# Project: Ordinary Differential Equations with Laboratory

2023/2024

Submission deadline: 02.06.2024

## Problem description

The system of equations describing the position $(x, y)$ of a satellite (with negligible mass compared to celestial bodies) on the Earth-Moon plane is as follows:

$$\ddot{x} = x + 2\dot{y} - \frac{(1-\mu)(x+\mu)}{A} - \frac{\mu(x-1+\mu)}{B},$$
$$\ddot{y} = y - 2\dot{x} - \frac{(1-\mu)y}{A} - \frac{\mu y}{B},$$

where:

$$A = \left((x+\mu)^2 + y^2\right)^{3/2}, \quad B = \left((x-1+\mu)^2 + y^2\right)^{3/2}, \quad \mu = 0.012277471.$$

In this coordinate system, Earth is located at the point $(-\mu, 0)$, and the Moon at the point $(1-\mu, 0)$. It can be shown that this problem has an analytical solution with the following initial conditions:

$$x(0) = 0.994, \quad \dot{x}(0) = 0, \quad y(0) = 0, \quad \dot{y}(0) = -2.00158510637908252240537862224,$$

which is periodic, and its period is approximately:

$$T_0 = 17.0652165601579625588917206249.$$

## Task 1

Present approximate trajectories of the satellite on the described plane, obtained through approximations in time steps $t_k = kh$:

1. Using Euler's method with a step size $h = \frac{T_0}{24000}$, 24000 steps;
2. Using the 4th-order Runge-Kutta method with a step size $h = \frac{T_0}{6000}$, 6000 steps.

## Task 2

Implement the Dormand-Prince numerical scheme defined by the following Butcher tableau:

$$
\begin{array}{c|ccccccc}
0 & & & & & & & \\
\frac{1}{5} & \frac{1}{5} & & & & & & \\
\frac{3}{10} & \frac{3}{40} & \frac{9}{40} & & & & & \\
\frac{4}{5} & \frac{44}{45} & -\frac{56}{15} & \frac{32}{9} & & & & \\
\frac{8}{9} & \frac{19372}{6561} & -\frac{25360}{2187} & \frac{64448}{6561} & -\frac{212}{729} & & & \\
1 & \frac{9017}{3168} & -\frac{355}{33} & \frac{46732}{5247} & \frac{49}{176} & -\frac{5103}{18656} & & \\
1 & \frac{35}{384} & 0 & \frac{500}{1113} & \frac{125}{192} & -\frac{2187}{6784} & \frac{11}{84} & 0 \\
\hline
 & \frac{35}{384} & 0 & \frac{500}{1113} & \frac{125}{192} & -\frac{2187}{6784} & \frac{11}{84} & 0
\end{array}
$$

Test this numerical scheme on selected, simple examples from the exercises and laboratory. Experimentally investigate the order of this scheme.

# Task 3

Apply the Dormand-Prince scheme to the described problem. Test different values of the step size $h > 0$. For a step size $h$ chosen by yourself, draw the approximate trajectory of the satellite. Be sure to justify the choice of step size in your report.

# Task 4

Improve the Dormand-Prince scheme by implementing adaptive step size control. To do this, calculate an additional value:

$$\hat{x}_{k+1} := x_k + h\left(\frac{5179}{57600}K_1 + \frac{7571}{16695}K_3 + \frac{393}{640}K_4 - \frac{92097}{339200}K_5 + \frac{187}{2100}K_6 + \frac{1}{40}K_7\right),$$

which is used to modify the step size. Define:

$$\text{err} := |x_{k+1} - \hat{x}_{k+1}|, \quad \text{toll} := 10^{-4},$$

and

$$\gamma := \left(\frac{\text{toll}}{\text{err}}\right)^{\frac{1}{p+1}},$$

where $p$ is the set order of the Dormand-Prince scheme. The new step size is then $\gamma h$. Intuitively, the step size is reduced if the estimated error is greater than the specified tolerance or increased otherwise. For the initial step size chosen in Task 3, compare the improved scheme with the one applied in Task 3.

# Task 5

Using the solution obtained in Task 3, draw an approximate plot showing the distance of the satellite from Earth and the Moon as a function of time. Express the distance in kilometers (to compute the scaling factor, note that the distance between Earth and the Moon on this plane is 1; find the actual distance in appropriate sources).

# Task 6

In all three methods, compute the distance between the point $(x(0), y(0))$ and the point $(x(T_0), y(T_0))$, where $(x(t), y(t))$ is the approximate solution.

## Solution

- A concise report in PDF format, including a discussion of the solution methods for each task in the project, presentation of the obtained results (including the plots mentioned in the tasks), and their discussion.
- A set of commented source files.