5n^2-n}{n^2+1}=\frac{5}{1}=\underset{\sim}{5}$$

$\lim\limits_{n\to\infty}\frac{(n^{12}+7n^5+1)^3}{(n^4+n^3-n^2-n)^{10}}=0$; най-високата степен в числителя е по-малка от тази в знаменателя $36<40$.

$$\lim_{n\to\infty}\left(\frac{2^n+4}{2^n+2}\right)^n=\lim_{n\to\infty}\left(\frac{2^n+2+2}{2^n+2}\right)^n=\lim_{n\to\infty}\left(1+\frac{2}{2^n+2}\right)^n=\lim_{n\to\infty}\left(1+\frac{\frac{2n}{2^n+2}}{n}\right)^n=$$

$$=e^{\lim\limits_{n\to\infty}\frac{2n}{2^n+2}}=e^0=\underset{\sim}{1}$$

$\lim\limits_{n\to\infty}\frac{2n}{2^n+2}=0$, $2n<<2^n$

$$\lim_{n\to\infty}\left(\frac{n^2+3n+5}{n^2+10n}\right)^{6n+42}=\lim_{n\to\infty}\left(1+\frac{-7n+5}{n^2+10n}\right)^{6n+42}=$$

$$=\lim_{n\to\infty}\left(1+\frac{\frac{(-7n+5)6n}{n^2+10n}}{6n}\right)^{6n}\cdot\underbrace{\lim_{n\to\infty}\left(1+\frac{-7n+5}{n^2+10n}\right)^{42}}_{\downarrow\atop 1}=$$

$$=\lim_{n\to\infty}\left(1+\frac{\frac{-42n^2+30n}{n^2+10n}}{6n}\right)^{6n}=\underset{\sim}{e^{-42}}$$

$\lim\limits_{n\to\infty}\frac{9n^2+\pi n3^n+9n!}{e4^n+12\ln(n)+6n^3}=+\infty$; $n!>>4^n>>n^3>>\ln(n)$

Зад. 4 $\quad a = 5, \quad b = 10 - 5 = 5$

$$\lim_{x \to 5} (5x + 6) = 31$$

Доказваме, че: $\forall \varepsilon > 0 \ \exists \delta_\varepsilon > 0: \ 0 < |x - 5| < \delta_\varepsilon \Rightarrow |5x + 6 - 31| < \varepsilon$

$|5x - 25| < \varepsilon$

Нека ε е произволно:

$|5x - 25| < \varepsilon$

$|5(x - 5)| < \varepsilon$

$5|x - 5| < \varepsilon \quad |:5$

$|x - 5| < \frac{\varepsilon}{5} \Rightarrow$ за $\boxed{\delta_\varepsilon \le \frac{\varepsilon}{5}}$ и $0 < |x - 5| < \delta_\varepsilon$ е изпълнено

$|5x + 6 - 31| < \varepsilon.$