Algorithm and Data Structure Analysis (ADSA)

Lecture 12: Skip Lists

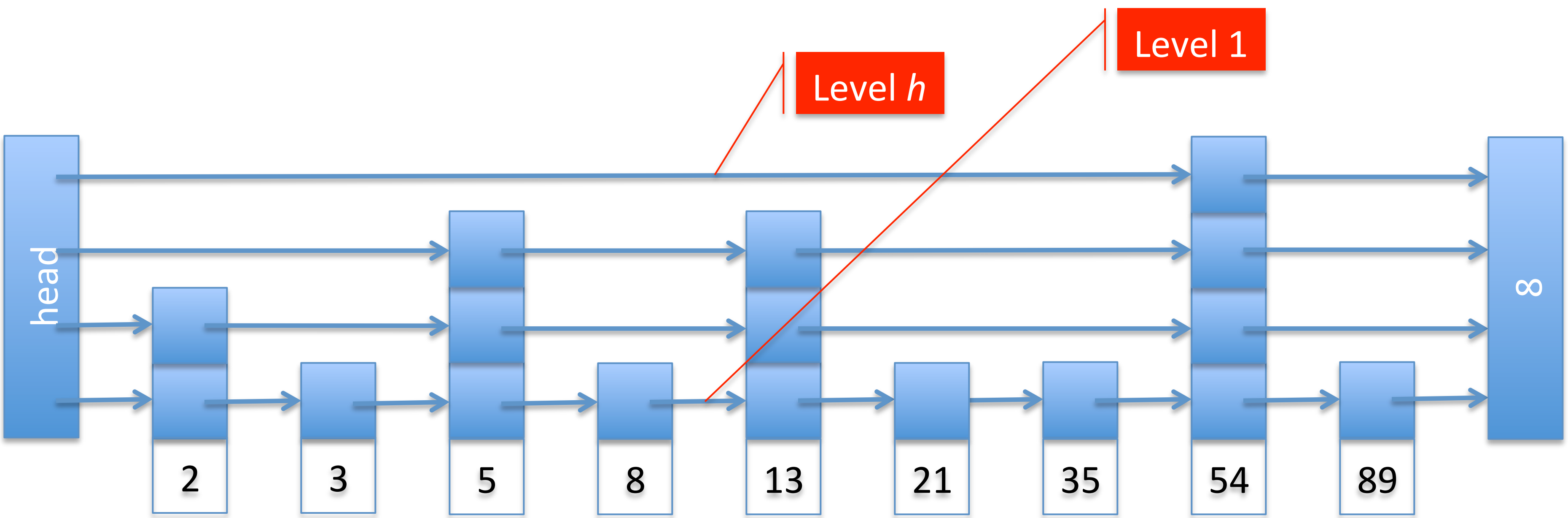
History

Invented by William Pugh (1990)

- A probabilistic data structure likely to replace balanced trees as the implementation method for many applications.
- Algorithms have the same asymptotic expected time bounds as balanced trees and are simpler.

Skip Lists

Sorted list of items, using a hierarchy of linked lists (bottom-top: S_0, S_1, …, S_h) that connect increasingly sparse subsequences of the items.



Skip Lists

Theorem (without proof) Let S be a skip list containing n elements.

The expected*…

- Runtime of a search is $O(\log n)$
- Height of the skip list is $O(\lfloor \log n \rfloor + 3)$
- Number of pointers is $O(2n + \lfloor \log n \rfloor + 3)$

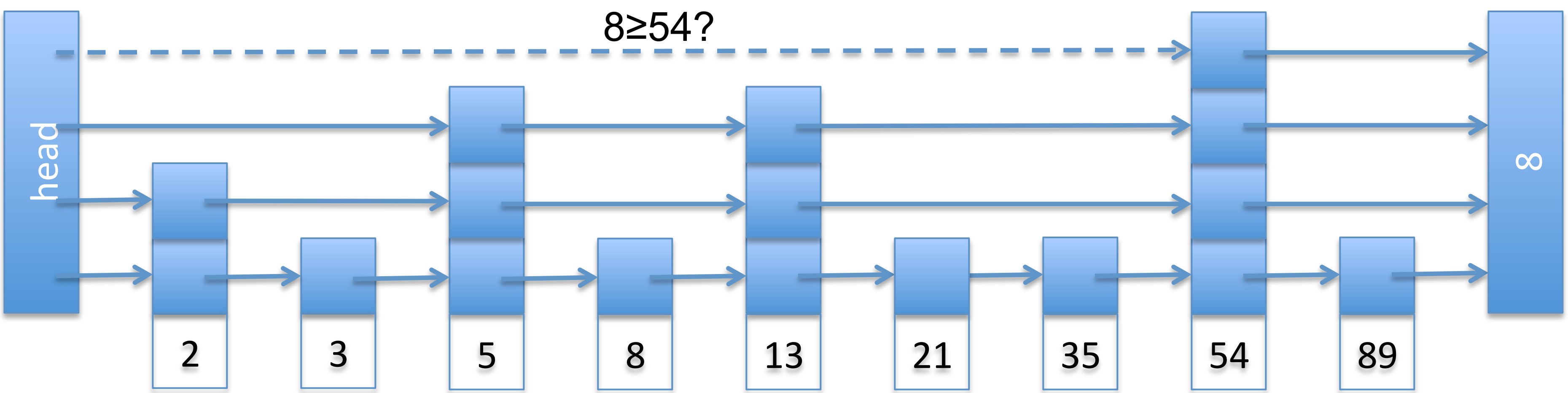
*large deviations extremely unlikely

Search(x)

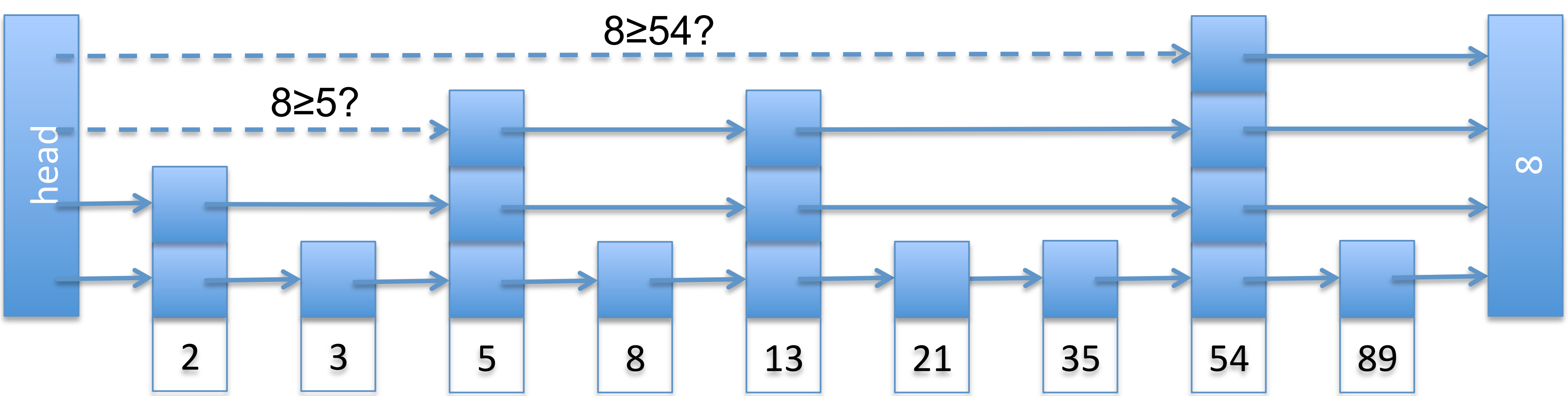
1. Start at highest level.
2. If next element $\leq x$ → go to next element
   else → descend one level

(similar to a search in a binary tree)

Example Search(8)



Example Search(8)



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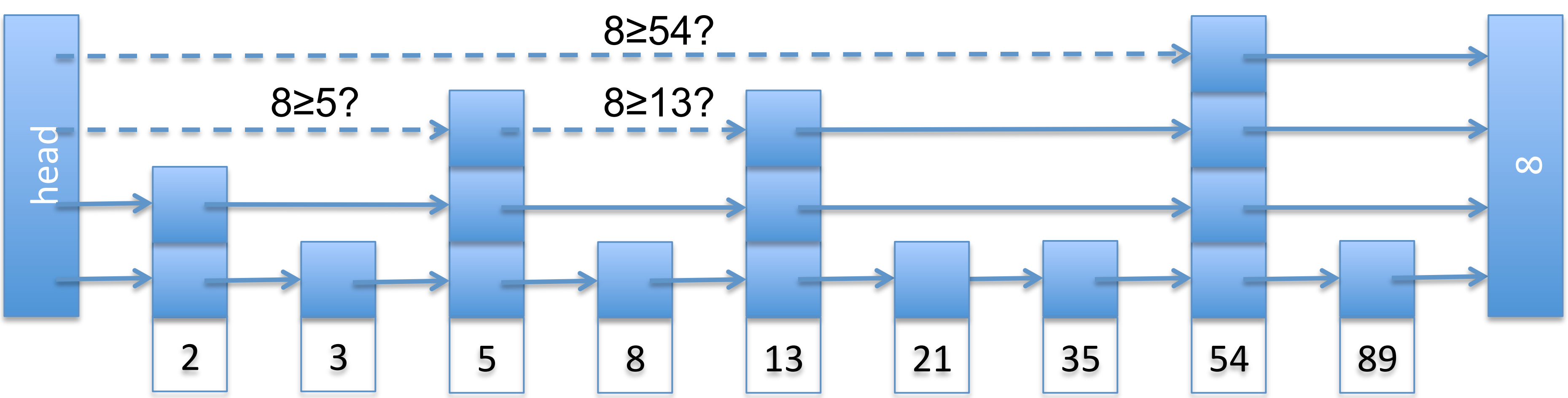

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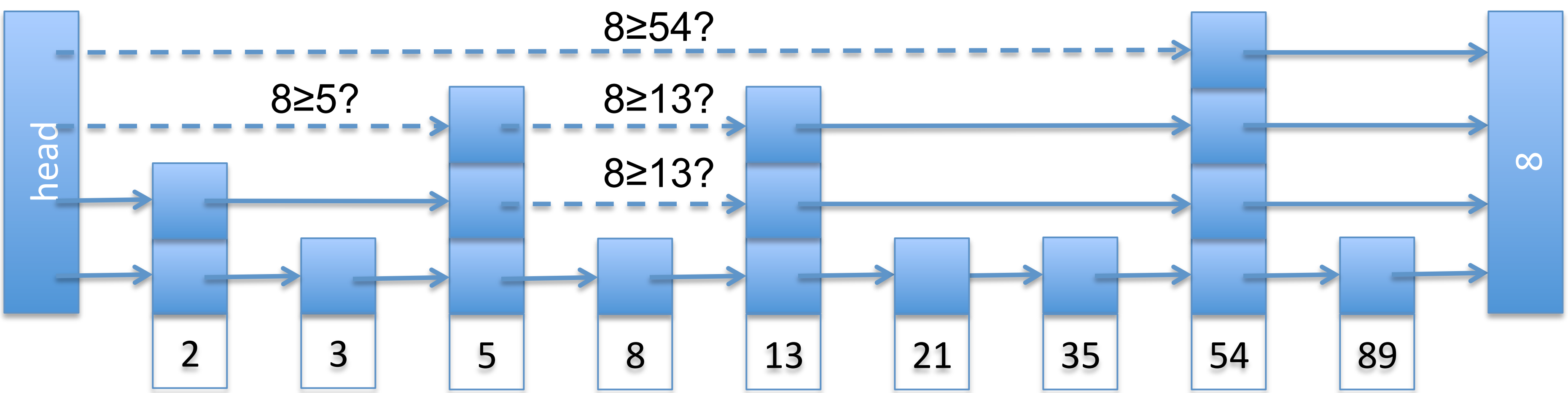

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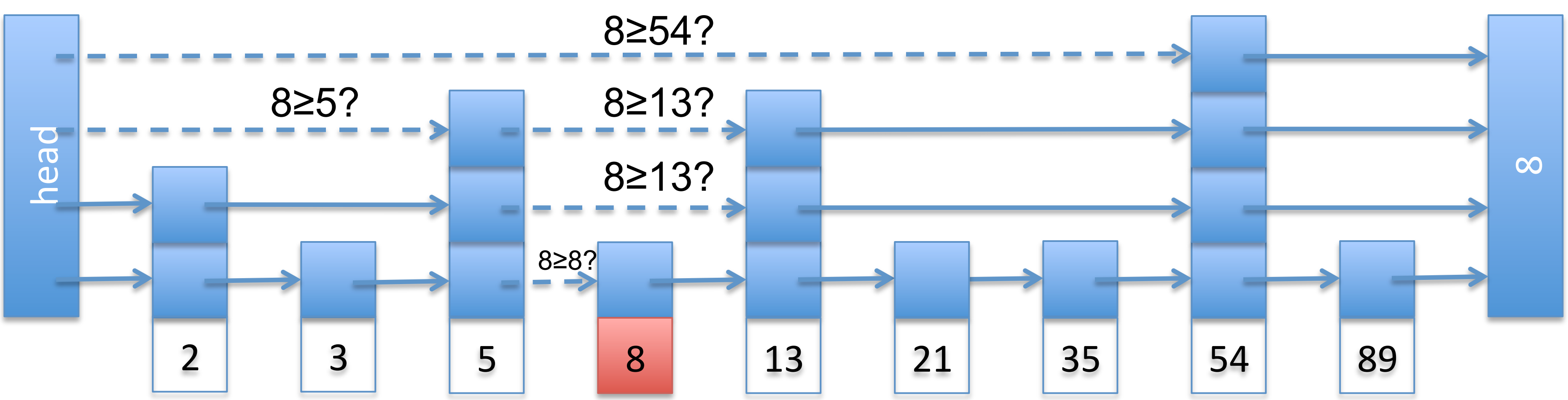
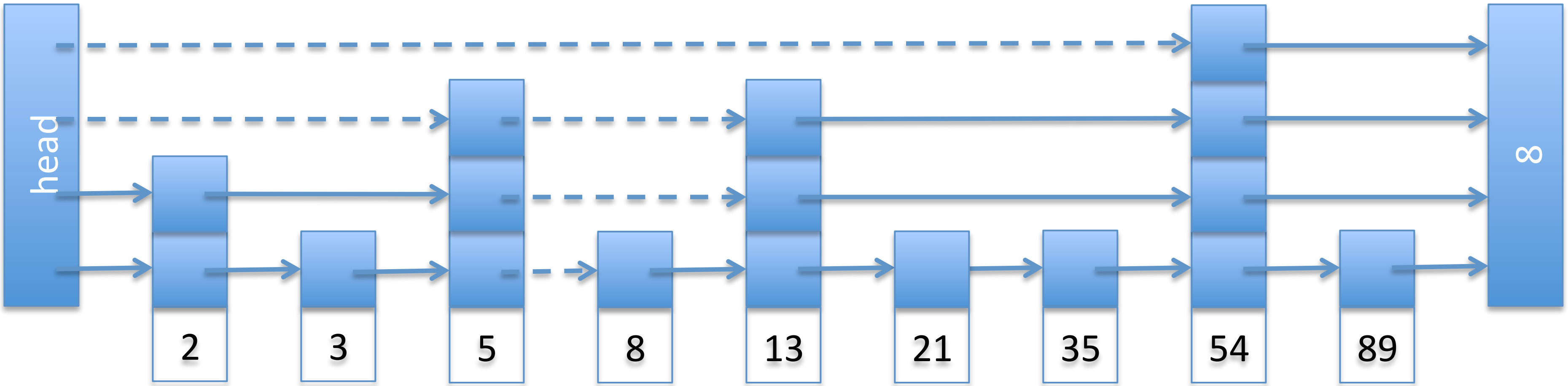


Split(x)

1. Search x.
2. Update pointers:
   - Pointers S_1 from x → point to ∞
   - Pointers S_2 ”over” x → point to ∞
   - Introduce new head for 2nd list, and have it point to where S_1 and S_2 pointed to.

Example Split(8)



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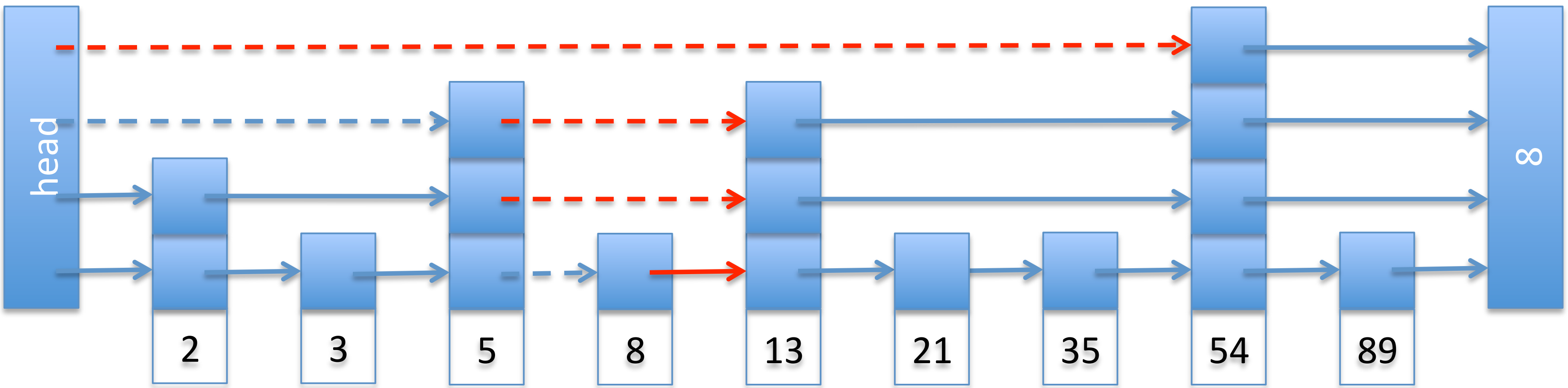

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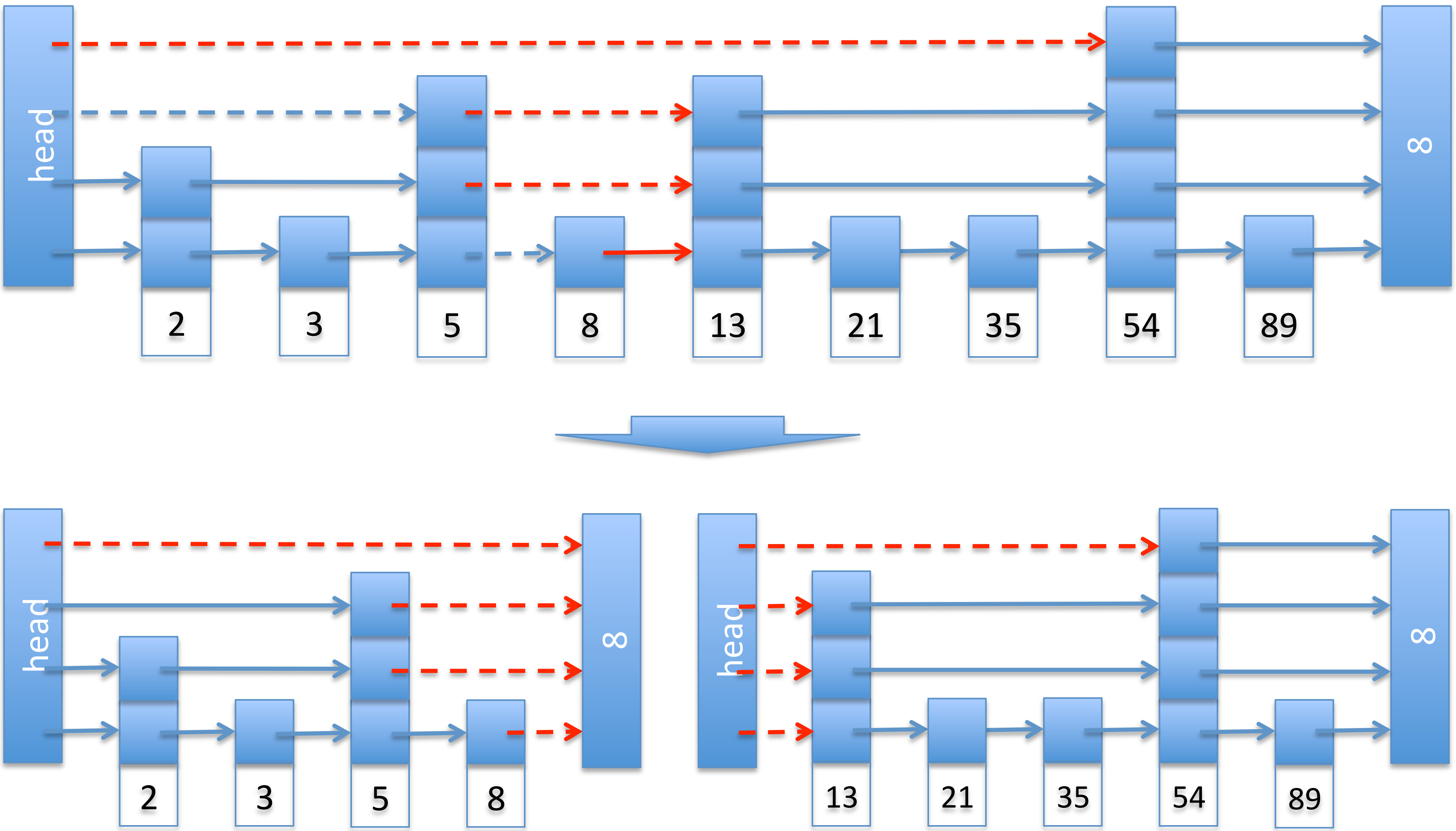

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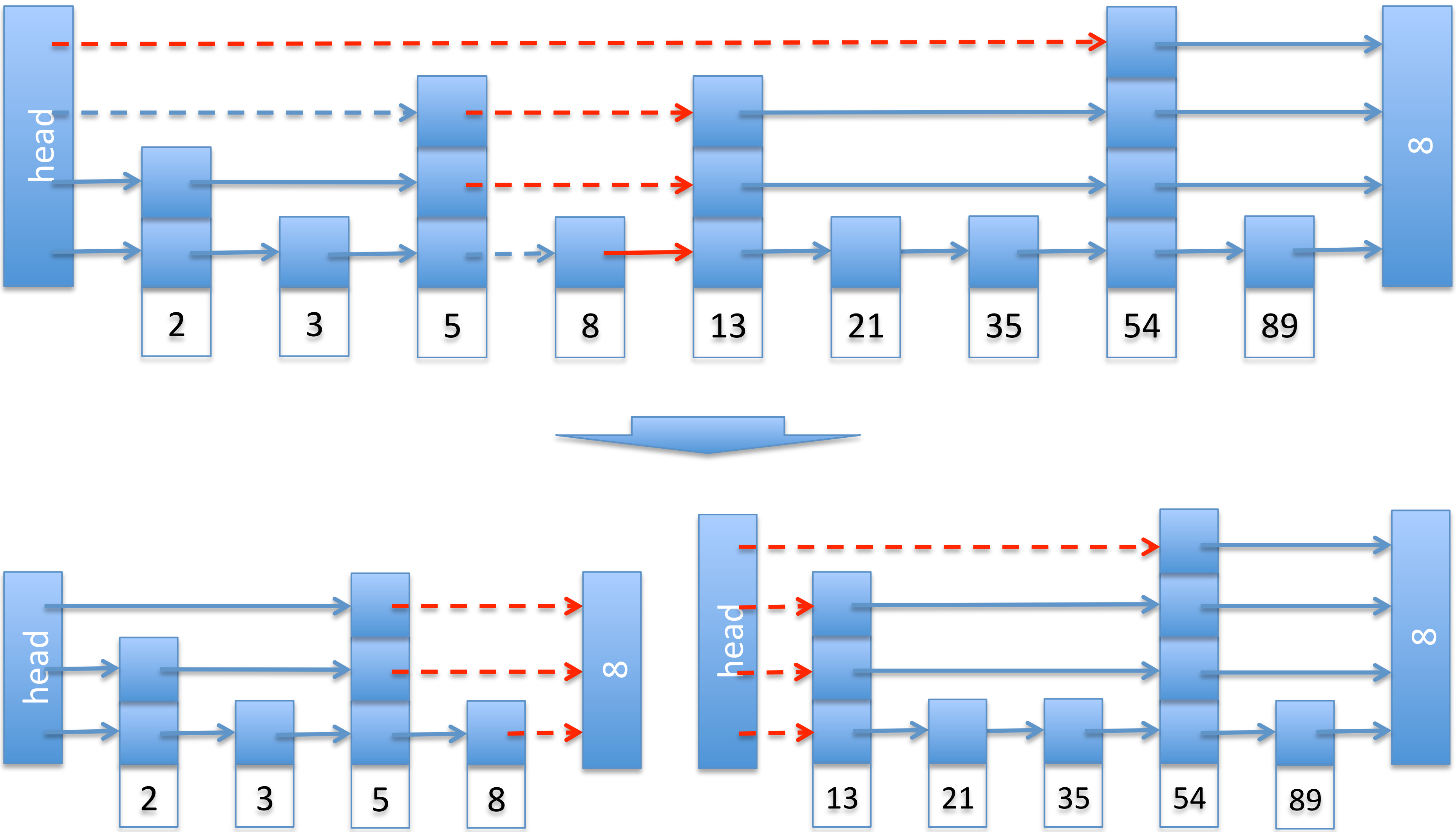
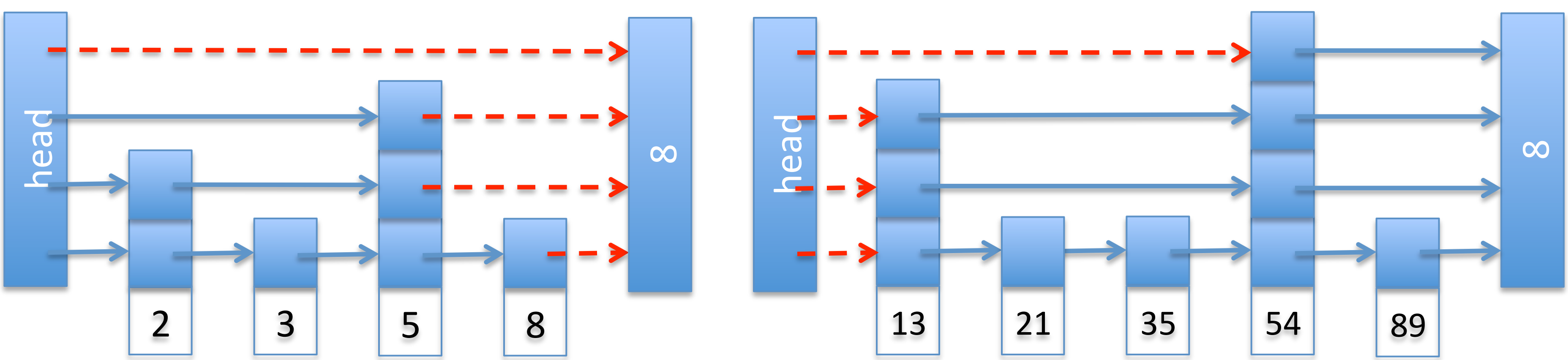
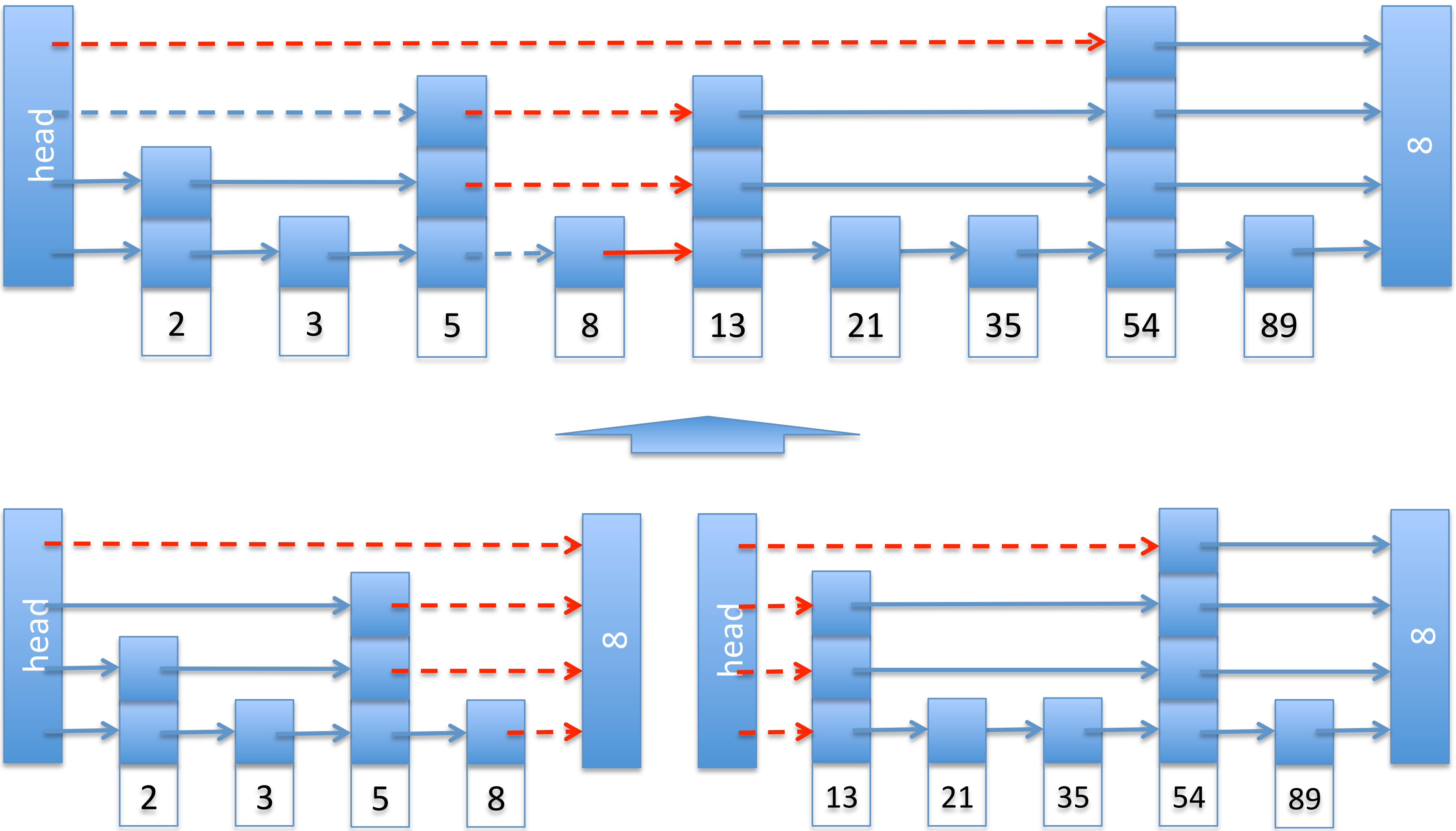


Example Concatenate



Example Concatenate

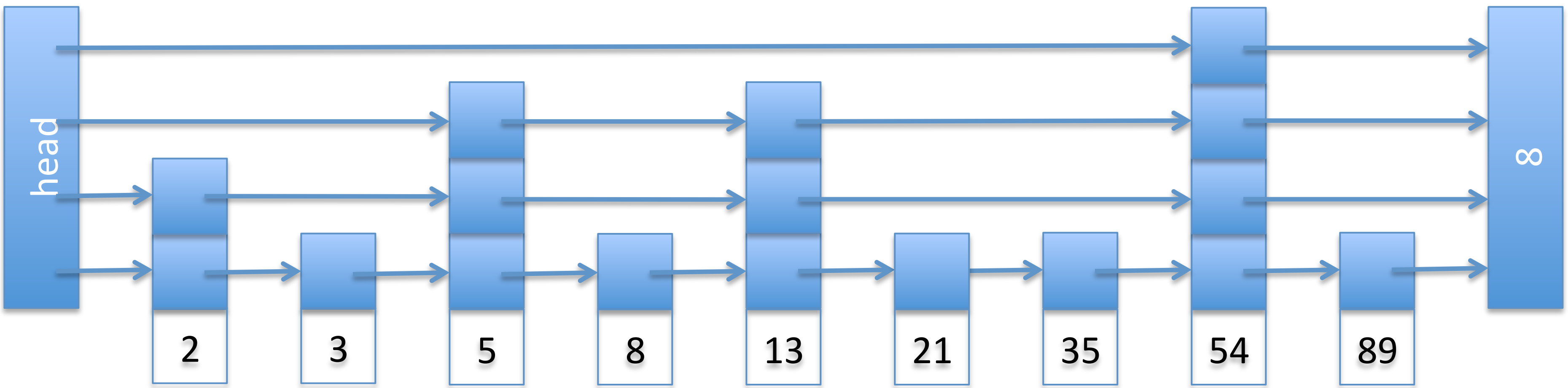


Delete(x)

1. Search x.
2. Update pointers:

   Pointers S through x → point to where x pointed to at the corresponding levels.

Example Delete(13)



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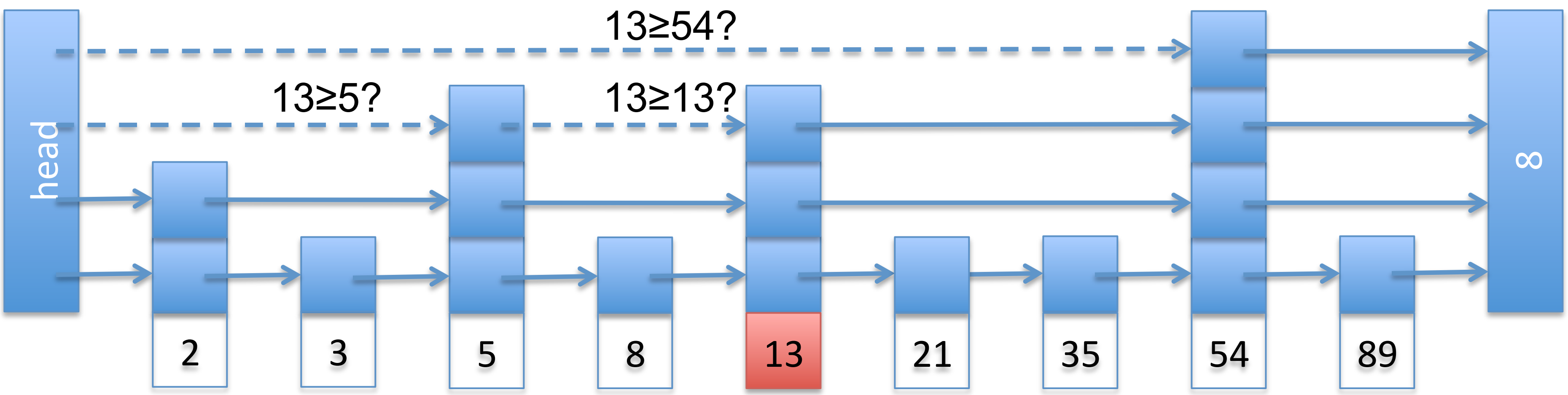
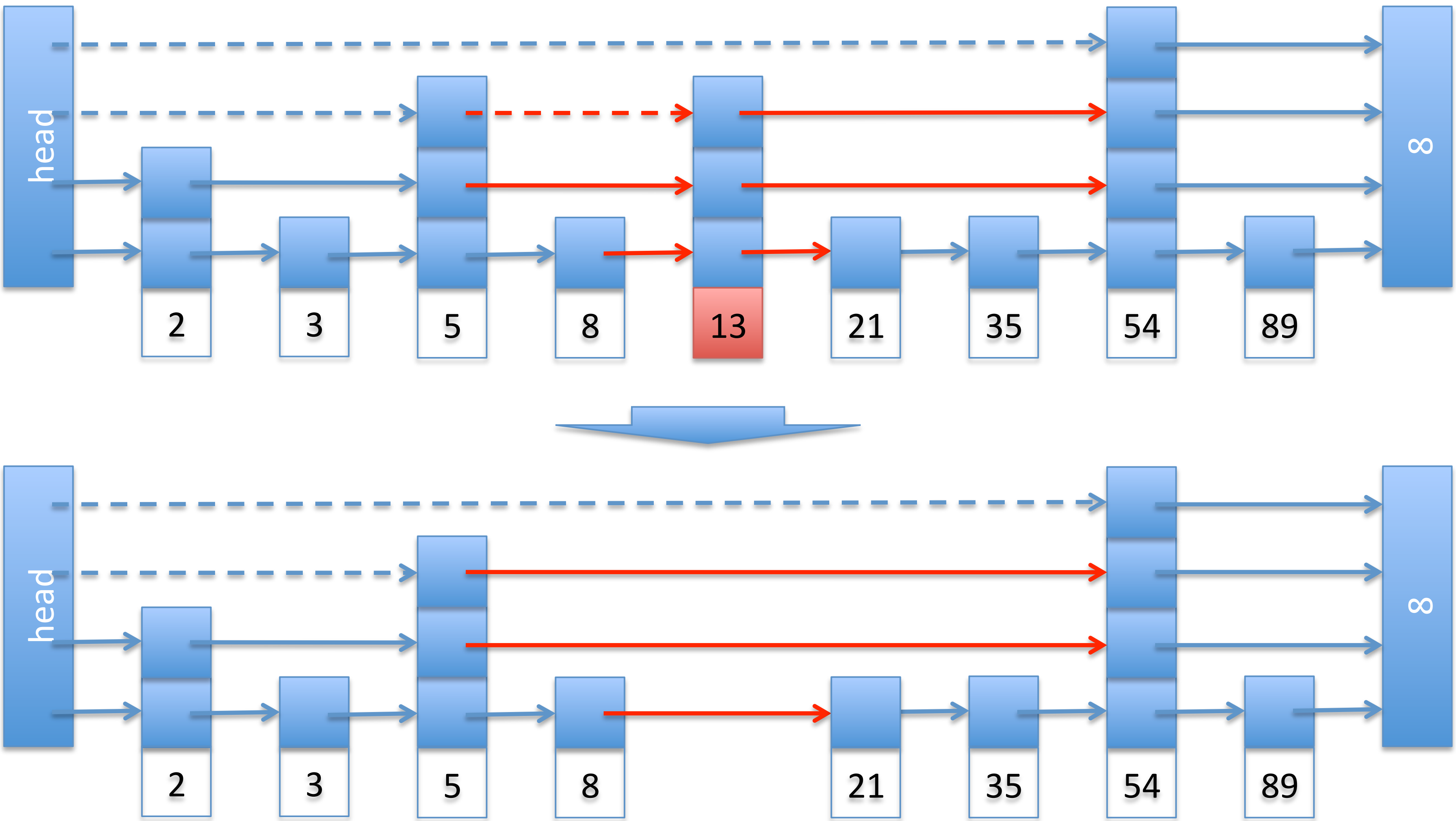


Example Delete(13)



Insert(x)

1. Search for x.
2. Flip coins to set the height h.

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Flip a coin until "head". If h trials are needed, the height of x is h. Thus, the probability for height h is $(1/2)^h$ and the expected height is

$$\sum_{1 \leq h \leq \infty} h \cdot (1/2)^h = 2$$

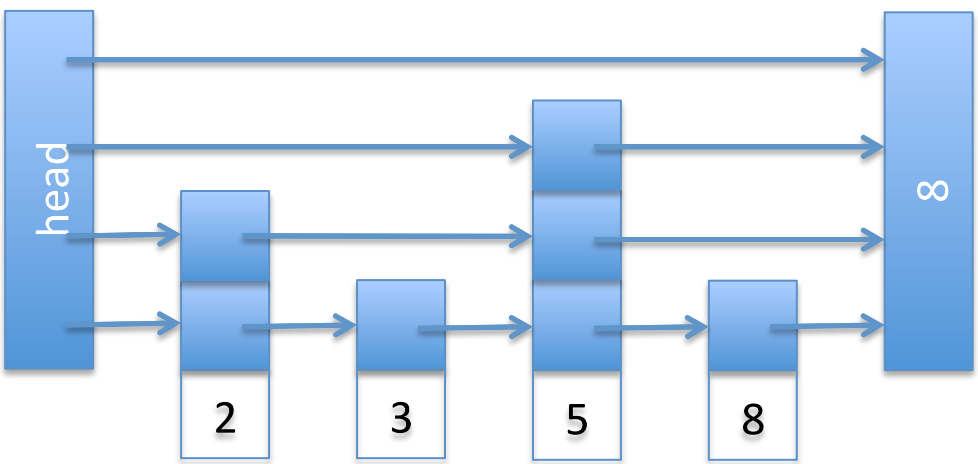
(geometric distribution)

Insert(x)

1. Search for x.
2. Flip coins to set the height h.
3. Update pointers. Note that element x with h will be present in layers $1, \ldots, h$.
   Then, for each layer l:
   If $l>h$ → do nothing,
   else → we know the pointers "going through x" and update those to point to x and from x to the subsequent element.

Example Insert(7)

$h=2$



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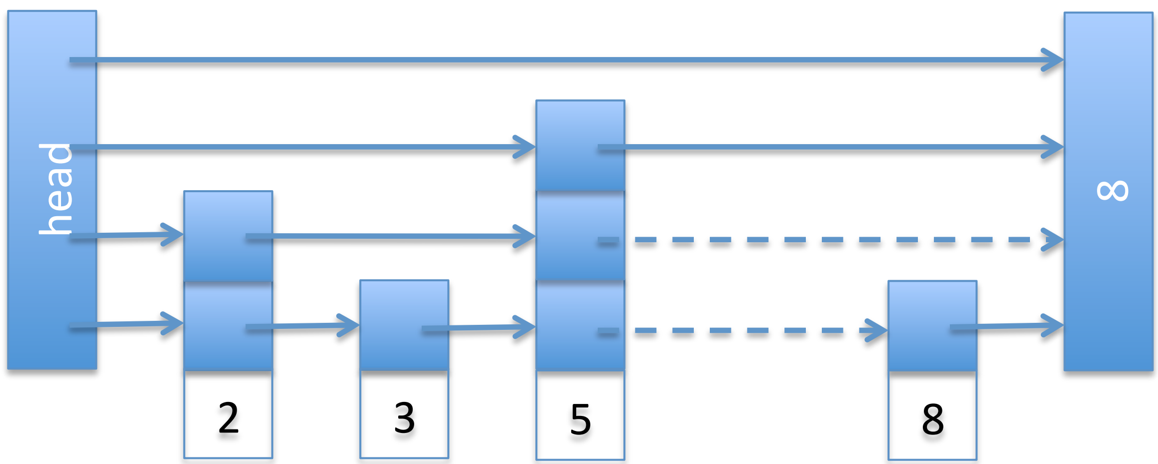
h=2



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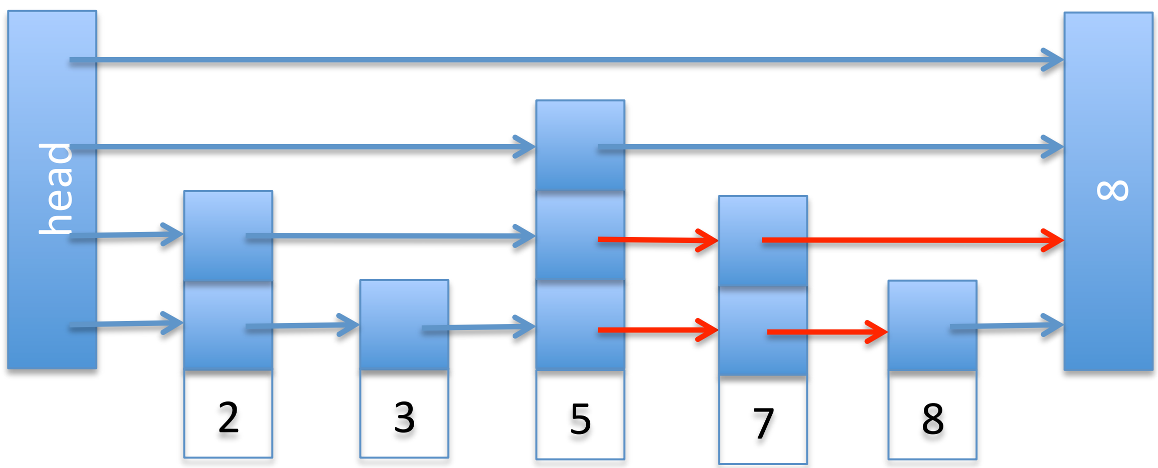
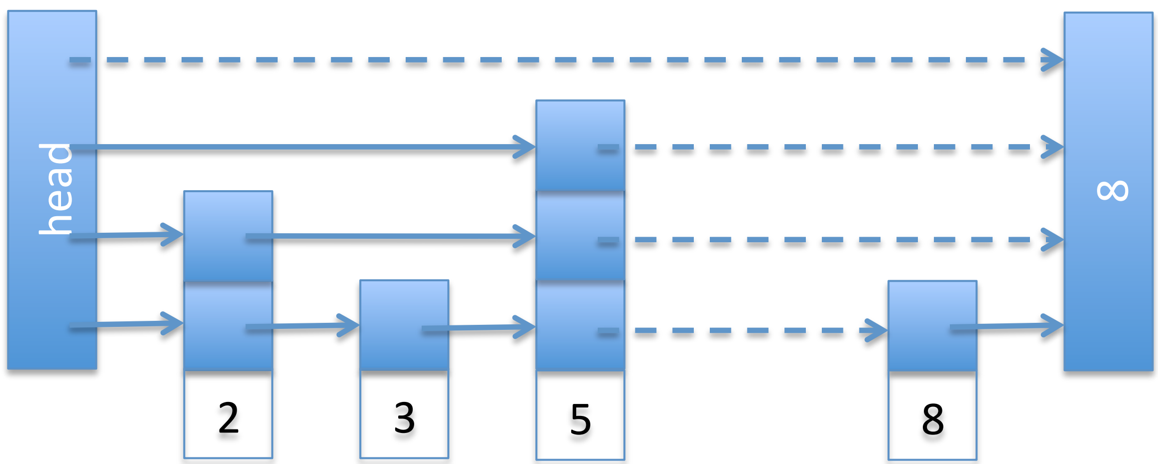


Example Insert(7)

$h=5$



Example Insert(7)

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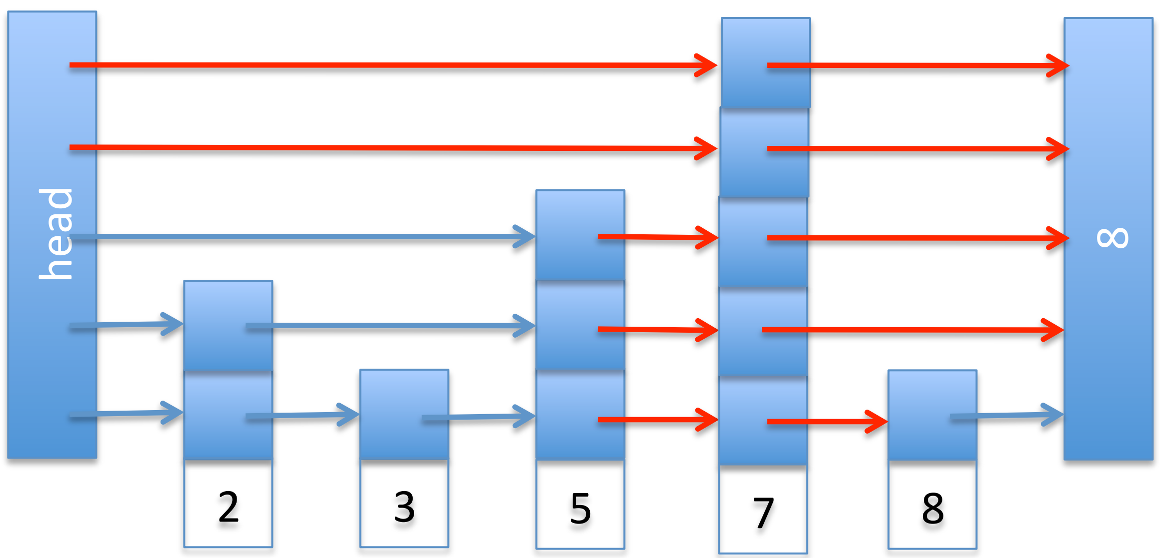

Space Usage

- The space used by a skip list depends on the random bits used by each invocation of the insertion algorithm
- We use the following two basic probabilistic facts:

Fact 1: The probability of getting i consecutive heads when flipping a coin is $1/2^i$

Fact 2: If each of n entries is present in a set with probability p, the expected size of the set is np

- Consider a skip list with n entries
  - By Fact 1, we insert an entry in list S_i with probability $1/2^i$
  - By Fact 2, the expected size of list S_i is $n/2^i$
- The expected number of nodes used by the skip list is

$$\sum_{i=0}^{h} \frac{n}{2^i} = n \sum_{i=0}^{h} \frac{1}{2^i} < 2n$$

- Thus, the expected space usage of a skip list with n items is $O(n)$

Height

- The running time of the search is affected by the height $\boldsymbol{h}$ of the skip lists
- We show that with high probability, a skip list with $\boldsymbol{n}$ items has height $\boldsymbol{O}(\log \boldsymbol{n})$
- We use the following additional probabilistic fact:

Fact 3: If each of $\boldsymbol{n}$ events has probability $\boldsymbol{p}$, the probability that at least one event occurs is at most $\boldsymbol{np}$

- Consider a skip list with $\boldsymbol{n}$ entires
  - By Fact 1, we insert an entry in list $\boldsymbol{S_i}$ with probability $1/2^{\boldsymbol{i}}$
  - By Fact 3, the probability that list $\boldsymbol{S_i}$ has at least one item is at most $\boldsymbol{n}/2^{\boldsymbol{i}}$
- By picking $\boldsymbol{i} = \mathrm{c}\log \boldsymbol{n}$, we have that the probability that $\boldsymbol{S}_{\mathrm{c}\log \boldsymbol{n}}$ has at least one entry is at most
  $$\boldsymbol{n}/2^{\mathrm{c}\log \boldsymbol{n}} = \boldsymbol{n}/\boldsymbol{n}^{\mathrm{c}} = 1/\boldsymbol{n}^{\mathrm{c}}$$
- Thus a skip list with $\boldsymbol{n}$ entries has height at most $\boldsymbol{O}(\log \boldsymbol{n})$ with probability at least $1 - 1/\boldsymbol{n}^{\mathrm{c}}$ that is asymptotically 1 for large constant c.