Notes on algebraic geometry

Contents

CHAPTER 1

scheme

1. Sheaves

DEFINITION 1.1. Let X be a topological space, presheaves of abelian groups on X are the category $\mathrm{Fun}(\mathrm{Top}(X)^{op},\ Ab)$, sometimes denoted by $\mathrm{Psh}(X,\ Ab)$.

REMARK 1.2. Even though presheves can be defined on other category, it is better to use an abelian category. In this case, the category of presheaves will also be an abelian category. In the note, we only need properties of presheaves on Ab.

PROP 1.3. Let Λ be a category, C be an abelian category, then $\mathrm{Fun}(\Lambda,\ C)$ is an abelian category, if $\mathcal{F}$, $\mathcal{G} \in \mathrm{Fun}(\Lambda,\ C)$, $\alpha : \mathcal{F} \to \mathcal{G}$ morphism, then ker and coker of α can be constructed as follows:

$$
\begin{tikzcd}
ker(\alpha_\lambda) \arrow[r] \arrow[d, dashed] & \mathcal{F}_\lambda \arrow[r, "\alpha_\lambda"] \arrow[d] & \mathcal{G}_\lambda \arrow[d] \\
ker(\alpha_\mu) \arrow[r] & \mathcal{F}_\mu \arrow[r, "\alpha_\mu"'] & \mathcal{G}_\mu
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{F}_\lambda \arrow[r, "\alpha_\lambda"] \arrow[d] & \mathcal{G}_\lambda \arrow[r] \arrow[d] & coker(\alpha_\lambda) \arrow[d, dashed] \\
\mathcal{F}_\mu \arrow[r, "\alpha_\mu"'] & \mathcal{G}_\mu \arrow[r] & coker(\alpha_\mu)
\end{tikzcd}
$$

It's easy to verify their universal property.

REMARK 1.4. In previous proposition, if $C = R$-mod, then ker and coker can be chosen (λ) point-wise as regular ker and coker in R-mod, and exactness in $\mathrm{Fun}(\Lambda,\ C)$ is equivalent to (λ) pointwise regular exactness in R-mod.

REMARK 1.5. Let $\mathcal{F} \in \mathrm{Fun}(\mathrm{Top}(X)^{op},\ Ab)$, $U \supset V$ open subsets of X, $s\in \mathcal{F}(U)$, $s|_V := \mathcal{F}(U \to V)(s)$.

DEFINITION 1.6. Let X, Y, Z be topological spaces, $f : X \to Y, g : Y \to Z$ be continuous maps, $\mathcal{F} \in \mathrm{Fun}(\mathrm{Top}(Y)^{op},\ Ab)$. The direct image of $\mathcal{F}$ w.r.t g is $g_*\mathcal{F}\in\mathrm{Fun}(\mathrm{Top}(Z)^{op},\ Ab)$. $(g_*\mathcal{F})(U \to V){:=}\mathcal{F}(g^{-1}(U) \to g^{-1}(V))$.

The inverse image of $\mathcal{F}$ w.r.t f is $f^{-1}\mathcal{F}\in\mathrm{Fun}(\mathrm{Top}(X)^{op},$

$$
\begin{tikzcd}
\varinjlim\limits_{W\supset f(U)} \mathcal{F}(W) \arrow[rr, dashed, "(f^{-1}\mathcal{F})(U\to V)"] & & \varinjlim\limits_{W\supset f(V)} \mathcal{F}(W) \\
\mathcal{F}(W) \arrow[u, "\alpha^{\mathcal{F}}_{f(U),W}"] \arrow[urr, "\alpha^{\mathcal{F}}_{f(V),W}"'] & &
\end{tikzcd}
$$

It's straight-forward to verify $f^{-1}\mathcal{F} \in \mathrm{Fun}(\mathrm{Top}(X)^{op},\ Ab)$.

REMARK 1.7. In order to make the definition of inverse image of presheaf to work, we need to fix $(\alpha^{\mathcal{F}}_{f(U),W}, \varinjlim\limits_{W\supset f(U)} \mathcal{F}(W))$ for given U.

DEFINITION 1.8. Let $f : X \to Y, g : Y \to Z$ be continuous maps, $\mathcal{F}, \mathcal{G} \in \mathrm{Fun}(\mathrm{Top}(Y)^{op},\ Ab)$, $\alpha : \mathcal{F} \to \mathcal{G}$ morphism. Then the direct image of α by g is $g_*\alpha : g_*\mathcal{F} \to g_*\mathcal{G}$, which is defined by $(g_*\alpha)(U) \coloneqq \alpha(g^{-1}(U))$, it's easy to verify it is morphism. The inverse image of α by f is $f^{-1}\alpha : f^{-1}\mathcal{F} \to f^{-1}\mathcal{G}$, which is defined by $(f^{-1}\alpha)(U) \coloneqq h_U$, where h_U is the unique map such that the following diagram is commmutative

$$\begin{tikzcd}
\varinjlim\limits_{W \supset f(U)} \mathcal{F}(W) \arrow[r, "h_U"] & \varinjlim\limits_{W \supset f(U)} \mathcal{G}(W) \\
\mathcal{F}(W) \arrow[u, "\alpha^{\mathcal{F}}_{f(U),W}"'] \arrow[r, "\alpha(W)"] & \mathcal{G}(W) \arrow[u, "\beta^{\mathcal{G}}_{f(U),W}"']
\end{tikzcd}$$

$$\begin{tikzcd}
\varinjlim\limits_{W \supset f(V)} \mathcal{F}(W) \arrow[r, "h_V"] & \varinjlim\limits_{W \supset f(V)} \mathcal{G}(W) \\
\varinjlim\limits_{W \supset f(U)} \mathcal{F}(W) \arrow[u, "(f^{-1}\mathcal{F})(U \to V)"'] \arrow[r, "h_U"] & \varinjlim\limits_{W \supset f(U)} \mathcal{G}(W) \arrow[u, "(f^{-1}\mathcal{G})(U \to V)"'] \\
\mathcal{F}(W) \arrow[u, "\alpha^{\mathcal{F}}_{f(U),W}"'] \arrow[r, "\alpha(W)"] \arrow[uu, bend left, "\alpha^{\mathcal{F}}_{f(V),W}"] & \mathcal{G}(W) \arrow[u, "\beta^{\mathcal{G}}_{f(U),W}"'] \arrow[uu, bend right, "\beta^{\mathcal{G}}_{f(V),W}"']
\end{tikzcd}$$

To show the upper box is commutative, it's equivalent to show it is commutative with composition $\alpha^{\mathcal{F}}_{f(U),W}$, then it's equivalent to the commutativity of the outer box, which is the definition of h_V.

DEFINITION 1.9. Let $\mathcal{F}$ be a presheaf, we say $\mathcal{F}$ is a sheaf if it has the following properties:

(1) (Uniqueness) Let U be an open subset of X, $s \in \mathcal{F}(U)$, U_i a covering of U by open subsets U_i. If $s|_{U_i} = 0$ for every i, then $s = 0$.
(2) (Glueing local sections) Let us keep the notation of (1). Let $s_i \in \mathcal{F}(U_i)$, $i \in I$, be sections such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$. Then there exists a section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ (this section s is unique by condition (1)).

PROP 1.10. Let $\mathcal{F}$ be a sheaf, then $\mathcal{F}(\varnothing) = \{0\}$.

PROOF. let $\varnothing = \bigcup\limits_{i \in \varnothing} U_i$, $s \in \mathcal{F}(\varnothing)$, $s|_{U_i} = 0$, thus $s = 0$, $\mathcal{F}(\varnothing) = \{0\}$. $\square$

PROP 1.11. Let $\mathcal{F} \in \mathrm{Fun}(\mathrm{Top}(X)^{op},\ Ab)$ be a sheaf, $f : X \to Y$ be continuous map, then $f_*\mathcal{F}$ in a sheaf.

DEFINITION 1.12. Let $\mathcal{F} \in \mathrm{Fun}(\mathrm{Top}(X)^{op},\ Ab)$, $U \subset X$ open subset, then we can define the restriction of $\mathcal{F}$ to U, $(\mathcal{F}|_U)(V \to W) := \mathcal{F}(V \to W)$. It's easy to check $\mathcal{F}|_U \in \mathrm{Fun}(\mathrm{Top}(U)^{op},\ Ab)$ and $\mathcal{F}$ is sheaf implys $\mathcal{F}|_U$ is sheaf.

DEFINITION 1.13. Let $\mathcal{F} \in \mathrm{Psh}(X,\ Ab)$, and let $x \in X$. The stalk of $\mathcal{F}$ at x is the group $\mathcal{F}_x = \varinjlim_{U \ni x} \mathcal{F}(U)$ with morphisms $\alpha_{x,U} : \mathcal{F}(U) \to \mathcal{F}_x$, the direct limit being taken over the open neighborhoods U of x. Let $s \in F(U)$ be a section, for any $x \in U$, we denote the image of s in $\mathcal{F}_x$ by s_x. We call s_x the germ of s at x.

DEFINITION 1.14. Let $\alpha : \mathcal{F} \to \mathcal{G}$ be morphism in $\mathrm{Fun}(\mathrm{Top}(X)^{op},\ Ab)$, then the stalk map of α at x is defined by the diagram:

$$\begin{tikzcd}
\mathcal{F}_x \arrow[r, "\alpha_x"] & \mathcal{G}_x \\
\mathcal{F}(U) \arrow[u, "\alpha^{\mathcal{F}}_{x,U}"] \arrow[r, "\alpha(U)"'] & \mathcal{G}(U) \arrow[u, "\alpha^{\mathcal{G}}_{x,U}"']
\end{tikzcd}$$

it's easy to verify $\{\alpha^{\mathcal{G}}_{x,U} \circ \alpha(U)\}_{U \ni x}$ are compatible, moreover we have:

$$\begin{tikzcd}
\mathcal{F}_x \arrow[r, "\alpha_x"] & \mathcal{G}_x \arrow[r, "\beta_x"] & \mathcal{H}_x \\
\mathcal{F}(U) \arrow[u, "\alpha^{\mathcal{F}}_{x,U}"] \arrow[r, "\alpha(U)"'] & \mathcal{G}(U) \arrow[u, "\alpha^{\mathcal{G}}_{x,U}"] \arrow[r, "\beta(U)"'] & \mathcal{H}(U) \arrow[u, "\alpha^{\mathcal{H}}_{x,U}"]
\end{tikzcd}$$

By uniqueness we have $(\beta \circ \alpha)_x = \beta_x \circ \alpha_x$.

LEMMA 1.15. Let $\mathcal{F}$ be a sheaf on X. Let $s, t \in \mathcal{F}(U)$ such that $\alpha_{x,U}(s) = \alpha_{x,U}(t)\ \forall\ x \in U$. Then $s = t$.

PROOF. $\forall\, x \in X, \exists\, U_x \ni x, U_x \subset U$ such that $\mathcal{F}(U \to U_x)(s) = \mathcal{F}(U \to U_x)(t)$, that is $s|_{U_x} = t|_{U_x}$ with previous notation. Then $s = t$ follows from definition of sheaf. $\square$

LEMMA 1.16. Let $\mathcal{F}$ be a presheaf on X, $\mathcal{G}$ be a sheaf on X, $\alpha : \mathcal{F} \to \mathcal{G}$ morphism, then $\alpha = 0$ if and only if $\alpha_x = 0\ \forall x \in X$.

PROOF. $s \in \mathcal{F}(U)$, $\alpha_x = 0, \forall x \iff \alpha_x(\alpha^{\mathcal{F}}_{x,U}(s)) = \alpha^{\mathcal{G}}_{x,U}(\alpha(U)s) = 0\ \forall x \in U, s \in \mathcal{F}(U) \iff \alpha(U)s = 0, \forall s \in \mathcal{F}(U)$(Lemma 1.15) $\iff \alpha = 0$ $\square$

PROP 1.17. Let $\alpha \in MorFun(\Lambda, \mathcal{C})$, C is abelian category. Then α is monomorphism (epimorphism) if and only if it is point-wise monomorphism (epimorphism). Or exactness is equivalent to pointwise exactness.

PROOF.

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\alpha"] & \mathcal{G} \arrow[r, "\beta"] & \mathcal{H} \\
Im\alpha \arrow[ur] \arrow[rr, dashed, "\theta"'] & & ker\beta \arrow[ul]
\end{tikzcd}$$

For a sequence in $MorFun(\Lambda, \mathcal{C})$, it's exact if $\beta \circ \alpha = 0$ and the induced θ is isomorphism. Notice that we have a choice such that

$$(Im\alpha)_\lambda = Im(\alpha_\lambda), (ker\beta)_\lambda = ker(\beta_\lambda).$$

(For exactness, arbitary choice of Im and ker is equivalent to special choice). So it's equiavlent to $\beta_\lambda \circ \alpha_\lambda = 0$ and the induced morhism, which is θ_λ due to uniqueness of θ_λ, is isomorphism.

$$\begin{tikzcd}
\mathcal{F}_\lambda \arrow[r, "\alpha_\lambda"] & \mathcal{G}_\lambda \arrow[r, "\beta_\lambda"] & \mathcal{H}_\lambda \\
(Im\alpha)_\lambda \arrow[ur] \arrow[rr, dashed, "\theta_\lambda"'] & & (ker\beta)_\lambda \arrow[ul]
\end{tikzcd}$$

The notation is good and bad in some sense. Using Im and ker for Ob makes less and more clear symbols, but technically they should used for arrows. $\square$

DEFINITION 1.18. Let $\mathcal{F}, \mathcal{G} \in$ Fun(Top$(X)^{op}$, Ab), $\alpha : \mathcal{F} \to \mathcal{G}$ morphism, if α_x is surjective(injective, isomorphism) for all x, we say α is stalk surjective(injective, isomorphism).

PROP 1.19. Let $\alpha : \mathcal{F} \to \mathcal{G}$ be morphism of sheaves on X. Then $\alpha(U)$ is injective $\forall\ U \subset X$ open subset if and only if α is stalk injective, α is isomorphism if and only if α is stalk isomorphism.

PROOF. (1) Assume α is stalk injective, let $\alpha(U)s = 0$ for $s \in \mathcal{F}$(U). $\alpha_x \circ \alpha_{x,U}(s) = \beta_{x,U} \circ \alpha(U)(s) = 0$. $\alpha_{x,U}(s) = 0$, $s|_{U_x} = 0$ $x \in U_x.U_x$ consists of open cover of U. Therefore $s = 0$.

(2) Assume $\alpha(W)$ is injective $\forall\, W \subset X$ open subset, $\alpha_x(s_x) = 0, s \in \mathcal{F}(U)$,$\beta_{x,U} \circ \alpha(U)(s)$=$\alpha_x \circ \alpha_{x,U}(s) = 0$, $\alpha(U)(s)|_V = \alpha(V)(s|_V)$,$s|_V = 0, s_x = (s|_V)_x = 0$.

(3) Assume α is stalk isomorphism, let $u \in \mathcal{G}(U)$,$\exists$ open cover U_i of $U, s_i \in \mathcal{F}(U_i)$ such that $\alpha(U_i)s_i = t|_{U_i}$.

$$\alpha(U_i \cap U_j)s_i|_{U_i \cap U_j} = \alpha(U_i)(s_i)|_{U_i \cap U_j} = t|_{U_i \cap U_j} = \alpha(U_i \cap U_j)s_j|_{U_i \cap U_j}$$

So $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$. $\exists! s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i.(\alpha(U)s)|_{U_i} = \alpha(U_i)(s|_{U_i}) = \alpha(U_i)(s_i) = t|_{U_i}$.Therefore $\alpha(U)s = t$.

(4) Assume α is isomorphism, $\exists \beta$ such that $\beta \circ \alpha = id_{\mathcal{F}}, \alpha \circ \beta = id_{\mathcal{G}}, \beta_x \circ \alpha_x = id_{\mathcal{F}_x}$,$\alpha_x \circ \beta_x = id_{\mathcal{G}_x}, \alpha_x$ is isomorphism. $\square$

LEMMA 1.20. Let $\mathcal{F}, \mathcal{H}$ be presheaves, G sheaf, $\theta : \mathcal{F} \to \mathcal{H}, \alpha : \mathcal{F} \to \mathcal{G}$. If θ is stalk surjective and $\ker(\theta(U)) \subset ker(\alpha(U))$ $\forall U$ open, then there exists a unique $\tilde{\alpha}$ such that $\alpha = \tilde{\alpha} \circ \theta$.

PROOF. For uniqueness of $\tilde{\alpha}$. For $s \in H(U)$, $\exists$ open cover U_i of U, $s_i \in \mathcal{F}(U_i)$ such that $\theta(U_i)s_i = s|_{U_i}$. $(\tilde{\alpha}(U)s)|_{U_i} = \tilde{\alpha}(U_i)(\theta(U_i)s_i) = \alpha(U_i)s_i$.

For existence, we need to show the above construction is well-defined. $\theta(U_i \cap U_j)(s_i|_{U_i \cap U_j} - s_j|_{U_i \cap U_j}) = (\theta(U_i)(s_i))|_{U_i \cap U_j} - (\theta(U_j)(s_j))|_{U_i \cap U_j} = (s|_{U_i})_{U_i \cap U_j} - (s|_{U_j})_{U_i \cap U_j} = 0$. $\alpha(U_i \cap U_j)(s_i|_{U_i \cap U_j} - s_j|_{U_i \cap U_j}) = 0$, $(\alpha(U_i)s_i)|_{U_i \cap U_j} = (\alpha(U_j)s_j)|_{U_i \cap U_j}$. we define $\tilde{\alpha}(U)s$ to be gluing $\alpha(U_i)s_i$. To see it's well-defined, let V_j be another open cover of $U, t_j \in \mathcal{F}(V_j)$, $\theta(V_j)t_j = s|_{V_j}$. $s|_{U_i \cap V_j} = \theta(U_i \cap V_j)(s_i|_{U_i \cap V_j}) = \theta(U_i \cap V_j)(t_j|_{U_i \cap V_j}).\alpha(U_i \cap V_j)(s_i|_{U_i \cap V_j}) = \alpha(U_i \cap V_j)(t_j|_{U_i \cap V_j})$. Gluing $\alpha(U_i)s_i$=Gluing $\alpha(U_i \cap V_j)(s_i|_{U_i \cap V_j})$, Gluing $\alpha(V_j)t_j$=Gluing $\alpha(U_i \cap V_j)(t_j|_{U_i \cap V_j})$.

To see it's morphism, let $V \subset U, s|_{U_i} = \alpha(U_i)s_i, (s|_V)|_{V \cap U_i} = \alpha(V \cap U_i)(s_i|_{V \cap U_i})$. ((Gluing $\alpha(U_i)s_i)|_V)|_{V \cap U_i} = \alpha(V \cap U_i)((s_i)|_{V \cap U_i}) =$ (gluing $\alpha(V \cap U_i)((s_i)|_{V \cap U_i}))|_{V \cap U_i}$. $\square$

DEFINITION 1.21. Let $\mathcal{F}$ be a presheaf on X. The sheafication of $\mathcal{F}$ is a pair $(\theta, \mathcal{F}^{sh})$, $\mathcal{F}^{sh}$ is a sheaf, $\theta : \mathcal{F} \to \mathcal{F}^{sh}$ such that for all such pair $(\alpha, \mathcal{G})$, $\exists! \tilde{\alpha} : \mathcal{F}^{sh} \to \mathcal{G}$ such that $\alpha = \tilde{\alpha} \circ \theta$.

PROP 1.22. Sheafication exists and is unique up to isomorphism.

PROOF. Uniqueness up to isomorphism is trivial, we only need to construct the sheafication of $\mathcal{F}$. $\mathcal{F}^{sh}(U) := \{f : U \to \coprod_{x \in U} \mathcal{F}_x | \exists$ open cover U_i of $U, s_i \in \mathcal{F}(U_i)$, such that $f(x) = (s_i)_x, \forall x \in U_i\}$.The operation on $\mathcal{F}^{sh}(U)$ is defined pointwise.$\mathcal{F}^{sh}(U \to V)(f) := f|_V$ (restriction in the usual sense), therefore $\mathcal{F}^{sh}(U)$ is presheaf. To see it is sheaf, let U_i be an open cover of $U, f|_{U_i} = 0$, then $f = 0$. Let $f_i \in \mathcal{F}(U_i)$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$, then $\exists! f : U \to \coprod_{x \in U} \mathcal{F}_x$ such that $f|_{U_i} = f_i$. Clearly $f \in \mathcal{F}(U)$. Now we define $\theta : \mathcal{F} \to \mathcal{F}^{sh}$ by $\theta(U)(s)(x) = s_x$. $(\mathcal{F}^{sh}(U \to V) \circ \theta(U))(s)(x) = s_x = (s|_V)_x = (\theta(V) \circ \mathcal{F}(U \to V))(s)(x)$. Thus θ is morphism.

If $\theta(U)s = 0$, then $s_x = 0 \ \forall x \in U$. $(\alpha(U)s)_x = \alpha_x(s_x) = 0$. $\alpha(U)s = 0$. And θ is obviously stalk surjective by definition. So by lemma 1.20, there is a unique $\tilde{\alpha}$ such that $\alpha = \tilde{\alpha} \circ \theta$.

□

REMARK 1.23. It's obvious that the sheafication morphism θ is stalk isomorphism.

EXAMPLE 1.24. Let A be an abelian group, X topological space, we define the presheaf A_X by $A_X(U) = \begin{cases} A & U \neq \varnothing \\ \{0\} & otherwise \end{cases}$, $A_X(U \to V) = \begin{cases} id_A & V \neq \varnothing \\ 0 & otherwise \end{cases}$, The sheafication of A is $A_X^{sh}(U) = \{f : U \to A | f$ is locally constant$\}$, with usual restriction of function. $\theta(U) : (A_X)(U) \to A_X^{sh}(U), a \mapsto Cons_a$.

DEFINITION 1.25. Let $\mathcal{F} \in Psh(U, Ab)$, $U \subset X$ open subset. Define extension of $\mathcal{F}$ by zero presheaf by $(j_{p!}\mathcal{F})(V) = \begin{cases} \mathcal{F}(V) & V \subset U \\ \{0\} & otherwise \end{cases}$, $(j_{p!}\mathcal{F})(V_1 \to V_2) = \begin{cases} F(V_1 \to V_2) & V_1 \subset U \\ 0 & otherwise \end{cases}$. The extension of $\mathcal{F}$ by zero sheaf $j_!\mathcal{F}$ is defined to be the sheafication of $j_{p!}\mathcal{F}$.

DEFINITION 1.26. Let $\mathcal{F}, \mathcal{F}'$ be presheaves, we say $\mathcal{F}'$ is subpresheaf of $\mathcal{F}$ if $\mathcal{F}'(U) \subset \mathcal{F}(U)$, and $\mathcal{F}'(U \to V) = \mathcal{F}(U \to V)|_{\mathcal{F}'(U)}^{\mathcal{F}'(V)}$. Then we can define the quotient presheaf $\mathcal{F}/\mathcal{F}'$ by $(\mathcal{F}/\mathcal{F}')(U \to V)([s]) := [\mathcal{F}(U \to V)(s)]$. let $\alpha : \mathcal{F} \to \mathcal{G}$ be morphism of presheaves, $(Ker\alpha)(U \to V) := \mathcal{F}(U \to V)|_{ker\alpha(U)}^{ker\alpha(V)}$. $(Im\alpha)(U \to V) := \mathcal{G}(U \to V)|_{Im\alpha(U)}^{Im\alpha(V)}$. It's easy to verify these three are presheaves.

REMARK 1.27. In above definition, if $\mathcal{F}, \mathcal{F}'$ are sheaves, then $\mathcal{F}/\mathcal{F}'$ satisfies the uniqueness axiom. If $\mathcal{F}, \mathcal{G}$ are sheaves, then $ker\alpha$ is sheaf.

REMARK 1.28. Even though using explicit construction for filtered direct limit is just as meaningless as that for tensor product, sometimes we use a relative construction just for convenience. Specifically, $\mathcal{F}, \mathcal{G} \in$ Fun(Λ, R-mod), Λ filtered, $\mathcal{G}_\lambda \subset \mathcal{F}_\lambda$,

$\mathcal{F}(\lambda\to\mu)(\mathcal{G}_\lambda)\subset\mathcal{G}_\mu$, $\mathcal{G}(\lambda\to\mu)=\mathcal{F}(\lambda\to\mu)|_{\mathcal{G}_\lambda}^{\mathcal{G}_\mu}$, $(\mathcal{F}/\mathcal{G})(\lambda\to\mu):=\mathcal{F}_\lambda/\mathcal{G}_\lambda\to\mathcal{F}_\mu/\mathcal{G}_\mu$, $\alpha_\lambda:\mathcal{F}_\lambda\to\varinjlim_{\lambda\in\Lambda}\mathcal{F}_\lambda$ direct limit, then $\alpha_\lambda|_{\mathcal{G}_\lambda}^{\bigcup_{\lambda\in\Lambda}\alpha_\lambda(\mathcal{G}_\lambda)}:\mathcal{G}_\lambda\to\bigcup_{\lambda\in\Lambda}\alpha_\lambda(\mathcal{G}_\lambda)$ is direct limit, and the direct limit of $\mathcal{F}/\mathcal{G}$ is given by the diagram:

$$\begin{tikzcd}
\mathcal{F}_\lambda \arrow[r, "\alpha_\lambda"] \arrow[d] & \varinjlim_{\lambda\in\Lambda}\mathcal{F}_\lambda \arrow[d] \\
\mathcal{F}_\lambda/\mathcal{G}_\lambda \arrow[r, dashed, "\widetilde{\alpha}_\lambda"] & \varinjlim_{\lambda\in\Lambda}\mathcal{F}_\lambda/\bigcup_{\lambda\in\Lambda}\alpha_\lambda(\mathcal{G}_\lambda)
\end{tikzcd}$$

PROP 1.29. Let $\alpha:\mathcal{F}\to\mathcal{G}$ be morphism in Psh(X,Ab), then $\ker(\alpha_x)=(\ker\alpha)_x$, $\mathrm{Im}(\alpha_x)=(\mathrm{Im}\alpha)_x$.

REMARK 1.30. Though it seems weird to use equality for an object that is unique up to isomorphism, it makes sense with the relative construction of direct limit in remark 1.26.

PROOF. It's basically using the definition of stalk map:

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\alpha(U)"] \arrow[d, "\alpha_{x,U}^{X,\mathcal{F}}"'] & \mathcal{G}(U) \arrow[d, "\alpha_{x,U}^{X,\mathcal{G}}"] \\
\mathcal{F}_x \arrow[r, "\alpha_x"'] & \mathcal{G}_x
\end{tikzcd}$$

$\mathrm{Im}(\alpha_x)=\alpha_x(\mathcal{F}_x)=\alpha_x(\bigcup_{U\ni x}\alpha_{x,U}^{\mathcal{F}}(\mathcal{F}(U)))=\bigcup_{U\ni x}\alpha_x(\alpha_{x,U}^{\mathcal{F}}(\mathcal{F}(U)))=\bigcup_{U\ni x}\alpha_{x,U}^{\mathcal{G}}(\alpha(U)(\mathcal{F}(U)))=\bigcup_{U\ni x}\alpha_{x,U}^{\mathcal{G}}((\mathrm{Im}\alpha)(U))=(\mathrm{Im}\alpha)_x$.

$\ker(\alpha_x)=\{\alpha_{x,U}^{\mathcal{F}}(s)|x\in U,s\in\mathcal{F}(U),\exists V\ni x$ such that $s|_V\in\ker(\alpha(V))\}=\{\alpha_{x,U}^{\mathcal{F}}(s)|x\in U,s\in\ker(\alpha(U))\}=(\ker\alpha)_x$

□

PROP 1.31. In the category of sheaves of abelian groups on X, monomorphism is equivalent to stalk injective, epimorphism is equivalent to stalk surjective.

PROOF. With prop 1.13 and lemma 1.15, it's easy to prove stalk injective (surjective) implys monomorphism (epimorphism).
For sheaves of abelian groups $\mathcal{F}$, $\mathrm{Hom}_{Sh(Ab,X)}(\underline{\mathbb{Z}},\mathcal{F})\cong\mathcal{F}$.(Here Hom stands for sheaf hom and $\underline{\mathbb{Z}}$ is constant sheaf on X). Let's prove this.

Assume $\alpha:\mathcal{F}\to\mathcal{G}$ is monomorphism. Fix $U\subset X$ open subset, $\mathcal{H}\in Sh(X,Ab)$, we have natural isomorphism:

$$Hom_{Sh(X,Ab)}(j_!(\mathbb{Z}_U),\mathcal{H})\xrightarrow{\sim}Hom_{Psh(X,Ab)}(j_{p!}(\mathbb{Z}_U),\mathcal{H})\longrightarrow Hom_{Psh(X,Ab)}(\mathbb{Z}_U,H_{|_U})\to H(U) \tag{1.1}$$

The last isomorhism maps $s\in H(U)$ to the morphism α, $\alpha(V)(1)=s|_V$. Notice that constant presheaf functor is left adjoint of restriction functor, which is the second isomorphism, the first is just the definition of sheafication.

$$\begin{tikzcd} Hom_{Sh(X,Ab)}(j_!(\mathbb{Z}_U),\mathcal{F}) \arrow[r, "-\circ Sh(j_{p!}(\mathbb{Z}_U))"] \arrow[d, "\alpha\circ-"'] & Hom_{Psh(X,Ab)}(j_{p!}(\mathbb{Z}_U),\mathcal{F}) \arrow[r] \arrow[d, "\alpha\circ-"'] & \mathcal{F}(U) \arrow[d, "\alpha(U)"] \\ Hom_{Sh(X,Ab)}(j_!(\mathbb{Z}_U),\mathcal{G}) \arrow[r, "-\circ Sh(j_{p!}(\mathbb{Z}_U))"'] & Hom_{Psh(X,Ab)}(j_{p!}(\mathbb{Z}_U),\mathcal{G}) \arrow[r] & \mathcal{G}(U) \end{tikzcd}$$

By assumption the left column is injective, thus $\alpha(U)$ is injective.

Assume $\alpha : \mathcal{F} \to \mathcal{G}$ is epimorphism. Fix $x \in X$, $\mathcal{H}(U) \coloneqq \begin{cases} \mathcal{G}_x/Im(\alpha_x) & x \in U \\ \{0\} & otherwise \end{cases}$, $\mathcal{H}(U \to V) \coloneqq \begin{cases} id_{\mathcal{G}_x/Im(\alpha_x)} & x \in V \\ 0 & otherwise \end{cases}$, $\mathcal{H} \in Sh(X,Ab)$. $\beta : \mathcal{G} \to \mathcal{H}$, $\beta(U) \coloneqq \begin{cases} [\alpha^{\mathcal{G}}_{x,U}] & x \in U \\ 0 & otherwise \end{cases}$, β is morphism. $\beta \circ \alpha = 0$, $\beta = 0$, $\mathcal{G}_x/Im(\alpha_x) = \{0\}$, α_x is surjective. □

PROP 1.32. Sheaves of abelian groups on X is an abelian category.

PROOF. (1) Zero object is the sheaf $\mathcal{F}$ defined by $\mathcal{F}(U) = \{0\}$, it's easy to verify it's a sheaf and zero object in the category.

(2) Biproduct of $\mathcal{F}$ and $\mathcal{G}$ is $(i_{\mathcal{F}}, i_{\mathcal{G}}, \pi_{\mathcal{F}}, \pi_{\mathcal{G}}, \mathcal{F}\oplus\mathcal{G})$, where $i_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}\oplus\mathcal{G}$, $i_{\mathcal{G}} : \mathcal{G} \to \mathcal{F}\oplus\mathcal{G}$, $\pi_{\mathcal{F}} : \mathcal{F}\oplus\mathcal{G} \to \mathcal{F}$, $\pi_{\mathcal{G}} : \mathcal{F}\oplus\mathcal{G} \to \mathcal{G}$, $(\mathcal{F}\oplus\mathcal{G})(U) \coloneqq \mathcal{F}(U)\oplus\mathcal{G}(U)$, $(\mathcal{F}\oplus\mathcal{G})(U \to V) \coloneqq \mathcal{F}(U \to V)\oplus\mathcal{G}(U \to V)$, $i_{\mathcal{F}}(U) \coloneqq l_{\mathcal{F}(U)}$, $i_{\mathcal{G}}(U) \coloneqq l_{\mathcal{G}(U)}$, $\pi_{\mathcal{F}}(U) \coloneqq \pi_{\mathcal{F}(U)}$, $\pi_{\mathcal{G}}(U) \coloneqq \pi_{\mathcal{G}(U)}$, easy to verify it's biproduct.

(3) $\alpha : \mathcal{F} \to \mathcal{G}$, $ker\alpha$ defined in def 1.20 with the inclusion tranformation is actually the kernel of α, cokernel of α is $\mathcal{G} \longrightarrow \mathcal{G}/Im(\alpha) \to S(\mathcal{G}/Im(\alpha))$

For the first statement,

$$\begin{tikzcd} ker\alpha \arrow[r] & \mathcal{F} \arrow[r, "\alpha"] & \mathcal{G} \\ \mathcal{H} \arrow[u, dashed, "\theta"] \arrow[ur, "\beta"'] & & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{H}(U) \arrow[r] \arrow[d] & ker(\alpha(U)) \arrow[r] \arrow[d] & \mathcal{F}(U) \arrow[d] \\ \mathcal{H}(V) \arrow[r] & ker(\alpha(V)) \arrow[r] & \mathcal{F}(V) \end{tikzcd}$$

$\alpha\circ\beta = 0$, $\mathrm{Im}(\beta(U)) \subset \ker(\alpha(U))$, $\theta(U) \coloneqq \beta(U)^{|ker(\alpha(U))}$, uniqueness is obvious.

For the second statement,

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\alpha"] & \mathcal{G} \arrow[r, "\pi"] \arrow[dr, "\beta"'] & \mathcal{G}/Im(\alpha) \arrow[r] \arrow[d, dashed, "\theta"'] & S(\mathcal{G}/Im(\alpha)) \arrow[dl, dashed, "\phi"] \\ & & \mathcal{H} & \end{tikzcd}$$

If there are ϕ_1, ϕ_2 then $\phi_1 \circ Sh(\mathcal{G}/Im(\alpha)) \circ \pi = \phi_2 \circ Sh(\mathcal{G}/Im(\alpha)) \circ \pi$, $\phi_1 \circ Sh(\mathcal{G}/Im(\alpha)) = \phi_2 \circ Sh(\mathcal{G}/Im(\alpha))$, $\phi_1 = \phi_2$.

(4) Assume α is monomorphism,

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\alpha"] \arrow[d, dashed, "\theta"'] & \mathcal{G} \arrow[r, "\pi"] & \mathcal{G}/Im\alpha \arrow[r] & S(\mathcal{G}/Im\alpha) \\ \mathcal{H} \arrow[ur, "\beta"'] & & & \end{tikzcd}$$

$\beta := \ker(\mathcal{G} \to S(\mathcal{G}/Im(\alpha)))$, α_x is injective(Prop 1.26), thus θ_x is injective. $\mathrm{Sh}(\mathcal{G}/Im\alpha) \circ \pi \circ \beta = 0$, $\mathrm{Sh}(\mathcal{G}/Im\alpha)_x \circ \pi_x \circ \beta_x = 0$, that is $\mathrm{Im}(\beta_x) \subset (\mathrm{Im}(\alpha))_x = \mathrm{Im}(\alpha_x)$, $\mathrm{Im}(\alpha_x) = \beta_x(\mathrm{Im}(\theta_x)) \subset \mathrm{Im}(\beta_x) \subset \mathrm{Im}(\alpha_x)$. Thus $\mathrm{Im}(\beta_x) = \beta_x(\mathrm{Im}(\theta_x))$, $\mathrm{Im}(\theta_x) = \mathcal{H}_x$ (β_x is injective). θ is isomorphism.
Assume α is epimorphism,

$$\begin{tikzcd} ker\alpha \arrow[r, "l"] & \mathcal{F} \arrow[r, "\alpha"] \arrow[d, "\pi"'] & \mathcal{G} \\ & \mathcal{F}/Iml \arrow[r] \arrow[ur, dashed, "\beta"'] & S(\mathcal{F}/Iml) \arrow[u, dashed, "\phi"'] \end{tikzcd}$$

α_x surjective implys β_x is surjective.

$$\begin{tikzcd} \mathcal{F}_x/(Iml)_x \arrow[dr, dashed, "\beta_x"] & \\ \mathcal{F}_x \arrow[u] \arrow[r, "\alpha_x"'] & \mathcal{G}_x \end{tikzcd}$$

$(Iml)_x = (ker\alpha)_x = ker(\alpha_x)$, therefore β_x is injective, β_x is isomorphism, ϕ_x is isomorphism, ϕ is isomorphism.

$\square$

PROP 1.33. Let $\alpha : \mathcal{F} \to \mathcal{G}, \beta : \mathcal{G} \to \mathcal{H}$ be morphisms of sheaves, then

$$\mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H}$$

is exact in the category of sheaves if and only if

$$\mathcal{F}_x \xrightarrow{\alpha_x} \mathcal{G}_x \xrightarrow{\beta_x} \mathcal{H}_x$$

is exact.

PROOF. First we show sheaficiation of the presheaf $\mathrm{Im}\alpha$ gives the image of α in the category of sheaves.

Let β be the kernel of $\mathcal{G} \to (\mathcal{G}/Im\alpha)^{sh}$

$$\begin{tikzcd} Im\alpha \arrow[r] \arrow[d] & \mathcal{G} \arrow[r] & (G/Im\alpha)^{sh} \\ (Im\alpha)^{sh} \arrow[ur] \arrow[r, dashed, "\theta"'] & \mathcal{H} \arrow[u, "\beta"'] & \end{tikzcd}$$

Using the same argument as in prop 1.32(4), θ is isomorphism. The first sequence is exact if and only if $\beta \circ \alpha = 0$ and ψ is isomorphism.

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\alpha"] & \mathcal{G} \arrow[r, "\beta"] & \mathcal{H} \\ Im\alpha \arrow[ur] \arrow[rr, dashed, "\phi"] \arrow[d] & & ker\beta \arrow[ul] \\ (Im\alpha)^{sh} \arrow[urr, dashed, "\psi"'] & & \end{tikzcd}$$

Notice that $\phi_x : (Im\alpha)_x \to (ker\beta)_x$ is inclusion from $Im(\alpha_x)$ to $ker(\beta_x)$. So ψ is isomorhism if and only if ϕ is isomorphism if and only if $Im(\alpha_x) = ker(\beta_x)$ $\forall x \in X$. That's just the corresponding stalk maps are exact. $\square$

2. Ringed topological spaces

DEFINITION 2.1. A ringed topological space (locally ringed in local rings) consists of a topological space X endowed with a sheaf of rings $\mathcal{O}_X$ on X such that $\mathcal{O}_{X,x}$ is a local ring for every $x \in X$. We denote it $(X, \mathcal{O}_X)$.(Actually in most case we treat $\mathcal{O}_{X,x}$ as sheaf of abelian groups because rings have bad categorical properties). The sheaf $\mathcal{O}_X$ is called the structure sheaf of $(X, \mathcal{O}_X)$. When there is no confusion possible, we will omit $\mathcal{O}_X$ from the notation. Let $\mathfrak{m}_x$ be the maximal ideal of $\mathcal{O}_{X,x}$; we call $\mathcal{O}_{X,x}/\mathfrak{m}_x$ the residue field of X at x, and we denote it $k(x)$.

DEFINITION 2.2. A morphism of ringed topological spaces

$$(f, f^{\#}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y) \tag{2.1}$$

consists of a continuous map $f : X \to Y$ and a morphism of sheaves of rings $f^{\#} : \mathcal{O}_Y \to f_*\mathcal{O}_X$ such that for every $x \in X$, the map $f^{\#}_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a local homomorphism (*i.e.*, ${f^{\#}_x}^{-1}(\mathfrak{m}_x) = \mathfrak{m}_{f(x)}$ or, equivalently, $f^{\#}_x(\mathfrak{m}_{f(x)}) \subset \mathfrak{m}_x$). $f^{\#}_x :=$ the stalk map of $f^{\#}$ at $f(x)$ composite h_x, where h_x is the unique map such that the following diagram commutes:

$$\begin{tikzcd}
(f_*\mathcal{O}_X)_{f(x)} \arrow[rr, "h_x"] & & \mathcal{O}_{X,x} \\
\mathcal{O}_X(f^{-1}(U)) \arrow[u, "\alpha^{f_*\mathcal{O}_X}_{f(x),U}"] \arrow[urr, "\alpha^{\mathcal{O}_X}_{x,f^{-1}(U)}"'] & &
\end{tikzcd}$$

It's also the unique map such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{O}_{Y,f(x)} \arrow[r, "f^{\#}_x"] & \mathcal{O}_{X,x} \\
\mathcal{O}_Y(U) \arrow[u, "\alpha^{\mathcal{O}_Y}_{f(x),U}"] \arrow[r, "f^{\#}(U)"] & \mathcal{O}_X(f^{-1}(U)) \arrow[u, "\alpha^{\mathcal{O}_X}_{x,f^{-1}(U)}"']
\end{tikzcd}$$

DEFINITION 2.3.

$$(f, f^{\#}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y), (g, g^{\#}) : (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z) \tag{2.2}$$

$$(g, g^{\#}) \circ (f, f^{\#}) := (g \circ f, (g_* f^{\#}) \circ g^{\#}) \tag{2.3}$$

we say $(g \circ f)^{\#} = g_* f^{\#} \circ g^{\#}$ (Which is not a good notation, because it may suggests the $f^{\#}$ is determined by f). $(g \circ f)^{\#}_x \circ \alpha^{\mathcal{O}_Z}_{g\circ f(x),V} = \alpha^{\mathcal{O}_X}_{x,(g\circ f)^{-1}(V)} \circ (g \circ f)^{\#}(V)$, $f^{\#}_x \circ g^{\#}_{f(x)} \circ \alpha^{\mathcal{O}_Z}_{g\circ f(x),V} = f^{\#}_x \circ \alpha^{\mathcal{O}_Y}_{f(x),g^{-1}(V)} \circ g^{\#}(V) = \alpha^{\mathcal{O}_X}_{x,f^{-1}(g^{-1}(V))} \circ f^{\#}(g^{-1}(V)) \circ g^{\#}(V)$, $(g \circ f)^{\#}_x \circ \alpha^{\mathcal{O}_Z}_{g\circ f(x),V} = f^{\#}_x \circ g^{\#}_{f(x)} \circ \alpha^{\mathcal{O}_Z}_{g\circ f(x),V}$.

The commutativity basically follows from the diagrams:

$$\begin{tikzcd}
\mathcal{O}_{Z,g\circ f(x)} \arrow[r, "(g\circ f)^{\#}_x"] & \mathcal{O}_{X,x} \\
\mathcal{O}_Z(V) \arrow[u, "\alpha^{\mathcal{O}_Z}_{g\circ f(x),V}"] \arrow[r, "(g\circ f)^{\#}(V)"] & \mathcal{O}_X((g\circ f)^{-1}(V)) \arrow[u, "\alpha^{\mathcal{O}_X}_{x,(g\circ f)^{-1}(V)}"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{O}_{Z,g\circ f(x)} \arrow[r, "g^{\#}_{f(x)}"] & \mathcal{O}_{Y,f(x)} \\
\mathcal{O}_Z(V) \arrow[u, "\alpha^{\mathcal{O}_Z}_{g\circ f(x),U}"] \arrow[r, "g^{\#}(V)"'] & \mathcal{O}_Y(g^{-1}(V)) \arrow[u, "\alpha^{\mathcal{O}_Y}_{f(x),g^{-1}(V)}"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{O}_{Y,f(x)} \arrow[r, "f^{\#}_x"] & \mathcal{O}_{X,x} \\
\mathcal{O}_Y(g^{-1}(V)) \arrow[u, "\alpha^{\mathcal{O}_Y}_{f(x),g^{-1}(V)}"] \arrow[r, "f^{\#}(g^{-1}(V))"'] & \mathcal{O}_X(f^{-1}(g^{-1}(V))) \arrow[u, "\alpha^{\mathcal{O}_X}_{x,f^{-1}(g^{-1}(V))}"']
\end{tikzcd}$$

Therefore $(g\circ f)^{\#}_x = f^{\#}_x \circ g^{\#}_{f(x)}$. It's straight-forward to verify ringed topological spaces form a category.

PROP 2.4. If $f : X \to Y$ is topological embedding, $\mathcal{F}$, $\mathcal{G} \in Psh(X, Ab)$ then the map $(f_*\mathcal{F})_{f(x)} \to \mathcal{F}_x$ is isomorphism and the following diagram commutes.

$$\begin{tikzcd}
(f_*\mathcal{F})_{f(x)} \arrow[r, "(f_*\alpha)_x"] \arrow[d] & (f_*\mathcal{G})_{f(x)} \arrow[d] \\
\mathcal{F}_x \arrow[r, "\alpha_x"'] & \mathcal{G}_x
\end{tikzcd}$$

PROOF.

$$\begin{tikzcd}
(f_*\mathcal{F})_{f(x)} \arrow[r, "h_x"] & \mathcal{F}_x \\
\mathcal{O}_X(f^{-1}(V)) \arrow[u, "\alpha^{f_*\mathcal{F}}_{f(x),V}"] \arrow[ur, "\alpha^{\mathcal{F}}_{x,f^{-1}(V)}"'] &
\end{tikzcd}$$

Let $f(X) = W$, for $U \ni x, s \in \mathcal{O}_X(U), f(U) = V \cap W, V$ open in $Y, f(x) \in V$. $f^{-1}(V) = f^{-1}(V \cap W) = U, h_x \circ \alpha^{f_*\mathcal{F}}_{f(x),V}(s) = \alpha^X_{x,U}(s)$. Thus h_x is surjective. For another $V \ni f(x), s \in \mathcal{F}(f^{-1}(V)), h_x \circ \alpha^{f_*\mathcal{F}}_{f(x),V}(s) = \alpha^{\mathcal{F}}_{x,f^{-1}(V)}(s) = 0$. $\exists U \ni x$, $U \subset f^{-1}(V)$, such that $\mathcal{F}(f^{-1}(V) \to U)(s) = 0$. $f(U) = V_1 \cap W, x \in f^{-1}(V_1 \cap V) \subset U, \alpha^{f_*\mathcal{F}}_{f(x),V}(s) = \alpha^{f_*\mathcal{F}}_{f(x),V\cap V_1} \circ (f_*\mathcal{F})(V \to V \cap V_1)(s) = \alpha^{f_*\mathcal{F}}_{f(x),V\cap V_1} \circ \mathcal{F}(U \to f^{-1}(V \cap V_1)) \circ \mathcal{F}(f^{-1}(V) \to U)(s) = 0$. h_x is injective.

Commutativity comes from

$$\begin{tikzcd}
(f_*\mathcal{F})(V) \arrow[r, "(f_*\alpha)(V)"] \arrow[d] & (f_*\mathcal{G})(V) \arrow[d] \\
(f_*\mathcal{F})_{f(x)} \arrow[r, "(f_*\alpha)_x"] \arrow[d] & (f_*\mathcal{G})_{f(x)} \arrow[d] \\
\mathcal{F}_x \arrow[r, "\alpha_x"'] & \mathcal{G}_x
\end{tikzcd}$$

$\square$

DEFINITION 2.5. Let $(X, \mathcal{O}_X) \in \mathrm{Fun}(\mathrm{Top}(X)^{op},\ Ab)$, $U \subset X$ open subset, $l_U : U \to X$ inclusion, $(\mathcal{O}_X)|_U$ is a sheaf, denoted by $\mathcal{O}_U$. $l^\# : \mathcal{O}_X \to (l_U)_*\mathcal{O}_U$ morphism defined by $l^\#(V) \coloneqq \mathcal{O}_X(V \to V \cap U)$. It's morphism because of the diagram:

$$\begin{tikzcd}
\mathcal{O}_X(V) \arrow[r, "\mathcal{O}_X(V \to V\cap U)"] \arrow[d, "\mathcal{O}_X(V \to W)"'] & ((l_U)_*\mathcal{O}_X)(V) \arrow[d, "\mathcal{O}_X(V\cap U \to W\cap U)"] \\
\mathcal{O}_X(W) \arrow[r, "\mathcal{O}_X(W \to W\cap U)"] & ((l_U)_*\mathcal{O}_X)(W)
\end{tikzcd}$$

we can define the map $(l_U)^\#_x$ by the diagram:

$$\begin{tikzcd}
\mathcal{O}_{X,x} \arrow[r, "(l_U)^\#_x"] & \mathcal{O}_{U,x} \\
\mathcal{O}_X(V) \arrow[r, "\mathcal{O}_X(V \to V\cap U)"'] \arrow[u, "\alpha^{\mathcal{O}_X}_{x,V}"] & \mathcal{O}_X(V \cap U) \arrow[u, "\alpha^{\mathcal{O}_U}_{x,V\cap U}"']
\end{tikzcd}$$

Again, it's easy to check $l^\#_x$ is isomorphism, therefore $(l_U, (l_U)^\#)$ is morphism of ringed topological space.

PROP 2.6. For $(f, f^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, $f(X) \subset V$, $V \subset Y$ open, $\exists!(f^{|_V}, (f^{|_V})^\#) : (X, \mathcal{O}_X) \to (V, \mathcal{O}_{Y|_V})$ such that $(l_V, (l_V)^\#) \circ (f^{|_V}, (f^{|_V})^\#) = (f, f^\#)$.

PROOF. Feed $W \subset Y$ open to the equation gives $(f^{|_V})^\#(V \cap W) \circ \mathcal{O}_Y(W \to W \cap V) = f^\#(W)$, if $W \subset V$, then $(f^{|_V})^\#(W) = f^\#(W)$. We need to verify $(f^{|_V})^\#$ defined in this way is morphism and preserves maximal ideal. It's trivial to check it's morphism, $l_V \circ f^{|_V} = f$, $(f^{|_V})^\#_x \circ (l_V)^\#_{f(x)} = f^\#_x$. Thus $(f^{|_V})^\#_x$ preserves maximal ideal. $\square$

PROP 2.7. For $(f, f^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, $(f, f^\#)$ is isomorphism if and only if f is homeomorphism and $f^\#$ is isomorphism if and only if f is homeomorphism and $f^\#_x$ is isomorphism $\forall x \in X$.

PROOF. (1) Assume $(f,f^{\#})$ is isomorphism, then $\exists(g,g^{\#}):(Y,\mathcal{O}_Y)\to(X,\mathcal{O}_X)$ such that $(g,g^{\#})\circ(f,f^{\#})=(id_X,id_{\mathcal{O}_X})$, $(f,f^{\#})\circ(g,g^{\#})=(id_Y,id_{\mathcal{O}_Y})$. That is $g\circ f=id_X$, $f\circ g=id_Y$, $g_*(f^{\#})\circ g^{\#}=id_{\mathcal{O}_X}$, $f_*(g^{\#})\circ f^{\#}=id_{\mathcal{O}_Y}$. For $U\subset X$ open, $V\subset Y$ open, $f^{\#}(g^{-1}(U))\circ g^{\#}(U)=id_{\mathcal{O}_X(U)}$, $g^{\#}(f^{-1}(V))\circ f^{\#}(V)=id_{\mathcal{O}_Y(V)}$. Let $U=g(V)$, then $f^{\#}(V)\circ g^{\#}(U)=id_{\mathcal{O}_X(U)}$, $g^{\#}(U)\circ f^{\#}(V)=id_{\mathcal{O}_Y(V)}$, $f^{\#}(V)$ is isomorphism.

(2) Assume f is homeomorphism and $f^{\#}$ is isomorphism, $g\coloneqq f^{-1}$, $g^{\#}\coloneqq(g_*f^{\#})^{-1}$. $(g,g^{\#})\circ(f,f^{\#})=(g\circ f,\ g_*f^{\#}\circ g^{\#})=(id_X,id_{\mathcal{O}_X})$, $(f,f^{\#})\circ(g,g^{\#})=(f\circ g,\ f_*g^{\#}\circ f^{\#})=(id_Y,f_*(g^{\#}\circ g_*f^{\#}))=(id_Y,f_*(id_{g_*\mathcal{O}_Y}))=(id_Y,id_{\mathcal{O}_Y})$.(Use the formula of $(g\circ f)^{\#}_x$ shows g is morphism).

(3) The latter 2 arguments are equivalent because of prop 1.19 and prop 2.4 .

□

DEFINITION 2.8. Let $(f,f^{\#}):(X,\mathcal{O}_X)\to(Y,\mathcal{O}_Y)$, we say $(f,f^{\#})$ is open immersion if f is topological open embedding and $f^{\#}_x$ is isomorphism $\forall x\in X$. $(f,f^{\#})$ is closed immersion if f is topological closed embedding and $f^{\#}_x$ is surjective $\forall x\in X$.

PROP 2.9. Let $(f,f^{\#}):(X,\mathcal{O}_X)\to(Y,\mathcal{O}_Y)$, $(f,f^{\#})$ is open immersion if and only if $\exists V\supset f(X)$ open such that $(f^{|_V},(f^{|_V})^{\#})$ is isomorphism if and only if f is topological open embedding and $f^{\#}_{s,f(x)}$ is isomorphism.

REMARK 2.10. For morphism between ringed topological spaces, we use $f^{\#}_{s,y}$ to denote the stalk map of $f^{\#}$ at $y\in Y$.

PROOF. (1) Assume $(f,f^{\#})$ is open immersion, then $V=f(X)$ open in Y and $f^{|_Y}$ is homeomorphism by definition, $f^{\#}_x=(f^{|_V})^{\#}_x\circ(l^{\#}_V)_{f(x)}$, $(l^{\#}_V)_{f(x)}$ is isomorphism by definition 2.5, thus $(f^{|_V})^{\#}_x$ is isomorphsim.

(2) Assume $\exists V\supset f(X)$ open such that $(f^{|_V},(f^{|_V})^{\#})$ is isomorphism, $V=f(X)$, f is topologcial open embedding, again, using $f^{\#}_x=(f^{|_V})^{\#}_x\circ(l^{\#}_V)_{f(x)}$, $(l^{\#}_V)_{f(x)}$ is isomorphism, we get $f^{\#}_x$ is isomorphism.

(3) $f^{\#}_x=h_x\circ f^{\#}_{s,f(x)}$, the fisrt and last statement has that f is homeomorphism to its image, therefore h_x is isomorphism, these two are equivalent.

□

PROP 2.11. Let $(f,f^{\#}):(X,\mathcal{O}_X)\to(Y,\mathcal{O}_Y)$, then f is closed immersion of ringed topological spaces if and only if f is topological closed embedding and $f^{\#}_{s,y}$ is surjective $\forall y\in Y$ if and only if f is topological closed embedding and $f^{\#}_{s,f(x)}$ is surjective $\forall x\in X$.

PROOF. The first and last statement are eqivalent because h_x is isomorphism, for $y\in\overline{f(X)}^c$, for $U\ni y$ open, and $U\cap Imf=\varnothing$, $(f_*\mathcal{O}_X)(U)=\mathcal{O}_X(\varnothing)=\{0\}$, let $\{\alpha^Y_V\}_{V\ni y}$ be insertions of the direct limit $(f_*\mathcal{O}_X)_y$, then $\alpha^Y_V(s)=\alpha^Y_{V\cap U}\circ(f_*\mathcal{O}_X)(V\to V\cap U)(s)=0$, thus $(f_*\mathcal{O}_X)_y=\{0\}$, $f^{\#}_{s,y}$ is automatically surjective for $y\notin\overline{f(X)}$. □

DEFINITION 2.12. Let $f : X \to Y, \mathcal{G} \in Psh(X, Ab)$, we can define $\phi_x^{\mathcal{G},f} : (f^{-1}\mathcal{G})_x \to \mathcal{G}_{f(x)}$ as follows:

$$\begin{tikzcd}
(f^{-1}\mathcal{G})_x \arrow[r, "\phi_x^{\mathcal{G},f}"] & \mathcal{G}_{f(x)} \\
(f^{-1}\mathcal{G})(V) \arrow[u, "\alpha_{x,V}^{f^{-1}\mathcal{G}}"] \arrow[ur] & \\
\mathcal{G}(W) \arrow[u, "\alpha_{f(V),W}^{\mathcal{G}}"] \arrow[uur, "\alpha_{f(x),W}^{\mathcal{G}}"'] &
\end{tikzcd}$$

PROP 2.13. $\phi_x^{\mathcal{G},f}$ is isomorphism and the diagram commutes.

$$\begin{tikzcd}
(f^{-1}\mathcal{G})_x \arrow[r] \arrow[d] & (f^{-1}\mathcal{H})_x \arrow[d] \\
\mathcal{G}_{f(x)} \arrow[r] & \mathcal{H}_{f(x)}
\end{tikzcd}$$

PROOF. Surjectivity is trivial,(you actually need to check it's well-defined, but it's also easy). To prove injectivity, let $s \in \mathcal{G}(W)$ such that $\phi_x^{\mathcal{G},f}(\alpha_{x,V}^{f^{-1}\mathcal{G}}(\alpha_{f(V),W}^{\mathcal{G}}(s))) = \alpha_{f(x),W}^{\mathcal{G}}(s) = 0,\ s|_{W_0} = 0,\ f(x) \in W_0 \subset W.\ \ V_1 := f^{-1}(W_0) \cap V, x \in V_1.$ $\alpha_{x,V}^{f^{-1}\mathcal{G}}(\alpha_{f(V),W}^{\mathcal{G}}(s)) = \alpha_{x,V_1}^{f^{-1}\mathcal{G}} \circ \alpha_{f(V_1),W_0}^{\mathcal{G}} \circ G(W \to W_0)(s) = 0$

These equations represent the process of "diagram chasing" as below:

$$\begin{tikzcd}
 & (f^{-1}G)_x \arrow[r] & G_{f(x)} \\
(f^{-1}G)(V_1) \arrow[ur] & (f^{-1}G)(V) \arrow[l] \arrow[u] \arrow[ur] & \\
G(W_0) \arrow[u] & G(W) \arrow[l] \arrow[ul] \arrow[u] &
\end{tikzcd}$$

where $s \in G(W),\ s|_{W_0} = 0,\ V_1 = f^{-1}(W_0) \cap V$. The commutativity of the diagram follows from the definition:

$$\begin{tikzcd}
\mathcal{G}(W) \arrow[r] \arrow[d] & \mathcal{H}(W) \arrow[d] \\
(f^{-1}\mathcal{G})(U) \arrow[r] \arrow[d] & (f^{-1}\mathcal{H})(U) \arrow[d] \\
(f^{-1}\mathcal{G})_x \arrow[r] \arrow[d] & (f^{-1}\mathcal{H})_x \arrow[d] \\
\mathcal{G}_{f(x)} \arrow[r] & \mathcal{H}_{f(x)}
\end{tikzcd}$$

$\square$

DEFINITION 2.14. Let $f : X \to Y, \mathcal{F} \in \mathrm{Fun}(\mathrm{Top}(X)^{op}, Ab)$, $\mathcal{G} \in \mathrm{Fun}(\mathrm{Top}(Y)^{op}, Ab)$, we can define $\alpha^f_{\mathcal{F}} : f^{-1}f_*\mathcal{F} \to \mathcal{F}$, $\beta^f_{\mathcal{G}} : \mathcal{G} \to f_*f^{-1}\mathcal{G}$ by the following diagrams:

$$\begin{tikzcd}
\mathcal{G}(U) \arrow[r, "\beta^f_{\mathcal{G}}(U)=\alpha^{Y,f^{-1}(U)}_{f(f^{-1}(U)),U}"] \arrow[d, "\mathcal{G}(U\to V)"'] & (f_*f^{-1}\mathcal{G})(U) \arrow[d] & \mathcal{G}(W) \arrow[l, "\alpha^{Y,f^{-1}(U)}_{f(f^{-1}(U)),W}"'] \arrow[dl, "\alpha^{Y,f^{-1}(V)}_{f(f^{-1}(V)),W}"] \\
\mathcal{G}(V) \arrow[r, "\beta^f_{\mathcal{G}}(V)=\alpha^{Y,f^{-1}(V)}_{f(f^{-1}(V)),V}"'] & (f_*f^{-1}\mathcal{G})(V) &
\end{tikzcd}$$

$$\begin{tikzcd}
(f^{-1}f_*\mathcal{F})(U) \arrow[r, "\alpha^f_{\mathcal{F}}(U)"] & \mathcal{F}(U) \\
\mathcal{F}(f^{-1}(W)) \arrow[u, "\alpha^{X,f(U)}_{f(U),W}"] \arrow[ur, "\mathcal{F}(f^{-1}(W)\to U)"] \arrow[d] & \\
\mathcal{F}(f^{-1}(T)) \arrow[uur, bend right, "\mathcal{F}(f^{-1}(T)\to U)"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & (f^{-1}f_*\mathcal{F})(V) \arrow[r, "\alpha^f_{\mathcal{F}}(V)"] & \mathcal{F}(V) \\
\mathcal{F}(f^{-1}(W)) \arrow[ur, "\alpha^{X,f(V)}_{f(V),W}"] \arrow[r, "\alpha^{X,f(U)}_{f(U),W}"] & (f^{-1}f_*\mathcal{F})(U) \arrow[u] \arrow[r, "\alpha^f_{\mathcal{F}}(U)"] & \mathcal{F}(U) \arrow[u]
\end{tikzcd}$$

PROP 2.15. If f is embedding, then $\alpha^f_{\mathcal{F}}$ is stalk isomorphism and $\beta^f_{\mathcal{G}}$ is stalk isomorphism at $\mathrm{Im} f$, if f is open embedding, then $\alpha^f_{\mathcal{F}}$ is isomorphism, if f is closed embedding, then $\beta^f_{\mathcal{G}}$ is stalk surjective.

PROOF. (1)

$$\begin{tikzcd}
(f_*\mathcal{F})(W) \arrow[d] \arrow[ddr] \arrow[ddrr] & & \\
(f^{-1}f_*\mathcal{F})(U) \arrow[d] & & \\
(f^{-1}f_*\mathcal{F})_x \arrow[r, "\phi_x^{f_*\mathcal{F},f}"'] & (f_*\mathcal{F})_{f(x)} \arrow[r, "h_x^{\mathcal{F},f}"'] & \mathcal{F}_x
\end{tikzcd}
\qquad
\begin{tikzcd}
(f_*\mathcal{F})(W) \arrow[d] \arrow[dr] & \\
(f^{-1}f_*\mathcal{F})(U) \arrow[r] \arrow[d] & \mathcal{F}(U) \arrow[d] \\
(f^{-1}f_*\mathcal{F})_x \arrow[r, "(\alpha^{\mathcal{F}}_f)_x"'] & \mathcal{F}_x
\end{tikzcd}$$

We get $(\alpha^{\mathcal{F}}_f)_x = h_x^{\mathcal{F},f} \circ \phi_x^{f_*\mathcal{F},f}$.

$$\begin{tikzcd}
\mathcal{G}_{f(x)} \arrow[r, "(\beta^f_{\mathcal{G}})_{f(x)}"] & (f_*f^{-1}\mathcal{G})_{f(x)} \arrow[r, "h_x^{f^{-1}\mathcal{G},f}"] & (f^{-1}\mathcal{G})_x \arrow[r, "\phi_x^{\mathcal{G},f}"] & \mathcal{G}_{f(x)} \\
\mathcal{G}(V) \arrow[u] \arrow[r, "\beta^f_{\mathcal{G}}(V)"'] & (f_*f^{-1}\mathcal{G})(V) \arrow[u] \arrow[r, "id"'] & (f^{-1}\mathcal{G})(f^{-1}(V)) \arrow[u] & \mathcal{G}(V) \arrow[u] \arrow[l, "\alpha^{\mathcal{G}}_{f(f^{-1}(V)),V}"]
\end{tikzcd}$$

Therefore $\phi_x^{\mathcal{G},f} \circ h_x^{f^{-1}\mathcal{G},f} \circ (\beta_{\mathcal{G}}^f)_{f(x)} = id_{\mathcal{G}_{f(x)}}$.

If f is embedding, $(\alpha_f^{\mathcal{F}})_x$ and $(\beta_{\mathcal{G}}^f)_{f(x)}$ are isomorphisms.

(2) Assume f is open embedding, let $W = f(U)$, $s \in \mathcal{F}(U)$, $s = \alpha_{\mathcal{F}}^f(U)(\alpha_{f(U),f(U)}^{X,f(U)}(s))$, $\alpha_{\mathcal{F}}^f(U)$ is surjective, let $s \in \mathcal{F}(f^{-1}(W))$, $\alpha_{\mathcal{F}}^f(U)(\alpha_{f(U),W}^{X,f(U)}(s)) = \mathcal{F}(f^{-1}(W) \to U)(s) = 0$, $\alpha_{f(U),W}^{X,f(U)}(s) = (\alpha_{f(U),f(U)}^{X,f(U)} \circ (f_*\mathcal{F})(W \to f(U)))(s) = (\alpha_{f(U),f(U)}^{X,f(U)} \circ \mathcal{F}(f^{-1}(W) \to U))(s) = 0$, $\alpha_{\mathcal{F}}^f(U)$ is injective.

(3) Assume f is closed embedding, as we have showed in prop 2.11 for $y \notin f(X)$, $(f_*f^{-1}\mathcal{G})_y = \{0\}$, so the stalk map of $\beta_{\mathcal{G}}^f$ at such y is surjective.

To summary the proposition, embedding implys the stalk map of $\beta_{\mathcal{G}}^f$ at Imf is isomorphism, Imf closed implys the stalk of $f_*f^{-1}\mathcal{G}$ outside Imf is $\{0\}$.

□

PROP 2.16. Let $f : X \to Y$, $f_* : Psh(X, Ab) \to Psh(Y, Ab)$, $f^{-1} : Psh(Y, Ab) \to Psh(X, Ab)$, we can define $\eta : id_{Psh(Y,Ab)} \to f_*f^{-1}$, $\varepsilon : f^{-1}f_* \to id_{Psh(X,Ab)}$ by $\eta_{\mathcal{G}} \coloneqq \beta_{\mathcal{G}}^f$, $\varepsilon_{\mathcal{F}} \coloneqq \alpha_{\mathcal{F}}^f$, f^{-1} is left adjoint of f_*, η is counit, ε is unit.

PROOF. It's obvious that f_* is functor, to see f^{-1} is functor, see the diagram of its definition:

$$\begin{tikzcd}
(f^{-1}\mathcal{F})(U) \arrow[r, "(f^{-1}\alpha)(U)"] & (f^{-1}\mathcal{G})(U) \arrow[r, "(f^{-1}\beta)(U)"] & (f^{-1}\mathcal{H})(U) \\
\mathcal{F}(V) \arrow[u, "\alpha^{\mathcal{F}}_{f(U),V}"] \arrow[r, "\alpha(V)"] & \mathcal{G}(V) \arrow[u, "\alpha^{\mathcal{G}}_{f(U),V}"] \arrow[r, "\beta(V)"] & \mathcal{H}(V) \arrow[u, "\alpha^{\mathcal{H}}_{f(U),V}"]
\end{tikzcd}$$

Either use uniqueness or compostion with insertions $\{\alpha_{f(U),V}^{\mathcal{F}}\}$ to get the result.

$$\begin{tikzcd}
\mathcal{G}_1(U) \arrow[r, "\eta_{\mathcal{G}_1}(U)"] \arrow[d] & (f_*f^{-1}\mathcal{G}_1)(U) \arrow[d] \\
\mathcal{G}_2(U) \arrow[r, "\eta_{\mathcal{G}_2}(U)"'] & (f_*f^{-1}\mathcal{G}_2)(U)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{F}_1(f^{-1}(W)) \arrow[r, "\alpha^{f_*\mathcal{F}_1}_{f(U),W}"] \arrow[d] & (f^{-1}f_*\mathcal{F}_1)(U) \arrow[r, "\varepsilon_{\mathcal{F}_1}(U)"] \arrow[d] & \mathcal{F}_1(U) \arrow[d] \\
\mathcal{F}_2(f^{-1}(W)) \arrow[r, "\alpha^{f_*\mathcal{F}_2}_{f(U),W}"'] & (f^{-1}f_*\mathcal{F}_2)(U) \arrow[r, "\varepsilon_{\mathcal{F}_2}(U)"'] & \mathcal{F}_2(U)
\end{tikzcd}$$

For the second diagram, to show the right box is commutative, it's equivalent to show the composition with $\alpha_{f(U),W}^{f_*\mathcal{F}_1}$, then it follows from that $\mathcal{F}_1 \to \mathcal{F}_2$ is natural transformation.

Finally we need to show that f^{-1} is left adjoint of f_*. That is $f_*(\varepsilon_{\mathcal{F}}) \circ \eta_{f_*\mathcal{F}} = id_{f_*\mathcal{F}}$,

$\varepsilon_{f^{-1}\mathcal{G}} \circ f^{-1}\eta_{\mathcal{G}} = id_{f^{-1}\mathcal{G}}$.

$$\begin{tikzcd}
(f_*\mathcal{F})(V) \arrow[r, "\alpha^{f_*\mathcal{F}}_{f(f^{-1}(V)),V}"] & (f_*f^{-1}f_*\mathcal{F})(V) \arrow[r, "(\eta_{\mathcal{F}})(f^{-1}(V))"] & \mathcal{F}(f^{-1}(V)) \\
 & (f_*\mathcal{F})(W) \arrow[u, "\alpha^{f_*\mathcal{F}}_{f(f^{-1}(V)),W}"] \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
(f^{-1}\mathcal{G})(U) \arrow[r, "(f^{-1}\eta_{\mathcal{G}})(U)"] & (f^{-1}f_*f^{-1}\mathcal{G})(U) \arrow[r, "(\varepsilon_{f^{-1}\mathcal{G}})(U)"] & (f^{-1}\mathcal{G})(U) \\
\mathcal{G}(V) \arrow[u, "\alpha^{\mathcal{G}}_{f(U),V}"] \arrow[r, "\alpha^{\mathcal{G}}_{f(f^{-1}(V)),V}"] & (f_*f^{-1}\mathcal{G})(V) \arrow[u, "\alpha^{f_*f^{-1}\mathcal{G}}_{f(U),V}"] \arrow[ur, "(f^{-1}\mathcal{G})(f^{-1}(V)\to U)"'] &
\end{tikzcd}$$

□

COROLLARY 2.17. Let $\mathcal{F} \in Psh(X, Ab)$, $\mathcal{G} \in Psh(Y, Ab)$, $\alpha : f^{-1}\mathcal{G} \to \mathcal{F}$, $\beta : \mathcal{G} \to f_*\mathcal{F}$ the corresponding morphism via the adjoint isomorphism $\phi_{\mathcal{G},\mathcal{F}}$: $\mathrm{Hom}(f^{-1}\mathcal{G}, \mathcal{F}) \to \mathrm{Hom}(\mathcal{G}, f_*\mathcal{F})$. Then $h^{\mathcal{F},f}_x \circ \beta_{f(x)} = \alpha_x \circ h^{f^{-1}\mathcal{G},f}_x \circ (\beta^{\mathcal{G}}_f)_{f(x)}$. If f is topological embedding, then the difference between $\beta_{f(x)}$ and α_x is composition with isomorphisms.

PROOF.

$$\begin{tikzcd}
(f_*f^{-1}\mathcal{G})_{f(x)} \arrow[r, "(f_*\alpha)_{f(x)}"] \arrow[d, "h^{f^{-1}\mathcal{G},f}_x"'] & (f_*\mathcal{F})_{f(x)} \arrow[d, "h^{\mathcal{F},f}_x"] \\
(f^{-1}\mathcal{G})_x \arrow[r, "\alpha_x"'] & \mathcal{F}_x
\end{tikzcd}$$

$\beta = f_*\alpha \circ \beta^{\mathcal{G}}_f$, $\beta_{f(x)} = (f_*\alpha)_{f(x)} \circ (\beta^{\mathcal{G}}_f)_{f(x)}$, $h^{\mathcal{F},f}_x \circ \beta_{f(x)} = h^{\mathcal{F},f}_x \circ (f_*\alpha)_{f(x)} \circ (\beta^{\mathcal{G}}_f)_{f(x)} = \alpha_x \circ h^{f^{-1}\mathcal{G},f}_x \circ (\beta^{\mathcal{G}}_f)_{f(x)}$. □

DEFINITION 2.18. Let$(X, \mathcal{O}_X)$ be a ringed topological space. Let $\mathcal{J}$ be a sheaf of ideals of $\mathcal{O}_X$ (i.e., $\mathcal{J}(U)$ is an ideal of $\mathcal{O}_X(U)$ for every open subset U and $\mathcal{O}(U \to V)(\mathcal{J}(U)) \subset \mathcal{J}(V)$). Let $\alpha_{x,U} : \mathcal{O}_X(U) \to \mathcal{O}_{X,x}$ be insertions, then $\alpha_{x,U}|^{A_x}_{\mathcal{J}(U)}$: $\mathcal{J}(U) \to A_x$ are insertions, where $A_x = \bigcup\limits_{U \ni x} \alpha_{x,U}(\mathcal{J}(U))$, $\widetilde{\alpha}_{x,U} : \mathcal{O}_X(U)/\mathcal{J}(U) \to \mathcal{O}_{X,x}/A_x$ are insertions.

PROP 2.19. Let $(X, \mathcal{O}_X)$ be a ringed topological space, $\mathcal{J}$ be a sheaf of ideals of $\mathcal{O}_X$, $V(\mathcal{J}) \coloneqq \{x \in X | A_x \neq \mathcal{O}_{X,x}\}$.Then $V(\mathcal{J})$ is a closed subset of X, $(V(\mathcal{J}), \mathcal{S}j^{-1}(\mathcal{O}_X/\mathcal{J}))$ is a ringed topological space, and we have a closed immersion $(j, j^{\#})$ of this space into $(X, \mathcal{O}_X)$, where $j^{\#}$ is the canonical surjection $\mathcal{O}_X \to \mathcal{O}_X/\mathcal{J} \simeq j_*(\mathcal{S}j^{-1}(\mathcal{O}_X/\mathcal{J}))$

PROOF. For $x \notin V(\mathcal{J})$, then $\exists U \in x$ such that $1 \in \mathcal{J}(U)$. For $y \in U$, $A_y = \mathcal{O}_{X,y}$, $y \notin V(\mathcal{J})$, $V(\mathcal{J})$ is closed.
$(\mathcal{S}j^{-1}(\mathcal{O}_X/\mathcal{J}))_x \simeq (j^{-1}(\mathcal{O}_X/\mathcal{J}))_x \simeq (\mathcal{O}_X/\mathcal{J})_x = \mathcal{O}_{X,x}/A_x$ (Def 1.17, Prop 2.13, Def 2.17), then $(V(\mathcal{J}), \mathcal{S}j^{-1}(\mathcal{O}_X/\mathcal{J}))$ is a ringed topological space.

$$\mathcal{O}_X/\mathcal{J} \to j_*j^{-1}(\mathcal{O}_X/\mathcal{J}) \to j_*(\mathcal{S}j^{-1}(\mathcal{O}_X/\mathcal{J}))$$

The first morphism is stalk isomorphism in Im j (Prop 2.15(2)), the second morphism is stalk isomorphism in Im j (j is topological embedding, Prop 2.4), the stalk of $\mathcal{O}_X/\mathcal{J}$ outside Im j is $\{0\}$ by definition of $V(\mathcal{J})$, the stalk of $j_*(\mathcal{S}j^{-1}(\mathcal{O}_X/\mathcal{J}))$ outside Im j is $\{0\}$ (Prop 2.11), therefore the composition is stalk isomorphism.Finally we can construct $(j, j^{\#})$ by

$$\mathcal{O}_X \to \mathcal{O}_X/\mathcal{J} \to j_*j^{-1}(\mathcal{O}_X/\mathcal{J}) \to j_*(\mathcal{S}j^{-1}(\mathcal{O}_X/\mathcal{J}))$$

The stalk map of the first morphism is projection, therefore $(j, j^{\#})$ is closed immersion.

$\square$

LEMMA 2.20. Let R be a ring, Λ a filtered category. Let C be the category of 3-term exact sequences of R-modules: its objects are the 3-term exact sequences, and its maps are the commutative diagrams

$$\begin{tikzcd}
L \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & N \arrow[d] \\
L' \arrow[r] & M' \arrow[r] & N'
\end{tikzcd}$$

Then, for any functor $\lambda \mapsto \left(L_\lambda \xrightarrow{\beta_\lambda} M_\lambda \xrightarrow{\gamma_\lambda} N_\lambda\right)$from Λ to C, the induced sequence $\varinjlim L_\lambda \xrightarrow{\beta} \varinjlim M_\lambda \xrightarrow{\gamma} \varinjlim N_\lambda$ is exact. Moreover, $(\alpha_\lambda^L, \alpha_\lambda^M, \alpha_\lambda^N)$ is filtered direct limit. From straight-forward diagram chasing

$$\begin{tikzcd}
 & L_\mu \arrow[rr] \arrow[dl, dashed] & & M_\mu \arrow[rr] \arrow[dl, dashed] & & N_\mu \arrow[dl, dashed] \\
L_\lambda \arrow[ur] \arrow[rr] \arrow[d] & & M_\lambda \arrow[ur] \arrow[rr] \arrow[d] & & N_\lambda \arrow[ur] \arrow[d] & \\
\varinjlim L_\lambda \arrow[rr] & & \varinjlim M_\lambda \arrow[rr] & & \varinjlim N_\lambda &
\end{tikzcd}$$

we get that $\varinjlim L_\lambda \xrightarrow{\beta} \varinjlim M_\lambda \xrightarrow{\gamma} \varinjlim N_\lambda$ is exact.

PROP 2.21. Let $f : X \to Y$ be closed immersion of ringed topological spaces, $Z = V(\mathcal{J})$ where $J = ker f^{\#}$, then there is $\phi : X \to Z$ isomorphism of ringed topological spaces such that $(f, f^{\#}) = (j, j^{\#}) \circ (\phi, \phi^{\#})$.

By definition we have the exact sequence in $Psh(Y, Ab)$

$$0 \to \mathcal{J} \to \mathcal{O}_Y \to f_*\mathcal{O}_X$$

Then use Cor 1.3 and lemma 2.19 to get the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{J}(U) \arrow[r, "l"] \arrow[d] & \mathcal{O}_Y(U) \arrow[r] \arrow[d] & (f_*\mathcal{O}_X)(U) \arrow[d] & \\
0 \arrow[r] & A_y \arrow[r, "l"'] & \mathcal{O}_{Y,y} \arrow[r, "f^{\#}_{s,y}"'] & (f_*\mathcal{O}_X)_y \arrow[r] & 0
\end{tikzcd}$$

$f^{\#}_{s,y}$ is surjective because f is closed immersion and prop 2.11. Then $A_y = \mathcal{O}_{Y,y}$ if and only if $(f_*\mathcal{O}_X)_y = 0$ if and only if $y \notin \mathrm{Im} f$. Thus $V(\mathcal{J}) = \mathrm{Im} f$.

We define $\phi^{\#} : \mathcal{O}_Z \to (f^{|Z})_*\mathcal{O}_X$ as follows:

In $Psh(Y, Ab)$ we have:

$$\begin{tikzcd}
\mathcal{O}_Y \arrow[r, "f^{\#}"] \arrow[d, "\pi"'] & f_*\mathcal{O}_X \\
\mathcal{O}_Y/\mathcal{J} \arrow[ur, dashed, "\overline{f^{\#}}"'] &
\end{tikzcd}$$

Then $\phi^{\#}$ is defined as

$$\begin{tikzcd}
j^{-1}(\mathcal{O}_Y/\mathcal{J}) \arrow[r, "(\phi_{\mathcal{O}_Y/\mathcal{J},(f^{|Z})_*\mathcal{O}_X})^{-1}(f^{\#})"] \arrow[d] & (f^{|Y})_*\mathcal{O}_X \\
Sj^{-1}(\mathcal{O}_Y/\mathcal{J}) \arrow[ur, dashed, "\phi^{\#}"'] &
\end{tikzcd}$$

Where $\phi_{\mathcal{O}_Y/\mathcal{J},(f^{|Z})_*\mathcal{O}_X} : \mathrm{Hom}(j^{-1}(\mathcal{O}_Y/\mathcal{J}), (f^{|Z})_*\mathcal{O}_X) \to \mathrm{Hom}((\mathcal{O}_Y/\mathcal{J}), j_*(f^{|Z})_*\mathcal{O}_X)$ is the adjoint isomorphism.

$(\overline{f^{\#}})_y : \mathcal{O}_{Y,y}/A_y \to (f_*\mathcal{O})_y$ is induced by $f^{\#}_{s,y}$, so it's isomorphism according to hte short exact sequence. Therefore $\phi^{\#}$ is stalk isomorphism(Cor 2.17), so ϕ is isomorphism of ringed topological space.

We only need to verify $(f, f^{\#}) = (j, j^{\#}) \circ (\phi, \phi^{\#})$. That is $j_*\phi^{\#} \circ j^{\#} = f^{\#}$. $(j, j^{\#}) \circ (\phi, \phi^{\#})$ is equal to

$$\mathcal{O}_Y \longrightarrow \mathcal{O}_Y/\mathcal{J} \longrightarrow j_*j^{-1}(\mathcal{O}_Y/\mathcal{J}) \longrightarrow j_*(Sj^{-1}(\mathcal{O}_Y/\mathcal{J})) \longrightarrow j_*(f^{|Z})_*\mathcal{O}_X$$

By definition the composition of the last two morphisms is $j_*(\phi_{\mathcal{O}_Y/\mathcal{J},(f^{|Z})_*\mathcal{O}_X})^{-1}(f^{\#})$, then the composition of the last three morphisms is $\overline{f^{\#}}$, and finally the whole composition is equal to $f^{\#}$.

3. Scheme

DEFINITION 3.1. Let X be a topological space, we say a collection of open subsets $\mathcal{B}$ is a basis if

(1) Every open subset of X can be written as union of elements in $\mathcal{B}$
(2) if $U, V \in \mathcal{B}$, then $U \cap V \in \mathcal{B}$.

DEFINITION 3.2. Let X be a topological space, $\mathcal{B}$ a basis of X, a $\mathcal{B}$-presheaf is a set of R-mod(R-algebra)$\{\mathcal{F}(U)\}_{U\in\mathcal{B}}$ with morphisms $\mathcal{F}(U \to V) : \mathcal{F}(U) \to \mathcal{F}(V)$ for each pair of $U \to V$ and satifies the condition $\mathcal{F}(U \to W) = \mathcal{F}(V \to W) \circ \mathcal{F}(U \to V)$

REMARK 3.3. Same with previous notation, for $s \in \mathcal{F}(U)$, $s|_V \coloneqq \mathcal{F}(U \to V)(s)$.

DEFINITION 3.4. Let X be a topological space, $\mathcal{B}$ a basis of X, $\mathcal{F}$ be a $\mathcal{B}$-presheaf, then $\mathcal{F}$ is called a $\mathcal{B}$-sheaf if

(1) For $U = \bigcup\limits_{i\in I} U_i$, $U, U_i \in \mathcal{B}$. $s, t \in \mathcal{F}(U)$, if $s|_{U_i} = t|_{U_i} \forall i \in I$, then $s = t$.
(2) For $U = \bigcup\limits_{i\in I} U_i$, $U, U_i \in \mathcal{B}$. $s_i \in \mathcal{F}(U_i)$ such that $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$, there exists $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ $\forall i \in I$.

THEOREM 3.5. *Let X be a topological space, $\mathcal{B}$ a basis of X, $\mathcal{F}$ a $\mathcal{B}$-sheaf, then there exists a sheaf $\bar{\mathcal{F}}$ on X unique up to isomorphism with identity on $\mathcal{B}$ that extends $\mathcal{F}$.*

PROOF. First we construct such $\bar{\mathcal{F}}$, let $\bar{\mathcal{F}}(U) = \varprojlim\limits_{\substack{V\subset U \\ V\in\mathcal{B}}} \mathcal{F}(V)$, the index category is given by $\#\mathrm{Hom}(U, V) = 1$ if $U \supset V$, $\#\mathrm{Hom}(U, V) = 0$ otherwise. Here we use the product construction for easy notation.

$$\bar{\mathcal{F}}(U) = \{(s_V) \in \prod_{\substack{V\subset U \\ V\in\mathcal{B}}} \mathcal{F}(U)|\ s_{V_1}|_{V_2} = s_{V_2} \forall V_1 \supset V_2\}$$

$\bar{\mathcal{F}}(U \to V)((s_V)) = (s_V)$(It's just projecting coordinates to a smaller index set). Clearly $\bar{\mathcal{F}}$ is a presheaf.

For $U = \bigcup\limits_{i\in I} U_i$, $(s_W), (t_W) \in \bar{\mathcal{F}}(U)$, $\bar{\mathcal{F}}(U \to U_i)(s_W) = \bar{\mathcal{F}}(U \to U_i)(t_W)$, that is $s_W = t_W$, $\forall W \subset U_i, W \in \mathcal{B}, i \in I$.

$$W = \bigcup_{i\in I}(W \cap U_i) = \bigcup_{i\in I} \bigcup_{\substack{\widetilde{W}\subset W\cap U_i \\ \widetilde{W}\in\mathcal{B}}} \widetilde{W}$$

$s_W|_{\widetilde{W}} = s_{\widetilde{W}} = t_{\widetilde{W}} = t_W|_{\widetilde{W}}$, $s_W = t_W$.

For $U = \bigcup\limits_{i\in I} U_i$, $\bar{\mathcal{F}}(U)$ can be identified as

$$\{f : \{W \in \mathcal{B}|W \subset U\} \to \coprod_{W\subset U} \mathcal{F}(W)|f(W) \in \mathcal{F}(W), f(W_1)|_{W_2} = f(W_2)\}$$

With the notation, let $f_i \in \bar{\mathcal{F}}(U_i)$, $f_i|_{H_{(U_i\cap U_j)}} = f_j|_{H_{(U_i\cap U_j)}}$, where $H_T = \{W \in \mathcal{B}|W \subset T\}$. $H_{U_i\cap U_j} = H_{U_i} \cap H_{U_j}$. There exists $f : \bigcup\limits_{i\in I} H_{U_i} \to \coprod\limits_{W\subset U} \mathcal{F}(W)$ such that $f|_{H_{U_i}} = f_i$. Clearly $f(W_1)|_{W_2} = f(W_2)$. $f(\widetilde{W}_1)_{|\widetilde{W}_1\cap\widetilde{W}_2} = f(\widetilde{W}_1 \cap \widetilde{W}_2) = f(\widetilde{W}_2)_{|\widetilde{W}_1\cap\widetilde{W}_2}$. Define $\tilde{f} : H_U \to \coprod\limits_{W\subset U} \mathcal{F}(W)$ by $\tilde{f}(W) = \mathit{gluing}\, f(\widetilde{W}) \in \mathcal{F}(W)$.(The index of the gluing is $\widetilde{W} \subset U_i \cap W, \widetilde{W} \in \mathcal{B}$) $\tilde{f}_{|H_{U_i}} = f_i$. $(\tilde{f}(W_1)_{|W_2})_{|\widetilde{W}} = \tilde{f}(W_1)_{|\widetilde{W}} = f(\widetilde{W})$, therefore $\tilde{f}(W_1)_{|W_2} = \tilde{f}(W_2)$, then $\tilde{f} \in \bar{\mathcal{F}}(U)$.

Now we show that with general inverse limit, $\mathcal{H}(U) = \varprojlim\limits_{\substack{V\subset U \\ V\in\mathcal{B}}} \mathcal{F}(V)$ is a sheaf. We only to need to show it's isomorphic to $\bar{\mathcal{F}}$, which follows from the diagram

$$\begin{tikzcd}
\bar{\mathcal{F}}(U_1) \arrow[r] & \bar{\mathcal{F}}(U_2) \arrow[dr] & \\
 & & \mathcal{F}(V) \\
\varprojlim\limits_{\substack{V\subset U_1 \\ V\in\mathcal{B}}} \mathcal{F}(V) \arrow[uu] \arrow[r] & \varprojlim\limits_{\substack{V\subset U_2 \\ V\in\mathcal{B}}} \mathcal{F}(V) \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

With lemma 3.7, we can choose $\mathcal{H}(U)=\mathcal{F}(U), \forall U\in\mathcal{B}$. With the diagram

$$\begin{tikzcd}
\varprojlim\limits_{\substack{V\subset U_1\\ V\in\mathcal{B}}} \mathcal{F}(V) \arrow[r] \arrow[d, "id"'] & \varprojlim\limits_{\substack{V\subset U_2\\ V\in\mathcal{B}}} \mathcal{F}(V) \arrow[d, "id"] \\
F(U_1) \arrow[r] & F(U_2)
\end{tikzcd}$$

we have $\mathcal{H}(U_1\to U_2)=\mathcal{F}(U_1\to U_2), U_1,U_2\in\mathcal{B}$. Therefore $\mathcal{H}$ extends $\mathcal{F}$.

For any sheaf $\mathcal{H}$ that extends $\mathcal{F}$, H is isomorphic to the presheaf $\varprojlim\limits_{\substack{V\subset *\\ V\in\mathcal{B}}} \mathcal{F}(V)$.

First we use the product construction for projective limit.

$$\begin{tikzcd}
\mathcal{H}(U) \arrow[r, "\tau_U"] \arrow[dr] & \varprojlim\limits_{\substack{V\subset U\\ V\in\mathcal{B}}} \mathcal{F}(V) \arrow[d] \\
& \mathcal{H}(V)
\end{tikzcd}$$

$\tau_U(s)=\{s_{|V}\}_{\substack{V\subset U\\ V\in\mathcal{B}}}$, clearly τ_U is injective. For $\{s_V\}\in\varprojlim\limits_{\substack{V\subset U\\ V\in\mathcal{B}}} \mathcal{F}(V)$, ${s_V}_{|_{V\cap V'}}=s_{V\cap V'}={s_{V'}}_{|_{V\cap V'}}$. There exist $s\in\mathcal{H}(U)$ such that $s_{|V}=s_V$, which is $\tau_U(s)=\{s_V\}$. τ_U is surjective. For the general projective limit, τ_U is is isomorphism and τ is natural transformation:

$$\begin{tikzcd}
\varprojlim\limits_{\substack{V\subset U_1\\ V\in\mathcal{B}}} \mathcal{F}(V) \arrow[r] & \varprojlim\limits_{\substack{V\subset U_2\\ V\in\mathcal{B}}} \mathcal{F}(V) \arrow[r] & \mathcal{H}(V) \\
\mathcal{H}(U_1) \arrow[r] \arrow[u, "\tau_{U_1}"] & \mathcal{H}(U_2) \arrow[u, "\tau_{U_2}"] \arrow[ur] &
\end{tikzcd}$$

With lemma 3.7, we can choose τ_U to be id for $U\in\mathcal{B}$, therefore all sheaves that extends $\mathcal{F}$ are isomorphic with identity on $\mathcal{B}$. $\square$

LEMMA 3.6. Let Λ and Λ' be small categories, $C:\Lambda'\to\Lambda$ a functor. Assume Λ' is filtered. Assume C is cofinal; that is,

(1) given $\lambda\in\Lambda$, there is a map $\lambda\to C\lambda'$ for some $\lambda'\in\Lambda'$, and
(2) given $\psi,\varphi:\lambda\rightrightarrows C\lambda'$, there is $\chi:\lambda'\to\lambda_1'$ with $(C\chi)\psi=(C\chi)\varphi$.

Let $\lambda\mapsto M_\lambda$ be a functor from Λ to $\mathcal{C}$ and direct limit of MC exists. Then $\varinjlim\limits_{\lambda'\in\Lambda'} M_{C\lambda'}=\varinjlim\limits_{\lambda\in\Lambda} M_\lambda$, more precisely, $M_\lambda\to M_{C\lambda'}\to\varinjlim\limits_{\lambda'\in\Lambda'} M_{C\lambda'}$ are insertions of MC.

PROOF. For $\lambda\to\lambda_1',\lambda\to\lambda_2'$, there exists λ_3' and $\lambda_1'\to\lambda_3',\lambda_2'\to\lambda_3'$.Then we have $\lambda\to\lambda_1'\to\lambda_3'\to\lambda_4'=\lambda\to\lambda_2'\to\lambda_3'\to\lambda_4'$. Thus we get the commutative diagram:

$$\begin{tikzcd}
 & M_{C\lambda'_1} \arrow[dr] \arrow[rrrd] & & & \\
M_\lambda \arrow[ur] \arrow[dr] & & M_{C\lambda'_3} \arrow[r] & M_{C\lambda'_4} \arrow[r] & \varinjlim_{\lambda' \in \Lambda'} F_{C\lambda'} \\
 & M_{C\lambda'_2} \arrow[ur] \arrow[rrru] & & &
\end{tikzcd}$$

Therefore $M_\lambda \to M_{C\lambda'} \to \varinjlim_{\lambda' \in \Lambda'} M_{C\lambda'}$ is well-defined.

For $\lambda_1 \to \lambda_2, \lambda_1 \to C\lambda'_1, \lambda_2 \to C\lambda'_2$ there exists λ'_3 and $\lambda'_1 \to \lambda'_3, \lambda'_2 \to \lambda'_3$ and

$$\lambda_1 \to C\lambda'_1 \to C\lambda'_3 \to C\lambda'_4 = \lambda_1 \to \lambda_2 \to C\lambda'_2 \to C\lambda'_3 \to C\lambda'_4$$

$$\begin{tikzcd}
M_{\lambda_1} \arrow[r] \arrow[dd] & M_{C\lambda'_1} \arrow[dr] \arrow[rrrd] & & & \\
 & & M_{C\lambda'_3} \arrow[r] & M_{C\lambda'_4} \arrow[r] & \varinjlim_{\lambda' \in \Lambda'} F_{C\lambda'} \\
M_{\lambda_2} \arrow[r] & M_{C\lambda'_2} \arrow[ur] \arrow[rrru] & & &
\end{tikzcd}$$

So the insertions are compatiable. Given $\alpha_\lambda : M_\lambda \to H$ such that $\alpha_\lambda = \alpha_\mu \circ \alpha^\lambda_\mu, \beta_{\lambda'} = \alpha_{C\lambda'}$, $\beta_{\lambda'} = \alpha_{C\mu'} \circ \alpha^{C\lambda'}_{C\mu'} = \beta_{\mu'} \circ \beta^{\lambda'}_{\mu'}$
There exists unique $f : \varinjlim_{\lambda' \in \Lambda'} F_{C\lambda'} \to H$ such that

$$\begin{tikzcd}
M_\lambda \arrow[d] & \\
M_{C\lambda'} \arrow[r] \arrow[d, "\beta_{\lambda'}"'] & \varinjlim_{\lambda' \in \Lambda'} F_{C\lambda'} \arrow[dl, dashed, "\exists! f"] \\
H &
\end{tikzcd}$$

$\square$

LEMMA 3.7. Let $F : \Lambda \to \mathcal{C}$ be functor, λ_0 initial object in Λ. Then $\alpha^{\lambda_0}_\mu : F_{\lambda_0} \to F_\mu$ is projective limit of F. Dually if μ_0 is terminal object of Λ, then $\alpha^\lambda_{\mu_0} : F_\lambda \to F_{\mu_0}$ is the direct limit of F.

PROOF. Clearly $\alpha^{\lambda_0}_\mu$ are compatible, given $\beta_\mu : H \to F_\mu$ compatiable, $\alpha^{\lambda_0}_\mu \circ \beta_{\lambda_0} = \beta_\mu$. If $f : H \to F_{\lambda_0}$ satisfies $\alpha^{\lambda_0}_\mu \circ f = \beta_\mu$, then $f = \beta_{\lambda_0}$. $\square$

LEMMA 3.8. Let $\Lambda, \Lambda', \mathcal{C}$ be categories, $F : \Lambda' \to \mathcal{C}$ functor, W_λ non-empty full subcategory of Λ' and $W_{\lambda_1} \supset W_{\lambda_2}$ if $\lambda_1 \to \lambda_2, H_\lambda$ initial object of $W_\lambda, H : \Lambda \to \Lambda'$ functor. Then the functor $\lambda \mapsto \varprojlim_{\mu \in W_\lambda} F_\mu$ is isomorphic to FH.

PROOF.

$$\begin{tikzcd}
FH_\lambda \arrow[r] & FH_{\lambda'} \arrow[dr] & \\
\varprojlim_{\mu\in W_\lambda} F_\mu \arrow[u] \arrow[r] & \varprojlim_{\mu\in W_{\lambda'}} F_\mu \arrow[u] \arrow[r] & F_\mu
\end{tikzcd}$$

$\square$

LEMMA 3.9. Let $\mathcal{B}$ be the set of principle open subsets of Spec(R), if $D(f) \supset D(g)$, then there is a morphism $L^R_{f\to g} : R_f \to R_g$, which sends $\frac{r}{1} \to \frac{r}{1}$, the insertion $R_f \to \varinjlim_{D(f)=V} R_f$ is isomorphism, $V \in \mathcal{B}$, the index category comes from an order where all elements are equal.

PROOF. $D(f) \supset D(g) \iff \sqrt{\langle f\rangle} \supset \sqrt{\langle g\rangle}$. Then $\frac{f}{1}$ is in invertible in R_g. According to the universal property of localization, there exists a morphism from R_f to R_g such that

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & R_g \\
R_f \arrow[ur, dashed, "L^R_{f\to g}"'] &
\end{tikzcd}$$

$\square$

It's obvious that if $D(f) = D(g)$, the induced morphism is isomorphism. $\varinjlim_{D(g)=V} R_g = \bigcup_{D(g)=V} \alpha^V_g(R_g) = \bigcup_{D(g)=V} \alpha^V_f \circ \alpha^g_f(R_g) = \bigcup_{D(g)=V} \alpha^V_f(R_f) = \alpha^V_f(R_f)$, α^V_f is surjective. If $\alpha^V_f(\frac{r}{f^n}) = 0$, then $\alpha^f_g(\frac{r}{f^n}) = 0$ for some g, $\frac{r}{f^n} = 0$, α^V_f is injective.

PROP 3.10. Let $\mathcal{B}$ be the set of principle open subsets of Spec(R), then there exists a $\mathcal{B}$-sheaf $\mathcal{O}$ on Spec(R) such that $\mathcal{O}(V) = \varinjlim_{D(f)=V} R_f$, the index category comes from an order where all elements are equal.

Moreover, there is a sheaf of rings $\bar{\mathcal{O}}$ on Spec(R) that extends $\mathcal{O}$, and the stalk $\bar{\mathcal{O}}_{\mathfrak{p}}$ is isomorphic to $R_{\mathfrak{p}}$, thus $(X, \bar{\mathcal{O}})$ is a ringed topological space, it's denoted by $\mathcal{O}_{Spec(R)}$.

PROOF. $O(V_1 \to V_2)$ is given by the diagram

$$\begin{tikzcd}
\varinjlim_{D(f)=V_1} R_f \arrow[r, "O(V_1\to V_2)"] & \varinjlim_{D(g)=V_2} R_g \\
R_f \arrow[u] \arrow[r] & R_g \arrow[u]
\end{tikzcd}$$

$\square$

It's easy to see $\mathcal{O}$ is a $\mathcal{B}$-presheaf by putting two diagram together

$$\begin{tikzcd}
\varinjlim_{D(f)=V_1} R_f \arrow[r, "O(V_1\to V_2)"] & \varinjlim_{D(g)=V_2} R_g \arrow[r, "O(V_2\to V_3)"] & \varinjlim_{D(h)=V_3} R_h \\
R_f \arrow[u] \arrow[r] & R_g \arrow[u] \arrow[r] & R_h \arrow[u]
\end{tikzcd}$$

Let $D(f) = \bigcup_{i\in I} D(f_i)$, that is $V(f) = \bigcap_{i\in I} V(f_i) = V(\langle f_i\rangle_{i\in I})$, $\sqrt{\langle f\rangle} = \sqrt{\langle f_i\rangle_{i\in I}}$. There exists $n\in\mathbb{N}$, $r_i\in R$, $i\in I_0$, $\#I_0<\infty$ such that $f^n = \sum_{i\in I_0} r_i f_i$.

Let $\alpha_f^{D(f)}(\frac{r}{f^m}) \in \mathcal{O}(D(f))$, $\alpha_f^{D(f)}(\frac{r}{f^m})_{|D(f_i)} = \alpha_{f_i}^{D(f_i)}(L^R_{f\to f_i}(\frac{r}{f^m})) = 0$. $\alpha_f^{D(f)}$ is isomorphism(lemma 3.9), thus $L^R_{f\to f_i}(\frac{r}{f^m})) = 0$, it's equivalent to $\frac{r}{1} = 0$ in R_{f_i}. There exists $k_i\in\mathbb{N}$ such that $f_i^{k_i} r = 0$. Let $k = \max_{i\in I_0} k_i$, $N = \#I_0$. Then $f^{nkN} r = 0$, $\frac{r}{f^m} = 0$ in R_f, $\alpha_f^{D(f)}(\frac{r}{f^m}) = 0$.

Let $\alpha_{f_i}^{D(f_i)}(\frac{r_i}{f_i^q}) \in \mathcal{O}(D(f_i))$, $\forall i\in I_0$. $D(f_i)\cap D(f_j) = D(f_if_j)$, $\alpha_{f_i}^{D(f_i)}(\frac{r_i}{f_i^q})_{|D(f_if_j)} = \alpha_{f_j}^{D(f_j)}(\frac{r_j}{f_j^q})_{|D(f_if_j)}$, which is $\alpha_{f_if_j}^{D(f_if_j)}(\frac{r_if_j^q}{(f_if_j)^q}) = \alpha_{f_if_j}^{D(f_if_j)}(\frac{r_jf_i^q}{(f_if_j)^q})$. Then $\frac{r_if_j^q}{(f_if_j)^q} = \frac{r_jf_i^q}{(f_if_j)^q}$ in $R_{f_if_j}$. There exists $k\geq q$ such that $(f_if_j)^k(r_if_j^q - r_jf_i^q) = 0$. From previous argument we have $D(f) = \bigcup_{i\in I_0} D(f_i) = \bigcup_{i\in I_0} D(f_i^{k+q})$, thus $f^n = \sum_{i\in I_0} \bar{r}_i f_i^{k+q}$. Let $s = \sum_{i\in I_0} \bar{r}_i r_i f_i^k$. $f_i^{k+q}s = \sum_{j\in I_0} \bar{r}_j r_j f_j^k f_i^{k+q} = \sum_{j\in I_0} \bar{r}_j r_j f_i^q (f_if_j)^k = \sum_{j\in I_0} \bar{r}_j r_i f_j^q (f_if_j)^k = r_i f_i^k f^n$, $f_i^k(f_i^q s - r_i f^n) = 0$, $\frac{s}{1} = \frac{r_if^n}{f_i^q}$ in R_{f_i}, $\frac{s}{1}(\frac{f}{1})^{-n} = \frac{r_i}{f_i^q}$ in R_{f_i}. Therefore $\alpha_f^{D(f)}(\frac{s}{f^n})_{|D(f_i)} = \alpha_{f_i}^{D(f_i)}(\frac{r_i}{f_i^q})$, $\forall i\in I_0$.

Generally $(\alpha_f^{D(f)}(\frac{s}{f^n})_{|D(f_i)})_{|D(f_if_j)} = (\alpha_f^{D(f)}(\frac{s}{f^n})_{|D(f_j)})_{|D(f_if_j)} = \alpha_{f_j}^{D(f_j)}(\frac{r_j}{f_j^q})_{|D(f_if_j)}$, $\forall j\in I_0$. $\bigcup_{j\in I_0} D(f_jf_i) = \bigcup_{j\in I_0} D(f_j)\cap D(f_i) = (\bigcup_{j\in I_0} D(f_j))\cap D(f_i) = D(f)\cap D(f_i) = D(f_i)$. Then $\alpha_f^{D(f)}(\frac{s}{f^n})_{|D(f_i)} = \alpha_{f_i}^{D(f_i)}(\frac{r_i}{f_i^q})$, $\forall i\in I$.

Let $\bar{\mathcal{O}}(U) = \varprojlim_{\substack{V\subset U\\ V\in\mathcal{B}}} \mathcal{O}(V)$, by theorem 3.5, $\bar{\mathcal{O}}$ is a sheaf that extends $\mathcal{O}$.

$$\bar{\mathcal{O}}_{\mathfrak{p}} = \varinjlim_{U\supset\mathfrak{p}} \bar{\mathcal{O}}(U) \simeq \varinjlim_{\substack{U\supset\mathfrak{p}\\ U\in\mathcal{B}}} \bar{\mathcal{O}}(U) = \varinjlim_{\substack{U\supset\mathfrak{p}\\ U\in\mathcal{B}}} \varinjlim_{D(f)=U} R_f$$

The isomorphism here is using lemma 3.6 with $\Lambda' = \{U\supset\mathfrak{p}, U\in\mathcal{B}\}$, $\Lambda = \{U\supset\mathfrak{p}\}$, both ordered by reverse inclusion, C inclusion functor.