# Lec7: Knowledge and logic reasoning 3: Inference in First-Order Logic

# Outline

- Grounded inference:
  - Turn FOL to propositional inference by assigning constants to variables (instantiating variables)
- Lifted inference
  - Unification
    - Making substitutions to make different statements look identical
  - Lifted Resolution

# Basic Setup

- We focus on a set of 1st-order clauses.
- All variables are universally quantified.
- Many knowledge bases can be converted to this format.
- Existential quantifiers are eliminated using function symbols

  ⇨Quantifier elimination, Skolemization.

# Two Basic Ideas for Inference in FOL

1. Grounding:
   I. Treat first-order sentences as a **template**.
   II. Instantiating all variables with all possible constants gives a set of ground propositional clauses.
   III. Apply efficient propositional solvers.

2. Lifted Inference:
   1. Generalize propositional methods for 1st-order methods.
   2. Unification: recognize instances of variables where necessary.

# Grounding: Universal instantiation (UI)

- Notation: Subst({v/g}, α) means the result of substituting g for v in sentence α

- Every instantiation of a universally quantified sentence is entailed by it:

$$\frac{\forall v\ \alpha}{\text{Subst}(\{v/g\}, \alpha)}$$

for any variable $v$ and ground term $g$

- E.g., ∀x $King(x) \wedge Greedy(x) \Rightarrow Evil(x)$ yields

$King(John) \wedge Greedy(John) \Rightarrow Evil(John)$, ***{x/John}***

$King(Richard) \wedge Greedy(Richard) \Rightarrow Evil(Richard)$, ***{x/Richard}***

$King(Father(John)) \wedge Greedy(Father(John)) \Rightarrow Evil(Father(John))$, ***{x/Father(John)}***

# Turn to propositional form

Suppose the KB contains the following:

$\forall x\ King(x) \wedge Greedy(x) \Rightarrow Evil(x)$
Father(x)
King(John)
Greedy(John)
Brother(Richard,John)

- Instantiating the universal sentence in all possible ways, we have:
  King(John) ∧ Greedy(John) ⇒ Evil(John)
  King(Richard) ∧ Greedy(Richard) ⇒ Evil(Richard)
  King(John)
  Greedy(John)
  Brother(Richard,John)

- The new KB is propositionalized: propositional symbols are
  - King(John), Greedy(John), Evil(John), King(Richard), etc

- Every FOL KB can be propositionalized so as to preserve entailment
  - A ground sentence is entailed by new KB iff entailed by original KB
- Idea for doing inference in FOL:
  - propositionalize KB and query
  - apply resolution-based inference
  - return result
- Problem: with function symbols, there are infinitely many ground terms,
  - e.g., *Father*(*Father*(*Father*(*John*))), etc

Theorem: Herbrand (1930). If a sentence α is entailed by a FOL KB, it is entailed by a finite subset of the propositionalized KB

Idea: For $n = 0$ to $\infty$ do
create a propositional KB by instantiating with depth-$n$ terms
see if α is entailed by this KB

Theorem: Turing (1936), Church (1936) Entailment for FOL is semidecidable (algorithms exist that say yes to every entailed sentence, but no algorithm exists that also says no to every non-entailed sentence.)

Example

$\forall x\ King(x) \wedge Greedy(x) \Rightarrow Evil(x)$
King(John)
$\forall x\ Greedy(x)$
Brother(Richard,John)

Query Evil(X)?

# Issues with grounding

1. Problem: works if α is entailed, loops if α is not entailed

2. Propositionalization generates lots of irrelevant sentences
   – So inference may be very inefficient.

3. With $p$ $k$-ary predicates and $n$ constants, there are $p \cdot n^k$ instantiations.

- Alternative: do inference directly with FOL sentences

# Unification

- Recall: Subst(θ, p) = result of substituting θ into sentence p

- Unify algorithm: takes 2 sentences p and q and returns a unifier if one exists.
  Unify(p,q) = θ  where Subst(θ, p) = Subst(θ, q)

- Example:
  p = Knows(John,x)
  q = Knows(John, Jane)

  Unify(p,q) = {x/Jane}

# Unification examples

- simple example: query = Knows(John,x), i.e., who does John know?

| p | q | θ |
|---|---|---|
| Knows(John,x) | Knows(John,Jane) | {x/Jane} |
| Knows(John,x) | Knows(y,OJ) | {x/OJ,y/John} |
| Knows(John,x) | Knows(y,Mother(y)) | {y/John,x/Mother(John)} |
| Knows(John,x) | Knows(x,OJ) | {fail} |

- Last unification fails: only because x can't take values John and OJ at the same time
- Problem is due to use of same variable x in both sentences
- Simple solution: Standardizing apart eliminates overlap of variables, e.g., Knows(z,OJ)

# Unification

- To unify *Knows(John,x)* and *Knows(y,z)*,

  θ = {y/John, x/z } or θ = {y/John, x/John, z/John}

- The first unifier is more general than the second.

- **Theorem:** There is a single most general unifier (MGU) that is unique up to renaming of variables.

  MGU = { y/John, x/z }

- General algorithm in Figure 9.1 in the textbook

# Recall our example...

$\forall x$ King(x) $\wedge$ Greedy(x) $\Rightarrow$ Evil(x)

King(John)

$\forall y$ Greedy(y)

Brother(Richard,John)

- We would like to infer Evil(John) without propositionalization.
- Basic Idea: Use Modus Ponens, Resolution when literals **unify.**

# Generalized Modus Ponens (GMP)

$p_1', p_2', \ldots, p_n', ( p_1 \wedge p_2 \wedge \ldots \wedge p_n \Rightarrow q)$

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$Subst(\theta,q)$

where we can unify $p_i'$ and $p_i$ for all $i$

Example:

*King*(*John*), *Greedy*(*John*) ,$\forall x$ King(x) $\wedge$ Greedy(x) $\Rightarrow$ Evil(x)

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*Evil*(*John*)

$p_1'$ is *King*(*John*) $p_1$ is *King*(*x*)

$p_2'$ is *Greedy*(*John*) $p_2$ is *Greedy*(*x*)

$\theta$ is {x/John} q is *Evil*(*x*)

Subst($\theta$,q) is *Evil*(*John*)

# Resolution in FOL

- Full first-order version:

$$\frac{\ell_1 \vee \cdots \vee \ell_k, \qquad m_1 \vee \cdots \vee m_n}{\text{Subst}(\theta, \ell_1 \vee \cdots \vee \ell_{i-1} \vee \ell_{i+1} \vee \cdots \vee \ell_k \vee m_1 \vee \cdots \vee m_{j-1} \vee m_{j+1} \vee \cdots \vee m_n)}$$

where $\texttt{Unify}(\ell_i, \neg m_j) = \theta$.

- The two clauses are assumed to be standardized apart so that they share no variables.

- For example,

$$\frac{\neg \mathit{Rich}(x) \vee \mathit{Unhappy}(x) \qquad \mathit{Rich}(\mathit{Ken})}{\mathit{Unhappy}(\mathit{Ken})}$$

with $\theta = \{x/\text{Ken}\}$

- Apply resolution steps to CNF(KB $\wedge \neg \alpha$); complete for FOL.

# Knowledge Base in FOL

- The law says that it is a crime for an American to sell weapons to hostile nations. The country Nono, an enemy of America, has some missiles, and all of its missiles were sold to it by Colonel West, who is American.
- Exercise: Formulate this knowledge in FOL.

# Knowledge Base in FOL

- The law says that it is a crime for an American to sell weapons to hostile nations. The country Nono, an enemy of America, has some missiles, and all of its missiles were sold to it by Colonel West, who is American.

… it is a crime for an American to sell weapons to hostile nations:

$American(x) \land Weapon(y) \land Sells(x,y,z) \land Hostile(z) \Rightarrow Criminal(x)$

Nono … has some missiles, i.e., $\exists x$ Owns(Nono,x) ∧ Missile(x):

$Owns(Nono,M_1)$ and $Missile(M_1)$

… all of its missiles were sold to it by Colonel West

$Missile(x) \land Owns(Nono,x) \Rightarrow Sells(West,x,Nono)$

Missiles are weapons:

$Missile(x) \Rightarrow Weapon(x)$

An enemy of America counts as "hostile":

$Enemy(x,America) \Rightarrow Hostile(x)$

West, who is American …

$American(West)$

The country Nono, an enemy of America …

$Enemy(Nono,America)$

# Example Knowledge Base in FOL (Hassan)

… it is a crime for an American to sell weapons to hostile nations:

*American(x) ∧ Weapon(y) ∧ Sells(x,y,z) ∧ Hostile(z) ⇒ Criminal(x)*

Nono … has some missiles, i.e., ∃x Owns(Nono,x) ∧ Missile(x):

*Owns(Nono,$M_1$) and Missile($M_1$)*

… all of its missiles were sold to it by Colonel West

*Missile(x) ∧ Owns(Nono,x) ⇒ Sells(West,x,Nono)*

Missiles are weapons:

*Missile(x) ⇒ Weapon(x)*

An enemy of America counts as "hostile“:

*Enemy(x,America) ⇒ Hostile(x)*

West, who is American …

*American(West)*

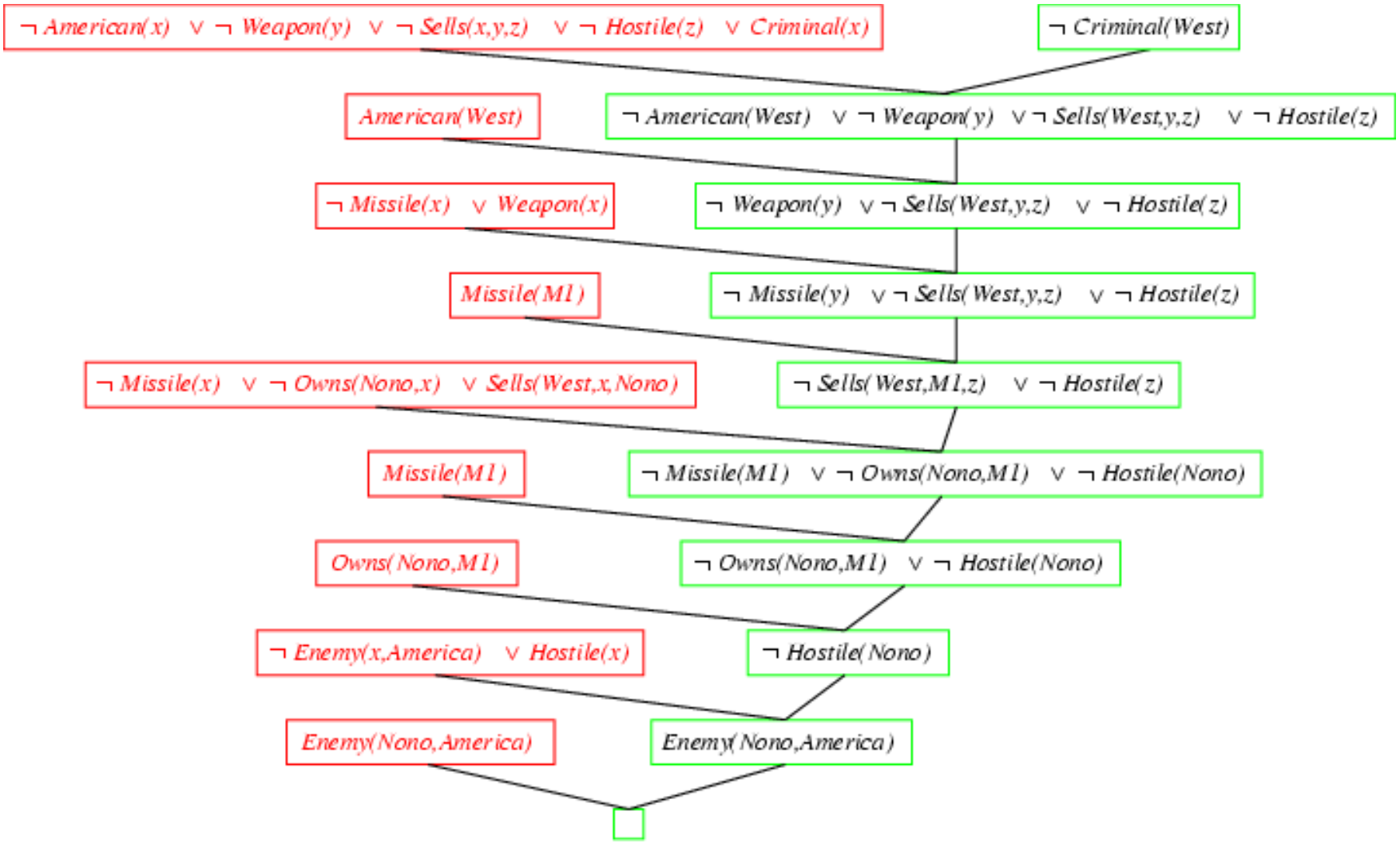
The country Nono, an enemy of America …

*Enemy(Nono,America)*

Can be converted to CNF

Query: Criminal(West)?

# Resolution proof



# Skolemization and Quantifier Elimination

- Replace existential quantifiers by **Skolem functions.**
- **E.g.** $\forall x \exists y \forall z.\, P(x, y, z)$

$$\forall x \forall z.\, P(x, f(x), z)$$

$$\forall x \Big( R(g(x)) \vee \exists y R(x, y) \Big) \iff \forall x \Big( R(g(x)) \vee R(x, f(x)) \Big)$$

where

$f(x)$ is a function that maps $x$ to $y$.

# The point of Skolemization

- Sentences with [forall thereis …] structure become [forall …].

⇒Can use unification of terms.

- Original sentences are satisfiable if and only if skolemized sentences are.

# Complex Skolemization Example

**KB:**

- *Everyone who loves all animals is loved by someone.*
- *Anyone who kills animals is loved by no-one.*
- *Jack loves all animals.*
- *Either Curiosity or Jack killed the cat, who is named Tuna.*

**Query:** *Did Curiosity kill the cat?*

**Inference Procedure:**

1. Express sentences in FOL.
2. Eliminate existential quantifiers.
3. Convert to CNF form and negated query.

A. $\forall x\,[\forall y\,\text{Animal}(y) \Rightarrow \text{Loves}(x, y)] \Rightarrow [\exists y\,\text{Loves}(y, x)]$

B. $\forall x\,[\exists y\,\text{Animal}(y) \wedge \text{Kills}(x,y)] \Rightarrow [\forall z\,\neg\text{Loves}(z,x)]$

C. $\forall x\,\text{Animal}(x) \Rightarrow \text{Loves}(\text{Jack}, x)$

D. $\text{Kills}(\text{Jack}, \text{Tuna}) \vee \text{Kills}(\text{Curiosity}, \text{Tuna})$

E. $\text{Cat}(\text{Tuna})$

F. $\forall x\,\text{Cat}(x) \Rightarrow \text{Animal}(x)$

¬G. $\neg\text{Kills}(\text{Curiosity}, \text{Tuna})$

A1. $\text{Animal}(F(x)) \vee \text{Loves}(G(x), x)$

A2. $\neg\text{Loves}(x, F(x)) \vee \text{Loves}(G(x), x)$

B. $\neg\text{Animal}(y) \vee \neg\text{Kills}(x,y) \vee \neg\text{Loves}(z,x)]$

C. $\neg\text{Animal}(x) \vee \text{Loves}(\text{Jack}, x)$

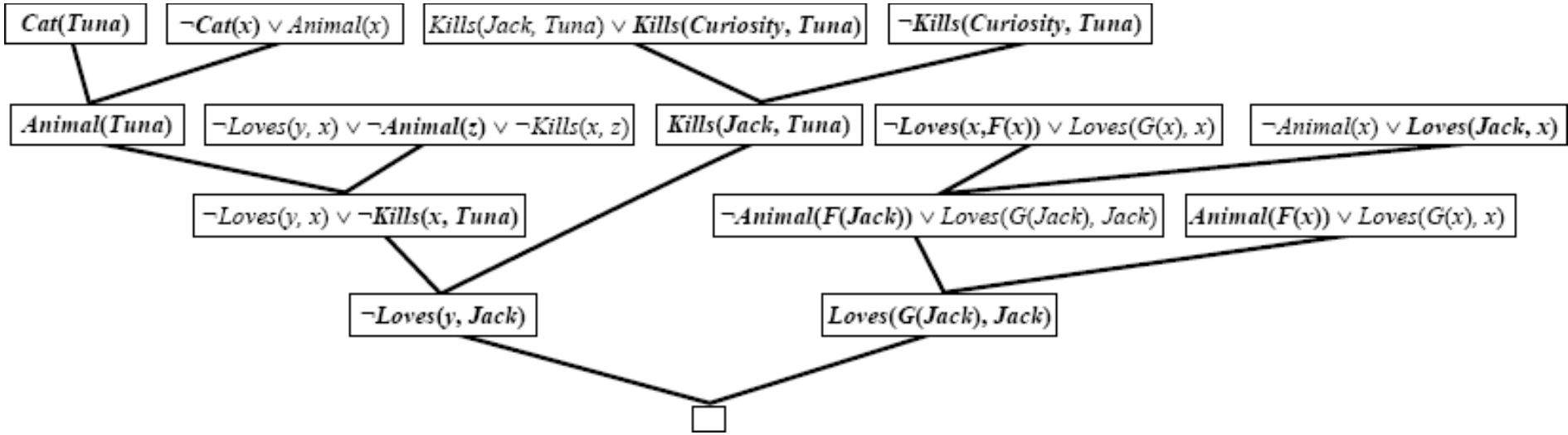
D. $\text{Kills}(\text{Jack}, \text{Tuna}) \vee \text{Kills}(\text{Curiosity}, \text{Tuna})$

E. $\text{Cat}(\text{Tuna})$

F. $\neg\text{Cat}(x) \vee \text{Animal}(x)$

$\neg$G. $\neg\text{Kills}(\text{Curiosity}, \text{Tuna})$

# Resolution-based Inference



# Summary

- Basic FOL inference algorithm (satisfiability check).

1. Use Skolemization to eliminate quantifiers
    1. Only universal quantifiers remain.
2. Convert to clausal form.
3. Use resolution + unification.

- This algorithm is **complete** (Gödel 1929).

# Summary

- Inference in FOL
  - Grounding approach: reduce all sentences to PL and apply propositional inference techniques.

- FOL/Lifted inference techniques
  - Propositional techniques + Unification.
  - Generalized Modus Ponens
  - Resolution-based inference.

- Many other aspects of FOL inference we did not discuss in class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