Algorithm and Data Structure Analysis (ADSA)

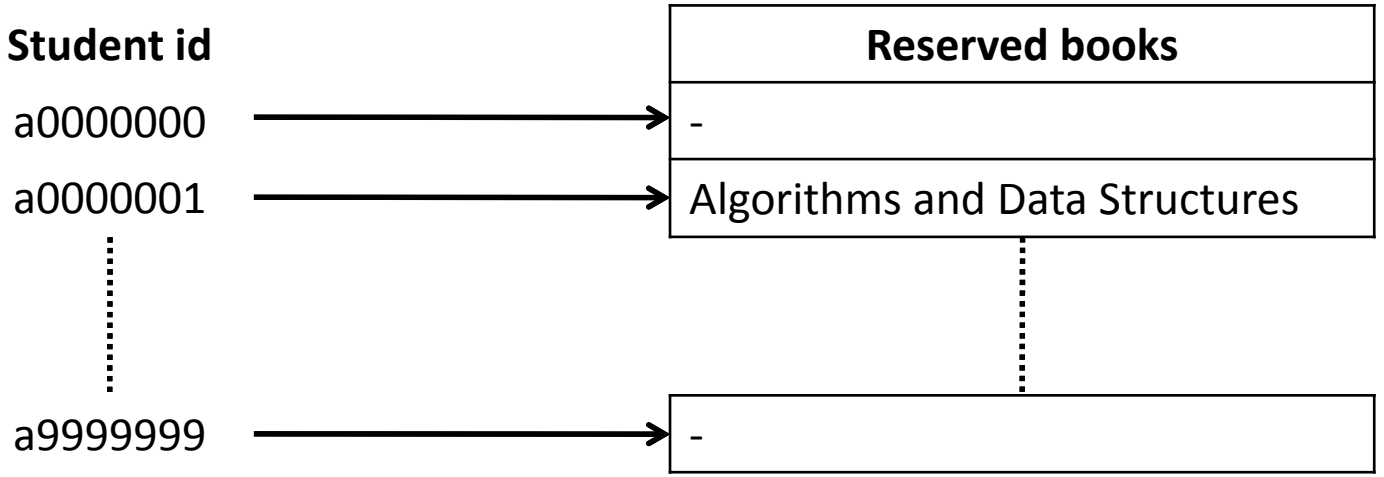
Hashing (2)

In previous lecture…

Previous Lecture: Symbols

- S = associative array
- t = hash table
- N = number of potential keys = $|S|$
- m = number of possible hash function values
  = $|t|$
- n = number of elements

Previous Lecture: Associative Arrays



- insert(x) is O(1)
- remove(x) is O(1)
- find(x) is O(1)

Problem: number of possible keys is MASSIVE.

How large is the table for a direct-address hashing?

a) n

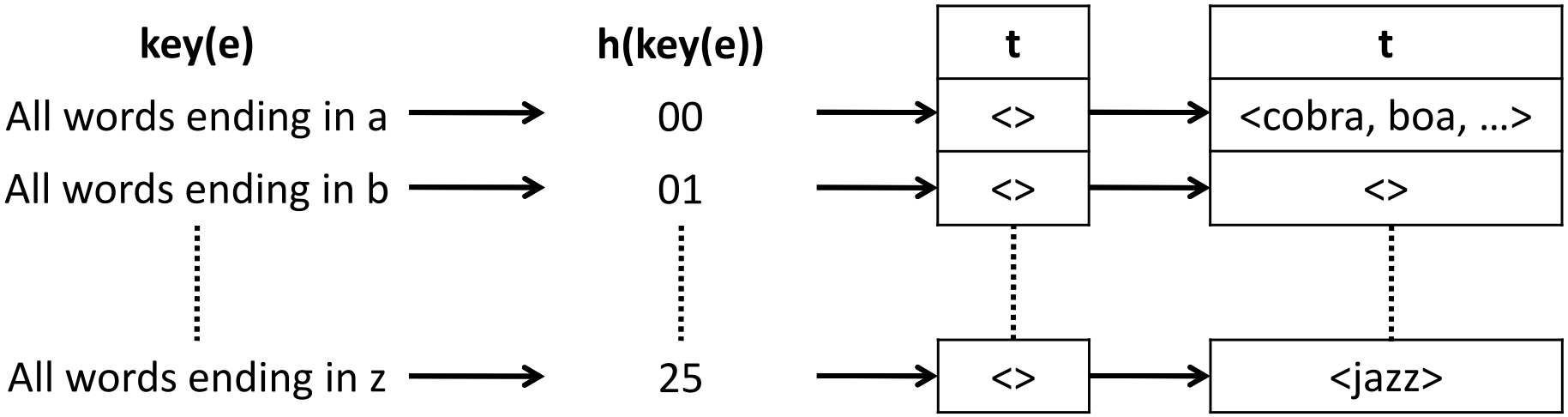
b) N

c) N^2

d) $n/|N|$

Previous Lecture: Hashing with chaining

Use hash function h(key(e)) to obtain index of element *e* in hash table *t*



What is the worst case performance for insert operation for hashing with chaining?

a) Θ(n)

b) O(m)

c) O(nlogn)

d) O(1)

What is the worst case performance for search operation for hashing with chaining?

a) $\Theta(n)$
b) $O(1+n/m)$
c) $O(\log n)$
d) $O(1)$

What is the expected execution time for search operation for hashing with chaining?

a) Θ(n)

b) O(1)

c) O(1+n/m)

d) O(logn)

What is the worst case performance for remove operation for hashing with chaining?

a) $\Theta(n)$
b) $O(\log n)$
c) $O(1+n/m)$
d) $O(m)$

What is the space requirement for hashing with chaining?

a) O(n)

What is the space requirement for hashing with chaining?

a) O(n)

b) O(nlogn)

What is the space requirement for hashing with chaining?

a) *O(n)*
b) *O(nlogn)*
c) *O(n+m)*

What is the space requirement for hashing with chaining?

a) O(n)

b) O(nlogn)

c) O(n+m)

d) O(m)

Previous Lecture: Average Case Analysis for Hashing with Chaining

Theorem: If n elements are stored in a hash table t with m entries using hashing with chaining and a random hash function is used, the expected execution time of remove or find is $O(1+ n/m)$.

Note: a random hash function maps e to all m table entries with the same probability.

Universal Hashing

Theorem 4.1 is unsatisfactory, as the class of "all hash functions" is too big to be useful: $|H|=m^N$, thus it requires $N \log m$ bits to specify a function in H.

This drawback can be overcome with much smaller classes of hash functions, and their members can be specified in constant space.

Universal Hashing

Definition 4.2 Let *c* be a positive constant. A family *H* of functions from *Key* to *0..m-1* is called c-universal if any two distinct keys collide with a probability of at most *c/m*:

$$\forall x, y \in Key, x \neq y :$$

$$\left|\left\{h \in H : h(x) = h(y)\right\}\right| \leq \frac{c}{m}|H|$$

Or, for a random h∈H: $prob(h(x) = h(y)) \leq \frac{c}{m}$

Universal Hashing

Theorem 4.3 If n elements are stored in a hash table with m entries using hashing with chaining and a random hash function from a c-universal family is used, the expected execution time of remove or find is $O(1+cn/m)$.

Proof

Follows the proof of Theorem 4.1.

Expected Execution Time remove/find

Proof:

Execution time for remove and find is constant time plus the time scanning the list *t[h(k)]*.

Let the random variable X be the length of the list *t[h(k)],* and let *E[X]* be the expected length of the list.

Thus the *expected* execution time = O(1 + E[X]).

Expected Execution Time remove/find

Proof (continued):

Let *S* be the set of *n* elements contained in *t*.

For each $e \in S$, let X_e be an indicator variable which indicates whether *e* hashes to the same value as *k*.

ie: **if** *h(key(e))* = *h(k)* **then** $X_e = 1$ **else** $X_e = 0$.

$$X = \sum_{e \in S} X_e$$

(ie how many e's are in table entry h(key(e)))

Expected Execution Time remove/find

Proof (continued):

$$E[X] = E\left[\sum_{e \in S} X_e\right]$$

$$= \sum_{e \in S} E[X_e]$$

$$= \sum_{e \in S} prob(X_e = 1)$$

Expected Execution Time remove/find

Proof (continued):

$$E[X] = \sum_{e \in S} prob(X_e = 1)$$

(From last slide)

$$= \sum_{e \in S} c/m$$

(As function h is chosen uniformly from a c-universal class: $prob(X_e = 1) \leq c/m$)

$$= c \cdot n/m$$

(Because n elements in S)

Expected Execution Time remove/find

Proof (continued):

Expected execution time = $O(1 + E[X])$,

$E[X] = c \cdot n/m$

Thus the expected execution time for remove and find under hashing with chaining is

$O(1 + c \cdot n/m)$.

□

C-universal families

For practical purposes: find c-universal families that are easy to construct and evaluate.

We will describe a simple and quite practical 1-universal family in detail…

Assumptions

- keys are bit strings of fixed length
- table size *m* is a prime number

1-universal family

Why prime? Arithmetic modulo prime is nice: the set $\mathbb{Z}_m=\{0,\ldots,m-1\}$ of numbers modulo m forms a field.

A field is a set with special elements 0 and 1, and with addition and multiplication operators, satisfying certain axioms (associative & commutative & distributive properties, existence of neutral elements, …)

1-universal family

Let $w = \lfloor \log m \rfloor$.

We subdivide the keys into pieces of *w* bits each (total *k* pieces). We interpret each piece as an integer in the range $0..2^w-1$ and keys as *k*-tuples of such integers.

For a key ***x***, we write ***x***$=(x_1,\ldots x_k)$ to denote its partition into pieces. Each x_i lies in $0..2^w-1$.

We can now define our class of hash functions.

1-universal family

For each $\boldsymbol{a}=(a_1,\ldots a_k) \in \{0..m-1\}^k$, we define a function h_a from *Key* to *0..m-1* as follows.

Let $\boldsymbol{x}=(x_1,\ldots,x_k)$ be a key and let $\boldsymbol{a} \cdot \boldsymbol{x} = \sum_{i=1}^{k} a_i x_i$ denote the scalar product of $\boldsymbol{a}$ and $\boldsymbol{x}$.

Then $h_a(\boldsymbol{x}) = \boldsymbol{a} \cdot \boldsymbol{x} \bmod m$.

1-universal family

Example

Let $m=17$, $k=4$. Then $w=4$ and we view keys as 4-tuples in the range $0..15$, for example $\boldsymbol{x}=(11,7,4,3)$.

A hash function is specified by a 4-tuple of integers in the range $0..16$, for example $\boldsymbol{a}=(2,4,7,16)$.

Then $h_a(\boldsymbol{x}) = (2 \cdot 11 + 4 \cdot 7 + 7 \cdot 4 + 16 \cdot 3) \bmod 17 = 7$.

1-universal family

Theorem 4.4

$$H = \{ h_a : \mathbf{a} \in \{0..m-1\}^k \}$$

is a 1-universal family of hash functions, if *m* is prime.

In other words, the scalar product between a tuple representation of a key and a random vector modulo m defines a good hash function.

Proof

Proof of Theorem 4.4:

- Consider two keys

$$x = (x_1, \ldots, x_k) \text{ and } y = (y_1, \ldots, y_k)$$

- Consider the number of choices of a such that

$$h_a(x) = h_a(y)$$

- Fix index j such that $x_j \neq y_j$
- Implies $(x_j - y_j) \neq 0 (mod\ m)$

- Equation $a_j(x_j - y_j) = b(mod\ m), b \in Z_m$ has unique solution

$$a_j = (x_j - y_j)^{-1} b(mod\ m)$$

Claim: For each choice of the $a_i, i \neq j$,there is exactly one choice of a_j such that

$$h_a(x) = h_a(y)$$

$$
\begin{aligned}
h_{\mathbf{a}}(\mathbf{x}) = h_{\mathbf{a}}(\mathbf{y}) \Leftrightarrow \quad & \sum_{1\le i\le k} a_i x_i \equiv \sum_{1\le i\le k} a_i y_i && \pmod m \\
\Leftrightarrow\ & a_j(x_j - y_j) \equiv \sum_{i\ne j} a_i(y_i - x_i) && \pmod m \\
\Leftrightarrow\ & a_j \equiv (y_j - x_j)^{-1} \sum_{i\ne j} a_i(x_i - y_i) && \pmod m \quad .
\end{aligned}
$$

m^{k-1} ways to choose a_i with $i \neq j$ and for each such choice there is a unique choice of a_j.

In total m^k choice which implies

$$
\text{prob}(h_{\mathbf{a}}(x) = h_{\mathbf{a}}(\mathbf{y})) = \frac{m^{k-1}}{m^k} = \frac{1}{m}
$$

□

Prime Table Sizes

Is it a serious restriction?

At first glance: yes.

- The user has to provide appropriate primes.
- While growing/shrinking: how to obtain new prime numbers for the new value of *m*?

Easy solution: consult a table of primes.

Analytical solution: not much harder.

Prime Table Sizes

From number theory:

- there is an infinite number of primes
- for any integer k there is a prime in the interval $[k^3,(k+1)^3]$

So, if we are aiming for a table size of about m, we determine k such that $k^3 \leq m \leq (k+1)^3$ and then search for a prime in this interval.

Prime Table Sizes

How does this search work?

Any nonprime in the interval must have a divisor which is at most $\sqrt{(k+1)^3} = (k+1)^{3/2}$.

We therefore iterate over the numbers from *2* to $(k+1)^{3/2}$, and for each such *j* remove its multiples in $[k^3,(k+1)^3]$.

For each fixed *j*, this takes time $((k+1)^3-k^3)/j=O(k^2/j)$.

Prime Table Sizes

The total time required is

$$\sum_{j \le (k+1)^{3/2}} O\left(\frac{k^2}{j}\right) = k^2 \sum_{j \le (k+1)^{3/2}} O\left(\frac{1}{j}\right)$$

$$= O\left(k^2 \ln\left((k+1)^{3/2}\right)\right) = O\left(k^2 \ln k\right) = o(m)$$

and hence is negligible compared with the cost of initializing a table of size *m*.

Alternative Approach to Hashing

Hashing with chaining is a closed hashing approach.

- Closed hashing: handles collision by storing all elements with the same hashed key in one table entry.
- Open hashing: handles collision by storing subsequent elements with the same hashed key in different table entries.

Hashing with Linear Probing

- Hashing with Linear Probing is an open hashing approach.
- All unused entries in *t* are set to $\perp$.
- When inserting, on a collision insert the element to the next free entry.

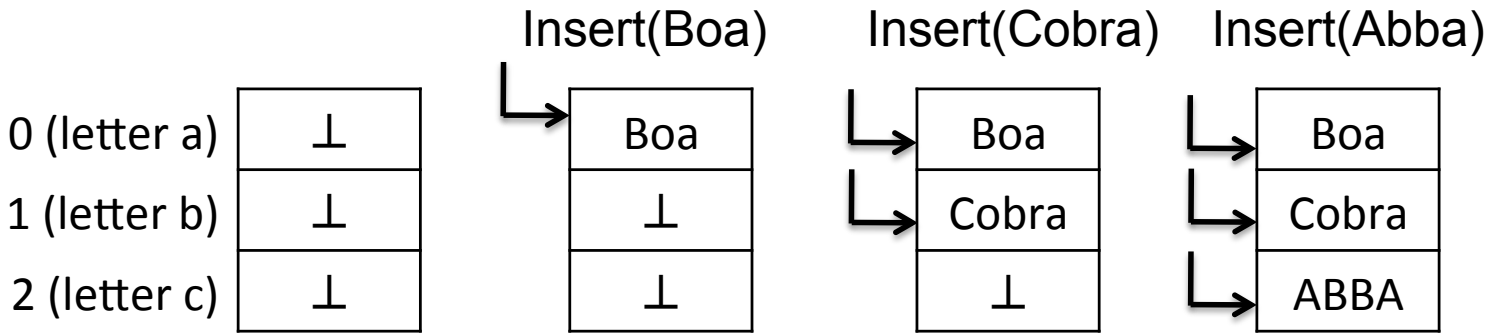
- What if the last entry is used?

Hashing with Linear Probing

- Trivial fix: allow more entries
- Make table *t* size $m + m'$ instead of m. Choose $m' < m$.
- Is this a good fix? Is there a better way?

Insert(e)

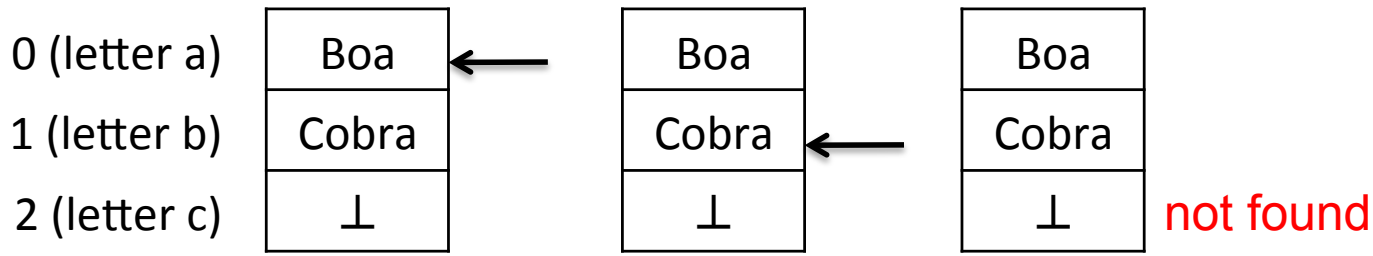
- insert(*e*: Element)
  1. Get index *i* = *h(key(e))*
  2. If *t*[*i*] == ⊥, store *e* at *t*[*i*]
  3. If *t*[*i*] is not empty, increase *i* by 1 and go to step 2.



Find(k)

- find(*k*: Key)
  1. Get index *i* = *h(k)*
  2. If *t*[*i*] == ⊥, return not found
  3. If element e at *t*[*i*] has *key(e)* == *k*, return found. Else increase *i* by 1 and go to step 2.

eg Find(ABBA)



Remove(k)

- Can't remove the element with *key(e)* == *k* and replace it with ⊥.
  - If we replace element *e1* at *t[i]* with ⊥, how do we find an element *e2* with the same *h(k)*?

- Instead, first remove the element with *key(e)* == *k* and then fix the invariant.

Remove(k)

- remove(*k*: Key)
  1. Get index *i* = *h(k)*

search (k)

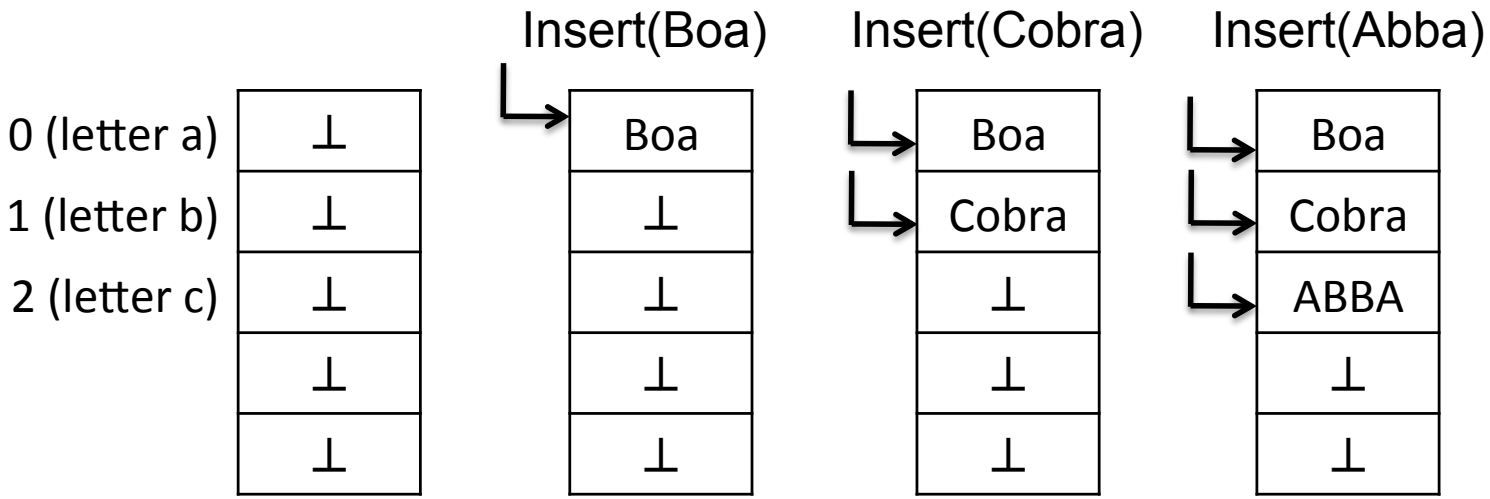
  2. If *t*[*i*] == ⊥, return
  3. If element *e* at *t*[*i*] has *key(e)* != *k*, increase *i* by 1 and go to step 2.

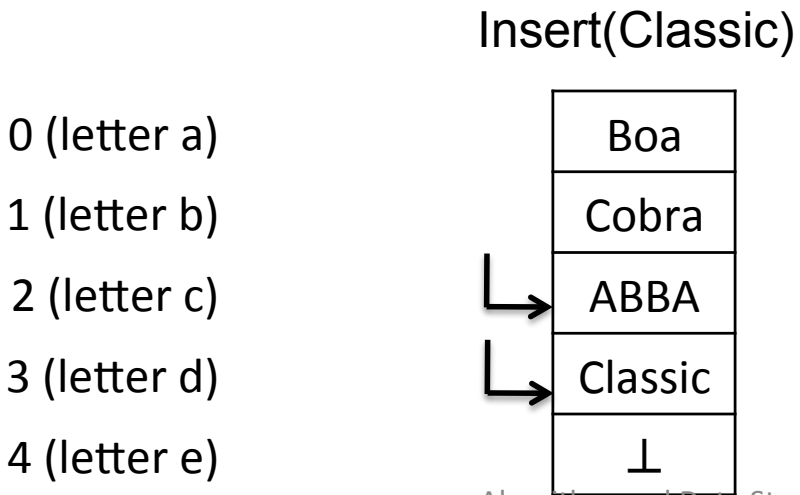
  4. Set *t*[*i*] = ⊥

repair

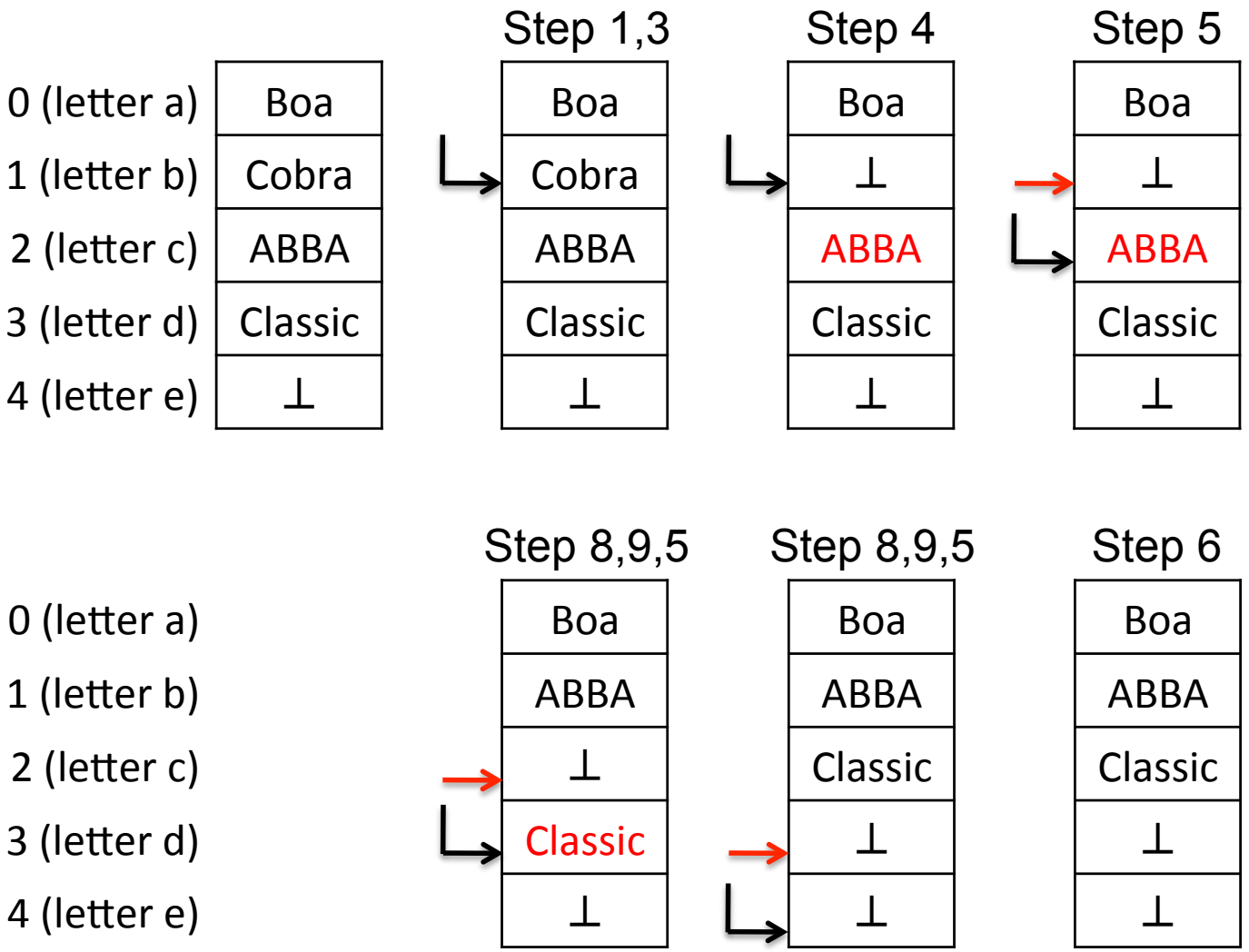
  5. Set index *j* = *i*+1
  6. If *t*[*j*] == ⊥, return
  7. If *h*(*t*[*j*]) > *i*, increase j by 1
  8. Else set *t*[*i*] = *t*[*j*] and *t*[*j*] = ⊥
  9. Set *i* = *j* and go to step 5.

Example Inserts





Example: Remove(Cobra)



Chaining vs. Linear Probing

Argumentation depends on the intended use and many technical parameters:

Chaining	Linear probing
+ referential integrity	+ use of contiguous memory
- waste of space	- gets slower as table fills up

A fair comparison must be based on space consumption, not only on the runtime.

Experimental results: so small differences that implementation details, used compiler, OS, … matter.