Entailment: A entail B means that in every model A is true, B is also true.

$$T \models T, \quad T \not\models F, \quad F \models everything$$

7.4 Which of the following are correct?

$(\alpha \land \beta) \equiv (\beta \land \alpha)$ commutativity of $\land$
$(\alpha \lor \beta) \equiv (\beta \lor \alpha)$ commutativity of $\lor$
$((\alpha \land \beta) \land \gamma) \equiv (\alpha \land (\beta \land \gamma))$ associativity of $\land$
$((\alpha \lor \beta) \lor \gamma) \equiv (\alpha \lor (\beta \lor \gamma))$ associativity of $\lor$
$\neg(\neg\alpha) \equiv \alpha$ double-negation elimination
$(\alpha \Rightarrow \beta) \equiv (\neg\beta \Rightarrow \neg\alpha)$ contraposition
$(\alpha \Rightarrow \beta) \equiv (\neg\alpha \lor \beta)$ implication elimination
$(\alpha \Leftrightarrow \beta) \equiv ((\alpha \Rightarrow \beta) \land (\beta \Rightarrow \alpha))$ biconditional elimination
$\neg(\alpha \land \beta) \equiv (\neg\alpha \lor \neg\beta)$ de Morgan
$\neg(\alpha \lor \beta) \equiv (\neg\alpha \land \neg\beta)$ de Morgan
$(\alpha \land (\beta \lor \gamma)) \equiv ((\alpha \land \beta) \lor (\alpha \land \gamma))$ distributivity of $\land$ over $\lor$
$(\alpha \lor (\beta \land \gamma)) \equiv ((\alpha \lor \beta) \land (\alpha \lor \gamma))$ distributivity of $\lor$ over $\land$

a. $False \models True$.

True

b. $True \models False$.

False

c. $(A \land B) \models (A \Leftrightarrow B)$.

False

$A \land B \models False, (A \land B) \models everything$

$A \land B \models True\ means\ (A \Leftrightarrow B) = True, Hence\ (A \land B) \models (A \Leftrightarrow B)$

d. $A \Leftrightarrow B \models A \lor B$.

False

$If\ A = F\ and\ B = F, (A \Leftrightarrow B = T), but\ (A \lor B = F). T \not\models F$

e. $A \Leftrightarrow B \models \neg A \lor B$.

True

$If\ A \Leftrightarrow B = F, \quad A \Leftrightarrow B \models \neg A \lor B$

$If\ A \Leftrightarrow B = T, A = B = T\ or\ A = B = F\ and\ (\neg A \lor B = T), A \Leftrightarrow B \models \neg A \lor B$

f. $(A \land B) \Rightarrow C \models (A \Rightarrow C) \lor (B \Rightarrow C)$.

True

$If\ (A \land B \Rightarrow C) = F,\ (A \land B \Rightarrow C) \models everything$

$If\ (A \land B \Rightarrow C) = T$

$If\ A \land B = F,\ (A \Rightarrow C) = T\ OR\ (B \Rightarrow C) = T, (A \land B \Rightarrow C) \models (A \Rightarrow C) \lor (B \Rightarrow C)$

$If\ A \land B = T,\ (A \Rightarrow C) = T\ AND\ (B \Rightarrow C) = T, (A \land B \Rightarrow C) \models (A \Rightarrow C) \lor (B \Rightarrow C)$

g. $(C \lor (\neg A \land \neg B)) \equiv ((A \Rightarrow C) \land (B \Rightarrow C))$.

True

$$\begin{aligned} C \lor (\neg A \land \neg B) &\equiv (\neg A \lor C) \land (\neg B \lor C) \\ &\equiv (A \Rightarrow C) \land (B \Rightarrow C) \end{aligned}$$

h. $(A \lor B) \land (\neg C \lor \neg D \lor E) \models (A \lor B)$.

$LHS = T\ iff\ (A \lor B) = T, thus\ if\ LHS = T\ then\ RHS = T,\ T \models T$

i. $(A \lor B) \land (\neg C \lor \neg D \lor E) \models (A \lor B) \land (\neg D \lor E)$.

False

$If\ A = T, B = T, C = F, D = T, E = F\ then\ LHS = T\ and\ RHS = F, but\ T \nvDash F$

j. $(A \lor B) \land \neg(A \Rightarrow B)$ is satisfiable.

True, when $A = T\ and\ B = F$

k. $(A \Leftrightarrow B) \land (\neg A \lor B)$ is satisfiable.

True, when $(A = B = T)\ or\ (A = B = F)$

l. $(A \Leftrightarrow B) \Leftrightarrow C$ has the same number of models as $(A \Leftrightarrow B)$ for any fixed set of proposition symbols that includes A, B, C.

False

A	B	C		A	B
T	T	T		T	T
F	F	T		F	F
T	F	F			
F	T	F			

$(A \Leftrightarrow B) \Leftrightarrow C$ has 4 models

$(A \Leftrightarrow B)$ has 2 models

$4 \neq 2$

7.5 Prove each of the following assertions:

$$A \Leftrightarrow B \equiv (A \Rightarrow B) \land (A \Leftarrow B)$$

a. α is valid if and only if $True \models \alpha$.

α is valid, hence $\alpha = T$ is all models

$\alpha = T \Rightarrow T \vDash \alpha$

$T \vDash \alpha \Rightarrow \alpha = T$

Hence α is valid

b. For any α, $False \models \alpha$.

True by the definition of entailment. False entails everything

c. $\alpha \models \beta$ if and only if the sentence $(\alpha \Rightarrow \beta)$ is valid.

$(\alpha \vDash \beta) \Rightarrow (\alpha \Rightarrow \beta)$

If $(\alpha \vDash \beta) = F\ then\ (\alpha \vDash \beta) \Rightarrow everything$

If $(\alpha \vDash \beta) = T\ then\ (\alpha = F)\ or\ (\alpha = T\ and\ B = T)$

If $\alpha = F\ then\ (\alpha \Rightarrow \beta = T)\ and\ (\alpha \vDash \beta) \Rightarrow (\alpha \Rightarrow \beta)$

If $(\alpha = T\ and\ B = T)\ then\ (\alpha \Rightarrow \beta = T)\ and\ (\alpha \vDash \beta) \Rightarrow (\alpha \Rightarrow \beta)$

$(\alpha \Rightarrow \beta) \Rightarrow (\alpha \vDash \beta)$

If $(\alpha \Rightarrow \beta) = F\ then\ (\alpha \Rightarrow \beta)\ \Rightarrow everything$

If $(\alpha \Rightarrow \beta) = T\ then\ (\alpha = F)\ or\ (\alpha = T\ and\ B = T)$

If $\alpha = F\ then\ (\alpha \vDash \beta = T)\ and\ (\alpha \Rightarrow \beta) \Rightarrow (\alpha \vDash \beta)$

If $(\alpha = T\ and\ B = T)\ then\ (\alpha \vDash \beta = T)\ and\ (\alpha \Rightarrow \beta) \Rightarrow (\alpha \vDash \beta)$

Hence, $(\alpha \vDash \beta)\ \Leftrightarrow (\alpha \Rightarrow \beta)$

d. $\alpha \equiv \beta$ if and only if the sentence $(\alpha\ \Leftrightarrow\ \beta)$ is valid.

$$\alpha \equiv \beta\ means\ \big((\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)\big)$$

$\big((\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)\big) \Rightarrow (\alpha \Leftrightarrow \beta)$

If $\big((\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)\big) = F, then\ \big((\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)\big) \Rightarrow everything$

If $\big((\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)\big) = T, then\ \ (\alpha = B = T)\ OR\ (\alpha = B = F)$

If $\ (\alpha = T\ and\ B = T), then\ (\alpha \Leftrightarrow \beta = T)\ and\ \big((\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)\big) \Rightarrow (\alpha \Leftrightarrow \beta)$

If $\ (\alpha = F\ and\ B = F), then\ (\alpha \Leftrightarrow \beta = T)\ and\ \big((\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)\big) \Rightarrow (\alpha \Leftrightarrow \beta)$

$(\alpha \Leftrightarrow \beta) \Rightarrow \big((\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)\big)$

If $(\alpha \Leftrightarrow \beta) = F, then\ (\alpha \Leftrightarrow \beta) \Rightarrow everything$

If $(\alpha \Leftrightarrow \beta) = T, then\ (\alpha = \beta = T)\ OR\ (\alpha = B = F)$

If $\ (\alpha = T\ and\ B = T), then\ \big((\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha) = T\big)\ and\ (\alpha \Leftrightarrow \beta) \Rightarrow \big((\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)\big)$

If $\ (\alpha = F\ and\ B = F), then\ \big((\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha) = T\big)\ and\ (\alpha \Leftrightarrow \beta) \Rightarrow \big((\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)\big)$

Hence $(\alpha \equiv \beta) \Leftrightarrow (\alpha \Leftrightarrow \beta)$

e. $\alpha \models \beta$ if and only if the sentence $(\alpha \wedge \neg\beta)$ is unsatisfiable.

$(\alpha \wedge \neg\beta)$ cannot be True (unsatisfiable) means $(\neg(\alpha \wedge \neg\beta) = (\neg\alpha \wedge \beta) = (\alpha \Rightarrow \beta))$ must be True. Reduces to proving $(\alpha \vDash \beta)\ \Leftrightarrow (\alpha \Rightarrow \beta)$, same as c)

7.6 Prove, or find a counterexample to, each of the following assertions:

a. If $\alpha \models \gamma$ or $\beta \models \gamma$ (or both) then $(\alpha \wedge \beta) \models \gamma$

True

If $(\alpha \wedge \beta) = F, (\alpha \wedge \beta) \vDash \gamma$

If $(\alpha \wedge \beta) = T\ \ then (\alpha = \beta = T)\ and\ \ (\alpha \vDash \gamma) or\ (\beta \vDash \gamma)\ ensures\ \gamma = T$

b. If $\alpha \models (\beta \wedge \gamma)$ then $\alpha \models \beta$ and $\alpha \models \gamma$.

True

If $\alpha = F, then\ \alpha \vDash anything$

If $\alpha = T, then\ (\beta \wedge \gamma) = T, \beta = \gamma = T.\ \ \alpha \vDash \beta\ and\ \alpha \vDash \gamma$

c. If $\alpha \models (\beta \vee \gamma)$ then $\alpha \models \beta$ or $\alpha \models \gamma$ (or both).

True

If $\alpha = F, then\ \alpha \vDash anything$

If $\alpha = T, then\ (\beta \vee \gamma) = T, \beta = T\ \vee\ \gamma = T,\ \alpha \vDash \beta\ \vee\ \ \alpha \vDash \gamma$

7.10 Decide whether each of the following sentences is valid, unsatisfiable, or neither. Verify your decisions using truth tables or the equivalence rules of Figure 7.11 (page 249).

Valid: Always True (always satisfiable)

Unsatisfiable: Always False (never satisfiable)

Neither: Sometimes True and sometimes False

a. $Smoke \Rightarrow Smoke$

Valid

S	S	
T	F	Satisfied
F	F	Satisfied

b. $Smoke \Rightarrow Fire$

Neither

S	F	
T	T	Satisfied
T	F	Not satisfied
F	T	Satisfied
F	F	Sarisfied

c. $(Smoke \Rightarrow Fire) \Rightarrow (\neg Smoke \Rightarrow \neg Fire)$

Neither

$(LHS = (S \Rightarrow F)) \Rightarrow (RHS = (\neg S \Rightarrow \neg F))$

LHS	S	F	RHS	Not S	Not F	
T	T	T	**T**	F	F	Satisfied
F	T	F	**T**	F	T	Satisfied
T	F	T	**F**	T	F	Not satisfied
T	F	F	**T**	T	T	Satisfied

d. $Smoke \vee Fire \vee \neg Fire$

Valid

$S \vee F \vee \neg F = S \vee T = T$

e. $((Smoke \wedge Heat) \Rightarrow Fire) \Leftrightarrow ((Smoke \Rightarrow Fire) \vee (Heat \Rightarrow Fire))$

Valid

LHS: $((S \wedge H) \Rightarrow F) \equiv \neg(S \wedge H) \vee F) \equiv \neg S \vee \neg H \vee F$

RHS: $((S \Rightarrow F) \vee (H \Rightarrow F)) \equiv (\neg S \vee F) \vee (\neg H \vee F) \equiv \neg S \vee \neg H \vee F$

$((S \wedge H) \Rightarrow F) \equiv ((S \Rightarrow F) \vee (H \Rightarrow F))$

f. $(Smoke \Rightarrow Fire) \Rightarrow ((Smoke \wedge Heat) \Rightarrow Fire)$

Valid

When $H = T, (S \Rightarrow F) \Rightarrow (S \Rightarrow F)$

When $H = F, \big((S \wedge H) \Rightarrow F\big) = T, (S \Rightarrow F) \Rightarrow T$

g. $Big \vee Dumb \vee (Big \Rightarrow Dumb)$

Valid

$(B \vee D \vee (B \Rightarrow D) \equiv (B \vee D \vee (\neg B \vee D) \equiv D \vee T \equiv T$

7.14 According to some political pundits, a person who is radical (R) is electable (E) if he/she is conservative (C), but otherwise is not electable.

a. Which of the following are correct representations of this assertion?

(i) $(R \wedge E) \iff C$
(ii) $R \Rightarrow (E \iff C)$
(iii) $R \Rightarrow ((C \Rightarrow E) \vee \neg E)$

English: If they're radical, they're electable if they're conservative but otherwise not electable.

R	E	C
T	T	T
T	F	F
F	*	*

i: If they're conservative, they must be both radical and electable. False, does not match

R	E	C
T	T	T
T	F	F
F	*	**F**

ii: If they're radical, they're electable if they're conservative but otherwise not electable. True, matches

R	E	C
T	T	T
T	F	F
F	*	*

iii: If they're radical they might be not electable, but they also might be. False, does not match

R	E	C
T	T	*
T	F	*
F	*	*