1 зад. ОКС $K = O\vec{e}_1\vec{e}_2\vec{e}_3$

$$c: \begin{cases} x^1 = \cosh q = ch q \\ x^2 = \sinh q = sh q \\ x^3 = q \end{cases}, q \in \mathbb{R}$$

Търсим $\vec{t}, \vec{n}, \vec{b}$

$\varkappa(q), \tau(q)$

$$\vec{t} = \frac{\dot{\vec{x}}}{|\dot{\vec{x}}|} = \frac{\dot{\vec{x}}}{\dot{s}}$$

$$\vec{b} = \frac{\dot{\vec{x}} \times \ddot{\vec{x}}}{|\dot{\vec{x}} \times \ddot{\vec{x}}|} \;!$$

$$\vec{n} = \vec{b} \times \vec{t}$$

$$\varkappa = \frac{|\dot{\vec{x}} \times \ddot{\vec{x}}|}{\dot{s}^3}$$

$$\tau = \frac{\langle \dot{\vec{x}}, \ddot{\vec{x}}, \dddot{\vec{x}} \rangle}{|\dot{\vec{x}} \times \ddot{\vec{x}}|^2} \;!$$

$\vec{x}(chq, shq, q)$

$\dot{\vec{x}}(shq, chq, 1)$

$\dot{s} = |\dot{\vec{x}}| = \sqrt{sh^2q + ch^2q + 1}$

$\dot{s} = \sqrt{2 \cdot ch^2q} = \sqrt{2} \cdot chq$

$\vec{t}\left(\frac{shq}{\sqrt{2}chq}, \frac{chq}{\sqrt{2}chq}, \frac{1}{\sqrt{2}chq}\right)$

$$chq = \frac{e^q + e^{-q}}{2}$$

$$shq = \frac{e^q - e^{-q}}{2}$$

$(chq)' = shq$

$(shq)' = chq$

$ch^2q - sh^2q = 1$

$sh^2q + 1 = ch^2q$

$\dot{\vec{x}}(shq, chq, 1)$

$\ddot{\vec{x}}(chq, shq, 0)$

$\dot{\vec{x}} \times \ddot{\vec{x}}(-shq, chq, -1) \Rightarrow |\dot{\vec{x}} \times \ddot{\vec{x}}| = \sqrt{sh^2q + ch^2q + 1} = \sqrt{2 \cdot ch^2q} = \sqrt{2} \cdot chq$

$\vec{b}\left(\frac{-shq}{\sqrt{2} \cdot chq}, \frac{chq}{\sqrt{2}chq}, \frac{-1}{\sqrt{2}chq}\right)$

$\vec{t}\left(\frac{shq}{\sqrt{2}chq}, \frac{chq}{\sqrt{2}chq}, \frac{1}{\sqrt{2}chq}\right)$

$\Rightarrow \vec{n} = \vec{b} \times \vec{t} = \frac{1}{2ch^2q}(2chq, 0, -2shq \cdot chq)$

$\vec{n}\left(\frac{1}{chq}, 0, -\frac{shq}{chq}\right)$

Проверка: $\begin{cases} (\vec{b} \cdot \vec{t}) \stackrel{?}{=} 0 \\ (\vec{b} \cdot \vec{n}) \stackrel{?}{=} 0 \\ (\vec{t} \cdot \vec{n}) = 0 \end{cases}$

$$\varkappa = \frac{|\dot{\vec{x}} \times \ddot{\vec{x}}|}{\dot{s}^3} = \frac{\sqrt{2} \cdot chq}{(\sqrt{2}chq)^3} = \frac{1}{2ch^2q} > 0$$

$\dot{\vec{x}}(shq, chq, 1)$

$\ddot{\vec{x}}(chq, shq, 0)$

$\dot{\vec{x}} \times \ddot{\vec{x}}(-shq, chq, -1) \Big\} \Rightarrow \langle \dot{\vec{x}}, \ddot{\vec{x}}, \dddot{\vec{x}} \rangle = -sh^2q + ch^2q = 1$

$\dddot{x}\,\dot{x}\,\ddot{x}\,(-sh q,\ ch q,\ -1)$

$\dddot{\vec{x}}\,(sh q,\ ch q,\ 0)$

$\Rightarrow \langle \dot{\vec{x}}, \ddot{\vec{x}}, \dddot{\vec{x}} \rangle = -sh^2 q + ch^2 q = 1$

$\tau(q) = \dfrac{1}{(\sqrt{2}\, ch q)^2}$

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2 зад. ОКС $K = O\vec{e}_1\vec{e}_2\vec{e}_3$

$$C: \begin{cases} x^1 = \cos^3 q \\ x^2 = \sin^3 q \\ x^3 = \cos 2q \end{cases}, \quad q \in (0; \tfrac{\pi}{2})$$

$\vec{t} = ?,\ \vec{n} = ?,\ \vec{b} = ?\quad \varkappa(q) = ?,\ \tau(q) = ?$

$$\begin{cases} \vec{t}\,' = \varkappa \cdot \vec{n}\ ! \\ \vec{n}\,' = -\varkappa \cdot \vec{t} + \tau \cdot \vec{b} \\ \vec{b}\,' = -\tau \cdot \vec{n}\quad / \cdot \vec{n} \end{cases}$$

1) $\vec{t} = \dfrac{\dot{\vec{x}}}{\dot{s}}, \quad \dot{s} = |\dot{\vec{x}}|$

2) $\vec{t}\,' = ? \Rightarrow \varkappa = |\vec{t}\,'|, \quad \vec{n} = \dfrac{\vec{t}\,'}{|\vec{t}\,'|} = \dfrac{\vec{t}\,'}{\varkappa}$

3) $\vec{b} = \vec{t} \times \vec{n}$

4) $\vec{b}\,' = ? \Rightarrow -\tau = \langle \vec{b}\,'; \vec{n} \rangle$

$\vec{x}\,(\cos^3 q,\ \sin^3 q,\ \cos 2q)$

$\dot{\vec{x}}\,(-3.\cos^2 q.\sin q,\ 3.\sin^2 q.\cos q,\ \underbrace{-2.\sin 2q}_{-4\sin q.\cos q})$ $\quad \sin 2q = 2.\sin q.\cos q$

$|\dot{\vec{x}}|^2 = \dot{s}^2 = 9\cos^4 q.\sin^2 q + 9\sin^4 q.\cos^2 q + 16\sin^2 q.\cos^2 q = \sin^2 q.\cos^2 q.25$

$\dot{s} = 5.\sin q.\cos q > 0,\ q \in (0; \tfrac{\pi}{2})$

$\vec{t} = \dfrac{\dot{\vec{x}}}{\dot{s}} \Rightarrow \vec{t}\left(\dfrac{-3\cos^2 q.\sin q}{5.\sin q.\cos q},\ \dfrac{3\sin^2 q.\cos q}{5.\sin q.\cos q},\ \dfrac{-4.\sin q.\cos q}{5.\sin q.\cos q}\right)$

$\vec{t}\left(-\tfrac{3}{5}.\cos q,\ \tfrac{3}{5}.\sin q,\ -\tfrac{4}{5}\right)$

2) $\vec{t}\,' = \dfrac{\dot{\vec{t}}}{\dot{s}} \Rightarrow \dot{\vec{t}}\left(\tfrac{3}{5}\sin q,\ \tfrac{3}{5}.\cos q,\ 0\right), \quad \dot{s} = 5.\sin q.\cos q$

$\vec{t}\,'\,(3\sin q, \quad 3\cos q \quad \ldots)$

$\vec{t}'\left(\frac{3\sin q}{25\sin q\cdot\cos q},\ \frac{3\cos q}{25\cdot\sin q\cdot\cos q},\ 0\right)$

$$\varkappa(q)=|\vec{t}'|=\left(\frac{9\sin^2 q+9\cos^2 q}{(25\cdot\sin q\cdot\cos q)^2}\right)^{\frac{1}{2}}=\left(\frac{9}{(25\cdot\sin q\cdot\cos q)^2}\right)^{\frac{1}{2}}$$

$$\varkappa(q)=\frac{3}{25\cdot\sin q\cdot\cos q}$$

$$\vec{n}=\frac{\vec{t}'}{|\vec{t}'|}=\frac{\vec{t}'}{\varkappa}$$

$\vec{t}'\left(\frac{3\sin q}{25\sin q\cdot\cos q},\ \frac{3\cos q}{25\cdot\sin q\cdot\cos q},\ 0\right)\Big/\cdot\frac{25\cdot\sin q\cdot\cos q}{3}$

$\vec{n}(\sin q,\ \cos q,\ 0)$

3) $\vec{b}=\vec{t}\times\vec{n}$

$\vec{t}\left(-\frac{3}{5}\cos q,\ \frac{3}{5}\sin q,\ -\frac{4}{5}\right)$

$\vec{n}(\sin q',\ \cos q'',\ 0''')$

$\vec{b}\left(+\frac{4}{5}\cos q,\ -\frac{4}{5}\cdot\sin q,\ -\frac{3}{5}\right)!$

4) $\tau=-\langle\vec{b}',\vec{n}\rangle$ $\quad\vec{b}'=\frac{\dot{\vec{b}}}{\dot{s}}\Rightarrow\dot{\vec{b}}\left(-\frac{4}{5}\cdot\sin q,\ -\frac{4}{5}\cos q,\ 0\right)\Big/\cdot\frac{1}{5\cdot\sin q\cdot\cos q}$

$\vec{b}'\left(\frac{-4\sin q}{25\sin q\cos q},\ \frac{-4\cos q}{25\cdot\sin q\cdot\cos q},\ 0\right)$

$\vec{n}(\sin q,\ \cos q,\ 0)$

$$\tau=-\langle\vec{b}',n\rangle=+\frac{4\cdot\sin^2 q+4\cdot\cos^2 q}{25\cdot\sin q\cdot\cos q}=\frac{4}{25\cdot\sin q\cdot\cos q}$$

Задача: $C: \begin{cases} x^1 = \cos^3 q \\ x^2 = \sin^3 q \\ x^3 = \cos 2q \end{cases}, \; q \in (0; \frac{\pi}{2})$

$K = O\vec{e}_1\vec{e}_2\vec{e}_3$

Да се намерят уравнения на линия $\bar{C}$ – геометричното място на ортогоналните проекции на т. O върху оскулачните равнини в точките на кривата C.

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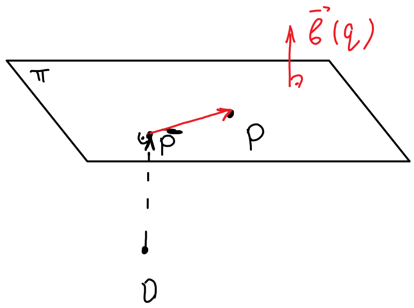

π – оскулачна р-на в т. $P \in C$

т. $P \in C \iff \vec{OP} = \vec{x}(q)$

$\vec{O\bar{P}} \perp \pi$, $\bar{P} \in \pi$

$\boxed{\vec{O\bar{P}} \Rightarrow \bar{x}^1, \bar{x}^2, \bar{x}^3 \Rightarrow \bar{C}}$ $\quad \vec{O\bar{P}} = ?$

1) $\vec{O\bar{P}} \parallel \vec{b}(q) \Rightarrow \exists!\, \alpha(q): \vec{O\bar{P}} = \alpha(q).\vec{b}(q)$ $\quad \alpha(q) = ?$

2) $\vec{P\bar{P}} \perp \vec{b}(q) \iff \langle \vec{P\bar{P}}, \vec{b} \rangle = 0$

$\vec{P\bar{P}} = \vec{O\bar{P}} - \vec{OP} = \alpha.\vec{b} - \vec{OP} \Rightarrow \langle \alpha.\vec{b} - \vec{OP}, \vec{b} \rangle = 0$

$\alpha.\underbrace{\vec{b}^2}_{1} - \langle \vec{OP}, \vec{b} \rangle = 0$

$\alpha(q) = \langle \vec{OP}, \vec{b} \rangle$

$\vec{OP}(\cos^3 q, \sin^3 q, \cos 2q)$

$\vec{b}\left(\frac{4}{5}\cos q, -\frac{4}{5}\cdot\sin q, -\frac{3}{5}\right)$ $\Rightarrow \alpha(q) = \frac{4}{5}\cdot\cos^4 q - \frac{4}{5}\cdot\sin^4 q - \frac{3}{5}\cdot\cos 2q$

$\alpha(q) = \frac{4}{5}\cdot\underbrace{(\cos^2 q + \sin^2 q)}_{1}.\underbrace{(\cos^2 q - \sin^2 q)}_{\cos 2q} - \frac{3}{5}\cdot\cos 2q$

$\alpha(q) = \frac{\cos 2q}{5} \rightarrow \vec{O\bar{P}} = \frac{\cos 2q}{5}.\vec{b}(q)$

$\vec{O\bar{P}}\left(\frac{4\cos q.\cos 2q}{25}, -\frac{4\sin q.\cos 2q}{25}, -\frac{3\cos 2q}{25}\right)$

$\bar{C} \begin{cases} \bar{x}^1(q) = \frac{4}{25}\cos q.\cos 2q \\ \bar{x}^2(q) = \frac{-4}{25}\cdot\sin q.\cos 2q \\ \bar{x}^3(q) = \frac{-3}{25}\cdot\cos 2q \end{cases}, \; q \in (0; \frac{\pi}{2})$

Задача:

ОКС

$K = O\vec{e}_1\vec{e}_2\vec{e}_3$

$C: \begin{cases} x^1 = a.(q - \sin q) \\ x^2 = a.(1 - \cos q), q \in \mathbb{R} \\ x^3 = 4a \;\; \sin\frac{q}{2}, \;\; a > 0 \end{cases}$

OXC

$K = O\vec e_1 \vec e_2 \vec e_3$

$$c: \begin{cases} x^1 = a.(q - \sin q) \\ x^2 = a.(1-\cos q), \; q \in \mathbb{R} \\ x^3 = 4.a.\sin q/2 \quad a>0 \end{cases}$$

От всяка точка P на линията c по главната нормала n към вдлъбнатата част на c е нанесена отсечка $P\bar P$ с дължина $d = 4a^2.\varkappa$. Да се намерят уравнения на линията $\bar c$, описана от точките $\bar P$. Да се докаже, че $\bar c$ е равнинна линия и да се намери уравнение на равнината, която я съдържа.

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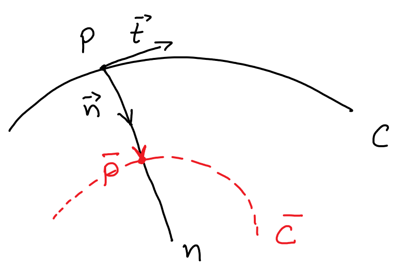
Търсим коорд. на $\overrightarrow{O\bar P}(\bar x^1, \bar x^2, \bar x^3) \to \bar c$



$\overrightarrow{P\bar P} \uparrow\uparrow \vec n$, $|\overrightarrow{P\bar P}| = 4a^2.\varkappa(q)$

$\overrightarrow{P\bar P} = 4.a^2.\varkappa.\vec n$

$\overrightarrow{O\bar P} - \overrightarrow{OP} = 4a^2.\varkappa.\vec n$

$\overrightarrow{O\bar P} = \overrightarrow{OP} + 4a^2 \underbrace{\boxed{\varkappa.\vec n}}_{\vec t\,'}$ $\qquad \vec t\,' = \varkappa.\vec n$

$\overrightarrow{OP} = \vec x\,(a.(q-\sin q),\; a.(1-\cos q),\; 4a.\sin\frac{q}{2})$ $\qquad \dot{\vec x} \Rightarrow |\dot{\vec x}| \Rightarrow \vec t = \frac{\dot{\vec x}}{\dot s}$

$\dot{\vec t} \Rightarrow \vec t\,' = \frac{\dot{\vec t}}{\dot s} \Rightarrow \overrightarrow{O\bar P}$

$\dot{\vec x}\,(a(1-\cos q),\; a.\sin q,\; 2a.\cos\frac{q}{2})$

$\dot s^2 = |\dot{\vec x}|^2 = a^2.\left[(1-\cos q)^2 + \sin^2 q + 4\cos^2\frac{q}{2}\right] = a^2.\left[1 - 2\cos q + \cos^2 q + \sin^2 q + 2.(1+\cos q)\right]$

$2\cos^2\frac{q}{2} = 1+\cos q$

$\dot s^2 = a^2.4 \Rightarrow \dot s = 2a$

$\vec t = \frac{\dot{\vec x}}{2a} \Rightarrow \vec t\left(\frac{1-\cos q}{2},\; \frac{\sin q}{2},\; \cos q/2\right)$

$\dot{\vec t}\left(\frac{\sin q}{2},\; \frac{\cos q}{2},\; -\frac{\sin q/2}{2}\right) \Rightarrow \vec t\,' = \frac{\dot{\vec t}}{\dot s} = \frac{\dot{\vec t}}{2a}$

$\vec t\,'\left(\frac{\sin q}{4a},\; \frac{\cos q}{4a},\; -\frac{\sin q/2}{4a}\right)$

$\overrightarrow{O\bar P} = \overrightarrow{OP} + 4a^2.\vec t\,' \Rightarrow$

$$\bar c \begin{cases} \bar x^1 = a.(q - \sin q) + 4a^2.\frac{\sin q}{4a} = a.q \\ \bar x^2 = a.(1-\cos q) + 4a^2.\frac{\cos q}{\ldots} = a. \ldots \end{cases}$$

$$\bar{C} \begin{cases} \bar{x}^2 = a.(1-\cos q) + 4a^2 . \dfrac{\cos q}{4a} = a \\ \bar{x}^3 = 4a.\sin \frac{q}{2} - 4a^2 . \dfrac{\sin q/2}{4a} = 3a.\sin q/2 \end{cases} \quad q \in \mathbb{R}$$

От $\bar{x}^2 = a = const. \Rightarrow \bar{C}$ е равнинна и изцяло лежи в равнината с уравнение $\bar{x}^2 = a$.