Notes on algebraic geometry

Contents

CHAPTER 1

scheme

1. Sheaves

DEFINITION 1.1. Let X be a topological space, presheaves of abelian groups on X are the category $\mathrm{Fun}(\mathrm{Top}(X)^{op},\ Ab)$.

REMARK 1.2. Even though presheves can be defined on other category, it is better to use an abelian category. In this case, the category of presheaves will also be an abelian category. In the note, we only need properties of presheaves on Ab.

PROP 1.3. Let Λ be a category, C be an abelian category, then $\mathrm{Fun}(\Lambda,\ C)$ is an abelian category, if $\mathcal{F},\ \mathcal{G} \in \mathrm{Fun}(\Lambda,\ C)$, $\alpha : \mathcal{F} \to \mathcal{G}$ morphism, then ker and coker of α can be constructed as follows:

$$\begin{tikzcd} ker(\alpha_\lambda) \arrow[r] \arrow[d, dashed] & \mathcal{F}_\lambda \arrow[r, "\alpha_\lambda"] \arrow[d] & \mathcal{G}_\lambda \arrow[d] \\ ker(\alpha_\mu) \arrow[r] & \mathcal{F}_\mu \arrow[r, "\alpha_\mu"'] & \mathcal{G}_\mu \end{tikzcd} \qquad \begin{tikzcd} \mathcal{F}_\lambda \arrow[r, "\alpha_\lambda"] \arrow[d] & \mathcal{G}_\lambda \arrow[r] \arrow[d] & coker(\alpha_\lambda) \arrow[d, dashed] \\ \mathcal{F}_\mu \arrow[r, "\alpha_\mu"'] & \mathcal{G}_\mu \arrow[r] & coker(\alpha_\mu) \end{tikzcd}$$

It's easy to verify their universal property.

COROLLARY 1.4. In previous proposition, if $C = R$-mod, then ker and coker can be chosen (λ) point-wise as regular ker and coker in R-mod, and exactness is equivalent to (λ) pointwise regular exactness in R-mod (the isomorphism has to be inclusion, thus id).

REMARK 1.5. Let $\mathcal{F} \in \mathrm{Fun}(\mathrm{Top}(X)^{op},\ Ab)$, $U \supset V$ open subsets of X, $s \in \mathcal{F}(U)$, $s|_V := \mathcal{F}(U \to V)(s)$

DEFINITION 1.6. Let X, Y, Z be topological spaces, $f : X \to Y, g : Y \to Z$ be continuous maps, $\mathcal{F} \in \mathrm{Fun}(\mathrm{Top}(Y)^{op},\ Ab)$. The direct image of $\mathcal{F}$ w.r.t g is $g_*\mathcal{F} \in \mathrm{Fun}(\mathrm{Top}(Z)^{op},\ Ab)$. $(g_*\mathcal{F})(U \to V) := \mathcal{F}(g^{-1}(U) \to g^{-1}(V))$.

The inverse image of $\mathcal{F}$ w.r.t f is $f^{-1}\mathcal{F} \in \mathrm{Fun}(\mathrm{Top}(X)^{op}$,

$$\begin{tikzcd} \varinjlim_{W \supset f(U)} \mathcal{F}(W) \arrow[r, dashed, "(f^{-1}\mathcal{F})(U \to V)"] & \varinjlim_{W \supset f(V)} \mathcal{F}(W) \\ \mathcal{F}(W) \arrow[u, "\alpha^{Y,\mathcal{F}}_{f(U),W}"] \arrow[ur, "\alpha^{Y,\mathcal{F}}_{f(V),W}"'] & \end{tikzcd}$$

It's straight-forward to verify $f^{-1}\mathcal{F} \in \mathrm{Fun}(\mathrm{Top}(X)^{op},\ Ab)$.

REMARK 1.7. In order to make the definition of inverse image of presheaf to work, we need to fix $(\alpha^{Y,\mathcal{F}}_{f(U),W},\ \varinjlim_{W \supset f(U)} \mathcal{F}(W))$ for given U.

DEFINITION 1.8. Let $f : X \to Y, g : Y \to Z$ be continuous maps, $\mathcal{F}, \mathcal{G} \in \mathrm{Fun}(\mathrm{Top}(Y)^{op}, Ab)$, $\alpha : \mathcal{F} \to \mathcal{G}$ morphism. Then the direct image of α by g is $g_*\alpha : g_*\mathcal{F} \to g_*\mathcal{G}$, which is defined by $(g_*\alpha)(U) \coloneqq \alpha(g^{-1}(U))$, it's easy to verify it is morphism. The inverse image of α by f is $f^{-1}\alpha : f^{-1}\mathcal{F} \to f^{-1}\mathcal{G}$, which is defined by $(f^{-1}\alpha)(U) \coloneqq h_U$, where h_U is the unique map such that the following diagram is commmutative

$$\begin{tikzcd}
\varinjlim\limits_{W\supset f(U)} \mathcal{F}(W) \arrow[r, "h_U"] & \varinjlim\limits_{W\supset f(U)} \mathcal{G}(W) \\
\mathcal{F}(W) \arrow[u, "\alpha^{Y,\mathcal{F}}_{f(U),W}"'] \arrow[r, "\alpha(W)"] & \mathcal{G}(W) \arrow[u, "\beta^{Y,\mathcal{G}}_{f(U),W}"']
\end{tikzcd}$$

$$\begin{tikzcd}
\varinjlim\limits_{W\supset f(V)} \mathcal{F}(W) \arrow[r, "h_V"] & \varinjlim\limits_{W\supset f(V)} \mathcal{G}(W) \\
\varinjlim\limits_{W\supset f(U)} \mathcal{F}(W) \arrow[u, "(f^{-1}\mathcal{F})(U\to V)"'] \arrow[r, "h_U"] & \varinjlim\limits_{W\supset f(U)} \mathcal{G}(W) \arrow[u, "(f^{-1}\mathcal{G})(U\to V)"'] \\
\mathcal{F}(W) \arrow[u, "\alpha^{Y,\mathcal{F}}_{f(U),W}"'] \arrow[uu, bend left, "\alpha^{Y,\mathcal{F}}_{f(V),W}"] \arrow[r, "\alpha(W)"] & \mathcal{G}(W) \arrow[u, "\beta^{Y,\mathcal{G}}_{f(U),W}"'] \arrow[uu, bend right, "\beta^{Y,\mathcal{G}}_{f(V),W}"']
\end{tikzcd}$$

To show the upper box is commutative, it's equivalent to show it is commutative with composition $\alpha^{Y,\mathcal{F}}_{f(U),W}$, then it's equivalent to the commutativity of the outer box, which is the definition of h_V.

DEFINITION 1.9. Let $\mathcal{F}$ be a presheaf, we say $\mathcal{F}$ is a sheaf if it has the following properties:

(1) (Uniqueness) Let U be an open subset of X, $s \in \mathcal{F}(U)$, U_i a covering of U by open subsets U_i. If $s|_{U_i} = 0$ for every i, then $s = 0$.

(2) (Glueing local sections) Let us keep the notation of (1). Let $s_i \in \mathcal{F}(U_i)$, $i \in I$, be sections such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$. Then there exists a section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ (this section s is unique by condition (1)).

PROP 1.10. Let $\mathcal{F}$ be a sheaf, then $\mathcal{F}(\varnothing) = \{0\}$

PROOF. let $\varnothing = \bigcup\limits_{i \in \varnothing} U_i$, $s \in \mathcal{F}(\varnothing)$, $s|_{U_i} = 0|_{U_i}$, thus $s = 0$, $\mathcal{F}(\varnothing) = \{0\}$. $\square$

PROP 1.11. Let $\mathcal{F} \in \mathrm{Fun}(\mathrm{Top}(X)^{op}, Ab)$ be a sheaf, $f : X \to Y$ be continuous map, then $f_*\mathcal{F}$ in a sheaf.

DEFINITION 1.12. Let $\mathcal{F} \in \mathrm{Fun}(\mathrm{Top}(X)^{op},\ Ab)$, $U \subset X$ open subset, then we can define the restriction of $\mathcal{F}$ to U, $(\mathcal{F}|_U)(V \to W) := \mathcal{F}(V \to W)$. It's easy to check $\mathcal{F}|_U \in \mathrm{Fun}(\mathrm{Top}(U)^{op},\ Ab)$ and $\mathcal{F}$ is sheaf implys $\mathcal{F}|_U$ is sheaf.

DEFINITION 1.13. Let $\mathcal{F} \in \mathrm{Psh}(X,\ Ab)$, and let $x \in X$. The stalk of $\mathcal{F}$ at x is the group $\mathcal{F}_x = \varinjlim_{U \ni x} \mathcal{F}(U)$ with morphisms $\alpha_{x,U} : \mathcal{F}(U) \to \mathcal{F}_x$, the direct limit being taken over the open neighborhoods U of x. Let $s \in F(U)$ be a section, for any $x \in U$, we denote the image of s in $\mathcal{F}_x$ by s_x. We call s_x the germ of s at x.

DEFINITION 1.14. Let $\alpha : \mathcal{F} \to \mathcal{G}$ be morphism in $\mathrm{Fun}(\mathrm{Top}(X)^{op},\ Ab)$, then the stalk map of α at x is defined by the diagram:

$$\begin{tikzcd}
\mathcal{F}_x \arrow[r, "\alpha_x"] & \mathcal{G}_x \\
\mathcal{F}(U) \arrow[u, "\alpha^{\mathcal{F}}_{x,U}"] \arrow[r, "\alpha(U)"'] & \mathcal{G}(U) \arrow[u, "\alpha^{\mathcal{G}}_{x,U}"']
\end{tikzcd}$$

it's easy to verify $\{\alpha^{\mathcal{G}}_{x,U} \circ \alpha(U)\}_{U \ni x}$ are compatible, moreover we have:

$$\begin{tikzcd}
\mathcal{F}_x \arrow[r, "\alpha_x"] & \mathcal{G}_x \arrow[r, "\beta_x"] & \mathcal{H}_x \\
\mathcal{F}(U) \arrow[u, "\alpha^{\mathcal{F}}_{x,U}"] \arrow[r, "\alpha(U)"'] & \mathcal{G}(U) \arrow[u, "\alpha^{\mathcal{G}}_{x,U}"] \arrow[r, "\beta(U)"'] & \mathcal{H}(U) \arrow[u, "\alpha^{\mathcal{H}}_{x,U}"]
\end{tikzcd}$$

By uniqueness we have $(\beta \circ \alpha)_x = \beta_x \circ \alpha_x$.

LEMMA 1.15. Let $\mathcal{F}$ be a sheaf on X. Let $s, t \in \mathcal{F}(U)$ such that $\alpha_{x,U}(s) = \alpha_{x,U}(t)\ \forall\ x \in U$. Then $s = t$.

PROOF. $\forall\, x \in X, \exists\, U_x \ni x, U_x \subset U$ such that $\mathcal{F}(U \to U_x)(s) = \mathcal{F}(U \to U_x)(t)$, that is $s|_{U_x} = t|_{U_x}$ with previous notation. Then $s = t$ follows from definition of sheaf. ☐

LEMMA 1.16. Let $\mathcal{F}$ be a presheaf on X, $\mathcal{G}$ be a sheaf on X, $\alpha : \mathcal{F} \to \mathcal{G}$ morphism, then $\alpha = 0$ if and only if $\alpha_x = 0$.

PROOF. $s \in \mathcal{F}(U)$, $\alpha_x = 0, \forall x \iff \alpha_x(\alpha^{\mathcal{F}}_{x,U}(s)) = \alpha^{\mathcal{G}}_{x,U}(\alpha(U)s) = 0\ \forall x \in U, s \in \mathcal{F}(U) \iff \alpha(U)s = 0, \forall s \in \mathcal{F}(U)$(Lemma 1.13) $\iff \alpha = 0$ ☐

PROP 1.17. Let $\alpha \in MorFun(\mathcal{A}, \mathcal{B})$, B has pullbacks or coproducts up to the size of its hom-sets. Then α is monomorphism if and only if it is point-wise monomorphism.

PROP 1.18. Let $\alpha \in MorFun(\mathcal{A}, \mathcal{B})$, B is finitely cocomplete. Then α is epimorphism if and only if it is point-wise epimorphism.

DEFINITION 1.19. Let $\mathcal{F}, \mathcal{G} \in \mathrm{Fun}(\mathrm{Top}(X)^{op},\ Ab)$, $\alpha : \mathcal{F} \to \mathcal{G}$ morphism, if α_x is surjective(injective, isomorphism) for all x, we say α is stalk surjective(injective, isomorphism).

PROP 1.20. Let $\alpha : \mathcal{F} \to \mathcal{G}$ be morphism of sheaves on X. Then $\alpha(U)$ is injective $\forall\ U \subset X$ open subset if and only if α is stalk injective, α is isomorphism if and only if α is stalk isomorphism.

PROOF. (1) Assume α is stalk injective, let $\alpha(U)s = 0$ for $s \in \mathcal{F}(\mathrm{U})$. $\alpha_x \circ \alpha_{x,U}(s) = \beta_{x,U} \circ \alpha(U)(s) = 0$. $\alpha_{x,U}(s) = 0$, $s|_{U_x} = 0$ $x \in U_x$.U_x consists of open cover of U. Therefore $s = 0$.

(2) Assume $\alpha(W)$ is injective $\forall\, W \subset X$ open subset, $\alpha_x(s_x) = 0, s \in \mathcal{F}(U)$,$\beta_{x,U} \circ \alpha(U)(s)$=$\alpha_x \circ \alpha_{x,U}(s) = 0$, $\alpha(U)(s)|_V = \alpha(V)(s|_V)$,$s|_V = 0, s_x = (s|_V)_x = 0$

(3) Assume α is stalk isomorphism, let $u \in \mathcal{G}(U)$, for all $x, \exists U_x \ni x, s^x \in \mathcal{F}(U_x)$ such that $\alpha(U_x)s^x = u|_{U_x}$.If $x, y \in U, \alpha(U_x \cap U_y)s^x|_{U_x \cap U_y} = \alpha(U_x)(s^x)|_{U_x \cap U_y} = (u|_{U_x})|_{U_x \cap U_y} = u|_{U_x \cap U_y} = \alpha(U_x \cap U_y)s^y|_{U_x \cap U_y}$,$s^x|_{U_x \cap U_y} = s^y|_{U_x \cap U_y}$.$\exists! s \in \mathcal{F}(U)$ such that $s|_{U_x} = s^x$.$(\alpha(U)s)_x = \alpha_x(s_x) = \alpha_x((s|_{U_x})_x) = \alpha_x(s^x|_x) = (\alpha(U_x)s^x)_x = (u|_{U_x})_x = u_x$.Therefore $\alpha(U)s = u$.

(4) Assume α is isomorphism, $\exists \beta$ such that $\beta \circ \alpha = id_{\mathcal{F}}, \alpha \circ \beta = id_{\mathcal{G}}, \beta_x \circ \alpha_x = id_{\mathcal{F}_x}$,$\alpha_x \circ \beta_x = id_{\mathcal{G}_x}, \alpha_x$ is isomorphism. □

DEFINITION 1.21. Let $\mathcal{F}$ be a presheaf on X. The sheafication of $\mathcal{F}$ is a pair $(\alpha, \mathcal{G}), \mathcal{G}$ is a sheaf, $\alpha : \mathcal{F} \to \mathcal{G}$ such that for all such pair $(\beta, \mathcal{H}), \exists! \theta : \mathcal{G} \to \mathcal{H}$ such that $\beta = \theta \circ \alpha$.

PROOF. Uniqueness up to isomorphism is trivial, we only need to construct the sheafication of $\mathcal{F}$. $\mathcal{F}^+(U) := \{f : U \to \coprod\limits_{x \in U} \mathcal{F}_x | \forall x \in U, \exists U_x \ni x, s^x \in \mathcal{F}(U_x)$, such that $f(y) = s^x_y, \forall y \in U_x\}$.The operation on $\mathcal{F}^+(U)$ is defined point-wise.$\mathcal{F}^+(U \to V)(f) := f|_V$ (restriction in the usual sense), therefore $\mathcal{F}^+(U)$ is presheaf. To see it is sheaf, let U_i be an open cover of $U, f|_{U_i} = 0$, then $f = 0$. Let $f_i \in \mathcal{F}(U_i)$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$, then $\exists! f : U \to \coprod\limits_{x \in U} \mathcal{F}_x$ such that $f|_{U_i} = f_i$. Clearly $f \in \mathcal{F}(U)$. Now we define $\alpha : \mathcal{F} \to \mathcal{F}^+$ by $\alpha(U)(s)(x) = s_x$. $(\mathcal{F}^+(U \to V) \circ \alpha(U))(s)(x) = s_x = (s|_V)_x = (\alpha(V) \circ \mathcal{F}(U \to V))(s)(x)$. Thus α is morphism.
Let $(\beta, \mathcal{G})$ be another pair, for $f \in \mathcal{F}^+(U)$, by definition of $\mathcal{F}^+(U)$, $\exists U_x \ni x, s^x \in \mathcal{F}(U_x)$, such that $f|_{U_x} = \alpha(U_x)s^x$.$\beta(U_x)(s^x)|_{U_x \cap U_y} = \beta(U_x \cap U_y)(s^x|_{U_x \cap U_y})$, $f|_{U_x \cap U_y} = (f|_{U_x})|_{U_x \cap U_y} = (\alpha(U_x)s^x)|_{U_x \cap U_y} = \alpha(U_x \cap U_y)(s^x|_{U_x \cap U_y}) = \alpha(U_x \cap U_y)(s^y|_{U_x \cap U_y})$.$\alpha(V)$ is injective (That is just lemma 1.7.), $s^x|_{U_x \cap U_y} = s^y|_{U_x \cap U_y}$.$\beta(U_x)(s^x)|_{U_x \cap U_y} = \beta(U_x \cap U_y)(s^x|_{U_x \cap U_y}) = \beta(U_x \cap U_y)(s^y|_{U_x \cap U_y}) = \beta(U_y)(s^y)|_{U_x \cap U_y}$. $\exists!$ $h \in \mathcal{G}(U)$ such that $h|_{U_x} = \beta(U_x)(s^x)$, then $\theta(U)(f) := h$. To see it's well-defined, choose another cover W_y, for (x, y) pair such that $(U_x \cap W_y) \neq \emptyset$, $f|_{U_x \cap W_y} = (f|_{U_x})|_{U_x \cap W_y} = (\alpha(U_x)s^x)|_{U_x \cap W_y} = \alpha(U_x \cap W_y)(s^x|_{U_x \cap W_y})$, from previous arguments we know that $\beta(U_x \cap W_y)(s^x|_{U_x \cap W_y})$ can be glued to a element in $\mathcal{G}(U)$.$h|_{U_x \cap W_y} = (h_{U_x})_{U_x \cap W_y} = (\beta(U_x)s^x)|_{U_x \cap W_y} = \beta(U_x \cap W_y)(s^x|_{U_x \cap W_y})$. $f|_{\mathcal{W}_y} = \alpha(\mathcal{W}_y)t^y$, $f|_{U_x \cap W_y} = (f|_{W_y})|_{U_x \cap W_y} = \alpha(U_x \cap W_y)t^y|_{U_x \cap W_y}$, $s^x|_{U_x \cap W_y} = t^y|_{U_x \cap W_y}$. Therefore $\beta(W_y)t^y$ glue to the same element as $\beta(U_x)s^x$ glue to. To see θ is morphism, let $f \in \mathcal{F}^+(U), U_x \subset V, ((\theta(U)f)|_V)|_{U_x} = \theta(U)f)|_{U_x} = \beta(U_x)s^x = \theta(V)f|_V$. $\theta \circ \alpha = \beta$ is obvious, for $\phi : \mathcal{F}^+ \to \mathcal{G}$ such that $\phi \circ \alpha = \beta$, let U_x, s^x be the same setting as above, $(\phi(U)f)|_{U_x} = \phi(U_x)(f|_{U_x}) = \phi(U_x)\alpha(U_x)(s^x) = \beta(U_x)(s^x)$, therefore $\phi = \theta$. □

EXAMPLE 1.22. Let A be an abelian group, X topological space, we define the presheaf A_X by $A_X(U) = \begin{cases} A & U \neq \varnothing \\ \{0\} & otherwise \end{cases}$, $A_X(U \to V) = \begin{cases} id_A & V \neq \varnothing \\ 0 & otherwise \end{cases}$, the sheafication of A is $S(A)(U) = \{f : U \to A | f$ is locally constant$\}$, $S(A)(U \to V)$=usual restriction of function from U to V. $Sh(A)(U) : (A_X)(U) \to S(A_X)(U), a \mapsto Cons_a$.

DEFINITION 1.23. Let $\mathcal{F} \in Psh(U, Ab)$, $U \subset X$ open subset. Define extension of $\mathcal{F}$ by zero presheaf by $(j_{p!}\mathcal{F})(V) = \begin{cases} \mathcal{F}(V) & V \subset U \\ \{0\} & otherwise \end{cases}$, $(j_{p!}\mathcal{F})(V_1 \to V_2) = \begin{cases} F(V_1 \to V_2) & V_1 \subset U \\ 0 & otherwise \end{cases}$. The extension of $\mathcal{F}$ by zero sheaf $j_!\mathcal{F}$ is defined to be the sheafication of $j_{p!}\mathcal{F}$.

DEFINITION 1.24. Let $\mathcal{F}, \mathcal{F}'$ be presheaves, we say $\mathcal{F}'$ is subpresheaf of $\mathcal{F}$ if $\mathcal{F}'(U) \subset \mathcal{F}(U)$, and $\mathcal{F}'(U \to V) = \mathcal{F}(U \to V)|_{\mathcal{F}'(U)}^{\mathcal{F}'(V)}$. Then we can define the quotient presheaf $\mathcal{F}/\mathcal{F}'$ by $(\mathcal{F}/\mathcal{F}')(U \to V)([s]) := [\mathcal{F}(s)]$. let $\alpha : \mathcal{F} \to \mathcal{G}$ be morphism of presheaves, $(Ker\alpha)(U \to V) := \mathcal{F}(U \to V)|_{ker\alpha(U)}^{ker\alpha(V)}$. $(Im\alpha)(U \to V) := \mathcal{G}(U \to V)|_{Im\alpha(U)}^{Im\alpha(V)}$. It's easy to verify these three are presheaves.

REMARK 1.25. In above definition, if $\mathcal{F}, \mathcal{F}'$ are sheaves, then $\mathcal{F}/\mathcal{F}'$ is sheaf. If $\mathcal{F}, \mathcal{G}$ are sheaves, then $ker\alpha$ is sheaf.

REMARK 1.26. Even though using explicit construction for filtered direct limit is just as meaningless as that for tensor product, sometimes we use a relative construction just for convenience. Specifically, $\mathcal{F}, \mathcal{G} \in$ Fun(Λ, R-mod), Λ filtered, $\mathcal{G}_\lambda \subset \mathcal{F}_\lambda$, $\mathcal{F}(\lambda \to \mu)(\mathcal{G}_\lambda) \subset \mathcal{G}_\mu$, $\mathcal{G}(\lambda \to \mu) = \mathcal{F}(\lambda \to \mu)|_{\mathcal{G}_\lambda}^{\mathcal{G}_\mu}$, $(\mathcal{F}/\mathcal{G})(\lambda \to \mu) := \mathcal{F}_\lambda/\mathcal{G}_\lambda \to \mathcal{F}_\mu/\mathcal{G}_\mu$, $\alpha_\lambda : \mathcal{F}_\lambda \to \varinjlim_{\lambda\in\Lambda} \mathcal{F}_\lambda$ direct limit, then $\alpha_\lambda|_{\mathcal{G}_\lambda}^{\bigcup_{\lambda\in\Lambda}\alpha_\lambda(\mathcal{G}_\lambda)} : \mathcal{G}_\lambda \to \bigcup_{\lambda\in\Lambda} \alpha_\lambda(\mathcal{G}_\lambda)$ is direct limit, and the direct limit of $\mathcal{F}/\mathcal{G}$ is given by the diagram:

$$\begin{tikzcd} \mathcal{F}_\lambda \arrow[rr, "\alpha_\lambda"] \arrow[d] & & \varinjlim_{\lambda\in\Lambda} \mathcal{F}_\lambda \arrow[d] \\ \mathcal{F}_\lambda/\mathcal{G}_\lambda \arrow[rr, dashed, "\widetilde{\alpha}_\lambda"] & & \varinjlim_{\lambda\in\Lambda} \mathcal{F}_\lambda / \bigcup_{\lambda\in\Lambda} \alpha_\lambda(\mathcal{G}_\lambda) \end{tikzcd}$$

PROP 1.27. Let $\alpha : \mathcal{F} \to \mathcal{G}$ be morphism in Psh(X,Ab), then $\ker(\alpha_x) = (\ker\alpha)_x$, $\mathrm{Im}(\alpha_x) = (\mathrm{Im}\alpha)_x$.

REMARK 1.28. Though it seems weird to use equality for an object that is unique up to isomorphism, it makes sense with the relative construction of direct limit in remark 1.26.

PROOF. It's basically using the definition of stalk map:

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\alpha(U)"] \arrow[d, "\alpha_{x,U}^{X,\mathcal{F}}"'] & \mathcal{G}(U) \arrow[d, "\alpha_{x,U}^{X,\mathcal{G}}"] \\ \mathcal{F}_x \arrow[r, "\alpha_x"'] & \mathcal{G}_x \end{tikzcd}$$

$\mathrm{Im}(\alpha_x) = \alpha_x(\mathcal{F}_x) = \alpha_x(\bigcup_{U\ni x} \alpha_{x,U}^{X,\mathcal{F}}(\mathcal{F}(U))) = \bigcup_{U\ni x} \alpha_x(\alpha_{x,U}^{X,\mathcal{F}}(\mathcal{F}(U))) = \bigcup_{U\ni x} \alpha_{x,U}^{X,\mathcal{G}}(\alpha(U)(\mathcal{F}(U))) = \bigcup_{U\ni x} \alpha_{x,U}^{X,\mathcal{G}}((\mathrm{Im}\alpha)(U)) = (\mathrm{Im}\alpha)_x$.

$\ker(\alpha_x) = \{\alpha_{x,U}^{X,\mathcal{F}}(s) | x \in U, s \in \mathcal{F}(U), \exists V \ni x$ such that $s|_V \in \ker(\alpha(V))\} = \{\alpha_{x,U}^{X,\mathcal{F}}(s) | x \in U, s \in \ker(\alpha(U))\} = (\ker\alpha)_x$

□

PROP 1.29. In the category of sheaves of abelian groups on X, monomorphism is equivalent to stalk injective, epimorphism is equivalent to stalk surjective.

PROOF. With prop 1.13 and lemma 1.15, it's easy to prove stalk injective (surjective) implys monomorphism (epimorphism).

Assume $\alpha : \mathcal{F} \to \mathcal{G}$ is monomorphism. Fix $U \subset X$ open subset, $\mathcal{H} \in Sh(X, Ab)$, we have natural isomorphism:

(1.1)
$$Hom_{Sh(X,Ab)}(j_!(\mathbb{Z}_U), \mathcal{H}) \xrightarrow{\sim} Hom_{Psh(X,Ab)}(j_{p!}(\mathbb{Z}_U), \mathcal{H}) \longrightarrow Hom_{Psh(X,Ab)}(\mathbb{Z}_U, H_{|U}) \to H(U)$$

$$\begin{tikzcd}
Hom_{Sh(X,Ab)}(j_!(\mathbb{Z}_U), \mathcal{F}) \arrow[r, "-\circ Sh(j_{p!}(\mathbb{Z}_U))"] \arrow[d, "\alpha\circ -"'] & Hom_{Psh(X,Ab)}(j_{p!}(\mathbb{Z}_U), \mathcal{F}) \arrow[r] \arrow[d, "\alpha\circ -"'] & \mathcal{F}(U) \arrow[d, "\alpha(U)"] \\
Hom_{Sh(X,Ab)}(j_!(\mathbb{Z}_U), \mathcal{G}) \arrow[r, "-\circ Sh(j_{p!}(\mathbb{Z}_U))"'] & Hom_{Psh(X,Ab)}(j_{p!}(\mathbb{Z}_U), \mathcal{G}) \arrow[r] & \mathcal{G}(U)
\end{tikzcd}$$

By assumption the left column is injective, thus $\alpha(U)$ is injective.

Assume $\alpha : \mathcal{F} \to \mathcal{G}$ is epimorphism. Fix $x \in X$, $\mathcal{H}(U) \coloneqq \begin{cases} \mathcal{G}_x/Im(\alpha_x) & x \in U \\ \{0\} & otherwise \end{cases}$, $\mathcal{H}(U \to V) \coloneqq \begin{cases} id_{\mathcal{G}_x/Im(\alpha_x)} & x \in V \\ 0 & otherwise \end{cases}$, $\mathcal{H} \in Sh(X, Ab)$. $\beta : \mathcal{G} \to \mathcal{H}$, $\beta(U) \coloneqq \begin{cases} [\alpha^{\mathcal{G}}_{x,U}] & x \in U \\ 0 & otherwise \end{cases}$, β is morphism. $(\beta, 0), (0, \beta) : \mathcal{G} \to \mathcal{H} \oplus \mathcal{H}$ and $(\beta, 0) \circ \alpha = (0, \beta) \circ \alpha$, $(\beta, 0) = (0, \beta)$, $\mathcal{G}_x/Im(\alpha_x) = \{0\}$, α_x is surjective. □

PROP 1.30. Sheaves of abelian groups on X is an abelian category.

PROOF. (1) Zero object is the sheaf $\mathcal{F}$ defined by $\mathcal{F}(U) = \{0\}$, it's easy to verify it's a sheaf and zero object in the category.
(2) Biproduct of $\mathcal{F}$ and $\mathcal{G}$ is $(i_{\mathcal{F}}, i_{\mathcal{G}}, \pi_{\mathcal{F}}, \pi_{\mathcal{G}}, \mathcal{F} \oplus \mathcal{G})$, where $i_{\mathcal{F}} : \mathcal{F} \to \mathcal{F} \oplus \mathcal{G}$, $i_{\mathcal{G}} : \mathcal{G} \to \mathcal{F} \oplus \mathcal{G}$, $\pi_{\mathcal{F}} : \mathcal{F} \oplus \mathcal{G} \to \mathcal{F}$, $\pi_{\mathcal{G}} : \mathcal{F} \oplus \mathcal{G} \to \mathcal{G}$, $(\mathcal{F} \oplus \mathcal{G})(U) \coloneqq \mathcal{F}(U) \oplus \mathcal{G}(U)$, $(\mathcal{F} \oplus \mathcal{G})(U \to V) \coloneqq \mathcal{F}(U \to V) \oplus \mathcal{G}(U \to V)$, $i_{\mathcal{F}}(U) \coloneqq l_{\mathcal{F}(U)}$, $i_{\mathcal{G}}(U) \coloneqq l_{\mathcal{G}(U)}$, $\pi_{\mathcal{F}}(U) \coloneqq \pi_{\mathcal{F}(U)}$, $\pi_{\mathcal{G}}(U) \coloneqq \pi_{\mathcal{G}(U)}$, easy to verify it's biproduct.
(3) $\alpha : \mathcal{F} \to \mathcal{G}$, $ker\alpha$ defined in def 1.20 with the inclusion tranformation is actually the kernel of α, cokernel of α is $\mathcal{G} \longrightarrow \mathcal{G}/Im(\alpha) \to S(\mathcal{G}/Im(\alpha))$
For the first statement,

$$\begin{tikzcd}
ker\alpha \arrow[r] & \mathcal{F} \arrow[r, "\alpha"] & \mathcal{G} \\
\mathcal{H} \arrow[u, dashed, "\theta"] \arrow[ur, "\beta"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{H}(U) \arrow[r] \arrow[d] & ker(\alpha(U)) \arrow[r] \arrow[d] & \mathcal{F}(U) \arrow[d] \\
\mathcal{H}(V) \arrow[r] & ker(\alpha(V)) \arrow[r] & \mathcal{F}(V)
\end{tikzcd}$$

$\alpha\circ\beta = 0$, $\mathrm{Im}(\beta(U)) \subset \ker(\alpha(U))$, $\theta(U) := \beta(U)|^{ker(\alpha(U))}$, uniqueness is obvious. For the second statement,

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\alpha"] & \mathcal{G} \arrow[r, "\pi"] \arrow[dr, "\beta"'] & \mathcal{G}/Im(\alpha) \arrow[r] \arrow[d, dashed, "\theta"'] & S(\mathcal{G}/Im(\alpha)) \arrow[dl, dashed, "\phi"] \\ & & \mathcal{H} & \end{tikzcd}$$

If there are ϕ_1, ϕ_2 then $\phi_1 \circ Sh(\mathcal{G}/Im(\alpha)) \circ \pi = \phi_1 \circ Sh(\mathcal{G}/Im(\alpha)) \circ \pi$, $\phi_1 \circ Sh(\mathcal{G}/Im(\alpha)) = \phi_1 \circ Sh(\mathcal{G}/Im(\alpha))$, $\phi_1 = \phi_2$.

(4) Assume α is monomorphism,

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\alpha"] \arrow[d, dashed, "\theta"'] & \mathcal{G} \arrow[r, "\pi"] & \mathcal{G}/Im\alpha \arrow[r] & S(\mathcal{G}/Im\alpha) \\ \mathcal{H} \arrow[ur, "\beta"'] & & & \end{tikzcd}$$

$\beta := \ker(\mathcal{G} \to S(\mathcal{G}/Im(\alpha)))$, α_x is injective(Prop 1.26), thus θ_x is injective. $\mathrm{Sh}(\mathcal{G}/Im\alpha) \circ \pi \circ \beta = 0$, $\mathrm{Sh}(\mathcal{G}/Im\alpha)_x \circ \pi_x \circ \beta_x = 0$, that is $\mathrm{Im}(\beta_x) \subset (\mathrm{Im}(\alpha))_x = \mathrm{Im}(\alpha_x)$, $\mathrm{Im}(\alpha_x) = \beta_x(\mathrm{Im}(\theta_x)) \subset \mathrm{Im}(\beta_x) \subset \mathrm{Im}(\alpha_x)$. Thus $\mathrm{Im}(\beta_x) = \beta_x(\mathrm{Im}(\theta_x))$, $\mathrm{Im}(\theta_x) = \mathcal{H}_x$ (β_x is injective). θ is isomorphism.

Assume α is epimorphism,

$$\begin{tikzcd} ker\alpha \arrow[r, "l"] & \mathcal{F} \arrow[r, "\alpha"] \arrow[d, "\pi"'] & \mathcal{G} \\ & \mathcal{F}/Iml \arrow[r] \arrow[ur, dashed, "\beta"'] & S(\mathcal{F}/Iml) \arrow[u, dashed, "\phi"'] \end{tikzcd}$$

α_x surjective implys β_x is surjective.

$$\begin{tikzcd} \mathcal{F}_x/(Iml)_x \arrow[dr, dashed, "\beta_x"] & \\ \mathcal{F}_x \arrow[u] \arrow[r, "\alpha_x"'] & \mathcal{G}_x \end{tikzcd}$$

$(Iml)_x = (ker\alpha)_x = ker(\alpha_x)$, therefore β_x is injective, β_x is isomorphism, ϕ_x is isomorphism, ϕ is isomorphism.

$\square$

2. Ringed topological spaces

DEFINITION 2.1. A ringed topological space (locally ringed in local rings) consists of a topological space X endowed with a sheaf of rings $\mathcal{O}_X$ on X such that $\mathcal{O}_{X,x}$ is a local ring for every $x \in X$. We denote it $(X, \mathcal{O}_X)$. The sheaf $\mathcal{O}_X$ is called the structure sheaf of $(X, \mathcal{O}_X)$. When there is no confusion possible, we will omit $\mathcal{O}_X$ from the notation. Let $\mathfrak{m}_x$ be the maximal ideal of $\mathcal{O}_{X,x}$; we call $\mathcal{O}_{X,x}/\mathfrak{m}_x$ the residue field of X at x, and we denote it $k(x)$.

DEFINITION 2.2. A morphism of ringed topological spaces

$$(f, f^{\#}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y) \tag{2.1}$$

consists of a continuous map $f : X \to Y$ and a morphism of sheaves of rings $f^{\#} : \mathcal{O}_Y \to f_*\mathcal{O}_X$ such that for every $x \in X$, the map $f^{\#}_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a local homomorphism (*i.e.*, ${f^{\#}_x}^{-1}(\mathfrak{m}_x) = \mathfrak{m}_{f(x)}$ or, equivalently, $f^{\#}_x(\mathfrak{m}_{f(x)}) \subset \mathfrak{m}_x$).

$f_x^{\#} :=$ the stalk map of $f^{\#}$ at $f(x)$ composite h, where h is the unique map such that the following diagram commutes:

$$\begin{tikzcd}
(f_*\mathcal{O}_X)_{f(x)} \arrow[r, "h_x"] & O_{X,x} \\
O_X(f^{-1}(U)) \arrow[u, "\alpha_{f(x),U}"] \arrow[ur, "\beta_{x,f^{-1}(U)}"'] &
\end{tikzcd}$$

It's also the unique map such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{O}_{Y,f(x)} \arrow[r, "f_x^{\#}"] & \mathcal{O}_{X,x} \\
\mathcal{O}_Y(U) \arrow[u, "\alpha^Y_{f(x),U}"] \arrow[r, "f^{\#}(U)"] & \mathcal{O}_X(f^{-1}(U)) \arrow[u, "\alpha^X_{x,f^{-1}(U)}"']
\end{tikzcd}$$

DEFINITION 2.3.

$$(f, f^{\#}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y), (g, g^{\#}) : (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z) \tag{2.2}$$

$$(g, g^{\#}) \circ (f, f^{\#}) := (g \circ f, (g_* f^{\#}) \circ g^{\#}) \tag{2.3}$$

we say $(g \circ f)^{\#} = g_* f^{\#} \circ g^{\#}$ (Which is not a good notation.). $(g \circ f)_x^{\#} \circ \alpha^Z_{g \circ f(x),V} = \alpha^X_{x,(g \circ f)^{-1}(V)} \circ (g \circ f)^{\#}(V)$, $f_x^{\#} \circ g^{\#}_{f(x)} \circ \alpha^Z_{g \circ f(x),V} = f_x^{\#} \circ \alpha^Y_{f(x),g^{-1}(V)} \circ g^{\#}(V) = \alpha^X_{x,f^{-1}(g^{-1}(V))} \circ f^{\#}(g^{-1}(V)) \circ g^{\#}(V)$, $(g \circ f)_x^{\#} \circ \alpha^Z_{g \circ f(x),V} = f_x^{\#} \circ g^{\#}_{f(x)} \circ \alpha^Z_{g \circ f(x),V}$
The commutativity basically follows from the diagrams:

$$
\begin{tikzcd}
\mathcal{O}_{Z,g\circ f(x)} \arrow[r, "(g\circ f)^{\#}_x"] & \mathcal{O}_{X,x} \\
\mathcal{O}_Z(V) \arrow[u, "\alpha^Z_{g\circ f(x),V}"] \arrow[r, "(g\circ f)^{\#}(V)"] & \mathcal{O}_X((g\circ f)^{-1}(V)) \arrow[u, "\alpha^X_{x,(g\circ f)^{-1}(V)}"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathcal{O}_{Z,g\circ f(x)} \arrow[r, "g^{\#}_{f(x)}"] & \mathcal{O}_{Y,f(x)} \\
\mathcal{O}_Z(V) \arrow[u, "\alpha^Z_{g\circ f(x),U}"] \arrow[r, "g^{\#}(V)"'] & \mathcal{O}_Y(g^{-1}(V)) \arrow[u, "\alpha^Y_{f(x),g^{-1}(V)}"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathcal{O}_{Y,f(x)} \arrow[r, "f^{\#}_x"] & \mathcal{O}_{X,x} \\
\mathcal{O}_Y(g^{-1}(V)) \arrow[u, "\alpha^Y_{f(x),g^{-1}(V)}"] \arrow[r, "f^{\#}(g^{-1}(V))"'] & \mathcal{O}_X(f^{-1}(g^{-1}(V))) \arrow[u, "\alpha^X_{x,f^{-1}(g^{-1}(V))}"']
\end{tikzcd}
$$

Therefore $(g\circ f)^{\#}_x = f^{\#}_x \circ g^{\#}_{f(x)}$. It's straight-forward to verify ringed topological spaces is a category.

PROP 2.4. If f is topological embedding, then h_x in definition 2.2 is isomorphism.

PROOF. Let $f(X) = W$, for $U \ni x, s \in \alpha^X_{x,U}, f(U) = V \cap W, V$ open in $Y, f(x) \in V$. $f^{-1}(V) = f^{-1}(V \cap W) = U, h_x \circ \alpha^Y_{f(x),V}(s) = \alpha^X_{x,U}(s)$. Thus h_x is surjective. For another $V \ni f(x), s \in \mathcal{O}_x(f^{-1}(V)), h_x \circ \alpha^Y_{f(x),V} = \alpha^X_{x,f^{-1}(V)} = 0$. $\exists U \ni x$, such that $\mathcal{O}_X(f^{-1}(V) \to U)(s) = 0$. $f(U) = V_1 \cap W, f(U) = f(U) \cap V = V_1 \cap V \cap W, f^{-1}(V_1 \cap V) = U, \alpha^Y_{f(x),V}(s) = \alpha^Y_{f(x),V\cap V_1} \circ (f_*\mathcal{O}_X)(V \to V \cap V_1)(s) = \alpha^Y_{f(x),V\cap v_1} \circ \mathcal{O}_X(U \to f^{-1}(V \cap V_1)) \circ \mathcal{O}_X(f^{-1}(V) \to U)(s) = 0$. h_x is injective. $\square$

DEFINITION 2.5. Let $(X, \mathcal{O}_X) \in \mathrm{Fun}(\mathrm{Top}(X)^{op}, Ab)$, $U \subset X$ open subset, $l_U : U \to X$ inclusion, $(\mathcal{O}_X)|_U$ is a sheaf, denoted by $\mathcal{O}_{X|_U}$. $l^{\#} : \mathcal{O}_X \to (l_U)_*\mathcal{O}_{X|_U}$ morphism defined by $l^{\#}(V) := \mathcal{O}_X(V \to V \cap U)$. It's morphism because of the diagram:

$$
\begin{tikzcd}
\mathcal{O}_X(V) \arrow[r, "\mathcal{O}_X(V\to V\cap U)"] \arrow[d, "\mathcal{O}_X(V\to W)"'] & ((l_U)_*\mathcal{O}_X)(V) \arrow[d, "\mathcal{O}_X(V\cap U\to W\cap U)"] \\
\mathcal{O}_X(W) \arrow[r, "\mathcal{O}_X(W\to W\cap U)"] & ((l_U)_*\mathcal{O}_X)(W)
\end{tikzcd}
$$

we can define the map $(l_U)^{\#}_x$ by the diagram:

$$\begin{tikzcd} \mathcal{O}_{X,x} \arrow[r, "(l_U)^{\#}_x"] & \mathcal{O}_{X|_U,x} \\ \mathcal{O}_X(V) \arrow[u, "\alpha^X_{x,V}"] \arrow[r, "\mathcal{O}_X(V\to V\cap U)"'] & \mathcal{O}_X(V\cap U) \arrow[u, "\alpha^U_{x,V\cap U}"'] \end{tikzcd}$$

Again, it's easy to check $l^{\#}_x$ is isomorphism, therefore $(l_U, (l_U)^{\#})$ is morphism of ringed topological space.

PROP 2.6. For $(f, f^{\#}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, $f(X) \subset V$, $V \subset Y$ open, $\exists!(f^{|_V}, (f^{|_V})^{\#}) : (X, \mathcal{O}_X) \to (V, \mathcal{O}_{Y|_V})$ such that $(l_V, (l_V)^{\#}) \circ (f^{|_V}, (f^{|_V})^{\#}) = (f, f^{\#})$.

PROOF. Feed $W \subset Y$ open to the equation gives $(f^{|_V})^{\#}(V \cap W) \circ \mathcal{O}_Y(W \to W \cap V) = f^{\#}(W)$, if $W \subset V$, then $(f^{|_V})^{\#}(W) = f^{\#}(W)$. We need to verify $(f^{|_V})^{\#}$ defined in this way is morphism and preserves maximal ideal. It's trivial to check it's morphism, $l_V \circ f^{|_V} = f$, $(f^{|_V})^{\#}_x \circ (l_V)^{\#}_{f(x)} = f^{\#}_x$. Thus $(f^{|_V})^{\#}_x$ preserves maximal ideal. $\square$

PROP 2.7. For $(f, f^{\#}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, $(f, f^{\#})$ is isomorphism if and only if f is homeomorphism and $f^{\#}$ is isomorphism if and only if f is homeomorphism and $f^{\#}_x$ is isomorphism $\forall x \in X$.

PROOF. (1) Assume $(f, f^{\#})$ is isomorphism, then $\exists(g, g^{\#}) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ such that $(g, g^{\#}) \circ (f, f^{\#}) = (id_X, id_{\mathcal{O}_X})$, $(f, f^{\#}) \circ (g, g^{\#}) = (id_Y, id_{\mathcal{O}_Y})$. That is $g \circ f = id_X$, $f \circ g = id_Y$, $g_*(f^{\#}) \circ g^{\#} = id_{\mathcal{O}_X}$, $f_*(g^{\#}) \circ f^{\#} = id_{\mathcal{O}_Y}$. For $U \subset X$ open, $V \subset Y$ open, $f^{\#}(g^{-1}(U)) \circ g^{\#}(U) = id_{\mathcal{O}_X(U)}$, $g^{\#}(f^{-1}(V)) \circ f^{\#}(V) = id_{\mathcal{O}_Y(V)}$. Let $U = g(V)$, then $f^{\#}(V) \circ g^{\#}(U) = id_{\mathcal{O}_X(U)}$, $g^{\#}(U) \circ f^{\#}(V) = id_{\mathcal{O}_Y(V)}$, $f^{\#}(V)$ is isomorphism.

(2) Assume f is homeomorphism and $f^{\#}$ is isomorphism, $g := f^{-1}$, $g^{\#} := (g_* f^{\#})^{-1}$. $g^{\#} \circ (g_* f^{\#}) = id_{g_* \mathcal{O}_Y}$,$(g_* f^{\#}) \circ g^{\#} = id_{\mathcal{O}_X}$. $g^{\#}_{s,x} \circ (g_* f^{\#}_{s,x}) = id_{(g_* \mathcal{O}_Y)_x}$, $(g_* f^{\#})_{s,x} \circ g^{\#}_{s,x} = id_{\mathcal{O}_{X,x}}$. $g^{\#}_{s,x}$ is isomorphism , therefor $g^{\#}_x$ is isomorphism.(Definition 2.2 and prop 2.4) $(g, g^{\#})$ is morphism. $(g, g^{\#}) \circ (f, f^{\#}) = (g \circ f,\ g_* f^{\#} \circ g^{\#}) = (id_X, id_{\mathcal{O}_X})$, $(f, f^{\#}) \circ (g, g^{\#}) = (f \circ g,\ f_* g^{\#} \circ f^{\#}) = (id_Y, f_*(g^{\#} \circ g_* f^{\#})) = (id_Y, f_*(id_{g_* \mathcal{O}_Y})) = (id_Y, id_{\mathcal{O}_Y})$.

(3) the latter 2 arguments are equivalent because of prop 1.15 and prop 2.4 .

$\square$

DEFINITION 2.8. Let $(f, f^{\#}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, we say $(f, f^{\#})$ is open immersion if f is topological open embedding and $f^{\#}_x$ is isomorphism $\forall x \in X$. $(f, f^{\#})$ is closed immersion if f is topological closed embedding and $f^{\#}_x$ is surjective $\forall x \in X$.

PROP 2.9. Let $(f, f^{\#}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, $(f, f^{\#})$ is open immersion if and only if $\exists V \supset f(X)$ open such that $(f^{|_V}, (f^{|_V})^{\#})$ is isomorphism if and only if f is topological open embedding and $f^{\#}_{s,f(x)}$ is isomorphism.

REMARK 2.10. For morphism between ringed topological spaces, we use $f^{\#}_{s,y}$ to denote the stalk map of $f^{\#}$ at $y \in Y$.

PROOF. (1) Assume $(f, f^{\#})$ is open immersion, then$V = f(X)$ open in Y and $f^{|_Y}$ is homeomorphism by definition, $f_x^{\#} = (f^{|_V})_x^{\#} \circ (l_V^{\#})_{f(x)}$, $(l_V^{\#})_{f(x)}$ is isomorphism by definition 2.5, thus $(f^{|_V})_x^{\#}$ is isomorphsim.

(2) Assume $\exists V \supset f(X)$ open such that $(f^{|_V}, (f^{|_V})^{\#})$ is isomorphism, V=f(X), f is topologcial open embedding, again, using $f_x^{\#} = (f^{|_V})_x^{\#} \circ (l_V^{\#})_{f(x)}$, $(l_V^{\#})_{f(x)}$, we get $f_x^{\#}$ is isomorphism.

(3) $f_x^{\#} = h_x \circ f_{s,f(x)}^{\#}$, the fisrt and last statement has that f is homeomorphism to its image, therefore h_x is isomorphism, these two are equivalent.

□

PROP 2.11. Let $(f, f^{\#}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, then f is closed immersion of ringed topological spaces if and only if f is topological closed embedding and $f_{s,y}^{\#}$ is surjective $\forall y \in Y$ if and only if f is topological closed embedding and $f_{s,f(x)}^{\#}$ is surjective $\forall x \in X$.

PROOF. The first and last statement are eqivalent because h_x is isomorphism, for $y \in f(X)^c$, $U = f(X)^c$ is open if f is topological closed embedding, for $W \subset U$ open, $(f_*\mathcal{O}_X)(W) = \mathcal{O}_X(\varnothing) = \{0\}$, let $\{\alpha_V^Y\}_{V \ni y}$ be insertions of the direct limit $(f_*\mathcal{O}_X)_y$, then $\alpha_V^Y(s) = \alpha_{V\cap U}^Y \circ (f_*\mathcal{O}_X)(V \to V \cap U)(s) = 0$, thus $(f_*\mathcal{O}_X)_y = \{0\}$, $f_{s,y}^{\#}$ is automatically surjective for $y \notin f(X)$. □

DEFINITION 2.12. Let $f : X \to Y, \mathcal{G} \in \mathrm{Fun}(\mathrm{Top}(X)^{op}, Ab)$, we can define $\phi_{f,x}^{\mathcal{G}} : (f^{-1}\mathcal{G})_x \to \mathcal{G}_{f(x)}$ as follows:

$$\begin{tikzcd}
(f^{-1}\mathcal{G})_x \arrow[r, "\phi_{f,x}^{\mathcal{G}}"] & \mathcal{G}_{f(x)} \\
(f^{-1}\mathcal{G})(V) \arrow[u, "\alpha_{x,V}^X"] \arrow[ur] & \\
\mathcal{G}(W) \arrow[u, "\alpha_{f(V),W}^{Y,f(V)}"] \arrow[uur, "\alpha_{f(x),W}^Y"'] &
\end{tikzcd}$$

PROP 2.13. $\phi_{f,x}^{\mathcal{G}}$ is isomorphism.

PROOF. Surjectivity is trivial, to prove injectivity, let $s \in \mathcal{G}(W)$ such that $\phi_{f,x}^{\mathcal{G}}(\alpha_{x,V}^X(\alpha_{f(V),W}^{Y,f(V)}(s))) = \alpha_{f(x),W}^Y(s) = 0$, $s|_K = 0$, $f(x) \in K \subset W$. $U :=$ $f^{-1}(K), x \in U$. $\alpha_{x,V}^X(\alpha_{f(V),W}^{Y,f(V)}(s)) = (\alpha_{x,f^{-1}(K)\cap V}^X \circ (f^{-1}\mathcal{G})(V \to f^{-1}(K) \cap V))(\alpha_{f(V),W}^{Y,f(V)}(s)) =$ $(\alpha_{x,f^{-1}(K)\cap V}^X \circ \alpha_{f(V\cap f^{-1}(K)),W}^{Y,f(V\cap f^{-1}(K))})(s) = 0$ □

DEFINITION 2.14. Let $f : X \to Y, \mathcal{F} \in \mathrm{Fun}(\mathrm{Top}(X)^{op}, Ab)$, $\mathcal{G} \in \mathrm{Fun}(\mathrm{Top}(Y)^{op}, Ab)$, we can define $\alpha_{\mathcal{F}}^f : f^{-1}f_*\mathcal{F} \to \mathcal{F}$, $\beta_{\mathcal{G}}^f : \mathcal{G} \to f_*f^{-1}\mathcal{G}$ by the following diagrams:

$$\begin{tikzcd}
\mathcal{G}(U) \arrow[r, "\beta^f_{\mathcal{G}}(U)=\alpha^{Y,f^{-1}(U)}_{f(f^{-1}(U)),U}"] \arrow[d, "\mathcal{G}(U\to V)"'] & (f_*f^{-1}\mathcal{G})(U) \arrow[d] & \mathcal{G}(W) \arrow[l, "\alpha^{Y,f^{-1}(U)}_{f(f^{-1}(U)),W}"'] \arrow[dl, "\alpha^{Y,f^{-1}(V)}_{f(f^{-1}(V)),W}"] \\
\mathcal{G}(V) \arrow[r, "\beta^f_{\mathcal{G}}(V)=\alpha^{Y,f^{-1}(V)}_{f(f^{-1}(V)),V}"'] & (f_*f^{-1}\mathcal{G})(V) &
\end{tikzcd}$$

$$\begin{tikzcd}
(f^{-1}f_*\mathcal{F})(U) \arrow[r, "\alpha^f_{\mathcal{F}}(U)"] & \mathcal{F}(U) \\
\mathcal{F}(f^{-1}(W)) \arrow[u, "\alpha^{X,f(U)}_{f(U),W}"] \arrow[ur, "\mathcal{F}(f^{-1}(W)\to U)"] \arrow[d] & \\
\mathcal{F}(f^{-1}(T)) \arrow[uur, bend right, "\mathcal{F}(f^{-1}(T)\to U)"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
& (f^{-1}f_*\mathcal{F})(V) \arrow[r, "\alpha^f_{\mathcal{F}}(V)"] & \mathcal{F}(V) \\
\mathcal{F}(f^{-1}(W)) \arrow[ur, "\alpha^{X,f(V)}_{f(V),W}"] \arrow[r, "\alpha^{X,f(U)}_{f(U),W}"] & (f^{-1}f_*\mathcal{F})(U) \arrow[u] \arrow[r, "\alpha^f_{\mathcal{F}}(U)"] & \mathcal{F}(U) \arrow[u]
\end{tikzcd}$$

PROP 2.15. If f is open embedding, then $\alpha^f_{\mathcal{F}}$ is isomorphism, if f is closed embedding, then $\beta^f_{\mathcal{G}}$ is stalk surjective.

PROOF. (1) Assume f is open embedding, let $W = f(U)$, $s \in \mathcal{F}(U)$, $s = \alpha^f_{\mathcal{F}}(U)(\alpha^{X,f(U)}_{f(U),f(U)}(s))$, $\alpha^f_{\mathcal{F}}(U)$ is surjective, let $s \in \mathcal{F}(f^{-1}(W))$, $\alpha^f_{\mathcal{F}}(U)(\alpha^{X,f(U)}_{f(U),W}(s)) = \mathcal{F}(f^{-1}(W) \to U)(s) = 0$, $\alpha^{X,f(U)}_{f(U),W}(s) = (\alpha^{X,f(U)}_{f(U),f(U)} \circ (f_*\mathcal{F})(W \to f(U)))(s) = (\alpha^{X,f(U)}_{f(U),f(U)} \circ \mathcal{F}(f^{-1}(W) \to U))(s) = 0$, $\alpha^f_{\mathcal{F}}(U)$ is injective.

(2) Assume f is closed embedding, as we have showed in prop 2.11 for $y \notin f(X)$, $(f_*f^{-1}\mathcal{G})_y = \{0\}$, so the stalk map of $\beta^f_{\mathcal{G}}$ at such y is surjective. For $y = f(x)$, let $(f, f^{\#}) = (f, \beta^f_{\mathcal{G}})$, $(\beta^f_{\mathcal{G}})_x = h_x \circ (\beta^f_{\mathcal{G}})_{s,f(x)}$, h_x is isomorphism, so we only need to show that $(\beta^f_{\mathcal{G}})_x$ is surjective.

In fact, we will prove that $(\beta^f_{\mathcal{G}})_x \circ \phi^{\mathcal{G}}_{f,x} = id_{(f^{-1}\mathcal{G})_x}$

$$\begin{tikzcd}
(f^{-1}\mathcal{G})_x \arrow[r, "\phi^{\mathcal{G}}_{f,x}"] & \mathcal{G}_{f(x)} \arrow[r, "(\beta^f_{\mathcal{G}})_x"] & (f^{-1}\mathcal{G})_x \\
(f^{-1}\mathcal{G})(V) \arrow[u, "\alpha^X_{x,V}"] \arrow[ur] & \mathcal{G}(U) \arrow[u, "\alpha^Y_{f(x),U}"'] \arrow[r, "\alpha^{Y,f(f^{-1}(U))}_{f(f^{-1}(U)),U}"] & (f^{-1}\mathcal{G})(f^{-1}(U)) \arrow[u, "\alpha^X_{x,f^{-1}(U)}"'] \\
\mathcal{G}(W) \arrow[u, "\alpha^{Y,f(V)}_{f(V),W}"] \arrow[uur, "\alpha^Y_{f(x),W}"'] & &
\end{tikzcd}$$

$(\beta^f_{\mathcal{G}})_x \circ \phi^{\mathcal{G}}_{f,x} \circ \alpha^X_{x,V} \circ \alpha^{Y,f(V)}_{f(V),W} = (\beta^f_{\mathcal{G}})_x \circ \alpha^Y_{f(x),W} = \alpha^X_{x,f^{-1}(W)} \circ \alpha^{Y,f(f^{-1}(W))}_{f(f^{-1}(W)),W} = \alpha^X_{x,V} \circ (f^{-1}\mathcal{G})(f^{-1}(W) \to V) \circ \alpha^{Y,f(f^{-1}(W))}_{f(f^{-1}(W)),W} = \alpha^X_{x,V} \circ \alpha^{Y,f(V)}_{f(V),W}$, thus $(\beta^f_{\mathcal{G}})_x \circ \phi^{\mathcal{G}}_{f,x} = id_{(f^{-1}\mathcal{G})_x}$, $(\beta^f_{\mathcal{G}})_x$ is surjective.
Actually, $\phi^{\mathcal{G}}_{f,x} \circ (\beta^f_{\mathcal{G}})_x = id_{\mathcal{G}_{f(x)}}$

$$\begin{tikzcd}
\mathcal{G}_{f(x)} \arrow[r, "(\beta^f_{\mathcal{G}})_x"] & (f^{-1}\mathcal{G})_x \arrow[r, "\phi^{\mathcal{G}}_{f,x}"] & \mathcal{G}_{f(x)} \\
\mathcal{G}(U) \arrow[u, "\alpha^Y_{f(x),U}"'] & (f^{-1}\mathcal{G})(V) \arrow[u, "\alpha^X_{x,V}"] \arrow[ur] & \\
 & \mathcal{G}(W) \arrow[u, "\alpha^{Y,f(V)}_{f(V),W}"] \arrow[uur, "\alpha^Y_{f(x),W}"'] &
\end{tikzcd}$$

$\phi^{\mathcal{G}}_{f,x} \circ (\beta^f_{\mathcal{G}})_x \circ \alpha^Y_{f(x),U} = \phi^{\mathcal{G}}_{f,x} \circ \alpha^X_{x,f^{-1}(U)} \circ \alpha^{Y,f(f^{-1}(U))}_{f(f^{-1}(U)),U} = \alpha^Y_{f(x),U}$, $\phi^{\mathcal{G}}_{f,x} \circ (\beta^f_{\mathcal{G}})_x = id_{\mathcal{G}_{f(x)}}$.
To summary the second proposition, embedding implys the stalk map of $\beta^f_{\mathcal{G}}$ at $\mathrm{Im} f$ is isomorphism, $\mathrm{Im} f$ closed implys the stalk of $f_* f^{-1}\mathcal{G}$ outside $\mathrm{Im} f$ is $\{0\}$.

$\square$

PROP 2.16. Let $f : X \to Y$, $f_* :$ Fun(Top$(X)^{op}$, Ab)$\to$ Fun(Top$(Y)^{op}$, Ab), f^{-1} :Fun(Top$(Y)^{op}$, Ab) $\to$ Fun(Top$(X)^{op}$, Ab), we can define $\eta : id_{Fun(Top(Y)^{op},Ab)} \to f_* f^{-1}, \varepsilon : f^{-1} f_* \to id_{Fun(Top(X)^{op},Ab)}$ by $\eta_{\mathcal{G}} := \beta^f_{\mathcal{G}}$, $\varepsilon_{\mathcal{F}} := \alpha^f_{\mathcal{F}}$, f^{-1} is left adjoint of f_*, η is counit, ε is unit.

PROOF. It's obvious that f_* is functor, to see f^{-1} is functor, see the diagram of its definition:

$$\begin{tikzcd}
(f^{-1}\mathcal{F})(U) \arrow[r, "(f^{-1}\alpha)(U)"] & (f^{-1}\mathcal{G})(U) \arrow[r, "(f^{-1}\beta)(U)"] & (f^{-1}\mathcal{H})(U) \\
\mathcal{F}(V) \arrow[u, "\alpha^{X,\mathcal{F}}_{f(U),V}"] \arrow[r, "\alpha(V)"] & \mathcal{G}(V) \arrow[u, "\alpha^{X,\mathcal{G}}_{f(U),V}"] \arrow[r, "\beta(V)"] & \mathcal{H}(V) \arrow[u, "\alpha^{X,\mathcal{H}}_{f(U),V}"]
\end{tikzcd}$$

Either use uniqueness or compostion with insertions $\{\alpha^{X,\mathcal{F}}_{f(U),V}\}$ to get the result.

$$\begin{tikzcd}
\mathcal{G}_1(U) \arrow[r, "\eta_{\mathcal{G}_1}(U)"] \arrow[d] & (f_*f^{-1}\mathcal{G}_1)(U) \arrow[d] \\
\mathcal{G}_2(U) \arrow[r, "\eta_{\mathcal{G}_2}(U)"'] & (f_*f^{-1}\mathcal{G}_2)(U)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{F}_1(f^{-1}(W)) \arrow[r, "\alpha^{Y,f_*\mathcal{F}_1}_{f(U),W}"] \arrow[d] & (f^{-1}f_*\mathcal{F}_1)(U) \arrow[r, "\varepsilon_{\mathcal{F}_1}(U)"] \arrow[d] & \mathcal{F}_1(U) \arrow[d] \\
\mathcal{F}_2(f^{-1}(W)) \arrow[r, "\alpha^{Y,f_*\mathcal{F}_2}_{f(U),W}"'] & (f^{-1}f_*\mathcal{F}_2)(U) \arrow[r, "\varepsilon_{\mathcal{F}_2}(U)"'] & \mathcal{F}_2(U)
\end{tikzcd}$$

For the second diagram, to show the right box is commutative, it's equivalent to show the composition with $\alpha^{Y,f_*\mathcal{F}_1}_{f(U),W}$, then it follows from that $\mathcal{F}_1 \to \mathcal{F}_2$ is natural transformation.

Finally we need to show that f^{-1} is left adjoint of f_*. That is $f_*(\varepsilon_{\mathcal{F}})\circ\eta_{f_*\mathcal{F}} = id_{f_*\mathcal{F}}$, $\varepsilon_{f^{-1}\mathcal{G}} \circ f^{-1}\eta_{\mathcal{G}} = id_{f^{-1}\mathcal{G}}$.

$$\begin{tikzcd}
(f_*\mathcal{F})(V) \arrow[r, "\alpha^{Y,f_*\mathcal{F}}_{f(f^{-1}(V)),V}"] & (f_*f^{-1}f_*\mathcal{F})(V) \arrow[r, "(\eta_{\mathcal{F}})(f^{-1}(V))"] & \mathcal{F}(f^{-1}(V)) \\
 & (f_*\mathcal{F})(W) \arrow[u, "\alpha^{Y,f_*\mathcal{F}}_{f(f^{-1}(V)),W}"] \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
(f^{-1}\mathcal{G})(U) \arrow[r, "(f^{-1}\eta_{\mathcal{G}})(U)"] & (f^{-1}f_*f^{-1}\mathcal{G})(U) \arrow[r, "(\varepsilon_{f^{-1}\mathcal{G}})(U)"] & (f^{-1}\mathcal{G})(U) \\
\mathcal{G}(V) \arrow[u, "\alpha^{Y,\mathcal{G}}_{f(U),V}"] \arrow[r, "\alpha^{Y,\mathcal{G}}_{f(f^{-1}(V)),V}"] & (f_*f^{-1}\mathcal{G})(V) \arrow[u, "\alpha^{Y,f_*f^{-1}\mathcal{G}}_{f(U),V}"] \arrow[ur, "(f^{-1}\mathcal{G})(f^{-1}(V)\to U)"'] &
\end{tikzcd}$$

$\square$

DEFINITION 2.17. Let$(X, \mathcal{O}_X)$ be a ringed topological space. Let $\mathcal{J}$ be a sheaf of ideals of $\mathcal{O}_X$ (i.e., $\mathcal{J}(U)$ is an ideal of $\mathcal{O}_X(U)$ for every open subset U). Let

$\alpha_{x,U} : \mathcal{O}_X(U) \to \mathcal{O}_{X,x}$ be insertions, then $\alpha_{x,U}|^{A_x}_{\mathcal{J}(U)} : \mathcal{J}(U) \to A_x$ are insertions, where $A_x = \bigcup\limits_{U \ni x} \alpha_{x,U}(\mathcal{J}(U))$, $\widetilde{\alpha}_{x,U} : \mathcal{O}_X(U)/\mathcal{J}(U) \to \mathcal{O}_{X,x}/A_x$ are insertions.

PROP 2.18. Let $(X, \mathcal{O}_X)$ be a ringed topological space, $\mathcal{J}$ be a sheaf of ideals of $\mathcal{O}_X$, $V(\mathcal{J}) := \{x \in X | A_x \neq \mathcal{O}_{X,x}\}$.Then $V(\mathcal{J})$ is a closed subset of X, $(V(\mathcal{J}), \mathcal{S}j^{-1}(\mathcal{O}_X/\mathcal{J}))$ is a ringed topological space, and we have a closed immersion $(j, j^\#)$ of this space into $(X, \mathcal{O}_X)$, where $j^\#$ is the canonical surjection $\mathcal{O}_X \to \mathcal{O}_X/\mathcal{J} \simeq j_*(\mathcal{S}j^{-1}(\mathcal{O}_X/\mathcal{J}))$

PROOF. For $x \notin V(\mathcal{J})$, then $\exists U \in x$ such that $1 \in \mathcal{J}(U)$. For $y \in U$, $A_y = \mathcal{O}_{X,y}$, $y \notin V(\mathcal{J})$, $V(\mathcal{J})$ is closed.
$(\mathcal{S}j^{-1}(\mathcal{O}_X/\mathcal{J}))_x \simeq (j^{-1}(\mathcal{O}_X/\mathcal{J}))_x \simeq (\mathcal{O}_X/\mathcal{J})_x = \mathcal{O}_{X,x}/A_x$ (Def 1.17, Prop 2.13, Def 2.17), then $(V(\mathcal{J}), \mathcal{S}j^{-1}(\mathcal{O}_X/\mathcal{J}))$ is a ringed topological space.

$$\mathcal{O}_X/\mathcal{J} \to j_*j^{-1}(\mathcal{O}_X/\mathcal{J}) \to j_*(\mathcal{S}j^{-1}(\mathcal{O}_X/\mathcal{J}))$$

The first morphism is stalk isomorphism in Im j (Prop 2.15(2)), the second morphism is stalk isomorphism in Im j (j is topological embedding, Prop 2.4), the stalk of $\mathcal{O}_X/\mathcal{J}$ outside Im j is $\{0\}$ by definition of $V(\mathcal{J})$, the stalk of $j_*(\mathcal{S}j^{-1}(\mathcal{O}_X/\mathcal{J}))$ outside Im j is $\{0\}$ (Prop 2.11), therefore the composition is stalk isomorphism.Finally we can construct $(j, j^\#)$ by

$$\mathcal{O}_X \to \mathcal{O}_X/\mathcal{J} \to j_*j^{-1}(\mathcal{O}_X/\mathcal{J}) \to j_*(\mathcal{S}j^{-1}(\mathcal{O}_X/\mathcal{J}))$$

The stalk map of the first morphism is projection, therefore $(j, j^\#)$ is closed immersion.

□

LEMMA 2.19. Let R be a ring, Λ a filtered category. Let C be the category of 3-term exact sequences of R-modules: its objects are the 3-term exact sequences, and its maps are the commutative diagrams

$$\begin{tikzcd}
L \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & N \arrow[d] \\
L' \arrow[r] & M' \arrow[r] & N'
\end{tikzcd}$$

Then, for any functor $\lambda \mapsto \left(L_\lambda \xrightarrow{\beta_\lambda} M_\lambda \xrightarrow{\gamma_\lambda} N_\lambda\right)$from Λ to C, the induced sequence $\varinjlim L_\lambda \xrightarrow{\beta} \varinjlim M_\lambda \xrightarrow{\gamma} \varinjlim N_\lambda$ is exact. Moreover, $(\alpha^L_\lambda, \alpha^M_\lambda, \alpha^N_\lambda)$ is filtered direct limit. From straight-forward diagram chasing

$$\begin{tikzcd}
 & L_\mu \arrow[rr] \arrow[dl, dashed, shift left] & & M_\mu \arrow[rr] \arrow[dl, dashed, shift left] & & N_\mu \arrow[dl, dashed, shift left] \\
L_\lambda \arrow[ur] \arrow[rr] \arrow[d] & & M_\lambda \arrow[ur] \arrow[rr] \arrow[d] & & N_\lambda \arrow[ur] \arrow[d] & \\
\varinjlim L_\lambda \arrow[rr] & & \varinjlim M_\lambda \arrow[rr] & & \varinjlim N_\lambda &
\end{tikzcd}$$

we get that $\varinjlim L_\lambda \xrightarrow{\beta} \varinjlim M_\lambda \xrightarrow{\gamma} \varinjlim N_\lambda$ is exact.

PROP 2.20. Let $f : Y \to X$ be closed immersion of ringed topological spaces, $Z = V(\mathcal{J})$ where $J = ker f^{\#}$, then there is $\phi : Y \to Z$ isomorphism of ringed topological spaces such that $(f, f^{\#}) = (j, j^{\#}) \circ (\phi, \phi^{\#})$.

By definition we have the exact sequence in Fun(Top$(X)^{op}$, Ab)

$$0 \to \mathcal{J} \to \mathcal{O}_X \to f_*\mathcal{O}_Y$$

Then use Cor 1.3 and lemma 2.19 to get the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{J}(U) \arrow[r, "l"] \arrow[d] & \mathcal{O}_X(U) \arrow[r] \arrow[d] & (f_*\mathcal{O}_Y)(U) \arrow[d] & \\
0 \arrow[r] & A_x \arrow[r, "l"'] & \mathcal{O}_{X,x} \arrow[r, "f^{\#}_{s,x}"'] & (f_*\mathcal{O}_Y)_x \arrow[r] & 0
\end{tikzcd}$$

$f^{\#}_{s,x}$ is surjective because f is closed immersion and prop 2.11. Then $A_x = \mathcal{O}_{X,x}$ if and only if $(f_*\mathcal{O}_Y)_x = 0$ if and only if $x \notin \mathrm{Im} f$. Thus $V(\mathcal{J}) = \mathrm{Im} f$.

We define $\phi^{\#} : \mathcal{O}_Z \to \phi_*\mathcal{O}_Y$ as follows:

$$\mathcal{O}_Z = \mathcal{S}j^{-1}{}^{\mathcal{O}_X}\!/_{\mathcal{J}} \to \mathcal{S}j^{-1}(\mathcal{S}\,{}^{\mathcal{O}_X}\!/_{\mathcal{J}}) \to \mathcal{S}j^{-1}f_*\mathcal{O}_Y = \mathcal{S}j^{-1}j_*\phi_*\mathcal{O}_Y \to \phi_*\mathcal{O}_Y$$

We have the exact sequence in Sh$(X$,$Ab)$:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{J} \arrow[r] & \mathcal{O}_X \arrow[r] \arrow[d] & f_*\mathcal{O}_Y \arrow[r] & 0 \\
& & \mathcal{O}_X/\mathcal{J} \arrow[r] & \mathcal{S}(\mathcal{O}_X/\mathcal{J}) \arrow[u, dashed] &
\end{tikzcd}$$

We only need to verify $(f, f^{\#}) = (j, j^{\#}) \circ (\phi, \phi^{\#})$. That is $j_*\phi^{\#} \circ j^{\#} = f^{\#}$.

We have the following commutative diagram:

$$\begin{tikzcd}
j_*\mathcal{S}j^{-1}(\mathcal{O}_X/\mathcal{J}) \arrow[r] & j_*\mathcal{S}j^{-1}(\mathcal{S}(\mathcal{O}_X/\mathcal{J})) \arrow[r] & j_*\mathcal{S}j^{-1}f_*\mathcal{O}_Y \arrow[r, "id"] & j_*\mathcal{S}j^{-1}j_*\phi_*\mathcal{O}_Y \\
j_*j^{-1}(\mathcal{O}_X/\mathcal{J}) \arrow[r] \arrow[u] & j_*j^{-1}(\mathcal{S}(\mathcal{O}_X/\mathcal{J})) \arrow[r] \arrow[u] & j_*j^{-1}f_*\mathcal{O}_Y \arrow[r, "id"] \arrow[u] & j_*j^{-1}j_*\phi_*\mathcal{O}_Y \arrow[u] \arrow[d, "j_*\alpha^j_{\phi_*\mathcal{O}_Y}"] \\
\mathcal{O}_X/\mathcal{J} \arrow[r] \arrow[u] & \mathcal{S}(\mathcal{O}_X/\mathcal{J}) \arrow[r] \arrow[u] & f_*\mathcal{O}_Y \arrow[u, "\beta^j_{f_*\mathcal{O}_Y}"'] & j_*\phi_*\mathcal{O}_Y \arrow[l, "id"'] \\
\mathcal{O}_X \arrow[u] & & &
\end{tikzcd}$$

The lower right corner box follows from Prop 2.16. So $j_*\phi^{\#} \circ j^{\#}$ is the outer path going in clockwise, it's equal to the path going in counterclockwise, which is just $f^{\#}$.