Algorithm and Data Structure Analysis (ADSA)

Depth first search / Strongly Connected Components

Overview

Depth-first-search

Strongly connected components

- Undirected graphs
- Directed graphs

Depth-First-Search

Idea for Depth-First-Search (DFS):

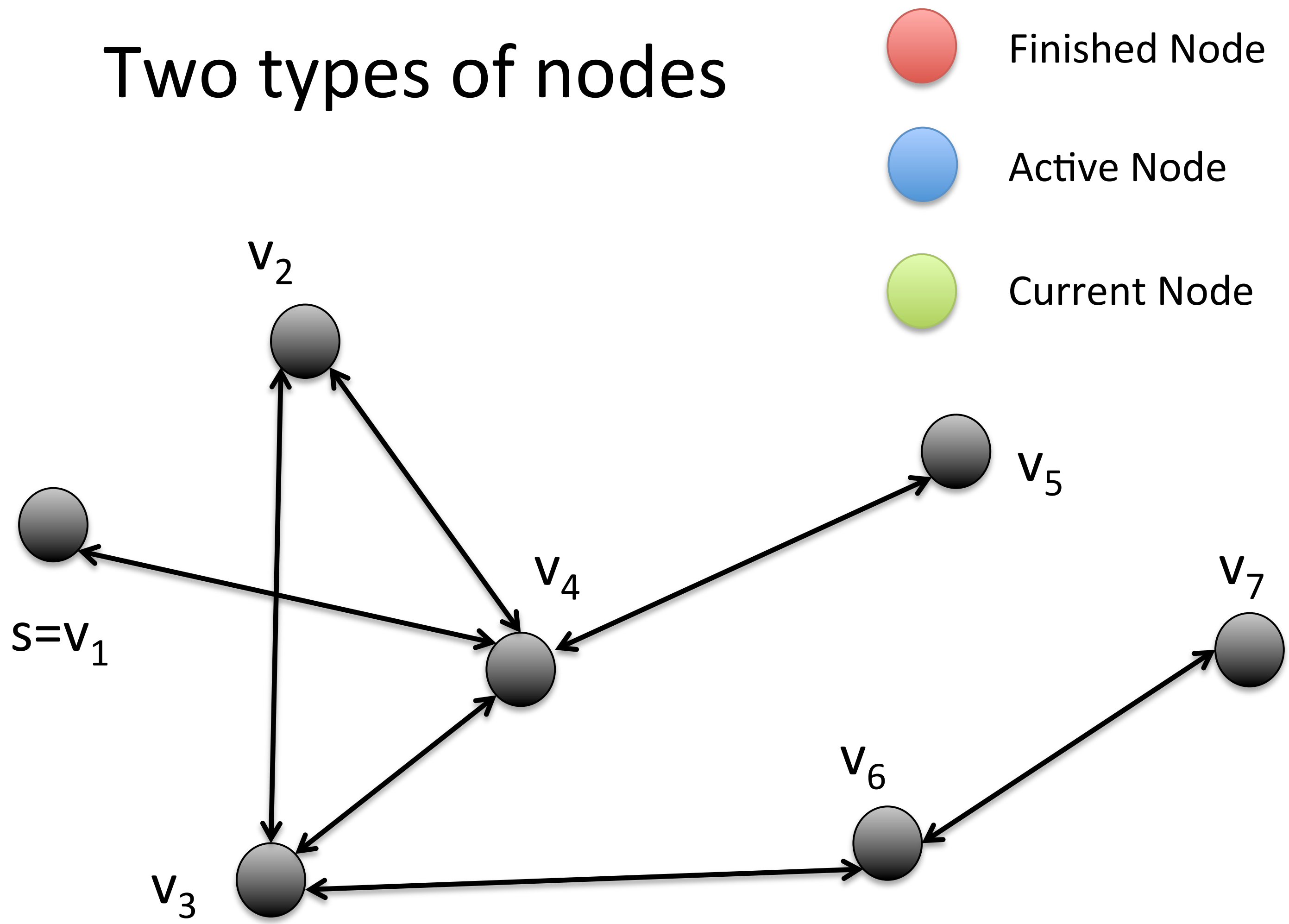
- Whenever you visit a vertex, explore in the next step one of its non-visited neighbours.

Implementation:

- When visiting a node, mark it as visited and recursively call DFS for one of its non-visited neighbors
- If there is no non-visited neighbor end recursive call.

Depth-first-search (DFS)

Two types of nodes



Depth-first-search (DFS)

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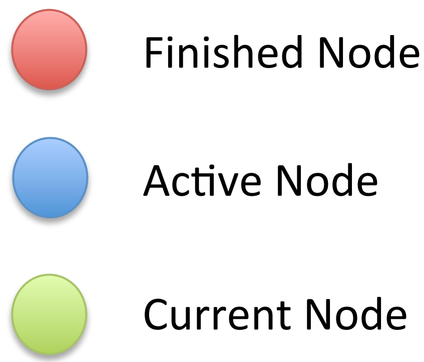

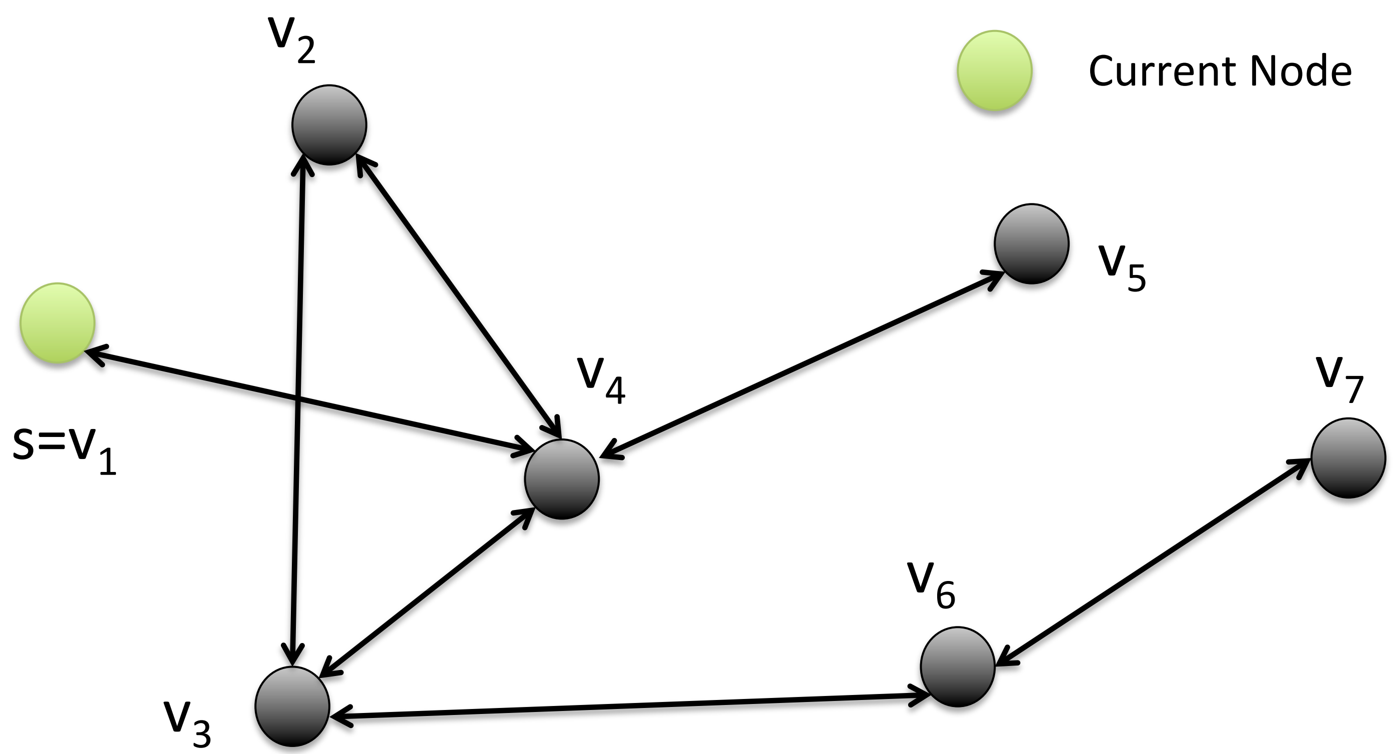

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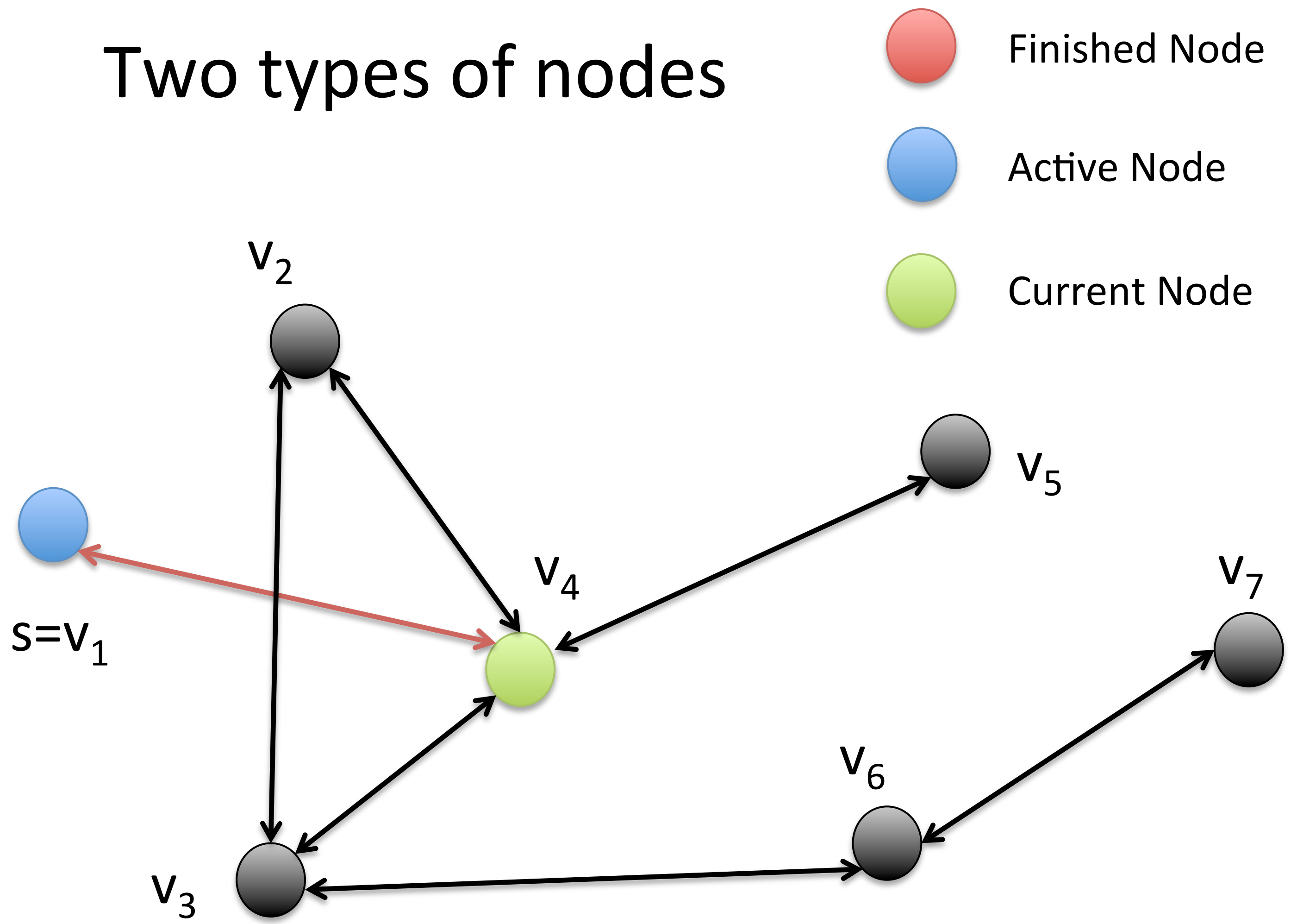

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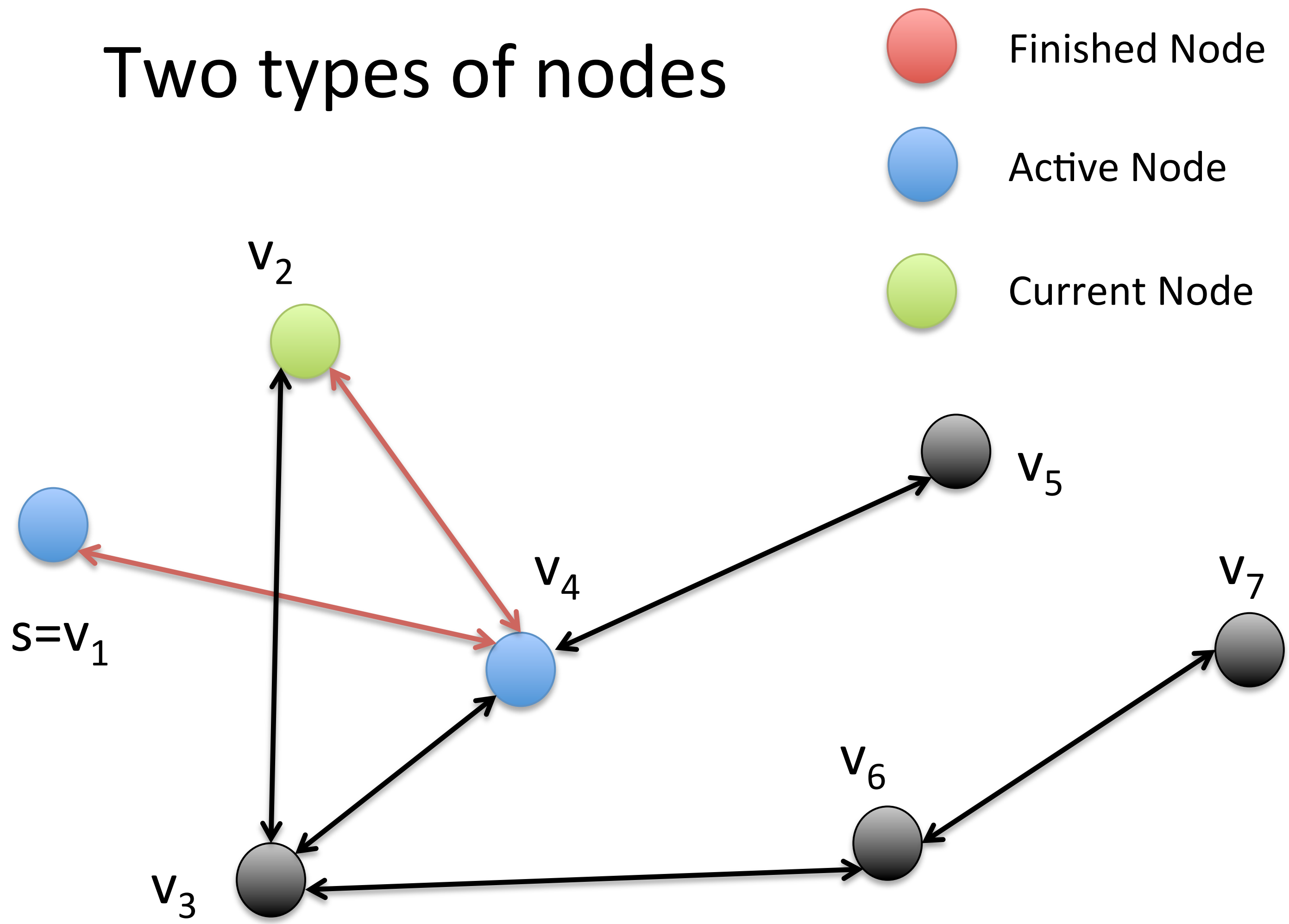

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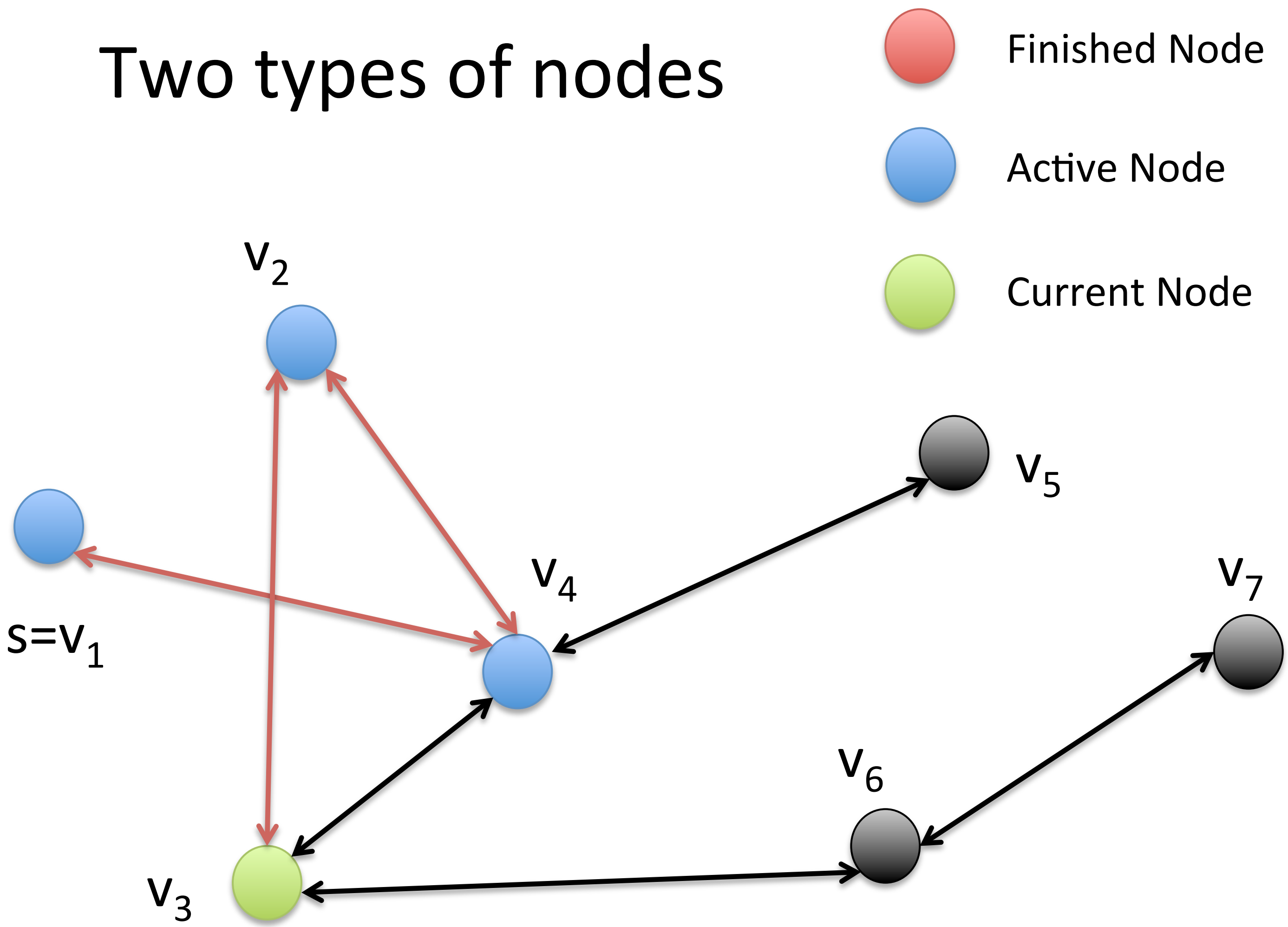

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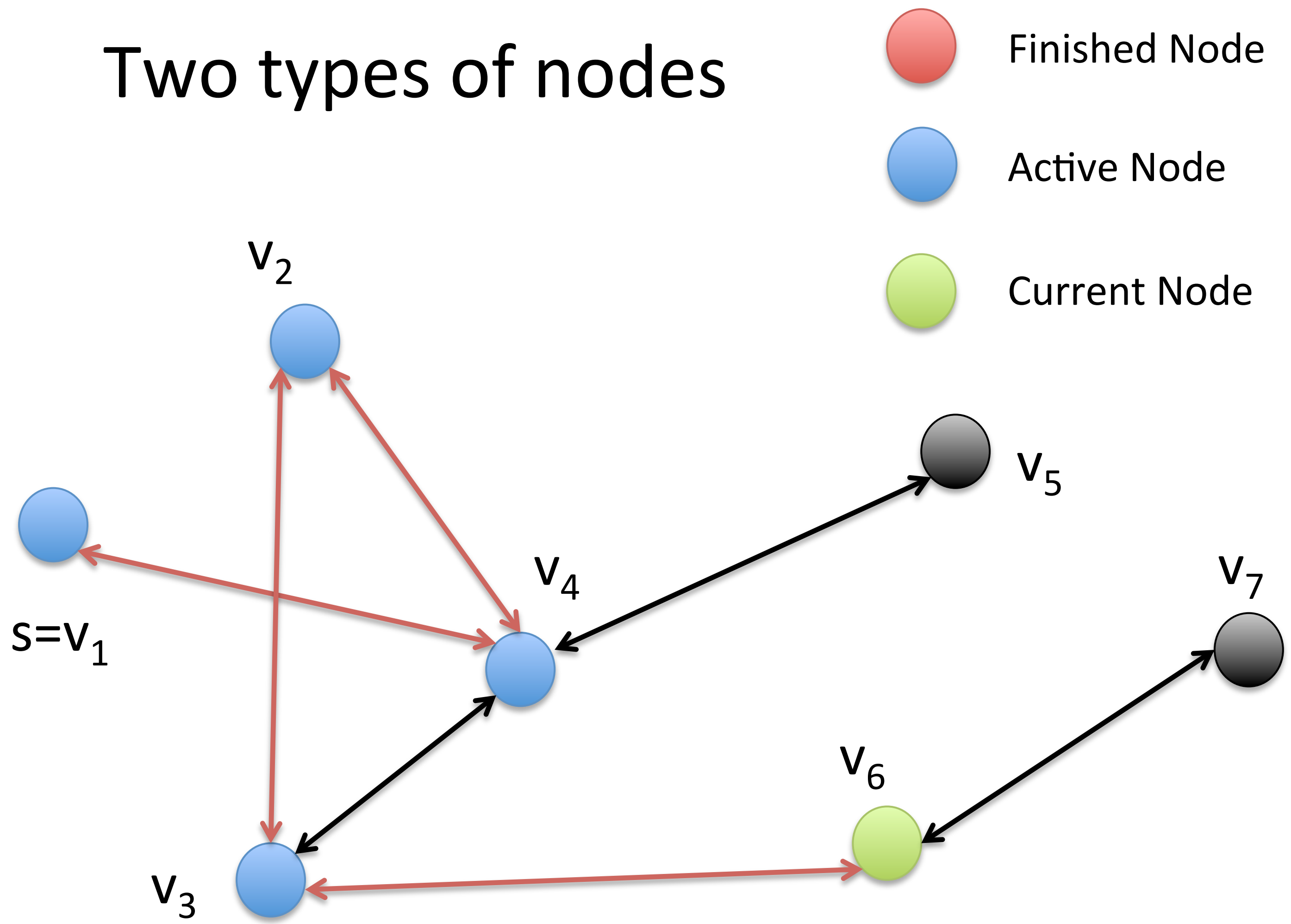

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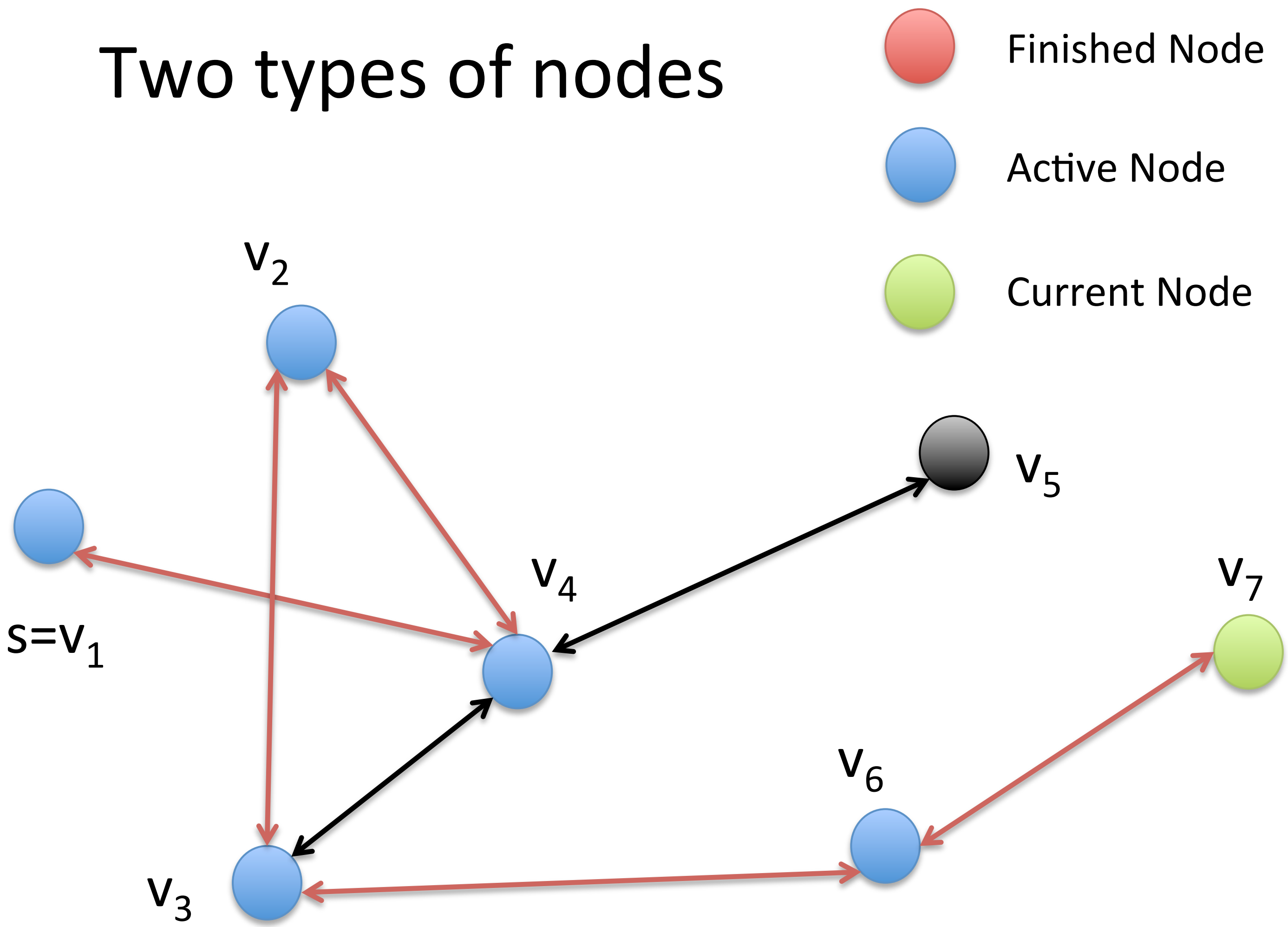

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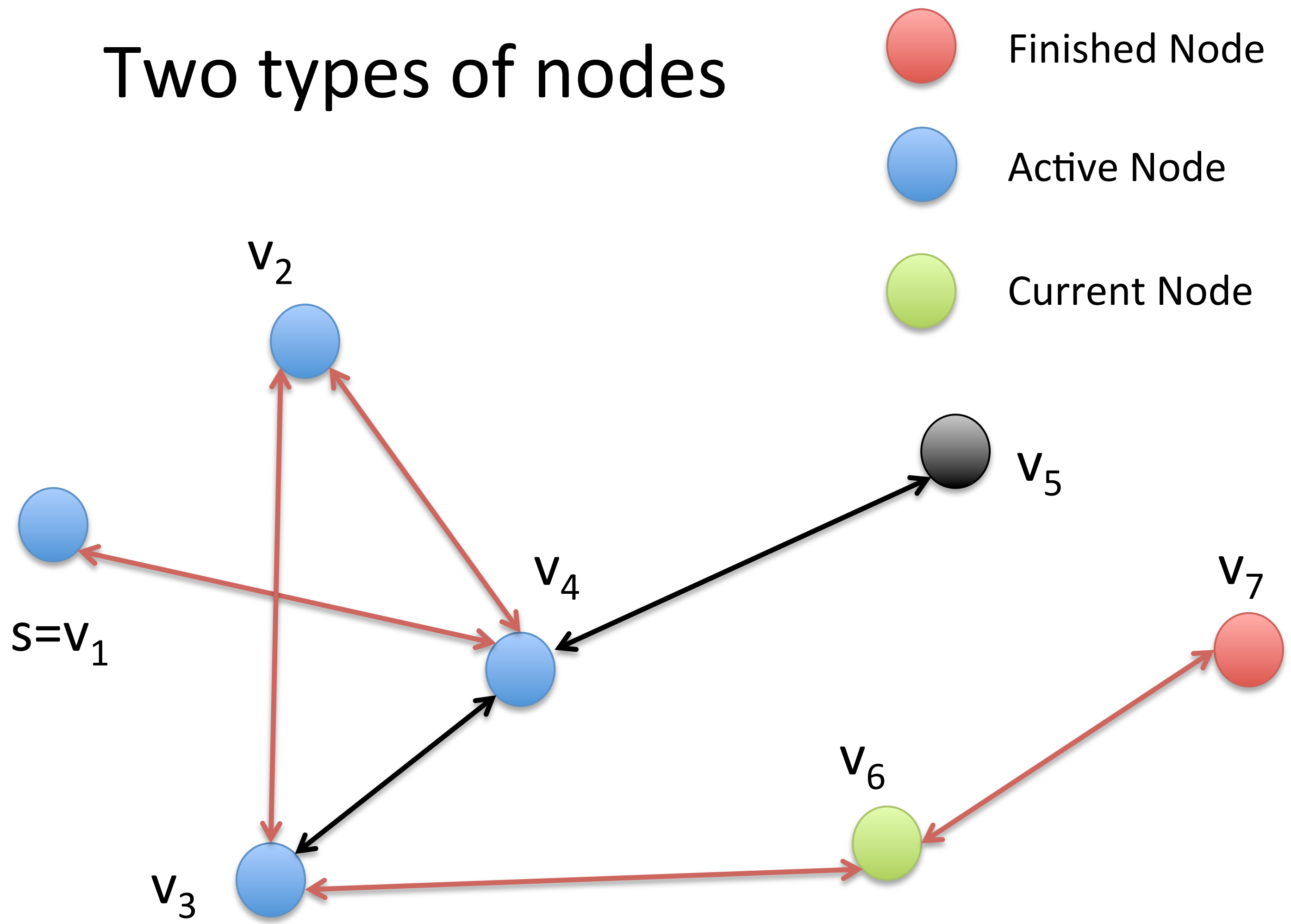

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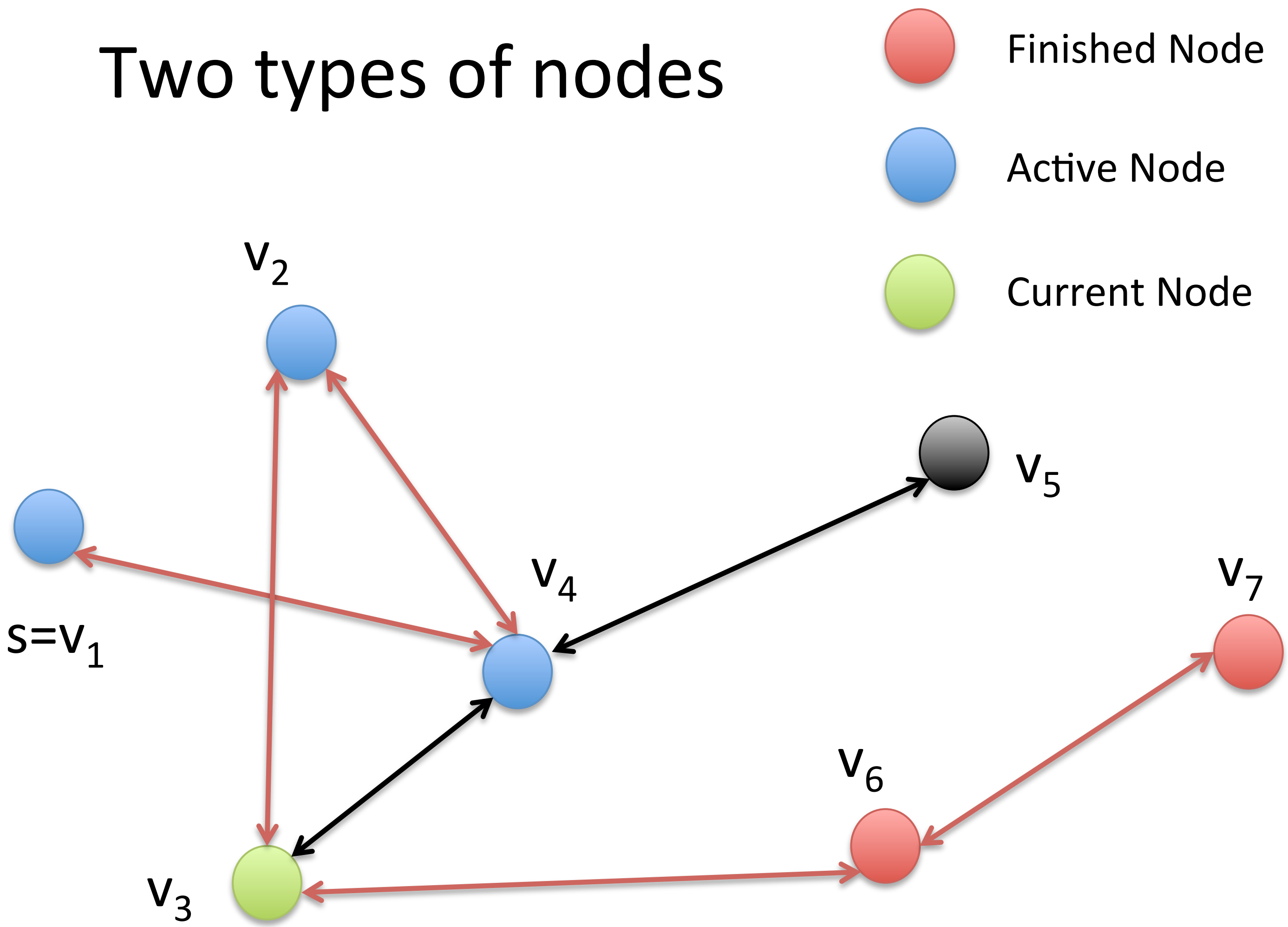
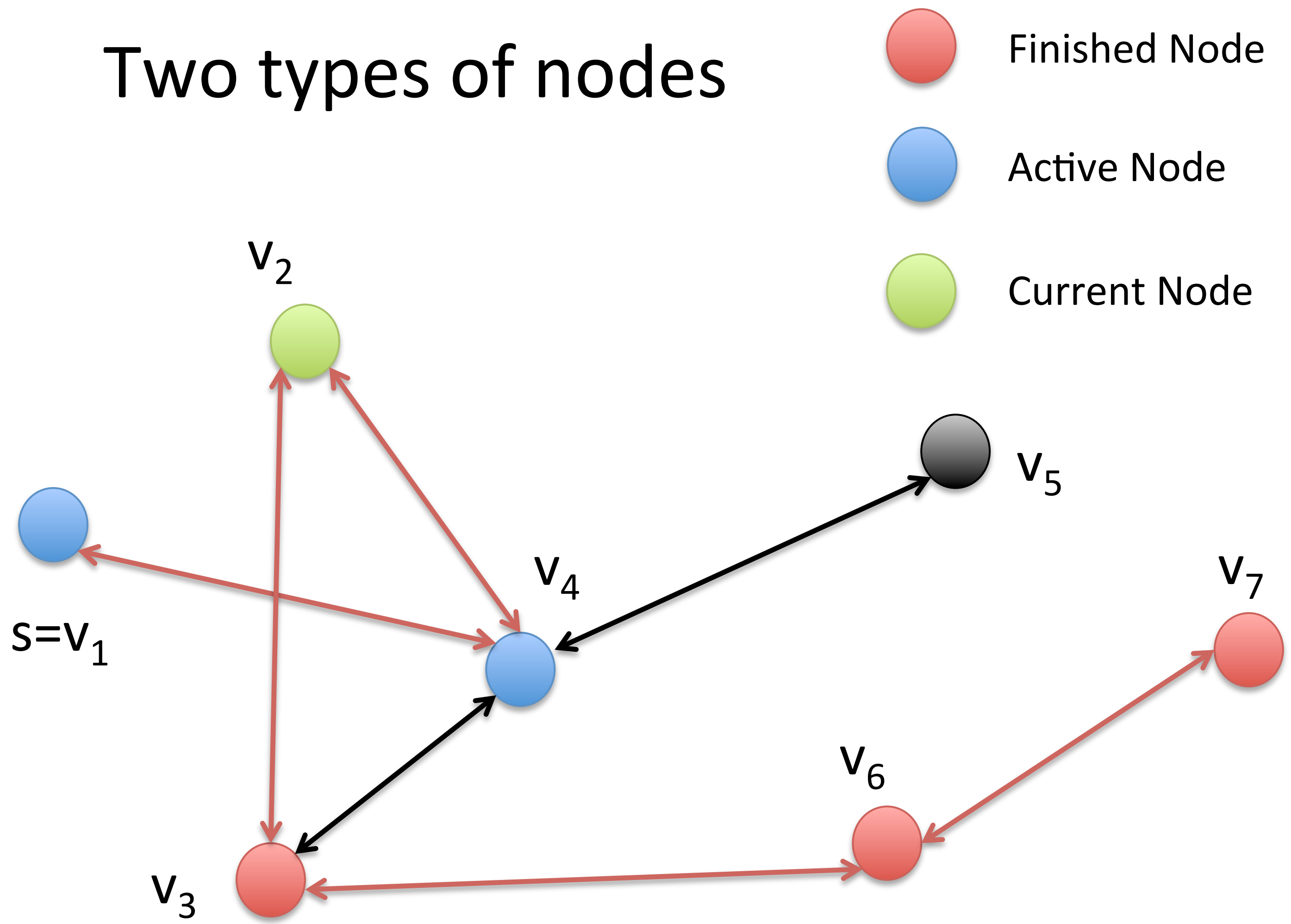


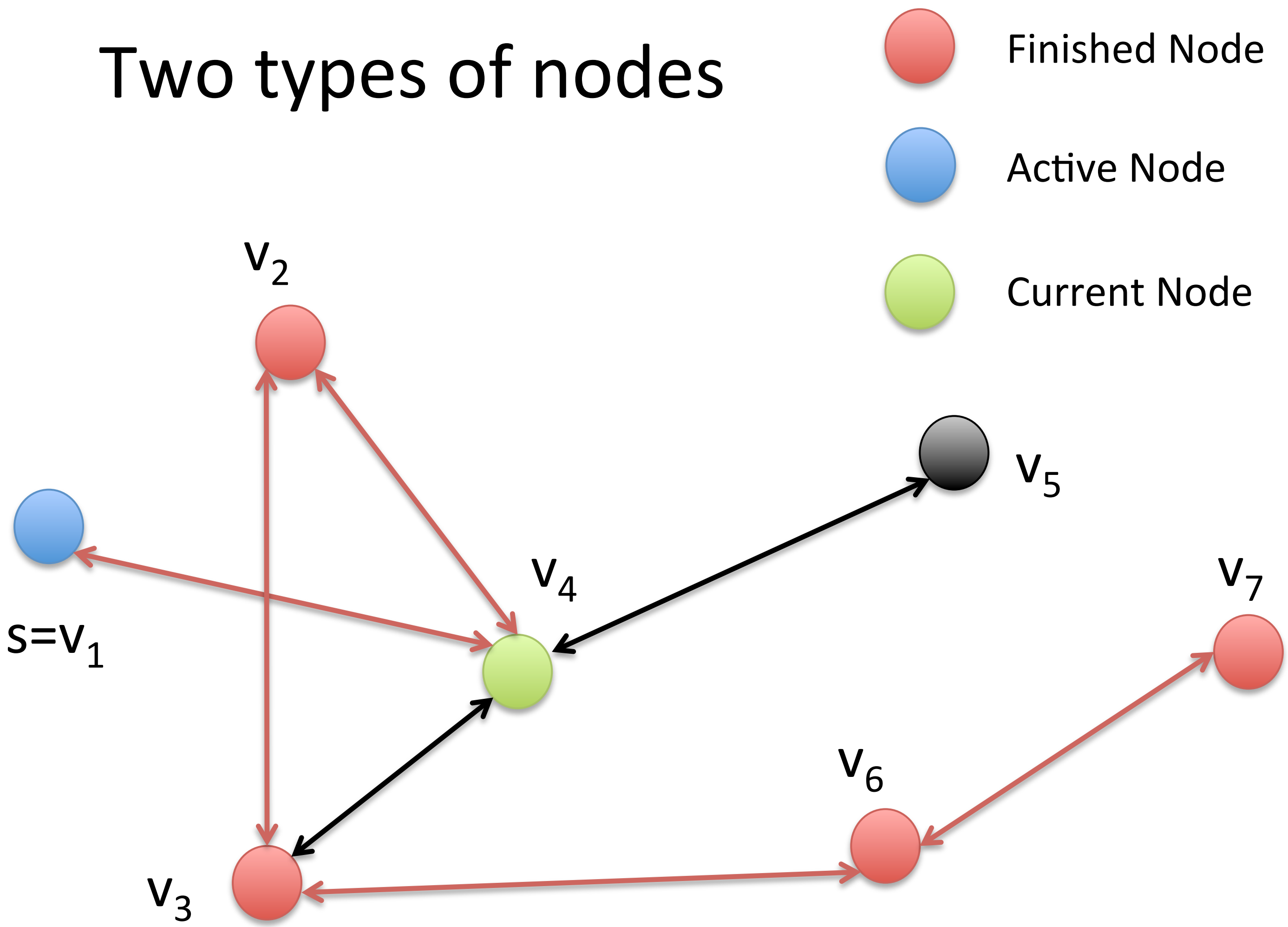
Depth-first-search (BFS)

Two types of nodes



Depth-first-search (DFS)

Two types of nodes



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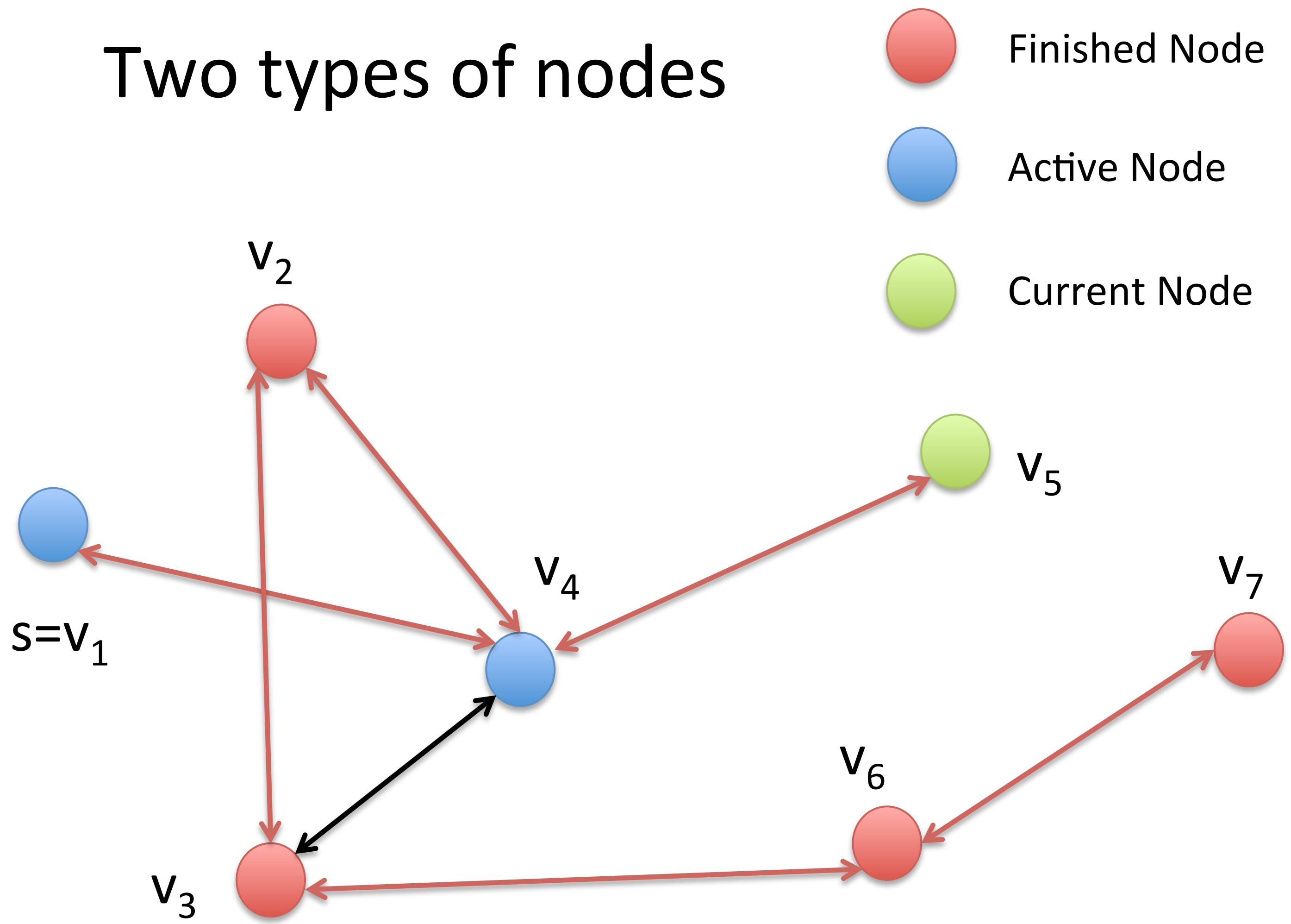

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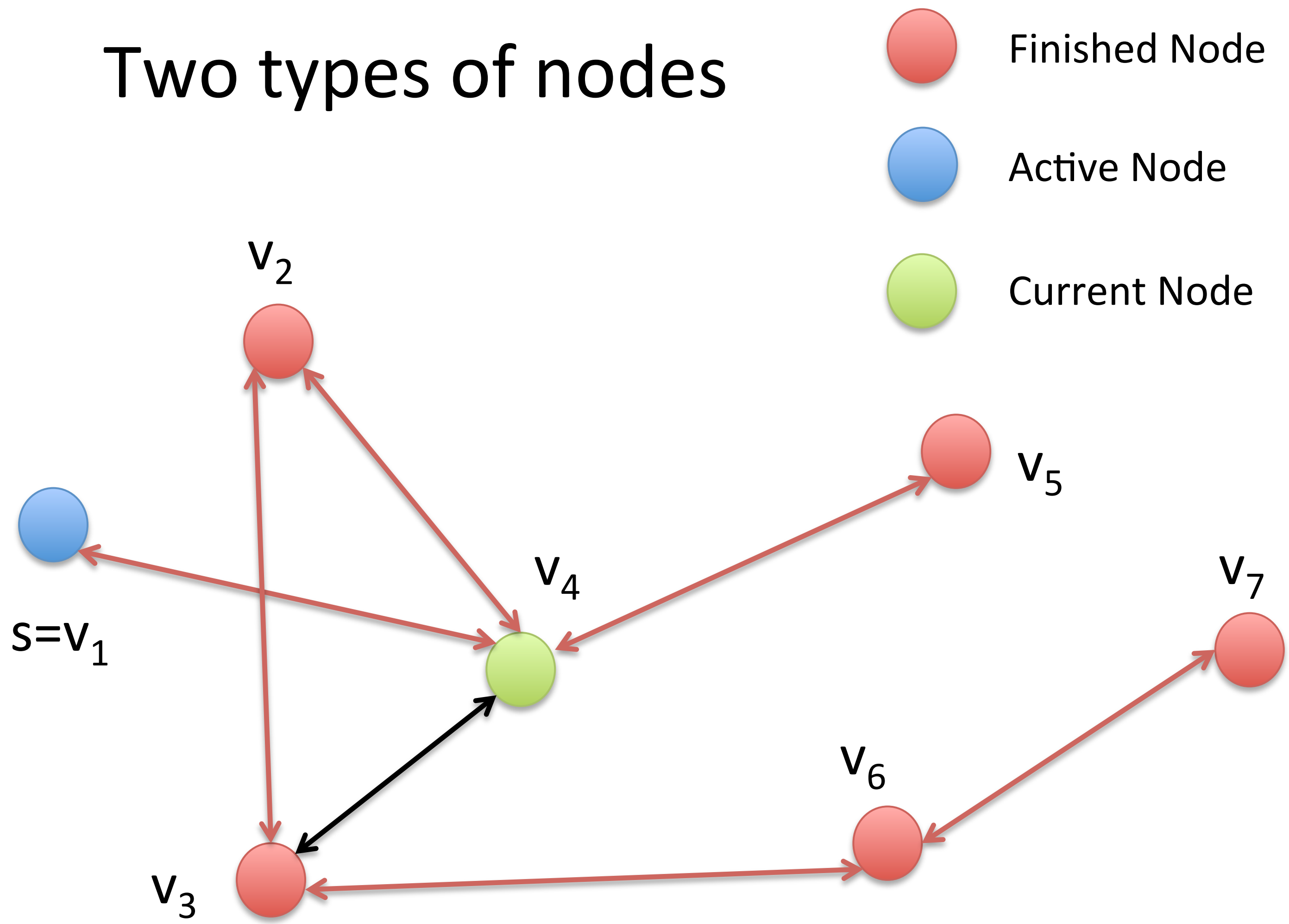

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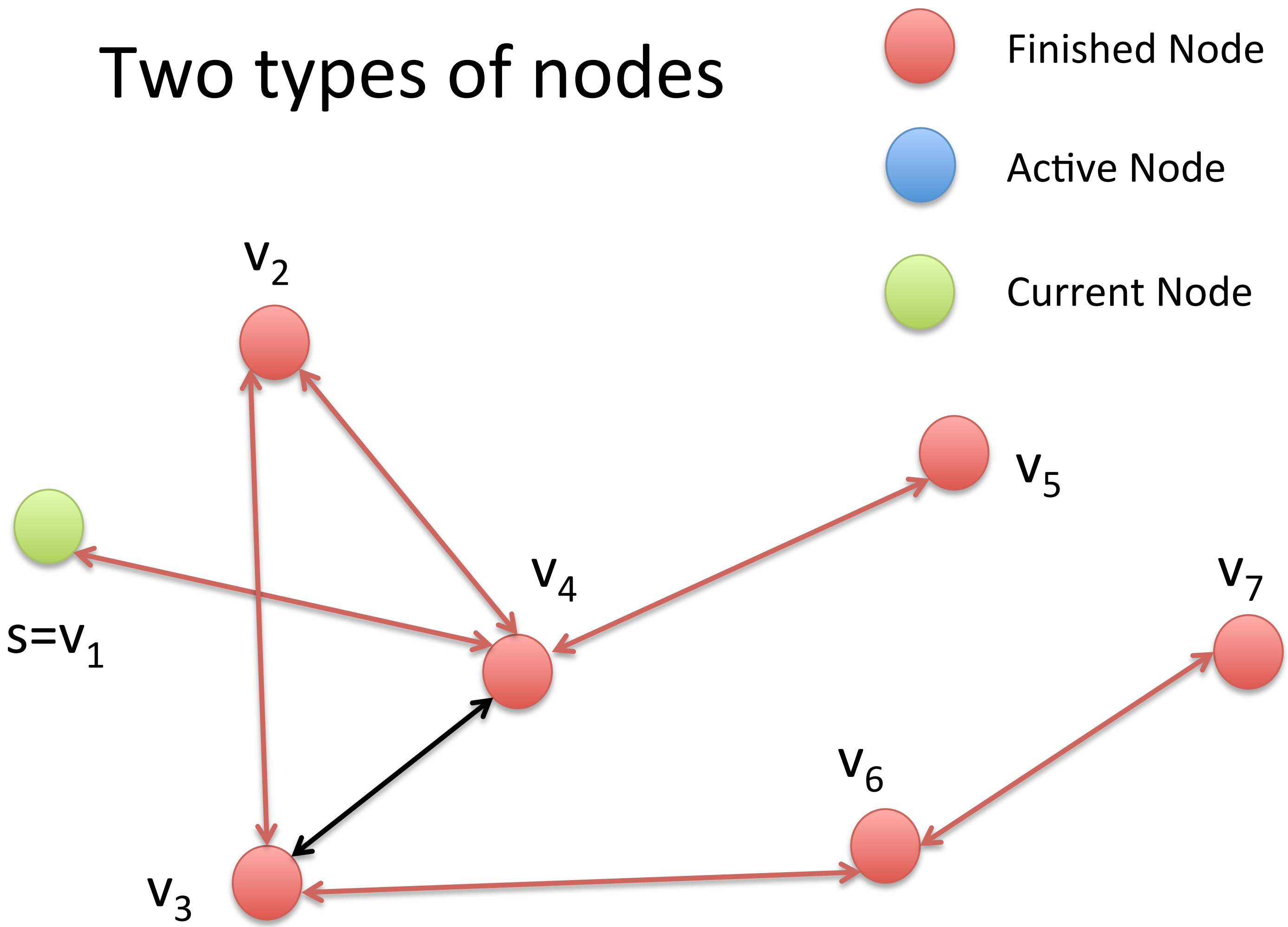

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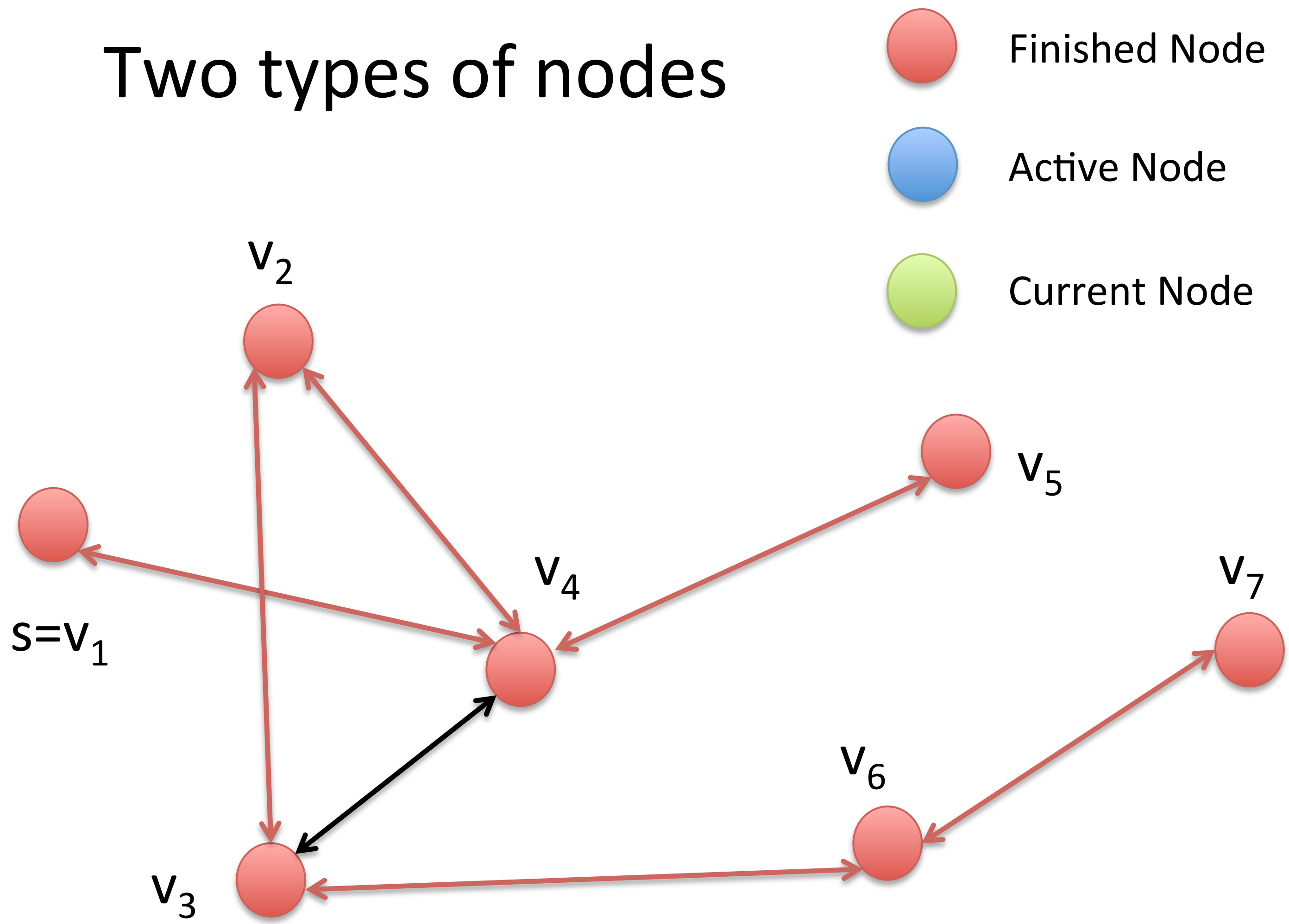

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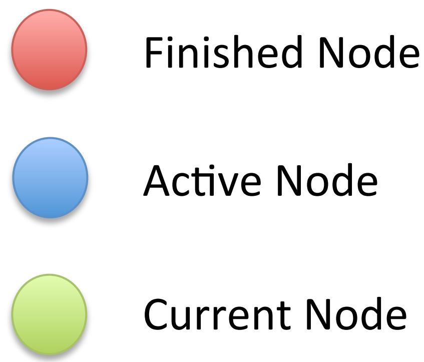
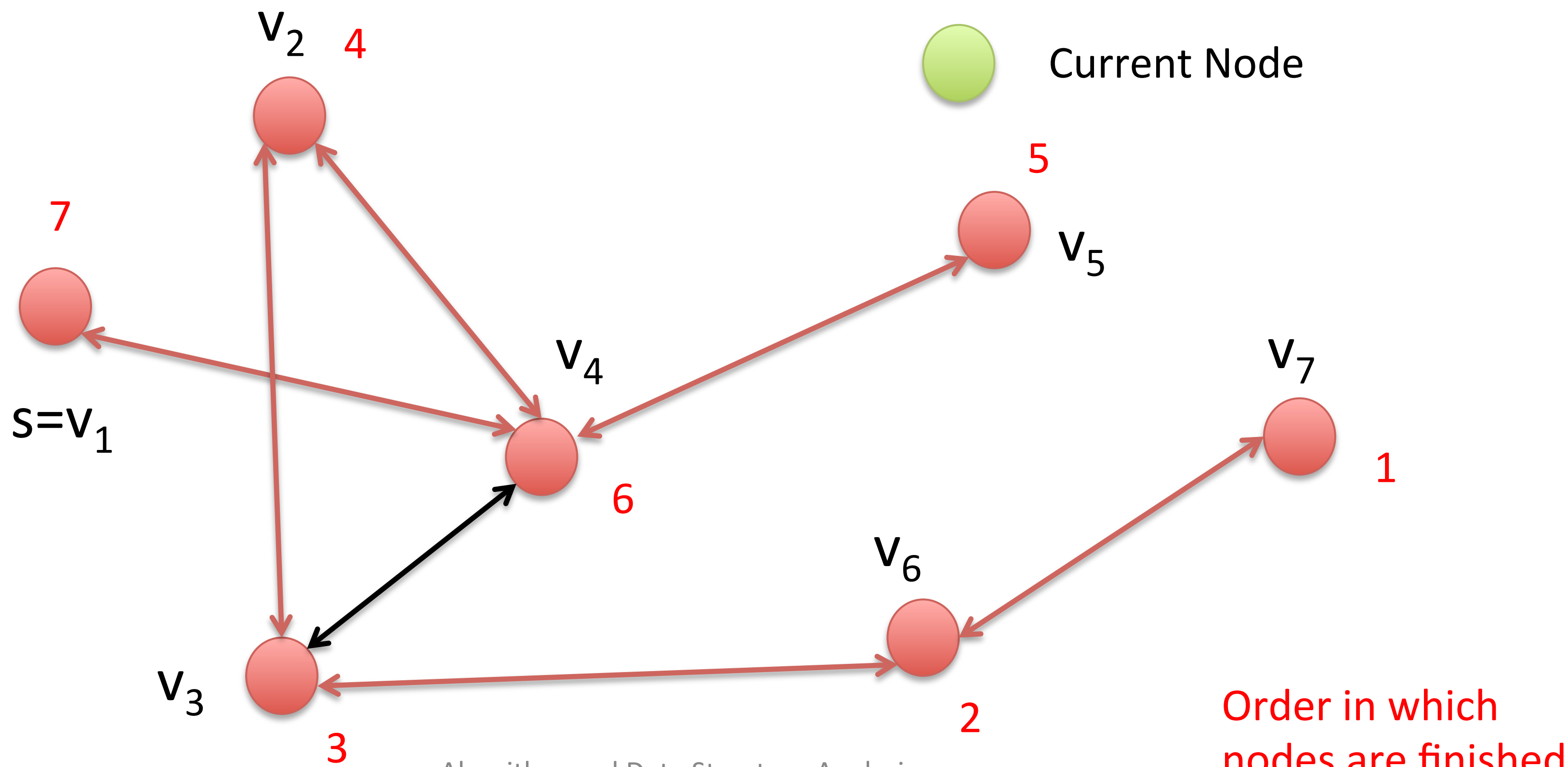
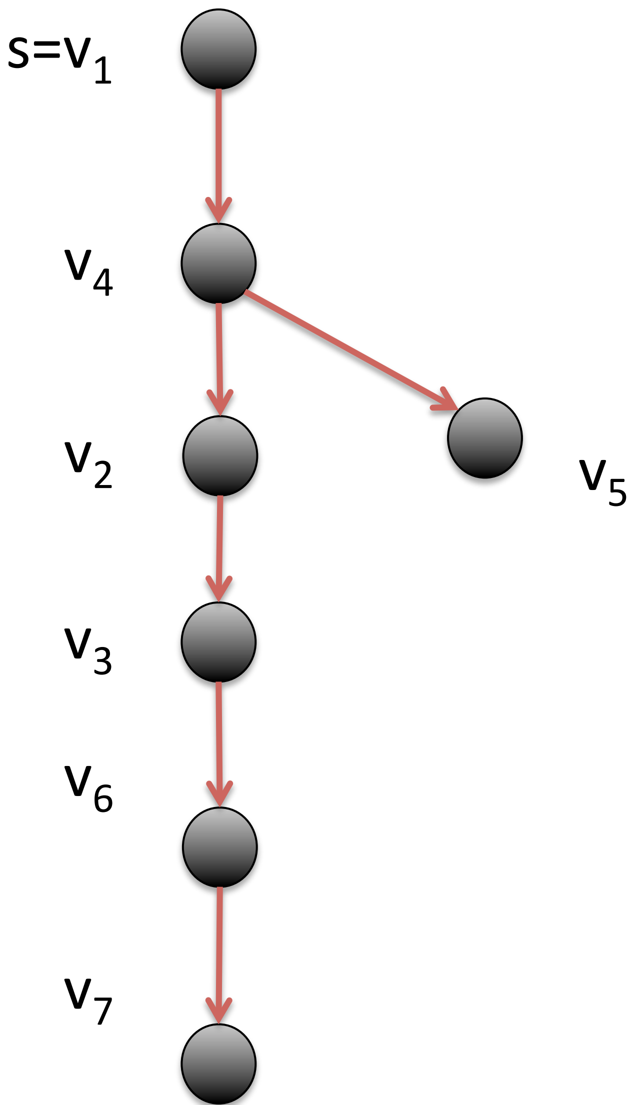




Depth-First-Search Tree



Depth-first search of a directed graph $G = (V, E)$

```
unmark all nodes
init
foreach s ∈ V do
    if s is not marked then
        mark s                                          // make s a root and grow
        root(s)                                         // a new DFS tree rooted at it.
        DFS(s, s)

Procedure DFS(u, v : NodeId)                            // Explore v coming from u.
    foreach (v, w) ∈ E do
        if w is marked then traverseNonTreeEdge(v, w)   // w was reached before
        else    traverseTreeEdge(v, w)                  // w was not reached before
                mark w
                DFS(v, w)
    backtrack(u, v)                                     // return from v along the incoming edge
```

 See Mehlhorn/Sanders, Fig. 9.4

Runtime DFS

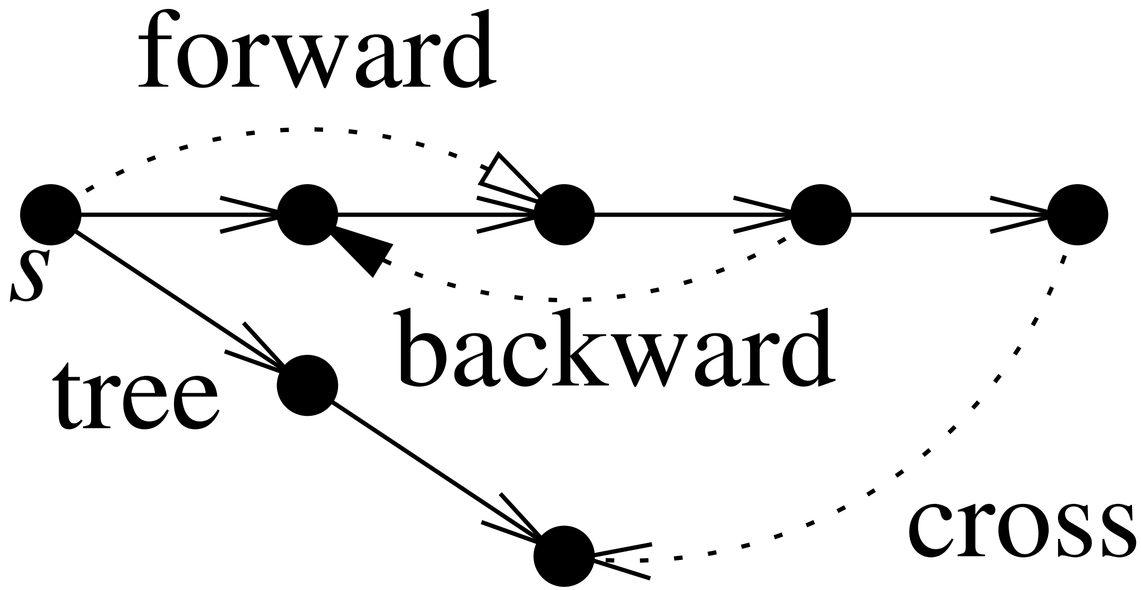
- DFS explores each node and its outgoing edges once.
- Use Adjacency List or an Adjacency Array for representing the graph and remember which edges have already been traversed.
- Runtime: $O(m+n)$

Depth-first-search

Considering a tree T of a given graph G, we can classify the edges into

- Tree edges
- Forward edges
- Backward edges
- Cross edges

Types of Edges



See Mehlhorn/Sanders, Fig. 9.1

Strongly connected components

Two nodes u and v belong to the same strongly connected component if there is a path from u to v and a path from v to u.

Task: Compute the strongly connected components of a given graph.

Undirected Graphs

Compute strongly connected components of a given undirected graph.

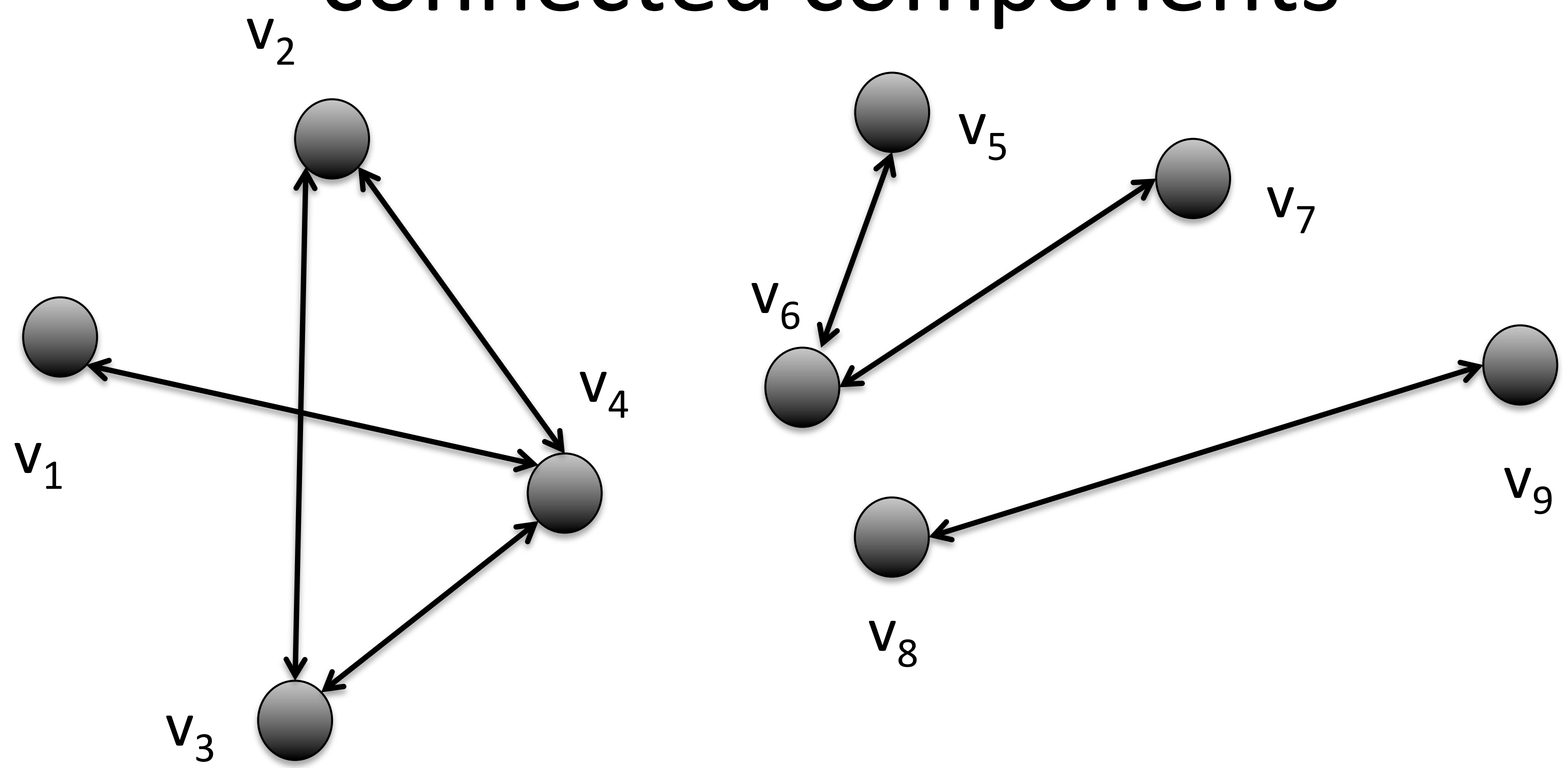
Observation:

- If there is a path from u to v then there is also a path from v to u.

Algorithmic approach:

- Use DFS (or BFS) to compute the different connected components of the given undirected graph.
- Runtime O(m+n).

Undirected graph with three strongly connected components



3 strongly connected components: $\{v_1, v_2, v_3, v_4\}$ $\{v_5, v_6, v_7\}$ $\{v_8, v_9\}$

Undirected Graphs

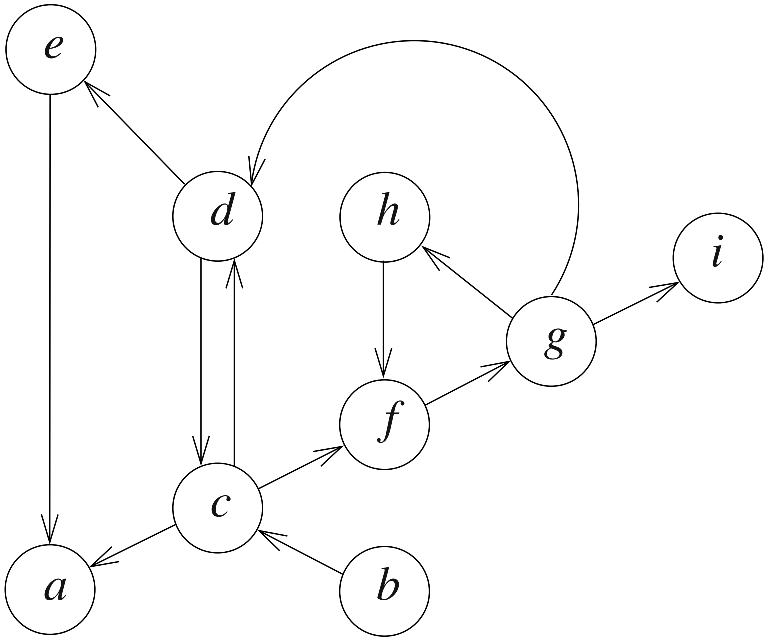
Each single tree corresponds to a connected component

Depth-first search of a directed graph $G = (V, E)$
unmark all nodes
init
foreach $s \in V$ **do**
  if s is not marked **then**
    mark s // make s a root and grow
    $root(s)$ // a new DFS tree rooted at it.
    $DFS(s, s)$

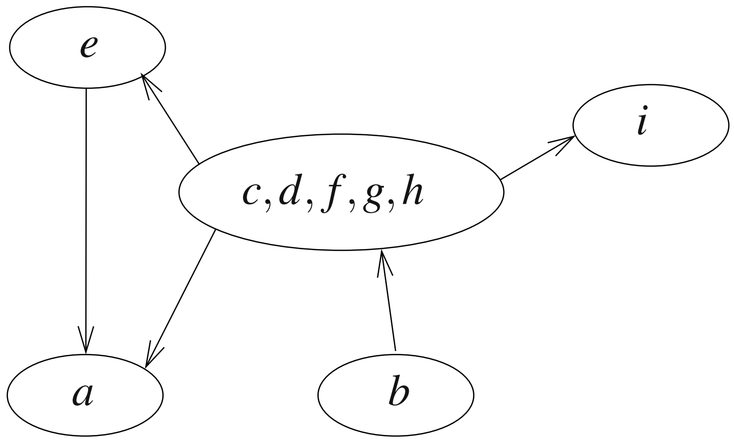
Procedure $DFS(u, v : NodeId)$ // Explore v coming from u.
  foreach $(v, w) \in E$ **do**
    if w is marked **then** $traverseNonTreeEdge(v, w)$ // w was reached before
    else $traverseTreeEdge(v, w)$ // w was not reached before
      mark w
      $DFS(v, w)$
  $backtrack(u, v)$ // return from v along the incoming edge

 See Mehlhorn/Sanders, Fig. 9.4

Directed Graphs



Directed graph



Strongly connected components

How to compute the strongly connected components?

See Mehlhorn/Sanders, Fig. 9.5

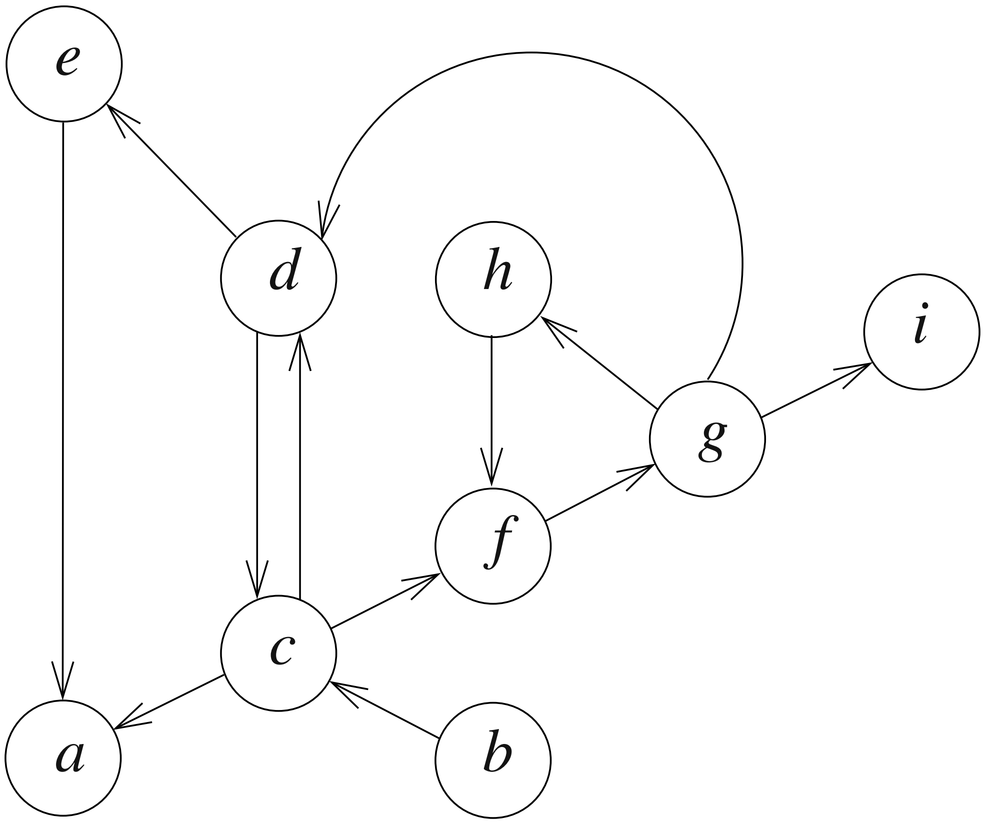
Algorithm (strongly connected components of directed graph G)

1. Run DFS on the given graph G. Number the nodes according to the termination of their recursive calls.
2. Compute the transpose graph G^T of G. It holds
   $(i,j) \in G^T$ if and only if $(j,i) \in G$

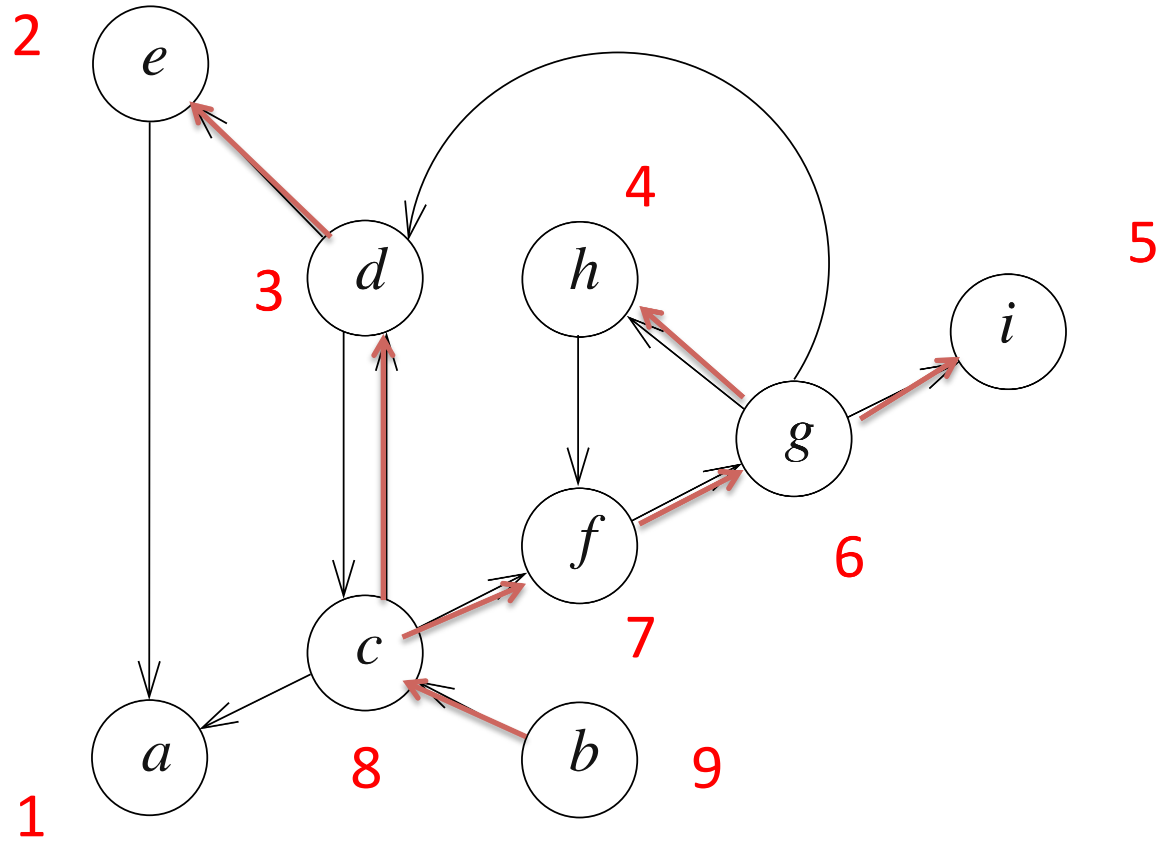
3. Use the numbering of step 1.) to run DFS on G^T. Start with the node that has the highest number. Whenever a tree is completed continue with the unvisited nodes that has the highest number.

4. The single trees computed in step 3) correspond to the node sets of the different strongly connected components.

Directed Graphs

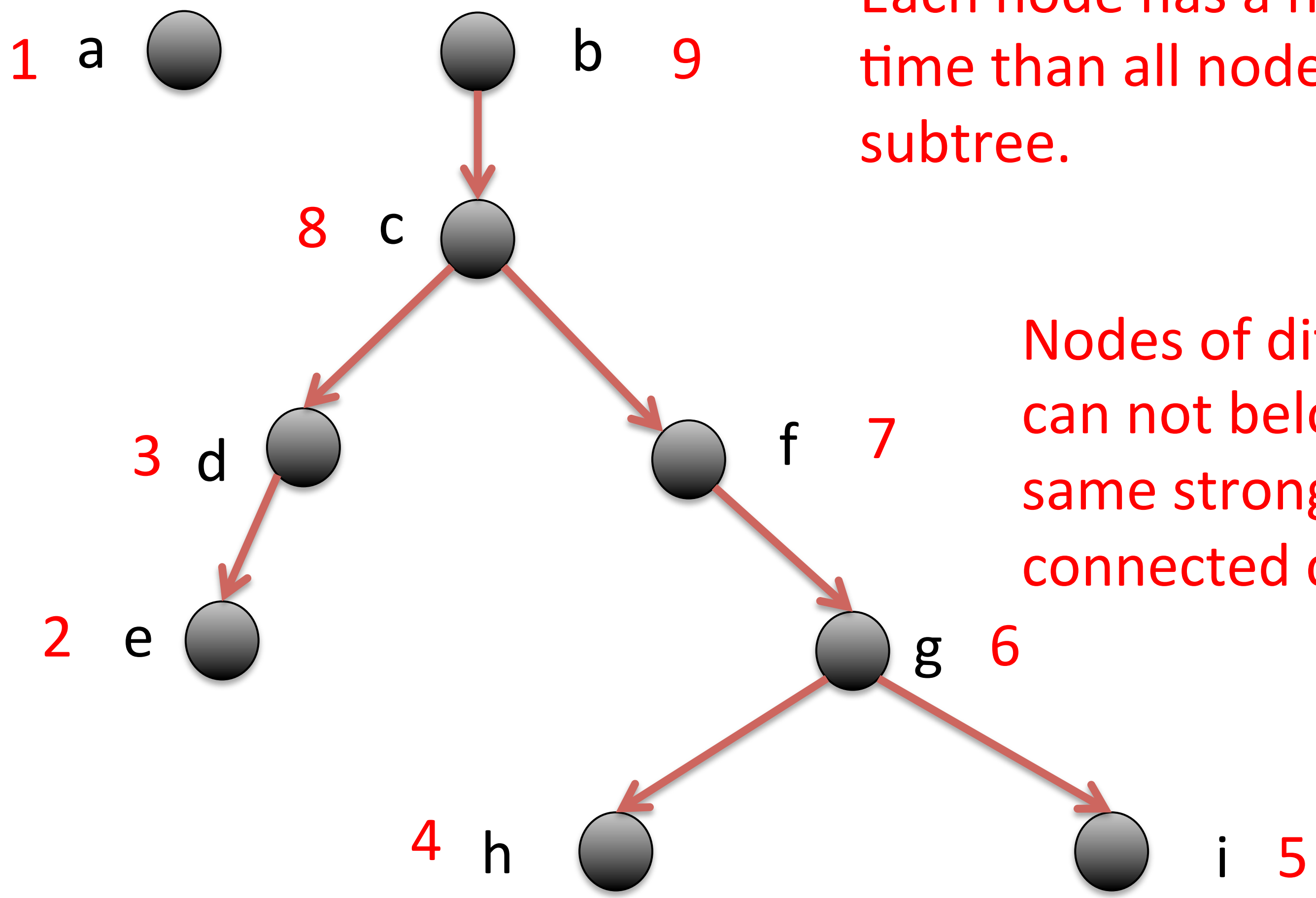


DFS-Tree and numbering



Nodes numbered according to termination of recursive calls

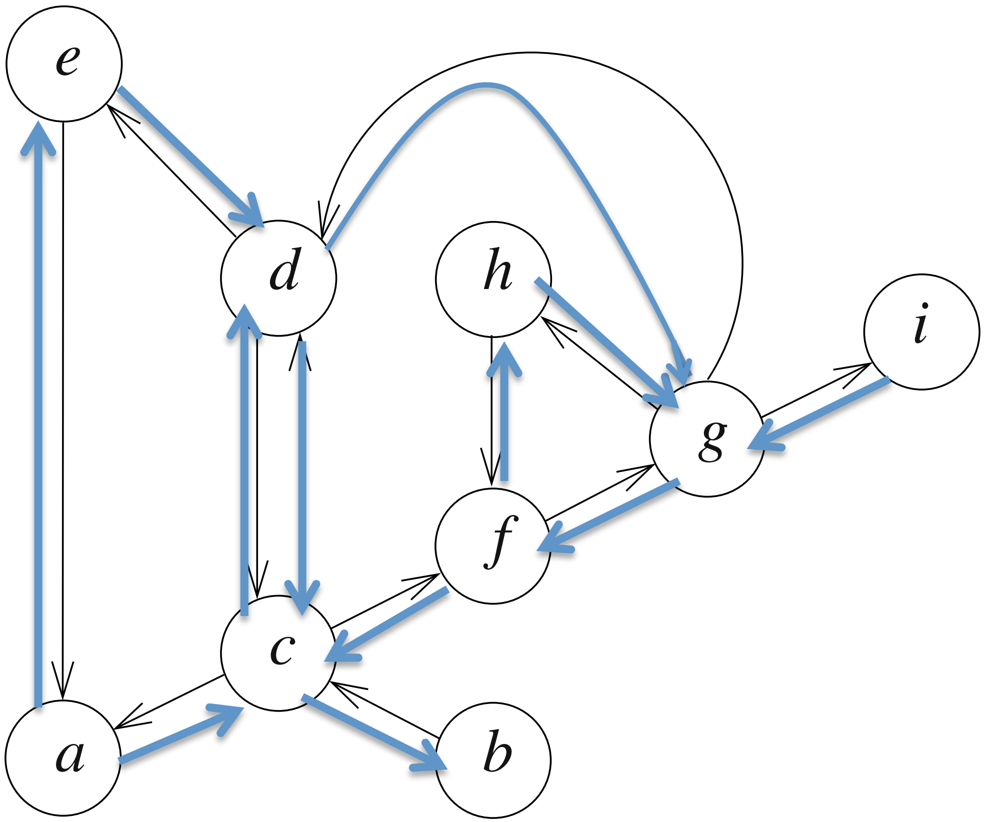
First run: DFS-Tree and finish times



Each node has a higher finish time than all nodes in its subtree.

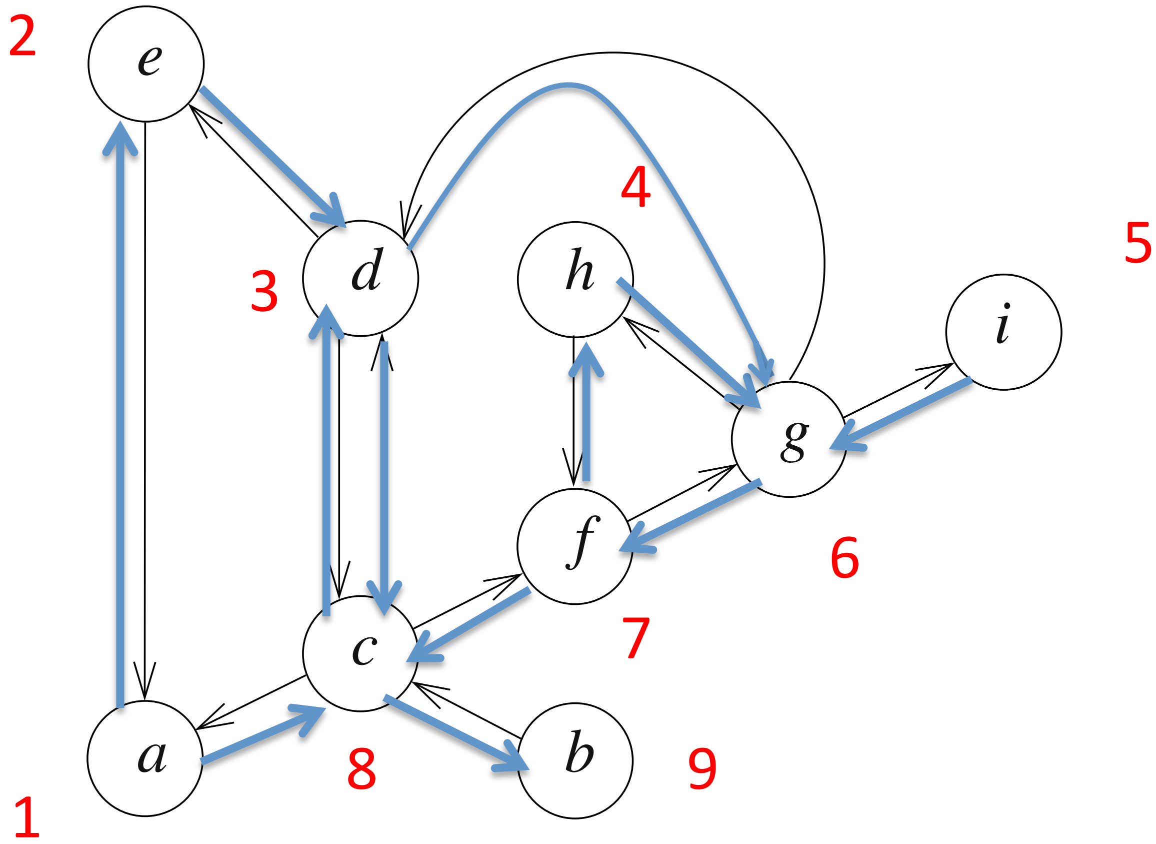
Nodes of different trees can not belong to the same strongly connected component

Directed Graphs

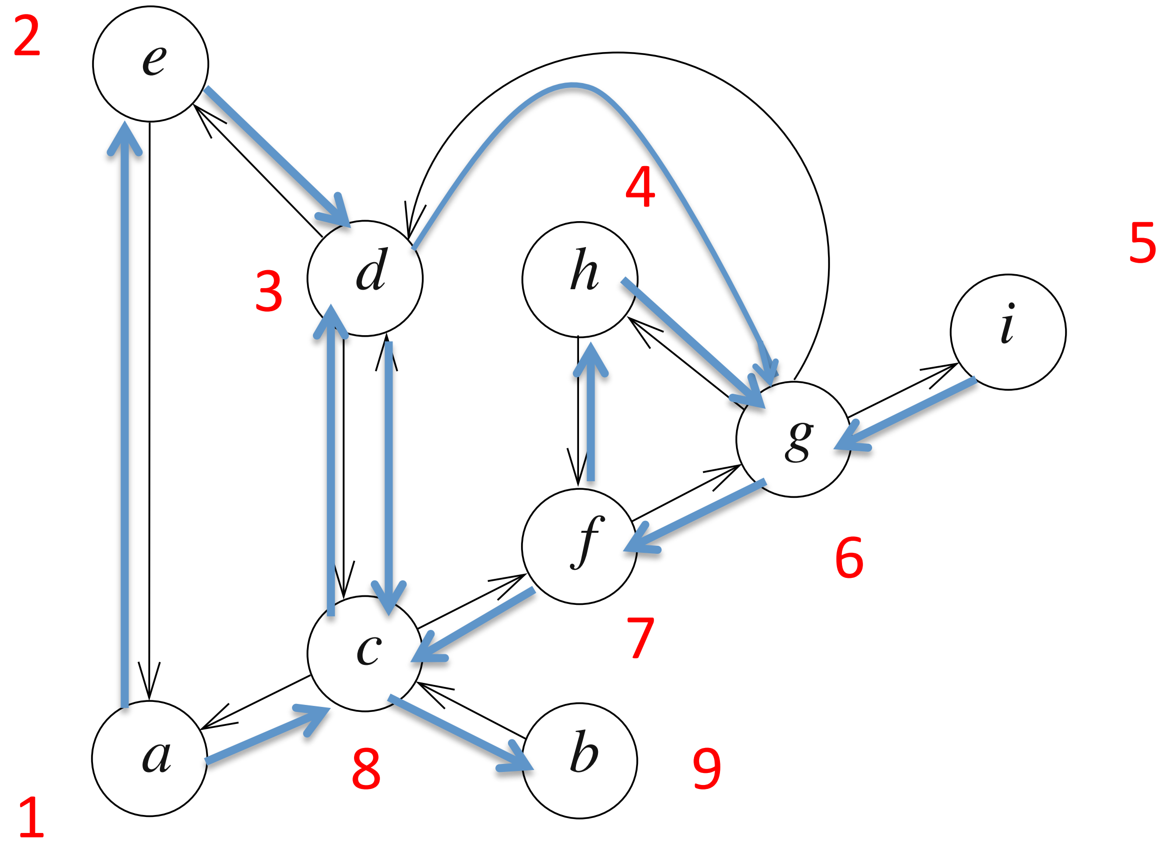


Transposed graph G^T

Directed Graphs



Directed Graphs



3. strongly connected component: {i}

1. strongly connected component: {b}

4. strongly connected component: {e}

2. strongly connected component: {c,d,g,f,h}

5. strongly connected component: {a}

Correctness

Consider DFS-Tree obtained in the first run.

First DFS-run:

- Each root of a (sub)-tree has higher finish time than its children.

Second DFS-run:

- Searches from each root r of a (sub)-tree for a backward path (traveling transposed edges) to its children.
- If a child v is reached then there is a path from v to the root r in that graph.
- This implies that r and v belong to the same strongly connected component.
- Second DFS run can only reach nodes with a smaller numbering.
- Traversing transposed edges implies that no node of another tree of the first DFS run is visited when starting at root r.

Runtime

- Use Adjacency Lists to represent the directed graph.
- We use DFS twice (time $O(m+n)$)
- Have to compute the transpose graph

  (time $O(m+n)$)
- Total runtime: $O(m+n)$