Algorithm and Data Structure Analysis (ADSA)

Lecture 2: Integer Arithmetics (Book Chapter 1)

Overview

- School Method of Addition
- School Method of Multiplication

What is an algorithm?

- An algorithm is a step by step list of directions that need to be followed to solve a problem.
- Algorithms are often used to describe how a computer might solve a problem.
- A recipe can be a type of algorithm. It tells what ingredients are needed to make the dish and what steps to follow. If the recipe tells exactly what to do without too much confusion, then it is an algorithm.

From Wikipedia

Integer Arithmetics

We want to have algorithms to carry out

- Addition
- Multiplication

of two numbers.

Recall what you have learned in school!

Example for Addition

$$a = 1289,\ b = 1342,\ B = 10$$

$$\begin{array}{llr} a & & 1289 \\ b & + & 1342 \\ c & & 00110 \\ s & = & 02631 \end{array}$$

Representation

- Want to investigate integer arithmetics
- Assume that integers are represented as digit strings
- Base B number system, B>1
- Digits 0, 1, …, B-1

$$a_{n-1}a_{n-2}\ldots a_1a_0 \qquad \text{represents}$$

$$\sum_{i=0}^{n-1} a_i B^i$$

Store in an array a of size n containing n digits.

Examples

- B=2

  "10101" represents

$$1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 = 21$$

- B=10

  "924" represents

$$9 \cdot 10^2 + 2 \cdot 10^1 + 4 \cdot 10^0 = 924$$

Primitive Operations

Assume that we have access to

- Addition of 3 digits with a 2 digits result (full adder)
- Multiplication of 2 digits with 2 digits result

Example (B=10):

Addition

$$\begin{array}{r} 3 \\ 5 \\ 5 \\ \hline 13 \end{array}$$

Multiplication

$$6 \cdot 7 = 42$$

Example for Addition

$$a = 1289,\ b = 1342,\ B = 10$$

$$\begin{array}{llr} a & & 1289 \\ b & + & 1342 \\ c & & 00110 \\ s & = & 02631 \end{array}$$

School Method for Addition

Input: Two integers $a = (a_{n-1} \cdots a_0)$ and $b = (b_{n-1} \cdots b_0)$
Compute: $s = a + b$,
where $s = (s_n \cdots s_0)$ is an $n + 1$ digit integer.

$$\begin{array}{cccccc}
 & a_{n-1} & \ldots & a_1 & a_0 & \text{first operand} \\
 & b_{n-1} & \ldots & b_1 & b_0 & \text{second operand} \\
c_n & c_{n-1} & \ldots & c_1 & 0 & \text{carries} \\
\hline
s_n & s_{n-1} & \ldots & s_1 & s_0 & \text{sum}
\end{array}$$

$c_n \cdots c_0$ is sequence of carries.

$c_0 = 0$, $c_{i+1} \cdot B + s_i = a_i + b_i + c_i$, $0 \leq i < n$,
$s_n = c_n$

Addition

Pseudo-code:

$c = 0;$

for $(i = 0,\ i < n,\ i++)\{$

add a_i, b_i and c to form s_i and a new carry c $\}$;

$s_n := c$

Theorem:

The addition of two n-digit integers requires exactly n primitive operations. The result is an $n+1$ digit integer.

Example for Multiplication

$$a = 342,\ b = 26,\ B = 10$$

1. Compute partial products

$$a \cdot b_1 = 342 \cdot 2$$
$$p_1 = 684$$

$$a \cdot b_0 = 342 \cdot 6$$
$$p_0 = 2052$$

2. Sum up aligned partial products

$$\begin{array}{r} 6840 \\ +2052 \\ = 8892 \end{array}$$

School Method for Multiplication

Input: Two integers $a = (a_{n-1} \cdots a_0)$ and $b = (b_{n-1} \cdots b_0)$

Compute: $p = a \cdot b$,

where $p = (p_{2n-1} \cdots p_0)$ is a $2n$ digit number.

Procedure:

1. Multiply n-digit integer a by a one-digit integer b_j to obtain partial product p_j, $0 \leq j \leq n-1$.

2. Sum up the aligned products $p_j \cdot B^j$.

$$p_j = a \cdot b_j$$

Compute for each i, $0 \leq i \leq n-1$, c_i and d_i such that

$$a_i \cdot b_j = c_i \cdot B + d_i$$

Form two integers $c = (c_{n-1} \cdots c_0 0)$ and $d = (d_{n-1} \cdots d_0)$.

Add c and d to obtain $p_j = a \cdot b_j$.

$$\begin{array}{ccccccc} c_{n-1} & c_{n-2} & \ldots & c_i & c_{i-1} & \ldots & c_0 \; 0 \\ & d_{n-1} & \ldots & d_{i+1} & d_i & \ldots & d_1 \; d_0 \\ \hline \multicolumn{7}{c}{\text{sum of } c \text{ and } d} \end{array}$$

Number of primitive operations:

n multiplications, n+1 additions, 2n+1 in total

Example

$$a = 342,\ b_j = 6,\ B = 10$$

Computing c’s and d’s:

$$a_0 \cdot b_j = 2 \cdot 6 = 12$$
$$c_0 = 1, d_0 = 2$$

$$a_1 \cdot b_j = 4 \cdot 6 = 24$$
$$c_1 = 2, d_1 = 4$$

$$a_2 \cdot b_j = 3 \cdot 6 = 18$$
$$c_2 = 1,\ d_2 = 8$$

Summing up:

$$\begin{array}{rr} & 1210 \\ + & 842 \\ = & 2052 \end{array}$$

School Method for Multiplication

Pseudo-code:

$$p = 0; \text{ for } (j = 0,\ j < n,\ \text{j++})\ \{$$
$$p = p + a \cdot b_j \cdot B^j\ \}$$

Theorem:

The school method multiplies two n-digit integers with $3n^2 + 2n = \Theta(n^2)$ primitive operations.

Analysis Multiplication

Number of primitive operations:

- $2n + 1$ operations for each p_j, $0 \leq j < n$,
  $\Longrightarrow 2n^2 + n$ operations.
- n summations of numbers having n-digits that are nonzero
  $\Longrightarrow n^2 + n$ operations.
- In total $3n^2 + 2n$ operations.

Summary

- Addition can be done using n primitive operations
- School method for multiplication needs $\Theta(n^2)$ primitive operations.
- Question: Are there faster algorithms for multiplication?
- Reading: Mehlhorn/Sanders ch 1.1/1.2