Метрични канонични уравнения на кривите от II степен - задачи

1 зад. ОКС $K = Oxy$

$k: \underline{9x^2 - 24xy + 16y^2} - 10x - 70y + 125 = 0$

Да се намери метрично канонично уравнение на кривата k и последователните координатни транформации, водещи до него.

I) E_2^*, ? брой безкрайни точки, брой особени точки

$k: \underbrace{9x^2 - 24.xy + 16y^2} - 10x.t - 70.y.t + 125.t^2 = 0$

1) $k \cap \omega = ?$
$\omega: t = 0$ $\Rightarrow$ $9x^2 - 24xy + 16y^2 = 0$

$a_{11} = 9 \quad a_{12} = -12 \quad a_{22} = 16$

$D = a_{12}^2 - a_{11}.a_{22} = (-12)^2 - 9.16 = 0$

k е от параболичен тип

2) $\det A = \begin{vmatrix} 9 & -12 & -5 \\ -12 & 16 & -35 \\ -5 & -35 & 125 \end{vmatrix} = \dots \neq 0 \Rightarrow$ k е парабола

$y^2 = 2p.x \qquad x^2 = 2.p.y$

II)

$k: 9x^2 - 24xy + 16y^2 - 10x - 70y + 125 = 0$

$A_1 = \begin{pmatrix} 9 & -12 \\ -12 & 16 \end{pmatrix}$ Търсим собст. стойности s_1 и s_2 на A_1
Търсим собст. вектори $\vec{b}_1$ и $\vec{b}_2$

1) $|A_1 - s.E| = 0$ $\quad \begin{vmatrix} 9-s & -12 \\ -12 & 16-s \end{vmatrix} = 0$

$(9-s).(16-s) - 144 = 0$

$\cancel{144} - 25.s + s^2 - \cancel{144} = 0$

$s.(s-25) = 0 \quad \Rightarrow \quad s_1 = 0 \qquad s_2 = 25$

2) За $s_1 = 0 \Rightarrow \vec{b}_1(\alpha_1, \beta_1)$ $\quad |\vec{b}_1| = 1 \Leftrightarrow \alpha_1^2 + \beta_1^2 = 1$

$$\begin{pmatrix} 9-0 & -12 \\ -12 & 16-0 \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \left| \begin{array}{l} 9\alpha_1 - 12\beta_1 = 0 \Rightarrow \alpha_1 = \frac{4}{3}\beta_1 \\ \alpha_1^2 + \beta_1^2 = 1 \end{array} \right.$$

$$\left| \begin{array}{l} \alpha_1 = \frac{4}{3}\beta_1 \\ \left(\frac{4}{3}\beta_1\right)^2 + \beta_1^2 = 1 \end{array} \right. \Rightarrow \left| \begin{array}{l} \alpha_1 = \frac{4}{5} \\ \beta_1 = +\frac{3}{5} \end{array} \right.$$

За $s_1 = 0$, $\vec{b}_1\left(\frac{4}{5}, \frac{3}{5}\right)$

За $s_2 = 25$ $\quad \vec{b}_2(\alpha_2, \beta_2)$, $|\vec{b}_2| = 1$

$$\begin{pmatrix} 9-25 & -12 \\ -12 & 16-25 \end{pmatrix} \cdot \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \left| \begin{array}{l} -16\alpha_2 - 12\beta_2 = 0 \\ \alpha_2^2 + \beta_2^2 = 1 \end{array} \right. \quad \left| \begin{array}{l} \alpha_2 = -\frac{3}{4}\cdot\beta_2 \\ \left(-\frac{3}{4}\beta_2\right)^2 + \beta_2^2 = 1 \end{array} \right.$$

За $s_1 = 0$, $\vec{b}_1\left(\frac{4}{5}, \frac{3}{5}\right)$ За $s_2 = 25$, $\vec{b}_2\left(-\frac{3}{5}, \frac{4}{5}\right)$

Изв. смяна на ОКС $K = Oxy \xrightarrow{T_1} K' = Ox'y'$:

$Ox' \uparrow\uparrow \vec{b}_1 \to s_1 \cdot x'^2$

$Oy' \uparrow\uparrow \vec{b}_2 \to s_2 \cdot y'^2$

$$T_1: \begin{cases} x = \frac{4}{5}\cdot x' - \frac{3}{5}\cdot y' \\ y = \frac{3}{5}\cdot x' + \frac{4}{5}\cdot y' \end{cases} \qquad A_1 = \begin{pmatrix} 9 & -12 \\ -12 & 16 \end{pmatrix} \to A_1' = \begin{pmatrix} 0 & 0 \\ 0 & 25 \end{pmatrix}$$

$$\text{Ур. } K': \underset{\substack{\parallel \\ s_1}}{0}\cdot x'^2 + 0\cdot x'y' + 25\cdot y'^2 - 10\cdot\underbrace{\left(\frac{4}{5}\cdot x' - \frac{3}{5}\cdot y'\right)}_{\text{от } T_1} - 70\cdot\underbrace{\left(\frac{3}{5}\cdot x' + \frac{4}{5}\cdot y'\right)}_{\text{от } T_1} + 125 = 0$$

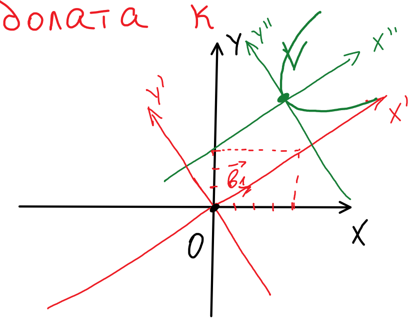
$K: 9x^2 - 24xy + 16y^2 - 10x - 70y + 125 = 0$ – ур. K

$\text{Ур. } K': 25\cdot y'^2 - 50\cdot x' - 50\cdot y' + 125 = 0 \quad | : 25$

$K: \boxed{y'^2 - 2x'} - 2y' + 5 = 0 \quad (*)$

III Търсим коорд. на $V(\alpha, \beta)$ – връх на параболата K $Y\ Y''$ X''

III Търсим коорд. на $V(\alpha,\beta)$ – връх на параболата k



От $K' = Ox'y' \xrightarrow{T_2} K'' = Vx''y''$

$$T_2: \begin{cases} x' = x'' + \alpha \\ y' = y'' + \beta \end{cases}$$

$V(\alpha,\beta)$ спр. K'

$T_2 \to (*)$

Спр. $K'' \Rightarrow k: (y''+\beta)^2 - 2.(x''+\alpha) - 2.(y''+\beta) + \cancel{125}\ 5 = 0$

$k: y''^2 + 2\beta.y'' + \beta^2 - 2x'' - 2\alpha - 2y'' - 2\beta + \cancel{125}\ 5 = 0$

$k: y''^2 - 2x'' + y''.\underbrace{(2\beta-2)}_{0} + \underbrace{\beta^2 - 2\alpha - 2\beta + \cancel{125}\ 5}_{0} = 0$

$\alpha = ?$ $\beta = ?$:
$$\left| \begin{array}{l} 2\beta - 2 = 0 \Rightarrow \beta = 1 \\ \beta^2 - 2\alpha - 2\beta + 5 = 0 \Rightarrow 1 - 2\alpha - 2 + 5 = 0 \\ \qquad\qquad\qquad\qquad\qquad \alpha = 2 \end{array} \right.$$

$V(1,2)$ спр. K' , $V(0,0)$ спр. K''

$V(\underset{x'}{1},\underset{y'}{2}) \xrightarrow{T_1} V$ спр. K

$k: y''^2 = 2x''$ $\qquad y^2 = 2p.x$

$p = 1$ $\qquad F(\frac{p}{2}, 0)$

$F(\frac{1}{2}, 0)$ спр. $K' \xrightarrow{T_2} \qquad \xrightarrow{T_1} F$ спр. K (Упр.)

2 задача: ОКС $K = Oxy$

$k: 6xy + 8y^2 - 12x - 26y + 11 = 0$

I E_2^*, безкр. точки + особени точки

$k: 6xy + 8.y^2 - 12xt - 26yt + 11t^2 = 0$

1) $D = a_{12}^2 - a_{11}.a_{22} = (3)^2 - 0.8 = 9 > 0 \Rightarrow k$ е от хиперболичен тип

2) $\det A = \begin{vmatrix} 0 & 3 & -6 \\ 3 & 8 & -13 \\ -6 & -13 & 11 \end{vmatrix} = 81 \neq 0 \Rightarrow K$ е хипербола

$K: \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

II $K: 6xy + 8y^2 - 12x - 26y + 11 = 0$ спр. K

1) $A_1 = \begin{pmatrix} 0 & 3 \\ 3 & 8 \end{pmatrix} \Rightarrow$ s_1 и s_2, $\vec{b}_1$ и $\vec{b}_2$

$\begin{vmatrix} 0-s & 3 \\ 3 & 8-s \end{vmatrix} = 0$ $\quad -s(8-s) - 9 = 0$

$s^2 - 8s - 9 = 0 \Rightarrow s_1 = 9 \qquad s_2 = -1$

За $s_1 = 9$ $\quad \vec{b}_1(\alpha_1, \beta_1)$, $|\vec{b}_1| = 1$

$\begin{pmatrix} 0-9 & 3 \\ 3 & 8-9 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{vmatrix} 3\alpha_1 - \beta_1 = 0 \\ \alpha_1^2 + \beta_1^2 = 1 \end{vmatrix} \quad \begin{vmatrix} \alpha_1 = \frac{1}{\sqrt{10}} \\ \beta_1 = \frac{3}{\sqrt{10}} \end{vmatrix}$

За $s_2 = -1 \Rightarrow \vec{b}_2(\alpha_2, \beta_2)$, $|\vec{b}_2| = 1$

$\begin{pmatrix} 0+1 & 3 \\ 3 & 8+1 \end{pmatrix} \cdot \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{vmatrix} \alpha_2 + 3\beta_2 = 0 \\ \alpha_2^2 + \beta_2^2 = 1 \end{vmatrix}$

За $s_2 = -1 \Rightarrow \vec{b}_2\left(\frac{-3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right)$ За $s_1 = 9 \Rightarrow \vec{b}_1\left(\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right)$

2) Изб. смяна на ОКС $K = Oxy \xrightarrow{T_1} K' = Ox'y'$

$Ox' \uparrow\uparrow \vec{b}_1 \to s_1 \cdot x'^2$

$Oy' \uparrow\uparrow \vec{b}_2 \to s_2 \cdot y'^2$

$T_1: \begin{cases} x = \frac{1}{\sqrt{10}} \cdot x' - \frac{3}{\sqrt{10}} \cdot y' \\ y = \frac{3}{\sqrt{10}} \cdot x' + \frac{1}{\sqrt{10}} \cdot y' \end{cases}$

Спр. K: $A_1 = \begin{pmatrix} 0 & 3 \\ 3 & 8 \end{pmatrix}$ $\qquad$ Спр. K': $A_1' = \begin{pmatrix} 9 & 0 \\ 0 & -1 \end{pmatrix}$

Спр. K' $k: 9 \cdot x'^2 + 0 \cdot x'y' - 1 \cdot y'^2 - 12 \cdot \left(\frac{x'-3y'}{\sqrt{10}}\right) - 26 \cdot \left(\frac{3x'+4y'}{\sqrt{10}}\right) + 11 = 0$

$k: 9x'^2 - y'^2 - 9\sqrt{10} \cdot x' + \sqrt{10} \cdot y' + 11 = 0 \quad (*)$

III Търсим коорд. на т. $C(\alpha, \beta)$ – центърът на k

$K' = Ox'y' \xrightarrow{T_2} K'' = Cx''y'' : \quad Cx'' \uparrow\uparrow Ox'$

$Cy'' \uparrow\uparrow Oy'$

$$T_2 : \begin{cases} x' = x'' + \alpha \\ y' = y'' + \beta \end{cases} \to (*)$$

$k: 9 \cdot (x''+\alpha)^2 - (y''+\beta)^2 - 9\sqrt{10} \cdot (x''+\alpha) + \sqrt{10} \cdot (y''+\beta) + 11 = 0$

$k: 9x''^2 + 18\alpha \cdot x'' + 9\alpha^2 - y''^2 - 2\beta \cdot y'' - \beta^2 - 9\sqrt{10} \cdot x'' - 9\sqrt{10} \cdot \alpha + \sqrt{10} \cdot y'' + \sqrt{10} \cdot \beta + 11 = 0$

$k: 9x''^2 - y''^2 + x'' \cdot \underbrace{(18\alpha - 9\sqrt{10})}_{0} + y'' \underbrace{(-2\beta + \sqrt{10})}_{0} + \underbrace{9\alpha^2 - \beta^2 - 9\sqrt{10}\alpha + \sqrt{10} \cdot \beta + 11}_{} = 0$

$$\left| \begin{array}{l} \alpha = \frac{\sqrt{10}}{2} \\ \beta = \frac{\sqrt{10}}{2} \end{array} \right. \qquad 9 \cdot \frac{10}{4} - \frac{10}{4} - 9 \cdot \sqrt{10} \cdot \frac{\sqrt{10}}{2} + \sqrt{10} \cdot \frac{\sqrt{10}}{2} + 11 = 20 - 45 + 5 + 11 = -9$$

$k: 9x''^2 - y''^2 - 9 = 0 \quad | : 9$

$k: \frac{x''^2}{1^2} - \frac{y''^2}{3^2} = 1 \qquad a = 1, \; b = 3$

Упр. Координати на върхове и фокуси спр. K
Уравнения на директриси и върхови допирателни.