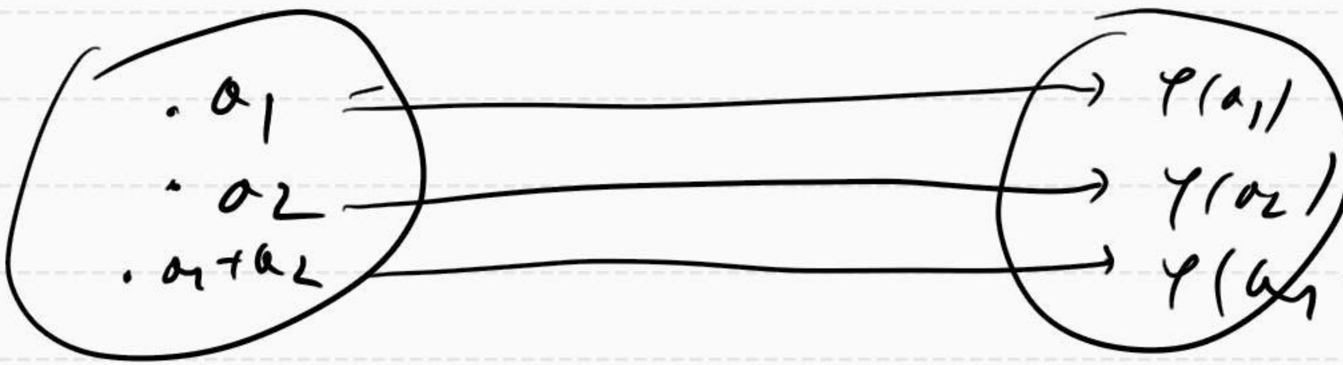
$\varphi: A \to B$

$+_A \qquad +_B$

$$\varphi(a_1 +_A a_2) = \varphi(a_1) +_B \varphi(a_2)$$



$\varphi(a_1+a_2) = \varphi(a_1) + \varphi(a_2)$

морфизъм
- гр., пр., полета — хомоморфизъм
- ЛП — линейно изображение $Hom(A,B)$

\+ биекция

изоморфизъм

$x=(x_1,\dots,x_n)\in F^n = \sum_{i=1}^n x_i e_i$

$\underbrace{v^*: F^n \to F \quad \text{ЛЧ}}_{v^*}$

Знаем $\Leftrightarrow$ знаем $v^*(e_i),\ i=1,\dots,n$, $v^*(e_i) = \alpha_i$

$v^*(x) = \sum x_i v^*(e_i) = \sum_{i=1}^n \alpha_i x_i$

$\dim V_1 = \dim V_2 \Rightarrow V_1 \cong V_2$

$e_1 \dots e_n$ – б. V_1

$f_1 \dots f_n$ – б. V_2

$\exists!\, \varphi: V_1 \to V_2 \quad (\varphi \in \mathrm{Hom}(V_1, V_2))$

$\forall i = 1,\dots,n \quad \varphi(e_i) = f_i$

$v = \sum \lambda_i e_i \in V_1$

$\varphi(v) = \sum \lambda_i f_i$ | e' – $e_1+e_2,\ e_1-e_2,\ e_3, \dots e_n$

φ': $e_1+e_2 \mapsto f_1$, $e_1-e_2 \mapsto f_2$

φ' ; $\varphi'(e_i') = \delta_i$

$e_1' = e_1 + e_2$; $e_2' = e_1 - e_2$; $e_3' = e_3$, ...

$\varphi(e_1) = f_1$
$\varphi'(e_1') = f_1$
$\varphi'(e_2') = \delta_2$

$\varphi'(e_1) = \frac{1}{2}(f_1 + f_2) \neq \varphi(e_1)$

$\Rightarrow \varphi \neq \varphi'$

$\theta : V \to V^{**}$

$v \mapsto \theta(v)$

$e_1 \dots e_n$ - б. V; $f_1 \dots f_n$ - д.б. на $\underline{\underline{e}}$ ($\in V^*$)

$g_1 \dots g_n$ - д.б на f ($\in V^{**}$)

$v = \sum \lambda_i e_i \to \theta(v) = \sum \lambda_i g_i$

Аналог., ако вземем бази e', f', g'

$v = \sum \lambda_i' e_i'$; $\theta(v) = \sum \lambda_i' g_i'$

$= \sum \lambda_i e_i$

Τότε $\sum_{i=1}^{n} \lambda_i g_i = \sum_{i=1}^{n} \lambda_i' g_i'$, ό.έ.δ.

θ να γράψουμε ισοδύναμα να δώσουμε

$((\theta(v))(v^*) = v^*(v))$

Απόδ. $\varphi : V \longrightarrow V^*$

$\sum \lambda_i e_i \longmapsto \sum \lambda_i f_i$

$\varphi' : V \longrightarrow V^*$

$\sum \lambda_i e_i' \longmapsto \sum \lambda_i f_i'$

$\varphi(v) \neq \varphi'(v)$

Pl. $V = F^n$; $v \in F^n$ $\quad v = \sum \lambda_i e_i$

$v^* \in (F^n)^*$ $\quad v^* = \sum \mu_i f_i$ $\quad$ ($\underline{f}$ – gyorsan az e)

– $v^*(v) = \sum_{i=1}^{n} \lambda_i \mu_i$

– $\theta(v) = \sum \lambda_i g_i$ $\quad$ ($\underline{g}$ – gyorsan az f)

$$\Big((\underset{\lambda}{\theta(v)})(\underset{\mu}{v^*}) = \underset{\mu}{v^*}(\underset{\lambda}{v}) \Big)$$

Тв. 1/ $\theta \in \mathrm{Hom}(V, V^{**})$ } произв. ЛП

2/ $\mathrm{Ker}\,\theta = \{0_V\}$ ($\Leftrightarrow \theta$ е инект.)

3/ Ако V е КМЛП, то θ е ЛИ

Зад. 1/ за КМЛП сме го доказали

$\theta : V \to V^{**}$

$e_i \longmapsto g_i$

Д-во (• θ е изобр. $V \to V^{**}$, т.е. $\forall v$ $\theta(v) \in V^{**}$

ЛИ $\theta(v) : V^* \to F$ /

$v^* \mapsto v^*(v) \in F$

• $\theta(v) \in \mathrm{Hom}(V^*, F)$

- $v_1^*, v_2^* \in V^*$

$$(\theta(v))(v_1^* + v_2^*) = (v_1^* + v_2^*)(v) =$$

$$= v_1^*(v) + v_2^*(v) = (\theta(v))(v_1^*) + (\theta(v))(v_2^*)$$

- $v^* \in V^*$; $\lambda \in F$

$$(\theta(v))(\underline{\lambda} v^*) = (\lambda . v^*)(v) = \lambda (v^*(v)) =$$

$$= \underline{\lambda} . (\theta(v))(v^*)$$

$\Rightarrow \theta(v)$ е ЛИ $\Rightarrow \theta$ е изобр. $\overline{\text{от}}$ $V \to V^{**}$

- θ е ЛИ

$- \; v_1, v_2 \in V \; ; \; \theta(v_1+v_2) \; ; \; v^* \in V^*$ - произв.

$$(\theta(v_1+v_2))(\underline{v^*}) = v^*(v_1+v_2) = v^*(v_1) + v^*(v_2) =$$

$$= (\theta(v_1))(v^*) + (\theta(v_2))(v^*) =$$

$$= [\theta(v_1) + \theta(v_2)](\underline{v^*}) \qquad \underline{\forall v^*} ;$$

$$\theta(v_1+v_2) = \theta(v_1) + \theta(v_2)$$

$- \; v \in V \; , \; \lambda \in F$

Аналог. $\theta(\lambda v) = \lambda \, \theta(v)$

$\Rightarrow \theta$ е ЛИ $\Rightarrow \theta \in \mathrm{Hom}(V, V^{**})$

2) $\operatorname{Ker}\theta = \{v \in V \mid \theta(v) = \mathcal{O}_{V^{**}} \in \operatorname{Hom}(V^*, F)\} =$

$= \{v \in V \mid \forall v^* \in V^* \ (\theta(v))(v^*) = 0\} =$

$= \{v \in V \mid \forall v^* \in V^* \ \ v^*(v) = 0\}$

- $\forall$ ЛП има базис
- $U \leq V \Rightarrow$ можем да допълним базис на U до базис V
  $\Uparrow$
- $U \leq V \Rightarrow \exists W \leq V : V = U \oplus W$

$\left(V = V_1 \oplus V_2 \oplus V_3 \iff \middle| \begin{array}{l} 1) \ V = V_1 + V_2 + V_3 \\ 2) \ V_1 \cap (V_2 + V_3) = V_2 \cap (V_1 + V_3) = V_3 \cap (V_1 + V_2) = \{0\} \end{array} \right.$

$V_1 + V_2 + V_3 = \{v_1 + v_2 + v_3 \mid v_1 \in V_1, v_2 \in V_2, v_3 \in V_3\}$

— $V = U \oplus W$

$$\varphi : U \to T$$
$$\psi : W \to T \quad \text{γ.α.}$$

$$\varphi \oplus \psi : V \to T \quad \text{γ.α.}$$
$$u + w \longmapsto \varphi(u) + \psi(w)$$

ευ. ορισμένη.

— $v \in V,\ v \neq 0$

$\exists\, U \leq V : \quad V = \ell(v) \oplus U$

$\varphi : \ell(v) \to F \qquad \psi : U \to F$

$v \longmapsto 1 \qquad \psi = 0_U \quad u \longmapsto 0$

$g = \varphi \oplus \psi$

$g \in \mathrm{Hom}(V, F)$

$g(v) = 1 \neq 0$