Math 140 **Tutorial 3**

January 20 - 23

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1. Which of the following functions is continuous on the given intervals?

   (a) $g(t) = \dfrac{t^2 + 5t}{2t + 1}$, $[0, \infty)$

   (b) $f(x) = \begin{cases} \frac{x-3}{x^2-9} & x \neq 3 \\ 0 & x = 3 \end{cases}$, $[0, \infty)$

   (c) $f(x) = \begin{cases} \frac{x-3}{x^2-9} & x \neq 3 \\ \frac{1}{6} & x = 3 \end{cases}$, $[0, \infty)$

   (d) $f(x) = \begin{cases} \sin(x) & x < \pi/4 \\ \cos(x) & x \geq \pi/4 \end{cases}$, $[0, \infty)$

Solution

(a) The function $\frac{t^2+5t}{2t+1}$ is continuous everywhere except where it has a vertical asymptote at $t = -\frac{1}{2}$, but $-\frac{1}{2}$ is not in the interval specified, therefore the function is continuous on the interval $[0, \infty)$

(b) This function is continuous everywhere except possibly at $x = 3$. Notice that $\frac{x-3}{x-9} = \frac{1}{x+3}$. So $\lim_{x \to 3} \frac{x-3}{x-9} = \frac{1}{6}$. So the function is not continuous at 3 since $f(3) = 0 \neq \frac{1}{6}$.

(c) This function is continuous by the same argument from part 2.

(d) Again this function is continuous everywhere except possibly at $\pi/4$. By checking the limit from the right and from the left at $\pi/4$, we see that they agree since $\sin(\pi/4) = \cos(\pi/4)$.

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2. Evaluate the following limits if the exist:

   (a) $\lim\limits_{x \to \pi} \sin(x + \sin x)$

   (b) $\lim\limits_{x \to 0} x \sin\left(\dfrac{\pi}{x^2}\right) + \cos x$

   (c) $\lim\limits_{t \to 0} \dfrac{1}{t\sqrt{t+1}} - \dfrac{1}{t}$

   (d) $\lim\limits_{x \to 0} \dfrac{\sqrt{3+x} - \sqrt{3}}{x}$

   (e) $\lim\limits_{x \to -1} \dfrac{x^2 - 4x}{x^2 - 3x - 4}$

   (f) $\lim\limits_{x \to 2} \arctan\left(\dfrac{x^2 - 4}{3x^2 - 6x}\right)$

Solution

(a) $\sin x$ is continuous therefore

$$\lim_{x\to\pi} \sin(x+\sin x) = \sin\left(\lim_{x\to\pi}(x+\sin x)\right)$$

and since x and $\sin x$ are continuous functions we can just plug π in to get

$$\sin(\pi + \sin(\pi)) = 0$$

(b) We know that

$$-1 \leq \sin\left(\frac{\pi}{x^2}\right) \leq 1$$

therefore

$$-x + \cos x \leq x\sin\left(\frac{\pi}{x^2}\right) + \cos x \leq x + \cos x$$

The left and right parts of the theorem both tend to 1 therefore by the squeeze theorem

$$\lim_{x\to 0} x\sin\left(\frac{\pi}{x^2}\right) + \cos x = 1$$

(c) By taking $\frac{1}{t}$ as a common factor we can rewrite the expression as

$$\frac{1-\sqrt{t+1}}{t\sqrt{t+1}}$$

Now we multiply the top and the bottom by $1+\sqrt{t+1}$ to get

$$\frac{-t}{t\sqrt{t+1}(1+\sqrt{t+1})} = \frac{-1}{\sqrt{t+1}(1+\sqrt{t+1})}$$

that final expression is continuous at 0 and has 0 in its domain so the limit is equal to $\frac{-1}{2}$

(d) We do a similar trick to the one we did in part (c) which is to rationalize one of the expressions by multiplying by $\sqrt{3+x}+\sqrt{3}$ at the top and bottom to get

$$\frac{x}{\sqrt{3+x}+\sqrt{3}} = \frac{1}{\sqrt{3+x}+\sqrt{3}}$$

and again the last expression clearly has 0 in its domain and is continuous, so the limit tends to $\frac{1}{2\sqrt{3}}$

(e) We begin by factorizing the numerator and denominator

$$\frac{x(x-4)}{(x-4)(x+1)} = \frac{x}{x+1}$$

so we can see that from the right the limit tends to $+\infty$ and from the left it tends to $-\infty$ therefore the limit does not exist.

(f) arctan is a continuous function therefore this limit is equal to

$$\arctan\left(\lim_{x\to 2}\frac{x^2-4}{3x^2-6x}\right)$$

we can again factorize the fraction and simplify to get

$$\frac{x+2}{3x}$$

which has limit $\frac{2}{3}$ as x tends to 2, therefore the final limit is aarctan $\left(\frac{2}{3}\right)$

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3. Determine if the statements below are true or false. If true, justify your answer; if not, provide a counterexample.

(a) If $f(x)$ is continuous on the interval $(0,1)$, then $f(x)^2$ is also continuous on the same interval.

(b) If $f(x)^2$ is continuous on the interval $(0,1)$, then $f(x)$ is also continuous on the same interval.

(c) If $f(x)$ and $g(x)$ are both continuous on the interval $(0,1)$, then $f(x)+g(x)$ is continuous on the same interval.

(d) If $f(x)+g(x)$ is continuous on the interval $(0,1)$, then so are $f(x)$ and $g(x)$ on the same interval.

(e) If both $f(x)+g(x)$ and $f(x)-g(x)$ are continuous on $(0,1)$, then so are $f(x)$ and $g(x)$ on the same interval.

Solution

(a) Yes, this is true by the composition law for continuity, since $f(x)^2$ is the result of composing $g(x)=x^2$ with $f(x)$ i.e

$$f(x)^2 = g(f(x))$$

and since both $g(x)$ and $f(x)$ are continuous then $g(f(x))$ is also continuous.

(b) This is not true, consider the function

$$f(x) = \begin{cases} -1 & x \leq \frac{1}{2} \\ 1 & x > \frac{1}{2} \end{cases}$$

This function is not continuous at $\frac{1}{2}$ but $f(x)^2 = 1$ on the interval $(0, 1)$ which is continuous, hence we have come up with a counterexample.

(c) This is true, and is one of our rules for continuous functions. The sum of two continuous functions is always continuous.

(d) This is not true. Consider

$$f(x) = \begin{cases} -1 & x \leq \frac{1}{2} \\ 1 & x > \frac{1}{2} \end{cases}$$

and

$$g(x) = \begin{cases} 1 & x \leq \frac{1}{2} \\ -1 & x > \frac{1}{2} \end{cases}$$

Then similarly to part (b), neither f or g are continuous at $\frac{1}{2}$ but $f(x)+g(x) = 0$, therefore $f(x) + g(x)$ is continuous.

(e) This is true, notice that

$$f(x) = \frac{(f(x) + g(x)) + (f(x) - g(x))}{2}$$

so since $f+g$ and $f-g$ are both continuous, then their sum is clearly continuous, and by dividing by 2 we get another continuous function by applying our rules for continuous functions. We can repeat the same argument for $g(x)$ by noting

$$f(x) = \frac{(f(x) + g(x)) - (f(x) - g(x))}{2}$$

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4. Use the Intermediate Value Theorem to show that the following equations have a root on the given intervals

(a) $\cos x = x$ on $(0, 1)$

(b) $\ln x = e^{-x}$ on $(1, 2)$

Solution

(a) We will apply the Intermediate Value Theorem to the function $f(x) = \cos x - x$. Notice $f(0) = 1 > 0$ and $f(1) = \cos(1) - 1 < 0$. So by the IVT there is a number c between 0 and 1 such that $f(c) = 0$, and that means that $\cos(c) = c$.

(b) We do the exact thing as above with the function $g(x) = \ln x - e^{-x}$. Notice $g(1) = -\frac{1}{e} < 0$ and $g(2) = \ln(2) - \frac{1}{e^2} > 0$. Therefore, there is a number c between 1 and 2 such that $g(c) = 0$.

5. In this question we will use the Intermediate Value Theorem to approximate a solution to the equation

$$x = e^{x-2}$$

(a) First, show that this equation has a solution in the interval $(0, 1)$.

(b) Now split this interval into two halves $(0, \frac{1}{2})$ and $(\frac{1}{2}, 1)$. Show that one of these two intervals has a solution to our equation. Which one is it?

(c) Split the interval you got in part (b) into two halves again to show that one of the four intervals $(0, \frac{1}{4}), (\frac{1}{4}, \frac{1}{2}), (\frac{1}{2}, \frac{3}{4}), (\frac{3}{4}, 1)$ has a solution to our equation. Which one is it?

(d) Repeat this process of splitting the intervals in halves until you have narrowed down the solution to 2 decimal places.

Solution

We consider the function $f(x) = x - e^{x-2}$. Notice $f(0) = -e^{-2} < 0$ and $f(1) = 1 - e^{-1} > 0$ therefore there's a root between 0 and 1. Then repeat this process: $f(\frac{1}{2}) > 0$ therefore there's a root betwee 0 and $\frac{1}{2}$, now we investigate the halfway point $f(\frac{1}{4}) > 0$ so we can conclude there's a root between 0 and $\frac{1}{4}$. We keep going: The midpoint is now $\frac{1}{8}$ and $f(\frac{1}{8}) < 0$ so there's a root between $\frac{1}{8}$ and $\frac{1}{4}$. The midpoint between them is $\frac{3}{16}$ and $f(\frac{3}{16}) > 0$ so there's a root between $\frac{1}{8}$ and $\frac{3}{16}$. The midpoint now is $\frac{5}{32}$ and $f(\frac{5}{32}) < 0$ so there's a root between $\frac{5}{32}$ and $\frac{3}{16}$. Narrowing down, there's a root between $\frac{5}{32}$ and $\frac{21}{128}$ and now we've narrowed it down to an interval of length less than 0.01.

6. A mountain climber starts climbing a mountain at 10am and they reach the top at 2pm where they stay for the night. The next day they begin their descent at 10am and reach the bottom at 2pm, they take the exact same path they took the day before. Is there necessarily a time between 10am and 2pm at which the climber was at the exact same spot on both days? (Note the the climber doesn't necessarily ascend and descend at the same speed)

Solution

Let $f(x)$ be the function describing the position of the climber during ascent, and $g(x)$ be the function describing the position during descent. Then at 10am the climber is at the bottom on the first day and at the top on the second day, therefore $f(10)-g(10)<0$, at 2pm it's switched, so $f(14)-g(14)>0$. Since the climber moves continuously, the functions f and g are continuous, therefore $f(x)-g(x)$ is continuous, therefore there is a time c between 8 and 14 such that $f(c)=g(c)$.

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7. Evaluate the following limits if they exist

   (a) $\lim_{x\to+\infty} \cos x$

   (b) $\lim_{x\to+\infty} \frac{x+1}{x}$

   (c) $\lim_{x\to+\infty} \frac{x}{x+1}$

   (d) $\lim_{x\to-\infty} \frac{x}{x^2+1}$

   (e) $\lim_{x\to+\infty} \frac{x^3-3x^2+3x-1}{3x^3+27x^2+9x+1}$

   (f) $\lim_{x\to1^+} \arctan\left(\frac{1}{x-1}\right)$

Solution

(a) The function $\cos x$ alternates between -1 and 1 therefore the limit does not exist.

(b) We can simplify this function to $1+\frac{1}{x}$. As x gets bigger $\frac{1}{x}$ gets much smaller and tends to 0, therefore $1+\frac{1}{x}$ tends to 1.

(c) We divide at the top and bottom by x to get

$$\frac{1}{1+1/x}$$

As in the previous part $\frac{1}{x}$ tends to 0 therefore the whole expression tends to $\frac{1}{1+0}=1$.

(d) Again we divide at the top and bottom by xx to get

$$\frac{1}{x+\frac{1}{x}}$$

again we note that $\frac{1}{x}$ tends to 0 and x tends to ∞ therefore $x+\frac{1}{x}$ tends to ∞ and so $\frac{1}{x+\frac{1}{x}}$ tends to 0.

(e) Here again we divide by the higher power of x in the numerator which is x^3 to get

$$\frac{1-\frac{3}{x}+\frac{3}{x^2}-\frac{1}{x^3}}{3+\frac{27}{x}+\frac{9}{x^2}+\frac{1}{x^3}}.$$

Notice that all the fractions $\frac{\alpha}{x^i}$ will just tend to 0 as x gets bigger. Therefore the numerator tends to 1 and the denominator tends to 3 so the ratio tends to $\frac{1}{3}$

(f) As x tends to 1 from the right $\frac{1}{x-1}$ tends to $+\infty$ so this limit is equal to

$$\lim_{t\to+\infty} \arctan(t)$$

which we know is equal to $\pi/2$.

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8. Find a rational function $f(x)$ such that $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x)$. Find another rational function $g(x)$ such that $\lim_{x\to\infty} g(x) \neq \lim_{x\to-\infty} g(x)$.

   Reminder: a rational function is a polynomial divided by a polynomial.