# Localization of triangulated category

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Abstract. Here is a summary of the results of each chapter.
Chapter 1 introduces $S$-roof and $S$-coroof, both induces localization of an additive category.
Chapter 2 gives definition of triangulated category and proves the localization of a triangulated category at a localizing class is triangulated if the localizing class is compatible with translation functor. For general properties of triangulated cateogory, see [**Nee01**].

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CHAPTER 1

# Localization of additive category

## 1. Localizing class

The localization of a category is basically "adding" inverse of some morphism. As we will see, some constructions involved in localization will make Hom-set between a pair of object a class. We still call them "category" here.

DEFINITION 1.1. Let $C$ be a category, $S$ class of morphisms in $C$, a functor $F : C \to D$ is called localization if $F(f)$ is isomorphism for $f$ in $S$ and for any functor $G : C \to E$ with $G(f)$ is isomorphism for $f$ in $S$, there exists a unique functor $H : D \to E$ such that $G = H \circ F$. For localization $F, D$ is usually denoted by $C[S^{-1}]$.

REMARK 1.2. The localization of a category always exist if we allow $C[S^{-1}]$ to be a generalized category [**Pan16**]. However, in such case, it has a complicated structure. So we consider a special type of S to make $C[S^{-1}]$ simple.

DEFINITION 1.3. Let $C$ be a category, $S$ is a class of morphisms in $C$. $S$ is called localizing class if it satisfies the following conditions

(1) $id_x \in$ S $\forall$ $x \in S$ $f, g$ in $S$ implys $g \circ f \in S$ (if $g \circ f$ is defined)
(2) Consider the following diagrams

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "s"'] & Y \\ Z & \end{tikzcd} \qquad \begin{tikzcd} & K \arrow[d, "s'"'] \\ T \arrow[r, "g"] & W \end{tikzcd}$$

where $s, s' \in S$ and $f, g \in C$ , they can be completed to

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "s"'] & Y \arrow[d, "s''"] \\ Z \arrow[r, "f'"'] & V \end{tikzcd} \qquad \begin{tikzcd} R \arrow[r, "g'"] \arrow[d, "s'''"'] & K \arrow[d, "s'"] \\ T \arrow[r, "g"'] & W \end{tikzcd}$$

where $s'', s''' \in S$
(3) Let $f, g : X \to Y$ be morphisms. Then there exists morpism $s \in S$ s.t. $sf = sg$ if and only if there exists morphism $s'$ s.t. $fs' = gs'$.

## 2. S-roof

DEFINITION 2.1. A $S-roof$ from $X \to Y$, denoted by$(s, f)$, is a diagram

$$\begin{tikzcd}
 & Z \arrow[dl, "s"'] \arrow[dr, "f"] & \\
X & & Y
\end{tikzcd}$$

where $s \in S$, $f \in C$. We say two roofs $(s, f)$ and $(t, g)$ are equivalent, denoted by $(s, f) \sim (t, g)$ if there is a such diagram

$$\begin{tikzcd}
 & & Z \arrow[dl, "s'"'] \arrow[dr, "h"] & & \\
 & X' \arrow[dl, "s"'] \arrow[drrr, "f"' near start] & & X'' \arrow[dlll, "t" near end] \arrow[dr, "g"] & \\
X & & & & Y
\end{tikzcd}$$

where $ss' \in S$, $h \in C$.

PROP 2.2. *Roof equibalence $\sim$ is equivalence relation.*

PROOF. Reflexivity and Symmetry is obvious. For transitivity, consider the following diagram

$$\begin{tikzcd}
 & & W \arrow[dl] \arrow[dr] & & \\
 & Z_1 \arrow[dl] \arrow[drrr] & & Z_2 \arrow[dlll] \arrow[dr] & \\
X & & & & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & T \arrow[dl] \arrow[dr] & & \\
 & Z_2 \arrow[dl] \arrow[drrr] & & Z_3 \arrow[dlll] \arrow[dr] & \\
X & & & & Y
\end{tikzcd}$$

use def 1.3 for

$$\begin{tikzcd}
 & W \arrow[d] \\
T \arrow[r] & X
\end{tikzcd}$$

we can compelete the diagram to

$$\begin{tikzcd}
M \arrow[r] \arrow[d] & W \arrow[d] \\
T \arrow[r] & X
\end{tikzcd}$$

where $M \to T$and $M \to W \in S$.

$$\begin{tikzcd}
 & & M \arrow[d] & & \\
 & & W \arrow[dl] \arrow[dr] & & \\
 & Z_1 \arrow[dl] \arrow[drrr] & & Z_2 \arrow[dlll] \arrow[dr] & \\
X & & & & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & M \arrow[d] & & \\
 & & T \arrow[dl] \arrow[dr] & & \\
 & Z_2 \arrow[dl] \arrow[drrr] & & Z_3 \arrow[dlll] \arrow[dr] & \\
X & & & & Y
\end{tikzcd}$$

$$M \longrightarrow W \longrightarrow Z_2 \longrightarrow X \qquad = \qquad M \longrightarrow T \longrightarrow Z_2 \longrightarrow X$$

use def 1.3 again, we have $M' \to M \in S$ s.t.

$$M' \longrightarrow M \longrightarrow W \longrightarrow Z_2 \qquad = \qquad M' \longrightarrow M \longrightarrow T \longrightarrow Z_2$$

$$\begin{tikzcd}
 & & M' \arrow[d] & & \\
 & & M \arrow[d] & & \\
 & & W \arrow[dl] \arrow[dr] & & \\
 & Z_1 \arrow[dl] \arrow[drrr] & & Z_2 \arrow[dlll] \arrow[dr] & \\
X & & & & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & M' \arrow[d] & & \\
 & & M \arrow[d] & & \\
 & & T \arrow[dl] \arrow[dr] & & \\
 & Z_2 \arrow[dl] \arrow[drrr] & & Z_3 \arrow[dlll] \arrow[dr] & \\
X & & & & Y
\end{tikzcd}$$

$$\begin{aligned}
& M' \longrightarrow M \longrightarrow W \longrightarrow Z_1 \longrightarrow X \\
= \quad & M' \longrightarrow M \longrightarrow T \longrightarrow Z_2 \longrightarrow X \\
= \quad & M' \longrightarrow M \longrightarrow T \longrightarrow Z_3 \longrightarrow X
\end{aligned}$$

$$\begin{aligned}
& M' \longrightarrow M \longrightarrow W \longrightarrow Z_1 \longrightarrow Y \\
= \quad & M' \longrightarrow M \longrightarrow W \longrightarrow Z_2 \longrightarrow Y \\
= \quad & M' \longrightarrow M \longrightarrow T \longrightarrow Z_2 \longrightarrow Y \\
= \quad & M' \longrightarrow M \longrightarrow T \longrightarrow Z_3 \longrightarrow Y
\end{aligned}$$

□

DEFINITION 2.3. For roofs $(s, f)$ from$X \to Y$ and $(s', g)$ from $Y \to Z$, their composition is defined to be the roof

$$\begin{tikzcd}
 & & W \arrow[dl, "s''"'] \arrow[dr, "h"] & & \\
 & T_1 \arrow[dl, "s"'] \arrow[dr, "f"] & & T_2 \arrow[dl, "s'"'] \arrow[dr, "g"] & \\
X & & Y & & Z
\end{tikzcd}$$

where $ss'' \in S$ and $fs'' = s'h$. The composition always exists due to def 1.3.

PROP 2.4. *Composition is well-defined for roof equivalent class.*

PROOF. First we consider the same representative for roofs.

$$\begin{tikzcd}[column sep=small]
 & & W \arrow[dl] \arrow[dr] & & \\
 & T_1 \arrow[dl] \arrow[dr] & & T_2 \arrow[dl] \arrow[dr] & \\
X & & Y & & Z
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & & W' \arrow[dl] \arrow[dr] & & \\
 & T_1 \arrow[dl] \arrow[dr] & & T_2 \arrow[dl] \arrow[dr] & \\
X & & Y & & Z
\end{tikzcd}$$

use def1.3 we have

$$\begin{tikzcd}
T \arrow[r] \arrow[d] & W \arrow[d] \\
W' \arrow[r] & X
\end{tikzcd}$$

where $T \to W$ and $T \to W' \in S$ use def 1.3 again

$$T' \to T \to W \to T_1 = T' \to T \to W' \to T_1$$

$$\begin{aligned}
&T' \to W \to T_2 \to Y \\
&= T' \to W \to T_1 \to Y \\
&= T' \to W' \to T_1 \to Y \\
&= T' \to W' \to T_2 \to Y
\end{aligned}$$

there exists $T'' \to T' \in S$ s.t. $T'' \to W \to T_2 = T'' \to W' \to T_2$
$(W \to X, W \to Z) \sim (T'' \to W \to X, T'' \to W \to Z) \sim (W' \to X, W' \to Z)$

$$\begin{tikzcd}
 & & W \arrow[dl, "s'''"'] \arrow[ddr, "h"] & & \\
 & T_3 \arrow[d, "s''"'] & & & \\
 & T_1 \arrow[dl, "s"'] \arrow[dr, "f"] & & T_2 \arrow[dl, "s'"'] \arrow[dr, "g"] & \\
X & & Y & & Z
\end{tikzcd}$$

where $ss'', ss''s''' \in S$.Therefore it's also composition of $(s, f)$ and $(s', g)$.
Let$(s, f) \sim (s_1, f_1)$,then $(s', g) \circ (s, f) \sim (s', g) \circ (s_1, f_1)$

$$\begin{tikzcd}
 & & W \arrow[ddl, "s'''"'] \arrow[dr, "h"] & & \\
 & & & T_3 \arrow[d, "s''"'] & \\
 & T_1 \arrow[dl, "s"'] \arrow[dr, "f"] & & T_2 \arrow[dl, "s'"'] \arrow[dr, "g"] & \\
X & & Y & & Z
\end{tikzcd}$$

where $s's'', ss''' \in S$.It is composition of $(s, f)$ and $(s', g)$.
Let $(s', g) \sim (s_2, g_2)$.Then $(s', g) \circ (s, f) \sim (s_2, g_2) \circ (s, f)$.
$(s', g) \circ (s, f) \sim (s_2, g_2) \circ (s, f) \sim (s_2, g_2) \circ (s_1, f_1)$ □

DEFINITION 2.5. Let $\tilde{C_S}$ be the category consisting of objects of C.
$Hom(X, Y) = \{[(s, f)]|s \in S, f \in C, cods = X, codf = Y\}$. Composition is defined from $S$-roof.

PROP 2.6. *Let $C$ be a category, $S$ localizing class, then $F : C \to \tilde{C_S}$ , which is id on object, send $f$ to $[(id, f)]$, is localization of $C$.*

PROOF. let $s \in S$

$$\begin{tikzcd}
 & & X \arrow[dl, "id"'] \arrow[dr, "id"] & & \\
 & X \arrow[dl, "id"'] \arrow[dr, "s"] & & X \arrow[dl, "s"'] \arrow[dr, "id"] & \\
{}'X & & Y & & X
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & X \arrow[dl, "id"'] \arrow[dr, "id"] & & \\
 & X \arrow[dl, "s"'] \arrow[dr, "id"] & & X \arrow[dl, "id"'] \arrow[dr, "s"] & \\
Y & & X & & Y
\end{tikzcd}$$

which shows $F$ maps morphisms in $S$ to isomorphisms in $\tilde{C_S}$.
Let $G : C \to D$ be a functor that send morphisms in $S$ to isomorphisms in $D$. Let $H : \tilde{C_S} \to D$ be a functor s.t. $G = H \circ F$.

$$H([(s, f)]) = H([(id, f)])H([(id, s)])^{-1} = G(f)G(s)^{-1}$$

we show $H$ is well-defined functor.
Let $(s, f)\sim (s', g)$. Then $(ss'', fs'') = (s's'', gs'')$ where $ss''\in S$
$G(f)G(s)^{-1} = G(fs'')G(ss'')^{-1} = G(g)G(s')^{-1}$. Thus $H$ is well-defined.
$H([(id, id)]) = G(id)G(id)^{-1} = id, [(s', g)][(s, f)] = [(ss'', gh)]$.
$H([(fs'', gh)]) = G(gh)G(ss'')^{-1} = G(g)(G(h)G(s'')^{-1})G(s)^{-1} =$
$G(g)(G(s')^{-1}G(f))G(s)^{-1}$ □

## 3. S-coroof

LEMMA 3.1. *Let $S$ be a category, $S$ localizing class. Then $S^{op}$ is localizing class of $C^{op}$.*

PROOF. Just reserve all arrows in def 1.3. □

DEFINITION 3.2. Let $C$ be a category, $S$ localizing class.A $S$-coroof is a diagram of the form

$$\begin{tikzcd}
X \arrow[dr, "f"'] & & Y \arrow[dl, "s"] \\
& Z &
\end{tikzcd}$$

where $s \in S$, denoted by $(f, s)$.Two coroofs $(f, s), (g, s')$ are equivalent if there is a diragram of the form

$$\begin{tikzcd}
X \arrow[dr, "f"'] \arrow[drrr, "s" near start] & & & & Y \arrow[dlll, "g"' near start] \arrow[dl, "s'"] \\
& Z_1 \arrow[dr, "h"'] & & Z_2 \arrow[dl, "s''"] & \\
& & W & &
\end{tikzcd}$$

where $s''s' \in S$ and $h \in C$, denoted by $(f, s) \sim^{\vee} (g, s')$
The composition of $(f, s)$ and $(g, s')$ is define to be the following diagram

$$\begin{tikzcd}
X \arrow[dr, "f"'] & & Y \arrow[dl, "s"'] \arrow[dr, "g"] & & Z \arrow[dl, "s'"] \\
& Z_1 \arrow[dr, "h"'] & & Z_2 \arrow[dl, "s''"] & \\
& & W & &
\end{tikzcd}$$

where $h \in C$ , $s''s' \in S$.

PROP 3.3. *$\sim^{\vee}$ is equivalence relation. Coroof composition exists and is well-defined for equivalent class of coroofs. There is a category with object the same as $C$, its morphism consists of equivalent class of coroofs, composition defined in this way, denoted by $\check{C}_S$, and it is isomorphic to $\tilde{C}_S$.*

PROOF. $S^{op}$-roof $Y \to X$ has a bijection to $S$-coroof $X \to Y$

$$\begin{tikzcd}
& Z \arrow[dl, "s^{op}"'] \arrow[dr, "f^{op}"] & \\
Y & & X
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[dr, "f"'] & & Y \arrow[dl, "s"] \\
& Z &
\end{tikzcd}$$

which send $(s^{op}, f^{op})$ to $(f, s)$. We carry the equivalent class of $S^{op}$-roof to $S$-coroof.

$$\begin{tikzcd}
& & W \arrow[dl, "s''^{op}"'] \arrow[dr, "h^{op}"] & & \\
& Z_1 \arrow[dl, "s^{op}"'] \arrow[drrr, "f^{op}"' near end] & & Z_2 \arrow[dlll, "s'^{op}" near end] \arrow[dr, "g^{op}"] & \\
Y & & & & X
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[dr, "g"'] \arrow[drrr, "f" near start] & & & & Y \arrow[dlll, "s'"' near start] \arrow[dl, "s"] \\
& Z_2 \arrow[dr, "h"'] & & Z_1 \arrow[dl, "s''"] & \\
& & W & &
\end{tikzcd}$$

The induced equivalence relation on $S$-coroof is $\sim^{\vee}$. $W(X, Y)$={$S$-coroof equivalent class from $X$ to $Y$}. Then $Hom_{(\tilde{C^{op}}_{S^{op}})^{op}}(X, Y) \simeq W(X, Y)$. Then the isomorphisms create a category with object the same as $C$, morphisms as equivalent classes of coroof, and composition as coroof composition.
Note that $G : C \to C^{op}[S^{op-1}]^{op}$ is localization of $C$, so the induced morphism

$R : C[S^{-1}] \to C^{op}[S^{op-1}]^{op}$ is isomorphism. Therefore the composite $\tilde{C_S} \to (\tilde{C^{op}}_{S^{op}})^{op} \to \check{C}_S$ is isomorphism. Which send $[(s,f)] \to [(f,id)][(s,id)]^{-1} = [(f',s')]$

$$\begin{tikzcd}
 & & & Z \arrow[dl, "s"'] \arrow[dr, "f"] & \\
Z \arrow[dr, "s"'] & & X \arrow[dl, "id"'] \arrow[dr, "f'"] & & Y \arrow[dl, "s'"] \\
 & X \arrow[dr, "f'"'] & & Z' \arrow[dl, "id"] & \\
 & & Z' & &
\end{tikzcd}$$

where $f', s'$ rise from def 1.3. So $[(f',s')][(s,id)] = [(f's,s')] = [(s'f,s')] = [(f,id)]$. ☐

## 4. Localization of additive category

THEOREM 4.1. *Let $A$ be additive category, $S$ localizing class. Then $A[S^{-1}]$ is additive category and the localization functor $Q : A \to A[S^{-1}]$ is additive functor.*

PROOF. (1) 0 (in $A$) is zero object. If we have two roofs

$$\begin{tikzcd}
 & Y \arrow[dl, "f"'] \arrow[dr] & & & Z \arrow[dl] \arrow[dr, "h"] & \\
X & & 0 & 0 & & T
\end{tikzcd}$$

where $f \in S$.

$$[(f,0)] = [(id \circ f, 0 \circ f)] = [(id, 0)]$$
$$[(0,h)] = [(0 \circ h), id \circ h] = [(0, id)].$$

(2) $Hom_{A[S^{-1}]}(X,Y)$ is abelian group.

$$[(s,f)] + [(s',f')] := [(ss'', fs'' + f's''')]$$

where $ss'' = s's'''$, $s''$ and $s''' \in S$.

To see it's well-defined, let $(s,f) \sim (s',f'), (s,g) \sim (s',g')$

$$(ss'', fs'') = (s's''', f's''')$$
$$(ss'', gs'') \sim (s's''', g's''')$$
$$(ss''s_1, gs''s_1) = (s's'''s_1, g's'''s_1) \ *$$

where $s'', s''', s_1 \in S$

$$(s, f+g) \sim (ss''s_1, (f+g)s''s_1) = (s's'''s_1, (f'+g')s'''s_1) \sim (s', f'+g')$$

(3) Existence of biproduct.

let $l_X, l_Y, \pi_X, \pi_Y$ be biproduct of $X$ and $Y$ in $A$. Then $[(id, l_X)], [(id, l_Y)], [(id, \pi_X)], [(id, \pi_Y)]$ is biproduct of $X$ and $Y$ in $C[S^{-1}]$

$$[(id,\pi_X)][(id,l_X)] = [(id,id)], [(id,\pi_Y)][(id,l_Y)] = [(id,id)]$$
$$[(id,\pi_Y)][(id,l_X)] = [(id,0)], [(id,\pi_X)][(id,l_Y)] = [(id,0)]$$
$$[(id,l_X)][(id,\pi_X)] + [(id,l_Y)][(id,\pi_Y)] = [(id, l_X\pi_X + l_Y\pi_Y)] = [(id,0)]$$

(4) Distributive law

$$\begin{tikzcd}[column sep=small]
 & & W \arrow[dl, "s''"'] \arrow[dr, "k"] & & \\
 & W_1 \arrow[dl, "s"'] \arrow[dr, "f"] & & W_2 \arrow[dl, "s'"] \arrow[dr, "h"] & \\
X & & Y & & Z
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & & W' \arrow[dl, "s'''"'] \arrow[dr, "t"] & & \\
 & W_1 \arrow[dl, "s"'] \arrow[dr, "g"] & & W_2 \arrow[dl, "s'"] \arrow[dr, "h"] & \\
X & & Y & & Z
\end{tikzcd}$$

where $s'', s''' \in S$

$$\begin{tikzcd}[column sep=small]
 & & W'' \arrow[d, "s_1"'] & & \\
 & & W \arrow[dl, "s''"'] \arrow[dr, "k"] & & \\
 & W_1 \arrow[dl, "s"'] \arrow[dr, "f"] & & W_2 \arrow[dl, "s'"] \arrow[dr, "h"] & \\
X & & Y & & Z
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & & W'' \arrow[d, "s_2"'] & & \\
 & & W' \arrow[dl, "s'''"'] \arrow[dr, "t"] & & \\
 & W_1 \arrow[dl, "s"'] \arrow[dr, "g"] & & W_2 \arrow[dl, "s'"] \arrow[dr, "h"] & \\
X & & Y & & Z
\end{tikzcd}$$

where $s_1, s_2 \in S$,$s''s_1 = s'''s_2$,$s_3 \coloneqq s''s_1$

$$\begin{tikzcd}
 & & W''' \arrow[dl, "s_3"'] \arrow[dr, "ks_1+ts_2"] & & \\
 & W_1 \arrow[dl, "s"'] \arrow[dr, "f+g"] & & W_2 \arrow[dl, "s'"] \arrow[dr, "h"] & \\
X & & Y & & Z
\end{tikzcd}$$

$[(s', h)]([(s, f)] + [(s, g)]) = [(s', h)][(s, f + g)] = [(ss_3, h(ks_1 + ts_2))] = [(ss_3, hks_1)] + [(ss_3, hts_2)] = [(s', h)][(s, f)] + [(s', h)][(s, g)]$
$([(s, f)]+[(s, g)])[(s', h)] = [(s, f+g)][(s', h)] = [(s's'', (f+g)t)] = [(s's'', ft)]+[(s's'', gt)] = [(s, f)][(s', h)] + [(s, g)][(s', h)]$
where $hs'' = st, s'' \in S$

(5) Q is additive.
$Q(f + g) = [(id, f + g)] = [(id, f)] + [(id, g)] = Q(f) + Q(g)$

$\square$

CHAPTER 2

# Triangulated category

## 1. Definition of triangulated category

DEFINITION 1.1. Let $A$ be an additive category, $T$ additive automorphism of $A$.A triangle in $A$ is

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

Denoted by $(X, Y, Z, u, v, w), where X[1] \coloneqq TX$, $T$ is sometimes called translation functor. Morphism from $(X, Y, Z, u, v, w)$ to $(X', Y', Z', u', v', w')$ is a triple $(f, g, h)$ s.t. the diagram commutes

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "w"] \arrow[d, "h"'] & X[1] \arrow[d, "f[1]"'] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1] \end{tikzcd}$$

where $f[1] \coloneqq Tf$. Then $(f, g, h)$ is isomorphism (in the category of some triangles) if and only if $f, g, h$ are isomorphisms.

DEFINITION 1.2. Let $A$ be an additive category, $T$ additive automorphism of $A, P$ a collection of triangles in $A$, called distinguished triangles. Then $A$ is called triangulated category w.r.t $T$ and $P$ if the following conditions are satisfied

(1) For any $X$ in $A$,$(X, X, 0, id, 0, 0) \in P$
(2) $(X, Y, Z, u, v, w) \in P$ if and only if $(Y, Z, X[1], v, w, -u[1]) \in P$.
(3) A triangle isomorphic a triangle in $P$ is in $P$.
(4) For any $f$ in $A$,there is a triangle $(X, Y, Z, u, v, w)$ in $P$ s.t. $u = f$.
(5) For two distinguished triangles $(X, Y, Z, u, v, w), (X', Y', Z', u', v', w')$ and $f, g$ s.t the diagram commutes

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "w"] & X[1] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1] \end{tikzcd}$$

Then there exist $h$ s.t the diagram commutes

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "w"] \arrow[d, dashed, "h"'] & X[1] \arrow[d, "f[1]"'] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1] \end{tikzcd}$$

(6) For distinguished triangles $(X, Y, Z, u, v, w), (Y, Y', Z'', u', v', w')$ and $(X, Y', Z', u'u, v'', w'')$, there exists distinguished triangle

$(Z, Z', Z'', u''', v''', w''')$ s.t. the diagram commutes

$$\begin{tikzcd}
 & & & & & X[1] \arrow[dddr, "id"] & \\
 & & & Z \arrow[urr, "w"] \arrow[ddr, dashed, "u'''"] & & & \\
 & Y \arrow[urr, "v"] \arrow[dr, "u'"] & & & & & \\
X \arrow[ur, "u"] \arrow[rr, "u'u"'] & & Y' \arrow[rr, "v''"'] \arrow[drrr, "v'"] & & Z' \arrow[rr, "w''"'] \arrow[dr, dashed, "v'''"] & & X[1] \arrow[dd, "u[1]"] \\
 & & & & & Z'' \arrow[dr, "w'"] \arrow[ddr, dashed, "w'''"'] & \\
 & & & & & & Y[1] \arrow[d, "v[1]"] \\
 & & & & & & Z[1]
\end{tikzcd}$$

[**SIG96**] provides the commutative condition in a octahedral, called "The octahedral axiom".

REMARK 1.3. Let $A$ be a triangulated category,$(X, Y, Z, u, v, w)$, $(X', Y', Z', u', v', w')$ distinguished triangles. If the diagram commutes

$$\begin{tikzcd}
X \arrow[r, "u"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & X[1] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & X'[1]
\end{tikzcd}$$

Then there exist $f$ s.t.

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, dashed, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & X[1] \arrow[d, dashed, "f[1]"] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & X'[1]
\end{tikzcd}$$

If the diagram commutes

$$\begin{tikzcd}
X \arrow[r, "u"] & Y \arrow[r, "v"] & Z \arrow[r, "w"] \arrow[d, "h"] & X[1] \arrow[d, "f[1]"] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & X'[1]
\end{tikzcd}$$

Then there exists g s.t.

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, dashed, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & X[1] \arrow[d, "f[1]"] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & X'[1]
\end{tikzcd}$$

## 2. Localization of triangulated category

DEFINITION 2.1. Let $A$ be a triangulated category, $S$ localizing class, $T$ translation functor. $T$ is called compatible with $S$ if

(1) $T(f) \in S$ if and only if $f \in S$.
(2) In condition 5 of triangulated category, for $f, g \in S$, there exists $h \in S$.

LEMMA 2.2. *Let $A$ be an additive category, $S$ localizing class, $Q$ localization functor. For the diagrams in $A[S^{-1}]$ and $A$ respectively*

$$\begin{tikzcd} X \arrow[r, "Qu"] \arrow[d, "t"'] & Y \arrow[d, "r"] \\ X' \arrow[r, "Qu'"] & Y' \end{tikzcd} \qquad \begin{tikzcd} & X \arrow[r, "u"] & Y \\ W_1 \arrow[ur, "s_1"] \arrow[r, "w"'] \arrow[dr, "f_1"'] & W_2 \arrow[ur, "s_2"] \arrow[dr, "f_2"] & \\ & X' \arrow[r, "u'"'] & Y' \end{tikzcd}$$

*The first diagram is commutative if and only if there exists $s_1, s_2 \in S, f_1, f_2, w \in A$ s.t. $t = [(s_1, f_1)], r = [(s_2, f_2)]$ and the second diagram is commutative.*

PROOF.

$$\begin{tikzcd} & & W \arrow[dl, "s_1"'] \arrow[dr, "w"] & & \\ & X \arrow[dl, "id"'] \arrow[dr, "u"] & & T \arrow[dl, "s_2"'] \arrow[dr, "f_2"] & \\ X & & Y & & Y' \end{tikzcd}$$

$r \circ Qu = [(s_2, f_2)][(id, u)] = [(s_1, f_2 w)] = [(s_1, u' f_1)] = Qu' \circ t$
Conversely, let $t = [(s_1, f_1)], r = [(s_2, f_2)]. Qu' \circ t = [(s_1, u' f_1)]. r \circ Qu = (s', f_2 g')$, where $us' = s_2 g', s' \in S. (s_1, u' f_1) \sim (s', f_2 g'). s' s'' = s_1 s'''$, where $s'', s''' \in S. (s' s'', f_2 g' s'') \sim (s_1 s''', u' f_1 s''')$. Then $f_2 g' s'' s_3 = u' f_1 s''' s_3 . s' s''' s_3 \in S. \tilde{f_1} = f_1 s''' s_3 . \tilde{s_1} = s_1 s''' s_3, w = g' s'' s_3$. Then $u \tilde{s_1} = s_2 w . u' \tilde{f_1} = f_2 w$. $[(s_1, f_1)] = [(\tilde{s_1}, \tilde{f_1})]$ □

THEOREM 2.3. *Let $A$ be a triangulated category, $S$ localizing class, $Q$ localization functor, $T$ translation functor compatible with $S$. Then $A[S^{-1}]$ is triangulated category with $\tilde{T}$ as translation functor and triangles isomorphic to $Q$ applied distinguished triangles in $A$ as distinguished triangles.*

PROOF. (1)

$$X \xrightarrow{id} X \xrightarrow{0} 0 \xrightarrow{0} X[1]$$

$$X \xrightarrow{[(id,id)]} X \xrightarrow{[(id,0)]} 0 \xrightarrow{[(id,0)]} X[1]$$

(2)

$$\begin{tikzcd}
X \arrow[r, "Qu"] \arrow[d, "f"'] & Y \arrow[r, "Qv"] \arrow[d, "g"'] & Z \arrow[r, "Qw"] \arrow[d, "h"'] & X[1] \arrow[d, "\tilde{T}(f)"'] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & X'[1]
\end{tikzcd}$$

$$\begin{tikzcd}
Y \arrow[r, "Qv"] \arrow[d, "g"'] & Z \arrow[r, "Qw"] \arrow[d, "h"'] & TX \arrow[r, "-\tilde{T}(Qu)"] \arrow[d, "\tilde{T}(f)"'] & TY \arrow[d, "\tilde{T}(g)"'] \\
Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & TX' \arrow[r, "-\tilde{T}(u')"'] & TY'
\end{tikzcd}$$

(3) Trivial

(4) Give $[(s,f)] : X \to Y$ in $A[S^{-1}]$. Let $(Z, Y, K, f, v, w)$ be a distinguished triangle in $A$.

$$\begin{tikzcd}
X \arrow[r, "{[(s,f)]}"] & Y \arrow[r, "Qv"] & K \arrow[r, "Q(Ts\circ w)"] & TX \\
Z \arrow[r, "Qf"'] \arrow[u, "Qs"] & Y \arrow[r, "Qv"'] \arrow[u, "Q(id)"] & K \arrow[r, "Qw"'] \arrow[u, "Q(id)"] & TZ \arrow[u, "Q(Ts)"]
\end{tikzcd}$$

(5) First we consider the case when two distinguished triangles are $Q$ applied triangles in $A$.

$$\begin{tikzcd}
X \arrow[r, "Qu"] \arrow[d, "f'"'] & Y \arrow[r, "Qv"] \arrow[d, "g'"'] & Z \arrow[r, "Qw"] & TX \\
X' \arrow[r, "Qu'"'] & Y' \arrow[r, "Qv'"'] & Z' \arrow[r, "Qw'"'] & TX'
\end{tikzcd}$$

We have the following commutative diagram in $A$

$$\begin{tikzcd}
 & X \arrow[rr, "u"] \arrow[dl, "s_1"'] & & Y \arrow[r, "v"] \arrow[dl, "s_2"'] & Z \arrow[rr, "w"] & & TX \arrow[dl] \\
W_1 \arrow[rr, "h_1"'] \arrow[dr, "f_1"'] & & W_2 \arrow[dr, "f_2"'] & & & TW_1 \arrow[dr] & \\
 & X' \arrow[rr, "u'"'] & & Y' \arrow[r, "v'"'] & Z' \arrow[rr, "w'"'] & & TX'
\end{tikzcd}$$

where $s_1, s_2 \in S$,$f' = [(s_1, f_1)], g' = [(s_2, f_2)]$. Then we have

$$\begin{tikzcd}
 & X \arrow[rr, "u"] & & Y \arrow[rr, "v"] & & Z \arrow[rr, "w"] & & TX \\
W_1 \arrow[ur, "s_1"] \arrow[rr, "h_1"'] \arrow[dr, "f_1"'] & & W_2 \arrow[ur, "s_2"] \arrow[rr, dashed, "h_2"'] \arrow[dr, "f_2"'] & & W_3 \arrow[ur, dashed, "s_3"] \arrow[rr, dashed, "h_3"'] \arrow[dr, dashed, "f_3"'] & & TW_1 \arrow[ur, "Ts_1"] \arrow[dr, "Tf_1"'] & \\
 & X' \arrow[rr, "u'"'] & & Y' \arrow[rr, "v'"'] & & Z' \arrow[rr, "w'"'] & & TX'
\end{tikzcd}$$

Therefore we have the following in $A[S^{-1}]$

$$\begin{tikzcd}
X \arrow[r, "Qu"] \arrow[d, "{[(s_1,f_1)]}"'] & Y \arrow[r, "Qv"] \arrow[d, "{[(s_2,f_2)]}"'] & Z \arrow[r, "Qw"] \arrow[d, dashed, "{[(s_3,f_3)]}"'] & TX \arrow[d, "{\tilde{T}[(s_1,f_1)]}"'] \\
X' \arrow[r, "Qu'"'] & Y' \arrow[r, "Qv'"'] & Z' \arrow[r, "Qw'"'] & TX'
\end{tikzcd}$$

For the general case

$$\begin{tikzcd}
X_1 \arrow[rr, "y"] \arrow[dr, "h_1"] \arrow[ddd, bend right, "\tilde{f}"'] & & Y_1 \arrow[rr, "p"] \arrow[dr, "h_2"] \arrow[ddd, bend right, "\tilde{g}"'] & & Z_1 \arrow[rr, "w"] \arrow[dr, "h_3"] \arrow[ddd, dashed, bend right, "\tilde{h}"'] & & TX_1 \arrow[dr, "\tilde{T}h_1"] \arrow[ddd, bend right, "\tilde{T}\tilde{f}"'] & \\
& X \arrow[rr, "Qu"] \arrow[d, "f'"'] & & Y \arrow[rr, "Qv"] \arrow[d, "g'"'] & & Z \arrow[rr, "Qw"] \arrow[d, dashed, "h'"'] & & TX \arrow[d, "\tilde{T}f'"'] \\
& X' \arrow[rr, "Qu'"] \arrow[dl, "h'_1"] & & Y' \arrow[rr, "Qv'"] \arrow[dl, "h'_2"] & & Z' \arrow[rr, "Qw'"] \arrow[dl, "h'_3"] & & TX' \arrow[dl, "\tilde{T}h'_1"] \\
X'_1 \arrow[rr, "y'"'] & & Y'_1 \arrow[rr, "p'"'] & & Z'_1 \arrow[rr, "w'"'] & & TX'_1 &
\end{tikzcd}$$

(6) First we consider the case when 3 distinguished triangles in $A[S^{-1}]$ is $Q$ applied distinguished triangles in $A$. We get the new distinguished triangle in $A$ and apply $Q$. We have $Q(Tf) = \tilde{T}(Qf)$, so it satisfies the commutative condition.

$\square$

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