

# Localization of triangulated category

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ABSTRACT. Here is a summary of the results of each chapter.

Chapter 1 introduces  $S$ -roof and  $S$ -coroof, both induces localization of an additive category.

Chapter 2 gives definition of triangulated category and proves the localization of a triangulated category at a localizing class is triangulated if the localizing class is compatible with translation functor. For general properties of triangulated category, see [Nee01].

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## CHAPTER 1

# Localization of additive category

### 1. Localizing class

The localization of a category is basically "adding" inverse of some morphism. As we will see, some constructions involved in localization will make Hom-set between a pair of object a class. We still call them "category" here.

DEFINITION 1.1. Let  $C$  be a category,  $S$  class of morphisms in  $C$ , a functor  $F : C \rightarrow D$  is called localization if  $F(f)$  is isomorphism for  $f$  in  $S$  and for any functor  $G : C \rightarrow E$  with  $G(f)$  is isomorphism for  $f$  in  $S$ , there exists a unique functor  $H : D \rightarrow E$  such that  $G = H \circ F$ . For localization  $F, D$  is usually denoted by  $C[S^{-1}]$ .

REMARK 1.2. The localization of a category always exist if we allow  $C[S^{-1}]$  to be a generalized category [Pan16]. However, in such case, it has a complicated structure. So we consider a special type of  $S$  to make  $C[S^{-1}]$  simple.

DEFINITION 1.3. Let  $C$  be a category,  $S$  is a class of morphisms in  $C$ .  $S$  is called localizing class if it satisfies the following conditions

- (1)  $id_x \in S \forall x \in S$   $f, g$  in  $S$  implies  $g \circ f \in S$  (if  $g \circ f$  is defined)
- (2) Consider the following diagrams

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow s & & \\ Z & & \end{array} \qquad \begin{array}{ccc} & & K \\ & & \downarrow s' \\ T & \xrightarrow{g} & W \end{array}$$

where  $s, s' \in S$  and  $f, g \in C$ , they can be completed to

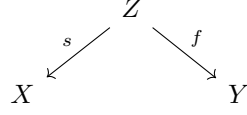
$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow s & & \downarrow s'' \\ Z & \xrightarrow{f'} & V \end{array} \qquad \begin{array}{ccc} R & \xrightarrow{g'} & K \\ \downarrow s''' & & \downarrow s' \\ T & \xrightarrow{g} & W \end{array}$$

where  $s'', s''' \in S$

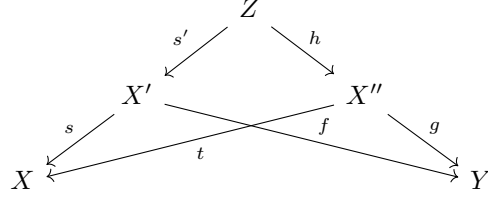
- (3) Let  $f, g : X \rightarrow Y$  be morphisms. Then there exists morphism  $s \in S$  s.t.  $sf = sg$  if and only if there exists morphism  $s'$  s.t.  $fs' = gs'$ .

### 2. S-roof

DEFINITION 2.1. A  $S$ -roof from  $X \rightarrow Y$ , denoted by  $(s, f)$ , is a diagram



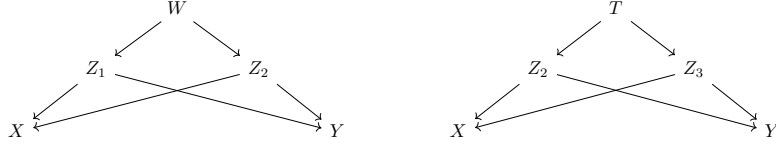
where  $s \in S$ ,  $f \in C$ . We say two roofs  $(s, f)$  and  $(t, g)$  are equivalent, denoted by  $(s, f) \sim (t, g)$  if there is a such diagram



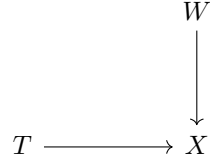
where  $ss' \in S$ ,  $h \in C$ .

PROP 2.2. *Roof equivalence  $\sim$  is equivalence relation.*

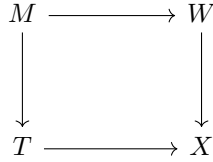
PROOF. Reflexivity and Symmetry is obvious. For transitivity, consider the following diagram



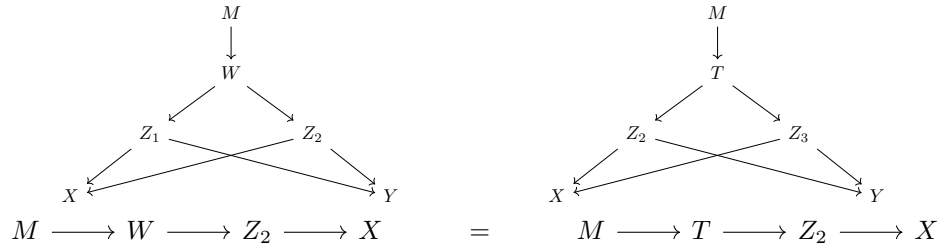
use def 1.3 for



we can complete the diagram to

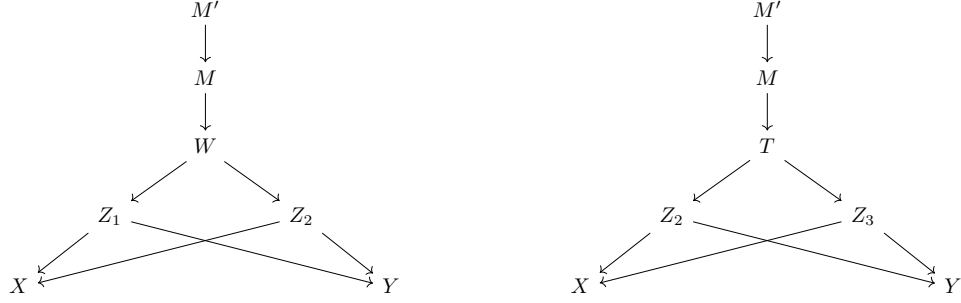


where  $M \rightarrow T$  and  $M \rightarrow W \in S$ .



use def 1.3 again, we have  $M' \rightarrow M \in S$  s.t.

$$M' \longrightarrow M \longrightarrow W \longrightarrow Z_2 \quad = \quad M' \longrightarrow M \longrightarrow T \longrightarrow Z_2$$



$$M' \longrightarrow M \longrightarrow W \longrightarrow Z_1 \longrightarrow X$$

$$= M' \longrightarrow M \longrightarrow T \longrightarrow Z_2 \longrightarrow X$$

$$= M' \longrightarrow M \longrightarrow T \longrightarrow Z_3 \longrightarrow X$$

$$M' \longrightarrow M \longrightarrow W \longrightarrow Z_1 \longrightarrow Y$$

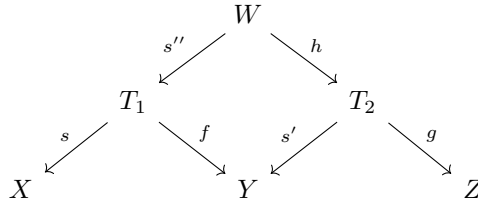
$$= M' \longrightarrow M \longrightarrow W \longrightarrow Z_2 \longrightarrow Y$$

$$= M' \longrightarrow M \longrightarrow T \longrightarrow Z_2 \longrightarrow Y$$

$$= M' \longrightarrow M \longrightarrow T \longrightarrow Z_3 \longrightarrow Y$$

□

DEFINITION 2.3. For roofs  $(s, f)$  from  $X \rightarrow Y$  and  $(s', g)$  from  $Y \rightarrow Z$ , their composition is defined to be the roof



where  $ss'' \in S$  and  $fs'' = s'h$ . The composition always exists due to def 1.3.

PROP 2.4. *Composition is well-defined for roof equivalent class.*

PROOF. First we consider the same representative for roofs.



use def1.3 we have

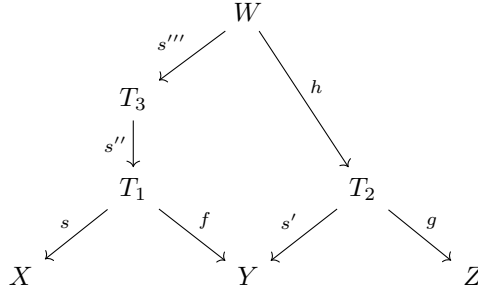
$$\begin{array}{ccc} T & \xrightarrow{\quad} & W \\ \downarrow & & \downarrow \\ W' & \xrightarrow{\quad} & X \end{array}$$

where  $T \rightarrow W$  and  $T \rightarrow W' \in S$  use def 1.3 again

$$T' \rightarrow T \rightarrow W \rightarrow T_1 = T' \rightarrow T \rightarrow W' \rightarrow T_1$$

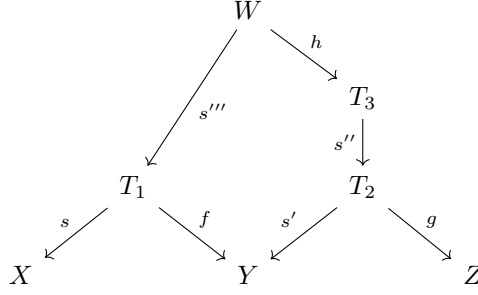
$$\begin{aligned} & T' \rightarrow W \rightarrow T_2 \rightarrow Y \\ &= T' \rightarrow W \rightarrow T_1 \rightarrow Y \\ &= T' \rightarrow W' \rightarrow T_1 \rightarrow Y \\ &= T' \rightarrow W' \rightarrow T_2 \rightarrow Y \end{aligned}$$

there exists  $T'' \rightarrow T' \in S$  s.t.  $T'' \rightarrow W \rightarrow T_2 = T'' \rightarrow W' \rightarrow T_2$   
 $(W \rightarrow X, W \rightarrow Z) \sim (T'' \rightarrow W \rightarrow X, T'' \rightarrow W \rightarrow Z) \sim (W' \rightarrow X, W' \rightarrow Z)$



where  $ss'', ss''s''' \in S$ . Therefore it's also composition of  $(s, f)$  and  $(s', g)$ .  
 Let  $(s, f) \sim (s_1, f_1)$ , then  $(s', g) \circ (s, f) \sim (s', g) \circ (s_1, f_1)$





where  $s's'', ss''' \in S$ . It is composition of  $(s, f)$  and  $(s', g)$ .

Let  $(s', g) \sim (s_2, g_2)$ . Then  $(s', g) \circ (s, f) \sim (s_2, g_2) \circ (s, f)$ .

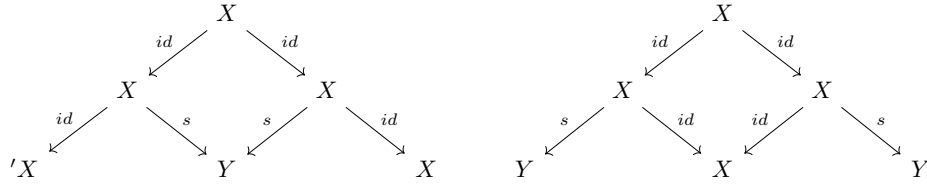
$(s', g) \circ (s, f) \sim (s_2, g_2) \circ (s, f) \sim (s_2, g_2) \circ (s_1, f_1)$

□

DEFINITION 2.5. Let  $\tilde{C}_S$  be the category consisting of objects of  $C$ .  
 $Hom(X, Y) = \{[(s, f)] | s \in S, f \in C, cods = X, codf = Y\}$ . Composition is defined from  $S$ -roof.

PROP 2.6. Let  $C$  be a category,  $S$  localizing class, then  $F : C \rightarrow \tilde{C}_S$ , which is  $id$  on object, send  $f$  to  $[(id, f)]$ , is localization of  $C$ .

PROOF. let  $s \in S$



which shows  $F$  maps morphisms in  $S$  to isomorphisms in  $\tilde{C}_S$ .

Let  $G : C \rightarrow D$  be a functor that send morphisms in  $S$  to isomorphisms in  $D$ . Let  $H : \tilde{C}_S \rightarrow D$  be a functor s.t.  $G = H \circ F$ .

$$H([(s, f)]) = H([(id, f)])H([(id, s)])^{-1} = G(f)G(s)^{-1}$$

we show  $H$  is well-defined functor.

Let  $(s, f) \sim (s', g)$ . Then  $(ss'', fs'') = (s's'', gs'')$  where  $ss'' \in S$

$G(f)G(s)^{-1} = G(fs'')G(ss'')^{-1} = G(g)G(s')^{-1}$ . Thus  $H$  is well-defined.

$H([(id, id)]) = G(id)G(id)^{-1} = id$ ,  $[(s', g)][(s, f)] = [(ss'', gh)]$ .

$H([(fs'', gh)]) = G(gh)G(ss'')^{-1} = G(g)(G(h)G(s'')^{-1})G(s)^{-1} = G(g)(G(s')^{-1}G(f))G(s)^{-1}$

□

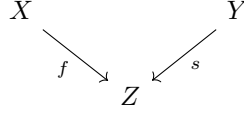
### 3. S-coroof

LEMMA 3.1. Let  $S$  be a category,  $S$  localizing class. Then  $S^{op}$  is localizing class of  $C^{op}$ .

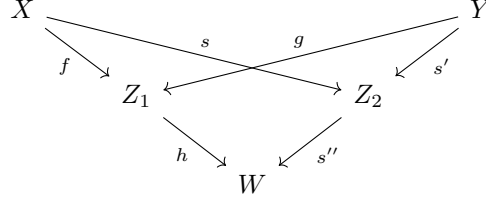
PROOF. Just reserve all arrows in def 1.3.

□

DEFINITION 3.2. Let  $C$  be a category,  $S$  localizing class. A  $S$ -coroof is a diagram of the form

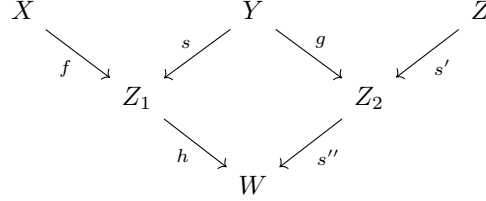


where  $s \in S$ , denoted by  $(f, s)$ . Two coroofts  $(f, s), (g, s')$  are equivalent if there is a diagram of the form



where  $s''s' \in S$  and  $h \in C$ , denoted by  $(f, s) \sim^\vee (g, s')$

The composition of  $(f, s)$  and  $(g, s')$  is define to be the following diagram



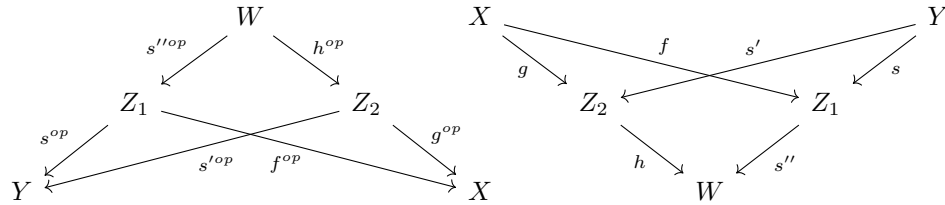
where  $h \in C, s''s' \in S$ .

PROP 3.3.  $\sim^\vee$  is equivalence relation. Corooft composition exists and is well-defined for equivalent class of coroofts. There is a category with object the same as  $C$ , its morphism consists of equivalent class of coroofts, composition defined in this way, denoted by  $\tilde{C}_S$ , and it is isomorphic to  $\tilde{C}_S$ .

PROOF.  $S^{op}$ -roof  $Y \rightarrow X$  has a bijection to  $S$ -corooft  $X \rightarrow Y$



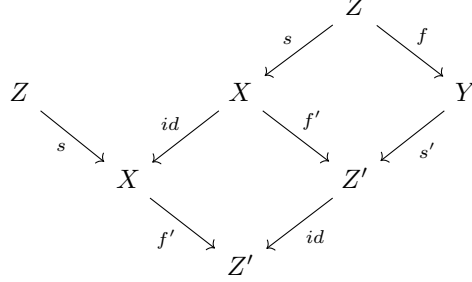
which send  $(s^{op}, f^{op})$  to  $(f, s)$ . We carry the equivalent class of  $S^{op}$ -roof to  $S$ -corooft.



The induced equivalence relation on  $S$ -corooft is  $\sim^\vee$ .  $W(X, Y) = \{S\text{-corooft equivalent class from } X \text{ to } Y\}$ . Then  $Hom_{(\tilde{C}^{op}_{S^{op}})^{op}}(X, Y) \simeq W(X, Y)$ . Then the isomorphisms create a category with object the same as  $C$ , morphisms as equivalent classes of corooft, and composition as corooft composition.

Note that  $G : C \rightarrow C^{op}[S^{op-1}]^{op}$  is localization of  $C$ , so the induced morphism

$R : C[S^{-1}] \rightarrow C^{op}[S^{op-1}]^{op}$  is isomorphism. Therefore the composite  $\tilde{C}_S \rightarrow (C^{op}_{S^{op}})^{op} \rightarrow \tilde{C}_S$  is isomorphism. Which send  $[(s, f)] \rightarrow [(f, id)][(s, id)]^{-1} = [(f', s')]$



where  $f', s'$  rise from def 1.3. So  $[(f', s')][(s, id)] = [(f's, s')] = [(s'f, s')] = [(f, id)]$ .  $\square$

#### 4. Localization of additive category

**THEOREM 4.1.** *Let  $A$  be additive category,  $S$  localizing class. Then  $A[S^{-1}]$  is additive category and the localization functor  $Q : A \rightarrow A[S^{-1}]$  is additive functor.*

**PROOF.** (1) 0 (in  $A$ ) is zero object. If we have two roofs



where  $f \in S$ .

$$[(f, 0)] = [(id \circ f, 0 \circ f)] = [(id, 0)]$$

$$[(0, h)] = [(0 \circ h, id \circ h)] = [(0, id)].$$

(2)  $Hom_{A[S^{-1}]}(X, Y)$  is abelian group.

$$[(s, f)] + [(s', f')] := [(ss'', fs'' + f's''')]$$

where  $ss'' = s's'''$ ,  $s''$  and  $s''' \in S$ .

To see it's well-defined, let  $(s, f) \sim (s', f')$ ,  $(s, g) \sim (s', g')$

$$(ss'', fs'') = (s's''', f's''')$$

$$(ss'', gs'') \sim (s's''', g's''')$$

$$(ss''s_1, gs''s_1) = (s's'''s_1, g's'''s_1) *$$

where  $s'', s''', s_1 \in S$

$$(s, f + g) \sim (ss''s_1, (f + g)s''s_1) = (s's'''s_1, (f' + g')s'''s_1) \sim (s', f' + g')$$

(3) Existence of biproduct.

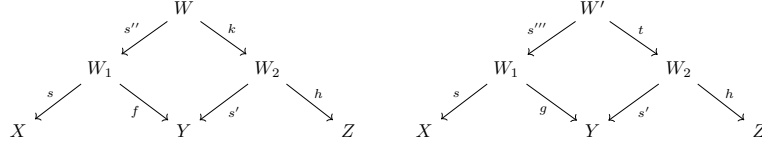
let  $l_X, l_Y, \pi_X, \pi_Y$  be biproduct of  $X$  and  $Y$  in  $A$ . Then  $[(id, l_X)], [(id, l_Y)], [(id, \pi_X)], [(id, \pi_Y)]$  is biproduct of  $X$  and  $Y$  in  $C[S^{-1}]$

$$[(id, \pi_X)][(id, l_X)] = [(id, id)], [(id, \pi_Y)][(id, l_Y)] = [(id, id)]$$

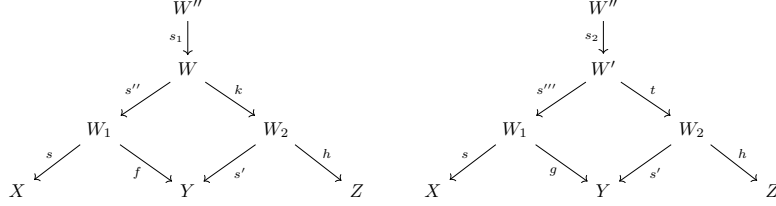
$$[(id, \pi_Y)][(id, l_X)] = [(id, 0)], [(id, \pi_X)][(id, l_Y)] = [(id, 0)]$$

$$[(id, l_X)][(id, \pi_X)] + [(id, l_Y)][(id, \pi_Y)] = [(id, l_X\pi_X + l_Y\pi_Y)] = [(id, 0)]$$

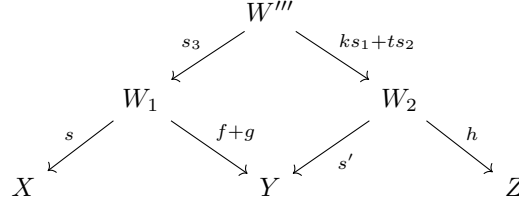
(4) Distributive law



where  $s'', s''' \in S$



where  $s_1, s_2 \in S, s''s_1 = s'''s_2, s_3 := s''s_1$



$$\begin{aligned}
 [(s', h)][(s, f)] + [(s, g)] &= [(s', h)][(s, f + g)] = [(ss_3, h(ks_1 + ts_2))] = \\
 &= [(ss_3, hks_1)] + [(ss_3, hts_2)] = [(s', h)][(s, f)] + [(s', h)][(s, g)] \\
 &= [(s, f) + (s, g)][(s', h)] = [(s, f + g)][(s', h)] = [(s's'', (f + g)t)] = [(s's'', ft)] + \\
 &+ [(s's'', gt)] = [(s, f)][(s', h)] + [(s, g)][(s', h)]
 \end{aligned}$$

where  $hs'' = st, s'' \in S$

(5)  $Q$  is additive.

$$Q(f + g) = [(id, f + g)] = [(id, f)] + [(id, g)] = Q(f) + Q(g)$$

□

## CHAPTER 2

# Triangulated category

### 1. Definition of triangulated category

DEFINITION 1.1. Let  $A$  be an additive category,  $T$  additive automorphism of  $A$ . A triangle in  $A$  is

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

Denoted by  $(X, Y, Z, u, v, w)$ , where  $X[1] := TX$ ,  $T$  is sometimes called translation functor. Morphism from  $(X, Y, Z, u, v, w)$  to  $(X', Y', Z', u', v', w')$  is a triple  $(f, g, h)$  s.t. the diagram commutes

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\ f \downarrow & & g \downarrow & & h \downarrow & & f[1] \downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \end{array}$$

where  $f[1] := Tf$ . Then  $(f, g, h)$  is isomorphism (in the category of some triangles) if and only if  $f, g, h$  are isomorphisms.

DEFINITION 1.2. Let  $A$  be an additive category,  $T$  additive automorphism of  $A$ ,  $P$  a collection of triangles in  $A$ , called distinguished triangles. Then  $A$  is called triangulated category w.r.t  $T$  and  $P$  if the following conditions are satisfied

- (1) For any  $X$  in  $A$ ,  $(X, X, 0, id, 0, 0) \in P$
- (2)  $(X, Y, Z, u, v, w) \in P$  if and only if  $(Y, Z, X[1], v, w, -u[1]) \in P$ .
- (3) A triangle isomorphic a triangle in  $P$  is in  $P$ .
- (4) For any  $f$  in  $A$ , there is a triangle  $(X, Y, Z, u, v, w)$  in  $P$  s.t.  $u = f$ .
- (5) For two distinguished triangles  $(X, Y, Z, u, v, w)$ ,  $(X', Y', Z', u', v', w')$  and  $f, g$  s.t the diagram commutes

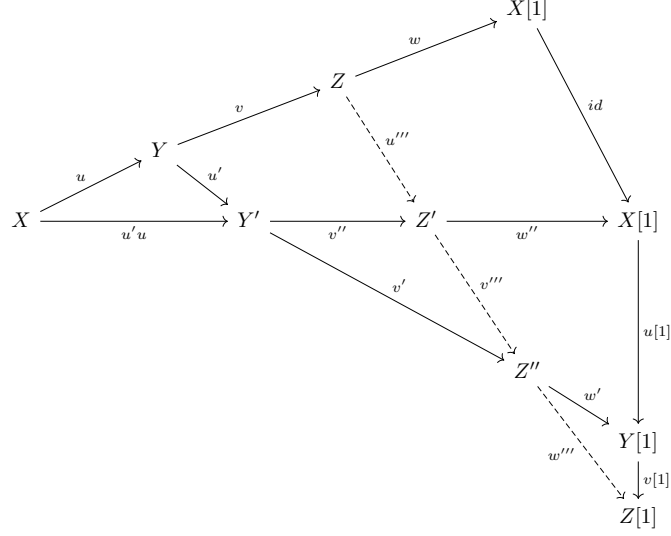
$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\ f \downarrow & & g \downarrow & & & & \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \end{array}$$

Then there exist  $h$  s.t the diagram commutes

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\ f \downarrow & & g \downarrow & & h \downarrow & & f[1] \downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \end{array}$$

- (6) For distinguished triangles  $(X, Y, Z, u, v, w)$ ,  $(Y, Y', Z'', u', v', w')$  and  $(X, Y', Z', u'u, v'', w'')$ , there exists distinguished triangle

$(Z, Z', Z'', u''', v''', w''')$  s.t. the diagram commutes



[SIG96] provides the commutative condition in a octahedral, called "The octahedral axiom".

REMARK 1.3. Let  $A$  be a triangulated category,  $(X, Y, Z, u, v, w)$ ,  $(X', Y', Z', u', v', w')$  distinguished triangles. If the diagram commutes

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\ & & g \downarrow & & h \downarrow & & \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & X'[1] \end{array}$$

Then there exist  $f$  s.t.

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\ f \downarrow & & g \downarrow & & h \downarrow & & f[1] \downarrow \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & X'[1] \end{array}$$

If the diagram commutes

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\ & & & & h \downarrow & & f[1] \downarrow \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & X'[1] \end{array}$$

Then there exists  $g$  s.t.

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\ f \downarrow & & g \downarrow & & h \downarrow & & f[1] \downarrow \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & X'[1] \end{array}$$

## 2. Localization of triangulated category

DEFINITION 2.1. Let  $A$  be a triangulated category,  $S$  localizing class,  $T$  translation functor.  $T$  is called compatible with  $S$  if

- (1)  $T(f) \in S$  if and only if  $f \in S$ .
- (2) In condition 5 of triangulated category, for  $f, g \in S$ , there exists  $h \in S$ .

LEMMA 2.2. Let  $A$  be an additive category,  $S$  localizing class,  $Q$  localization functor. For the diagrams in  $A[S^{-1}]$  and  $A$  respectively

$$\begin{array}{ccc}
 X & \xrightarrow{Qu} & Y \\
 \downarrow t & & \downarrow r \\
 X' & \xrightarrow{Qu'} & Y'
 \end{array}
 \qquad
 \begin{array}{ccccc}
 & & X & \xrightarrow{u} & Y \\
 & s_1 \nearrow & & \searrow s_2 & \\
 W_1 & \xrightarrow{w} & W_2 & & \\
 & f_1 \searrow & & \nearrow f_2 & \\
 & & X' & \xrightarrow{u'} & Y'
 \end{array}$$

The first diagram is commutative if and only if there exists  $s_1, s_2 \in S, f_1, f_2, w \in A$  s.t.  $t = [(s_1, f_1)], r = [(s_2, f_2)]$  and the second diagram is commutative.

PROOF.

$$\begin{array}{ccccc}
 & & W & & \\
 & s_1 \swarrow & & \searrow w & \\
 & X & & T & \\
 id \swarrow & & u \searrow & s_2 \swarrow & \searrow f_2 \\
 X & & Y & & Y'
 \end{array}$$

$r \circ Qu = [(s_2, f_2)][(id, u)] = [(s_1, f_2w)] = [(s_1, u'f_1)] = Qu' \circ t$   
 Conversely, let  $t = [(s_1, f_1)], r = [(s_2, f_2)]. Qu' \circ t = [(s_1, u'f_1)]. r \circ Qu = (s', f_2g')$ , where  $us' = s_2g', s' \in S. (s_1, u'f_1) \sim (s', f_2g'). s's'' = s_1s'''$ , where  $s'', s''' \in S. (s's'', f_2g's'') \sim (s_1s''', u'f_1s''')$ . Then  $f_2g's''s_3 = u'f_1s'''s_3. s's'''s_3 \in S. \tilde{f}_1 = f_1s'''s_3. \tilde{s}_1 = s_1s'''s_3, w = g's''s_3$ . Then  $u\tilde{s}_1 = s_2w. u'\tilde{f}_1 = f_2w. [(s_1, f_1)] = [(\tilde{s}_1, \tilde{f}_1)]$   $\square$

THEOREM 2.3. Let  $A$  be a triangulated category,  $S$  localizing class,  $Q$  localization functor,  $T$  translation functor compatible with  $S$ . Then  $A[S^{-1}]$  is triangulated category with  $\tilde{T}$  as translation functor and triangles isomorphic to  $Q$  applied distinguished triangles in  $A$  as distinguished triangles.

PROOF. (1)

$$\begin{array}{ccccccc}
 X & \xrightarrow{id} & X & \xrightarrow{0} & 0 & \xrightarrow{0} & X[1] \\
 X & \xrightarrow{[(id, id)]} & X & \xrightarrow{[(id, 0)]} & 0 & \xrightarrow{[(id, 0)]} & X[1]
 \end{array}$$

$$\begin{array}{ccccccc}
X & \xrightarrow{Qu} & Y & \xrightarrow{Qv} & Z & \xrightarrow{Qw} & X[1] \\
f \downarrow & & g \downarrow & & h \downarrow & & \tilde{T}(f) \downarrow \\
X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & X'[1]
\end{array}$$

(2)

$$\begin{array}{ccccccc}
Y & \xrightarrow{Qv} & Z & \xrightarrow{Qw} & TX & \xrightarrow{-\tilde{T}(Qu)} & TY \\
g \downarrow & & h \downarrow & & \tilde{T}(f) \downarrow & & \tilde{T}(g) \downarrow \\
Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & TX' & \xrightarrow{-\tilde{T}(u')} & TY'
\end{array}$$

(3) Trivial

(4) Give  $[(s, f)] : X \rightarrow Y$  in  $A[S^{-1}]$ . Let  $(Z, Y, K, f, v, w)$  be a distinguished triangle in  $A$ .

$$\begin{array}{ccccccc}
X & \xrightarrow{[(s, f)]} & Y & \xrightarrow{Qv} & K & \xrightarrow{Q(Tsow)} & TX \\
Qs \uparrow & & Q(id) \uparrow & & Q(id) \uparrow & & Q(Ts) \uparrow \\
Z & \xrightarrow{Qf} & Y & \xrightarrow{Qv} & K & \xrightarrow{Qw} & TZ
\end{array}$$

(5) First we consider the case when two distinguished triangles are  $Q$  applied triangles in  $A$ .

$$\begin{array}{ccccccc}
X & \xrightarrow{Qu} & Y & \xrightarrow{Qv} & Z & \xrightarrow{Qw} & TX \\
f' \downarrow & & g' \downarrow & & & & \\
X' & \xrightarrow{Qu'} & Y' & \xrightarrow{Qv'} & Z' & \xrightarrow{Qw'} & TX'
\end{array}$$

We have the following commutative diagram in  $A$

$$\begin{array}{ccccccc}
& X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX \\
s_1 \swarrow & & & s_2 \swarrow & & & & \swarrow \\
W_1 & \xrightarrow{h_1} & W_2 & & & & TW_1 & \\
f_1 \searrow & & f_2 \searrow & & & & \searrow & \\
& X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & TX'
\end{array}$$

where  $s_1, s_2 \in S, f' = [(s_1, f_1)], g' = [(s_2, f_2)]$ . Then we have

$$\begin{array}{ccccccc}
& X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX \\
s_1 \swarrow & & s_2 \swarrow & & s_3 \swarrow & & & \swarrow \\
W_1 & \xrightarrow{h_1} & W_2 & \xrightarrow{h_2} & W_3 & \xrightarrow{h_3} & TW_1 & \\
f_1 \searrow & & f_2 \searrow & & f_3 \searrow & & \searrow & \\
& X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & TX'
\end{array}$$

Therefore we have the following in  $A[S^{-1}]$



$$\begin{array}{ccccccc}
X & \xrightarrow{Qu} & Y & \xrightarrow{Qv} & Z & \xrightarrow{Qw} & TX \\
\downarrow [(s_1, f_1)] & & \downarrow [(s_2, f_2)] & & \downarrow [(s_3, f_3)] & & \downarrow \tilde{T}[(s_1, f_1)] \\
X' & \xrightarrow{Qu'} & Y' & \xrightarrow{Qv'} & Z' & \xrightarrow{Qw'} & TX'
\end{array}$$

For the general case

$$\begin{array}{ccccccc}
X_1 & \xrightarrow{y} & Y_1 & \xrightarrow{p} & Z_1 & \xrightarrow{w} & TX_1 \\
\downarrow h_1 & & \downarrow h_2 & & \downarrow h_3 & & \downarrow \tilde{T}h_1 \\
X & \xrightarrow{Qu} & Y & \xrightarrow{Qv} & Z & \xrightarrow{Qw} & TX \\
\downarrow f' & & \downarrow g' & & \downarrow h' & & \downarrow \tilde{T}f' \\
X' & \xrightarrow{Qu'} & Y' & \xrightarrow{Qv'} & Z' & \xrightarrow{Qw'} & TX' \\
\downarrow h'_1 & & \downarrow h'_2 & & \downarrow h'_3 & & \downarrow \tilde{T}h'_1 \\
X'_1 & \xrightarrow{y'} & Y'_1 & \xrightarrow{p'} & Z'_1 & \xrightarrow{w'} & TX'_1
\end{array}$$

$\tilde{f}$  (curved arrow from  $X_1$  to  $X'_1$ )  
 $\tilde{g}$  (curved arrow from  $Y_1$  to  $Y'_1$ )  
 $\tilde{h}$  (dashed curved arrow from  $Z_1$  to  $Z'_1$ )  
 $\tilde{T}\tilde{f}$  (curved arrow from  $TX_1$  to  $TX'_1$ )

- (6) First we consider the case when 3 distinguished triangles in  $A[S^{-1}]$  is  $Q$  applied distinguished triangles in  $A$ . We get the new distinguished triangle in  $A$  and apply  $Q$ . We have  $Q(Tf) = \tilde{T}(Qf)$ , so it satisfies the commutative condition.

□



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