Mathematical Exploration

Investigating complex functions

Contents

1 Introduction

When I first discovered complex numbers I was very confused but also interested. At the time they seemed to me like a very abstract idea. However, I later found out that they were used in different branches of science like engineerign and quantum mechanics and so I wanted to explore them further.In class we only found zeroes of complex polynomials with real coefficients. It was pretty straightforward. I wondered then why don't we consider functions with complex coefficients and powers. It can't be that hard, right? So I decided to go and ask Wolfram Alpha what the zeroes of the following function are equal to:

$f(z) = (a + bi)z^{(c+di)} + (f + gi) \quad a, b, c, d, f, g \in \mathbb{R} \setminus \{0\}, z \in \mathbb{C}$

And this is what I got:

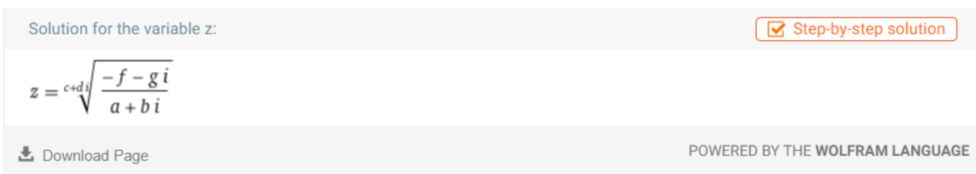

Figure 1: Wolfram Alpha's answer to my initial problem

That is not a sastisfactory answer. A complex number raised to a complex number is a very supreficial form of solution. After seeing that supposedly the most advanced computational intelligence on the internet cannot handle my function, I decided to obtain as much information about it as I can. The purpose of this exploration is to investingate the affromentioned function: find the general formula for all its zeroes and a way to effecitbely approximate them using computational technology.

2 Solving an example

For some people it is easier to understand mathematics on numbers, instead of variables, so first I will try to solve an example. The difficulties encountered during that process may also point towards where to focus when later generalising. Let $f(z) = (1 + \sqrt{3}i)z^{\left(\frac{1}{2+i}\right)} + (1 - \sqrt{3}i)$. I have choosen the exponent to be a fraction and coefficients to be conjugates to simplify further calculations.First we will set the funtiction to be equal to zero and rearrange to find an expression for z.

$$(1 + \sqrt{3}i)z^{\left(\frac{1}{2+i}\right)} + (1 - \sqrt{3}i) = 0$$
$$z^{\left(\frac{1}{2+i}\right)} = -\frac{1+\sqrt{3}i}{1-\sqrt{3}i}$$

Then I will rationalise the fraction.

$$z^{\left(\frac{1}{2+i}\right)} = -\frac{(1-\sqrt{3}i)^2}{1+3} = -\frac{1-3-2\sqrt{3}i}{1+3} = -\frac{-2(1+\sqrt{3}i)}{4} = \frac{1+\sqrt{3}i}{2}$$

Next I find an expression for z.

$$z = (\frac{1+\sqrt{3}i}{2})^{(2+i)}$$

I want to raise a complex number to a complex number. I have no idea how to do that in cartesian form. However if I change the base of the expression for z to Euler's form. This way I can distribiute the power to which it is raised to the number itself, and then convert once again into cartesian form. But first consider the Euler's formula.

$$e^{i\pi} = -1$$

square it

$e^{2i\pi} = 1$ raise it to some number n hence

$$(e^{2i\pi})^n = 1^n$$

Now in order to get the set of possible values of n we transform this into trigonometric form.

$$\cos(2\pi n) + i\sin(2\pi n) = 1^n$$

1 raised to any real n is 1, so we can omit that n. We can see that for this equation to be true sine must be equal to zero and cosine must be equal to one. This is possible only for integer values of n hence $n \in \mathbb{Z}$. This way we can multiply every complex number by $e^{2k\pi i}, k \in \mathbb{Z}$, at the same time not changing its value. However it will matter while exponentiating. Hence:

$$z = (\frac{1+\sqrt{3}i}{2})^{(2+i)} = \left(re^{i(\theta+2k\pi)}\right)^{(2+i)}, k \in \mathbb{Z}$$

$$r = \sqrt{(\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} = 1$$

$$\theta = \arctan(\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}) = \arctan\left(\frac{\sqrt{3}}{2} \times 2\right) = \arctan\sqrt{3} = \frac{\pi}{3}$$

Here it is important to mention that using simple arctan function does not always work properly to determine the argument of the number. Let's take the additive inverse of our original number as an example: $\frac{-1-\sqrt{3}i}{2}$, it would give the same result, beacause the minus signs would cancel out. This is obviously not true as the final arm of the argument of this number lies in the 3rd quarter of the Argand diagram and the argument of $\frac{+1+\sqrt{3}i}{2}$ lies in the first. The following Argand diagram from Geo Gebra potrays this phenomenon ($\sqrt{3}$ was approximated as 1.73 by GeoGebra).

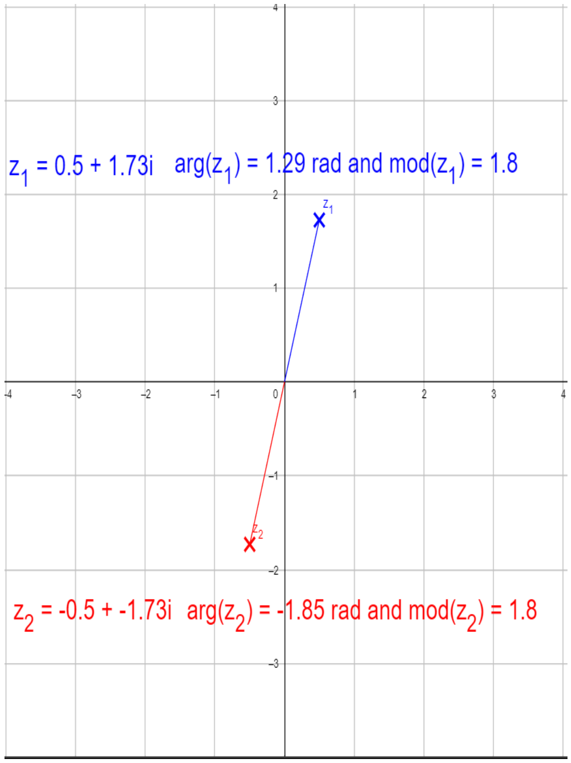

Figure 2: Geo Gebra Argand Diagram showing aforementioned phenomenon ($\sqrt{3}$ was approximated as 1.73)

As one can see the angles differ by π radians. It will be important to keep in mind while generalising.

Going further and hence substituting r and θ

$$z = \left(1e^{i(\frac{\pi}{3}+2k\pi)}\right)^{(2+i)}$$

I decided to leave 1 here as I am not sure what is 1 raised to a complex number. My intuition tells me that it is irrelevant and 1 raised to any number is 1, however I will leave it for now to chceck whether it matters. Now I will tranform this expression to obtain a single complex number.

$$z = \left(1e^{i(\frac{\pi}{3}+2k\pi)}\right)^{(2+i)} = 1^{2+i} \times e^{(\frac{2\pi}{3}+4k\pi)i-(\frac{\pi}{3}+2k\pi)}$$

Notice that $1 = e^{ln1} = e^0$ hence $1^{2+i} = e^{0(2+i)} = e^0 = 1$

One raised to a complex number is still one. However, if the modulus of the base of our expression was not 1 this would give us another complex number to multiply our expression by. This is important information to consider in the future generalisation.

Coming back to our example that means that

$$z = 1^{2+i} \times e^{(\frac{2\pi}{3}i-\frac{\pi}{3})} = e^{(\frac{2\pi}{3}+4k\pi)i-(\frac{\pi}{3}+2k\pi)} = e^{-\frac{\pi}{3}+2k\pi} \times e^{(\frac{2\pi}{3}+4k\pi)i}$$

It could be left in this form as an answer, however I will express it in terms of trigonometric functions and account for their periodicity to obtain generall solution or if possible in cartesian form as it seems as more intuitive interpretation of geometry.

$$z = e^{-\frac{\pi}{3}+2k\pi}(\cos(\frac{2\pi}{3} + 4k\pi) + i\sin(\frac{2\pi}{3} + 4k\pi)) = e^{-\frac{\pi}{3}+2k\pi}(-\frac{1}{2} + \frac{\sqrt{3}}{2}i)$$

Interestingly we found a family of solutions, that significantly varies from each other. The $2k\pi$ causes the expression to grow exponentialy, which was not expected. When solving equations we are used to solutions being very close to one another, but in this case, with increasing values of k the solutions are more and more distant from each other. That is quite a nice looking answer, however the generall solution is probably more complicated. This exploration of an example helped to showcase what aspects I will have to take into account when generalising the formula for the zero of this type of functions. I will have to consider different possibilities for arguments and consider rising modulus of a complex number to a complex power.

3 Generalisation

Let $f(z) = (a + bi)z^{(c+di)} + (f + gi)$ where $a, b, c, d, f, g \in \mathbb{R} \setminus \{0\}, z \in \mathbb{C}$

Thus to find a zero of this function equation was created and rearranged to express z in terms of coefficients.

$$(a + bi)z^{(c+di)} + (f + gi) = 0$$

$$z^{(c+di)} = -\frac{f+gi}{a+bi}$$

$$z = (-\frac{f+gi}{a+bi})^{\frac{1}{c+di}} \quad (1)$$

3.1 Rising a complex number to a complex number

To raise a complex number to a complex number and obtain a solution this equation, I will use Euler's formula and the properties of natural logarithm. For the sake of clarity first consider the expression:

$(x + yi)^{k+li}$ (2) $x, y, k, l \in \mathbb{R}$ in polar form,

then calculating the magnitude and argument of x+yi

$$r = \sqrt{x^2 + y^2} \ , \theta = \arg(x + yi)$$

we can use the Euler's formula to obtain the following equality

$$(x+yi)^{k+li} = (re^{i\theta})^k \times (re^{i\theta})^{li}$$

$$(x+yi)^{k+li} = r^k e^{i\theta k} \times r^{li} e^{-l\theta}$$

Then using the property of nautral logarithm that:

$$r = e^{ln(r)}$$

we obtain

$$(x+yi)^{k+li} = r^k e^{i\theta k} \times e^{ln(r)li} e^{-l\theta}$$

$$(x+yi)^{k+li} = r^k e^{-l\theta} e^{i(k\theta + l \times ln(r))}$$

using $e^{i\alpha} = cos(\alpha) + isin(\alpha)$

$$(x+yi)^{k+li} = r^k e^{-l\theta}(\cos(k\theta + l \times ln(r)) + i\sin(k\theta + l \times ln(r))) \quad (3)$$

Which eliminates complex numbers from the exponent effectively rising a complex number to a complex number.

3.2 Substitiuting the values to obtain a solution

Now we can combine expressions (1) and (2) and (3) to get a solution. I will transform the expresion (1) to (3) through expression (2).

................(1)...........................(2)..(3)..

$$z = (-\frac{f+gi}{a+bi})^{\frac{1}{c+di}} = (x+yi)^{k+li} = r^k e^{-l\theta}(\cos(k\theta + l \times ln(r)) + i\sin(k\theta + l \times ln(r))) \quad (4)$$

3.2.1 Calculating r

From (4) we conclude that $x + yi = -\frac{f+gi}{a+bi}$

To find real and imaginary parts I multiply numerator and the denominator by conjugate of the denominator $-\frac{f+gi}{a+bi} \times \frac{a-bi}{a-bi} = -\frac{gb+fa+(ga-fb)i}{a^2+b^2} = (-\frac{fa+bg}{a^2+b^2}) + (\frac{fb-ga}{a^2+b^2})i$ thus

$$x = (-\frac{fa+bg}{a^2+b^2}), y = (\frac{fb-ga}{a^2+b^2}) \quad (5)$$

hence I find r from (3)

$$r = \sqrt{x^2+y^2} = \sqrt{(-\frac{fa+bg}{a^2+b^2})^2 + (\frac{fb-ga}{a^2+b^2})^2}$$

$$r = \sqrt{\frac{f^2a^2+b^2g^2+f^2b^2+g^2a^2+2fabg-2fabg}{(a^2+b^2)^2}}$$

$$r = \sqrt{\frac{f^2(a^2+b^2)+g^2(a^2+b^2)}{(a^2+b^2)^2}}$$

$$r = \sqrt{\frac{f^2+g^2}{a^2+b^2}} \quad (A)$$

3.2.2 Calculating θ

$\theta = \arg(x + yi)$ hence

$$\theta = \begin{cases} \arctan(\frac{y}{x}) + 2k\pi & \text{when } x > 0 \\ \arctan(\frac{y}{x}) + \pi + 2k\pi & \text{when } x < 0, y \geq 0 \\ \arctan(\frac{y}{x}) - \pi + 2k\pi & \text{when } x < 0, y \leq 0 \end{cases}$$
$$k \in \mathbb{Z}$$

This is portrayed by this graph. Different values of k indicate different branches of this function.

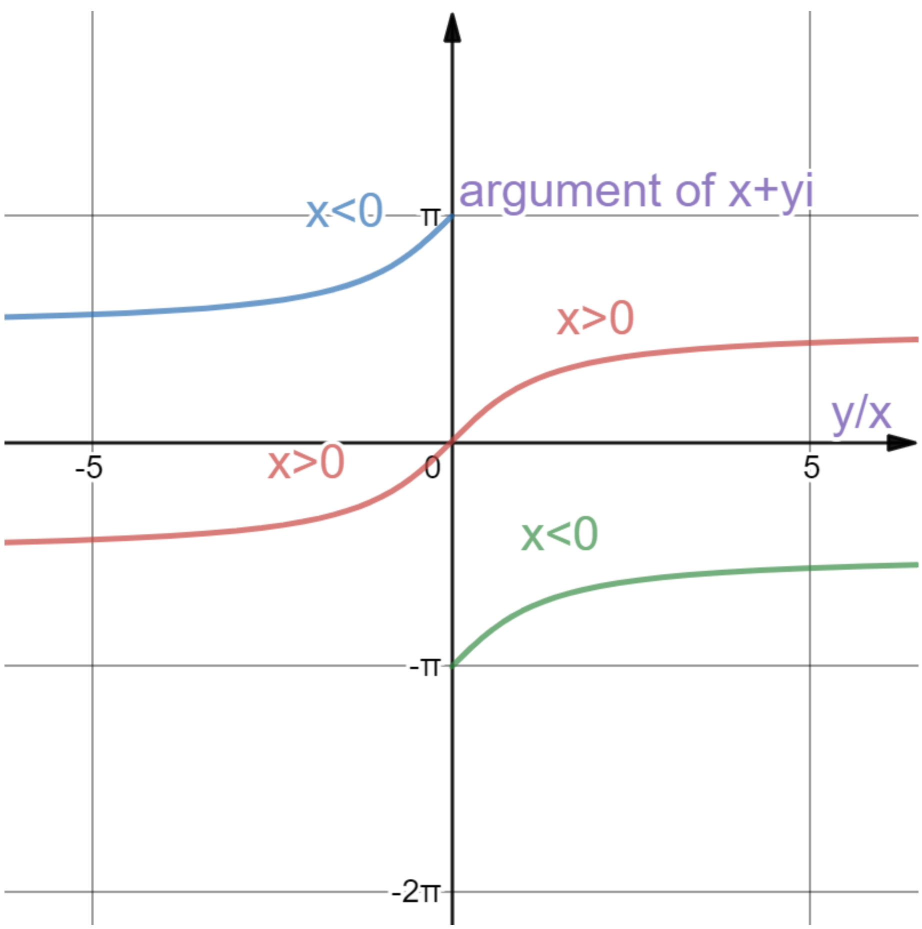

Figure 3: Grapg of the function $\theta = f(x, y)$, which represents the argument of the complex number based on its real (x) and imaginary (y) part

It was previously established at (5) that $x = (-\frac{fa+bg}{a^2+b^2}), y = (\frac{fb-ga}{a^2+b^2})$ Now substituting x and y we obtain:

$$\frac{y}{x} = \frac{\frac{fb-ga}{a^2+b^2}}{-\frac{fa+bg}{a^2+b^2}} = \frac{fb-ga}{a^2+b^2} \times (-\frac{a^2+b^2}{fa+bg})$$

$$\frac{y}{x} = \frac{ga-fb}{fa+bg}$$

I have distributed the minus sign from the denominator (x) to the numerator (y), hence I have to add or subtract π from every θ to account for change in the sign of denominator and numerator. This can be seen on a graph below.

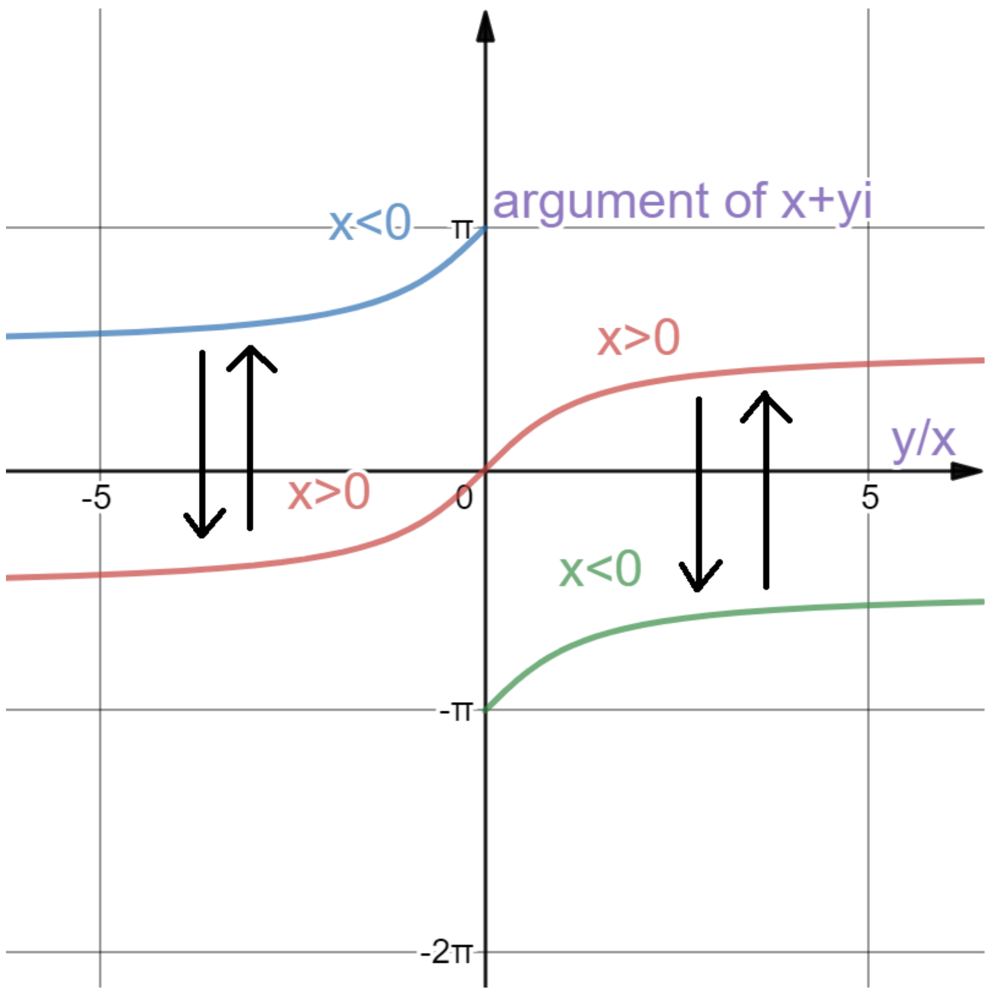

Figure 4: Same graph as in figure 3, but with arrows showing how the branches of the arctan function change after redistributing the minus sign.

However, this function have different branches, so instead of determining different conditions and where to add or where to substract π I can just add π to account for change in sign. This way a part of the function will just go to the higher branch. This does not matter as $k \in \mathbb{Z}$

$$\theta = \begin{cases} \arctan(\frac{fb-ga}{fa+bg}) + \pi + 2k\pi & \text{when } fa > -bg \\ \arctan(\frac{fb-ga}{fa+bg}) + 2\pi + 2k\pi & \text{when } fa < -bg, fb \geq ga \\ \arctan(\frac{fb-ga}{fa+bg}) + 2k\pi & \text{when } fa < -bg, fb < ga \end{cases} \quad (B)$$

$k \in \mathbb{Z}$

3.2.3 Calculating the value of exponent

I transform the exponent from (1) into another form

$\frac{1}{c+di} = \frac{1}{c+di} \times \frac{c-di}{c-di} = \frac{c}{c^2+d^2} + \frac{-di}{c^2+d^2}$

Then from (4) it can be seen that

$$k = \frac{c}{c^2+d^2}, l = \frac{-d}{c^2+d^2} \quad (C)$$

3.3 Combining the results

Combining all the results we obtain the general result for the equation:

$$(a+bi)z^{(c+di)} + (f+gi) = 0 \quad a, b, c, d, f, g \in \mathbb{R} \setminus \{0\}, z \in \mathbb{C}$$

From (4) it is known that $z = r^k e^{-l\theta}(\cos(k\theta + l \times ln(r)) + i\sin(k\theta + l \times ln(r)))$ Now substituting r

from (A) and k and r from (C) I obtain

z=$(\sqrt{\frac{f^2+g^2}{a^2+b^2}})^{\frac{c}{c^2+d^2}} \times e^{-(\frac{-d}{c^2+d^2})\theta} \times ([\cos(\frac{c}{c^2+d^2})\theta + (\frac{-d}{c^2+d^2})\ln(\sqrt{\frac{f^2+g^2}{a^2+b^2}})] + [i\sin(\frac{c}{c^2+d^2})\theta + (\frac{-d}{c^2+d^2})\ln(\sqrt{\frac{f^2+g^2}{a^2+b^2}})])$

And accounting for θ from (B), we obtain the general solution:

$$z=\begin{cases}(\sqrt{\frac{f^2+g^2}{a^2+b^2}})^{\frac{c}{c^2+d^2}}\times e^{-(\frac{-d}{c^2+d^2})(\arctan(\frac{fb-ga}{fa+bg})+\pi+2k\pi)}\times \\ (\cos[(\frac{c}{c^2+d^2})(\arctan(\frac{fb-ga}{fa+bg})+\pi+2k\pi)+(\frac{-d}{c^2+d^2})\ln(\sqrt{\frac{f^2+g^2}{a^2+b^2}})] \\ +i\sin[(\frac{c}{c^2+d^2})(\arctan(\frac{fb-ga}{fa+bg})+\pi+2k\pi)+(\frac{-d}{c^2+d^2})\ln(\sqrt{\frac{f^2+g^2}{a^2+b^2}})]) & \text{when } fa>-bg \\ \\ (\sqrt{\frac{f^2+g^2}{a^2+b^2}})^{\frac{c}{c^2+d^2}}\times e^{-(\frac{-d}{c^2+d^2})(\arctan(\frac{fb-ga}{fa+bg})+2\pi+2k\pi)}\times \\ (\cos[(\frac{c}{c^2+d^2})(\arctan(\frac{fb-ga}{fa+bg})+2\pi+2k\pi)+(\frac{-d}{c^2+d^2})\ln(\sqrt{\frac{f^2+g^2}{a^2+b^2}})] \\ +i\sin[(\frac{c}{c^2+d^2})(\arctan(\frac{fb-ga}{fa+bg})+2\pi+2k\pi)+(\frac{-d}{c^2+d^2})\ln(\sqrt{\frac{f^2+g^2}{a^2+b^2}})]) & \text{when } fa<-bg, fb\geq ga \\ \\ (\sqrt{\frac{f^2+g^2}{a^2+b^2}})^{\frac{c}{c^2+d^2}}\times e^{-(\frac{-d}{c^2+d^2})(\arctan(\frac{fb-ga}{fa+bg})+2k\pi)}\times \\ (\cos[(\frac{c}{c^2+d^2})(\arctan(\frac{fb-ga}{fa+bg})+2k\pi)+(\frac{-d}{c^2+d^2})\ln(\sqrt{\frac{f^2+g^2}{a^2+b^2}})] \\ +i\sin[(\frac{c}{c^2+d^2})(\arctan(\frac{fb-ga}{fa+bg})+2k\pi)+(\frac{-d}{c^2+d^2})\ln(\sqrt{\frac{f^2+g^2}{a^2+b^2}})]) & \text{when } fa<-bg, fb<ga\end{cases}$$

$$k\in\mathbb{Z}$$

4 Approximating the solution using computational technology

4.1 Basic examples

The above result is quite difficult to calculate by hand and I found no way to check it, so naturally I decided to write a piecie of code in R, that would automatically evaluate zeroes from the above formula given the coefficients of the function. Of course, it is not able to calculate exact values, but the margin of error with relatively small numbers should be minute enough, to show whether my formula is correct.The code can be found in appendix (Appendix A). What the code does is it calculates the approximate zero of a function and then substitutes it to evaluate it at supposed zero. Below is the table with examples of input and output. I will present five of my examples.

Table 1: Table with function, its approximated zero calculated by my algorithm and error term created by inputting this approximated zero into the function

	$f(z)$	$\sim z_0$	$Error\ term\ f(\sim z_0)$
1	$(1+i)z^{1+i}+(1+i)$	$0+4.810477i$	$-2.220446\times10^{-16}+1.110223i\times10^{-16}$
2	$(7+2i)z^{5+8i}+(13+9i)$	$1.415559+0.178107i$	$-7.10543\times10^{-15}+1.421085i\times10^{-14}$
3	$(2.25+70i)z^{3-8i}+(15-10i)$	$0.8413624-0.0903828i$	$5.329071\times10^{-15}-8.881784i\times10^{-15}$
4	$(0.1+999i)z^{-0.99+0.01i}+(0.01-50i)$	$0.2654524-0.1627231i$	$3.716645\times10^{-14}+3.552714i\times10^{-14}$
5	$(1-i)z^{2-3i}+5-8i$	$0.4320607+0.5294815i$	$0-7.105427i\times10^{-15}$

As presented on the table above, it seems that my solution is correct. The zeroses are exact enough to produce error rainging only from 10^{-16} to 10^{-14}.

4.2 Verifying accuracy with broader range of coefficients

However, five examples is a small amount to confirm that, so I generated 100000 of them and plottet the error in terms of magnitude on the graph. I generated the coefficients of functions using uniform distribution in range from -10000 to 10000:

$$(a, b, c, d, f, g) \sim U(-10000, 10000)$$

This is the result:

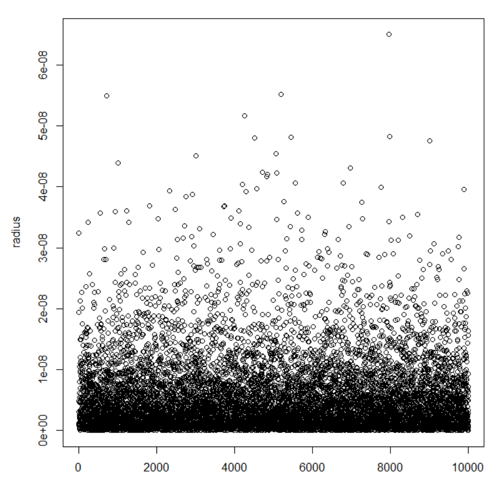

Figure 5: Error of approximated zeroes in terms of distance (magnitude) from the proper zero.

As can be seen the error in terms of magnitude is usually below 1×10^{-8} and rarely does it go beyond the boundary of 4×10^{-8}. Hence it is quite a good approximation, at least in the of variables used in the distribution.

4.3 Verifying accuracy for different values of k

I have yet to check the error for different values of k. I decided to run write a programme that generates one example from an uniform distribution in range from -10 to 10:

$$(a, b, c, d, f, g) \sim U(-10, 10)$$

and then it adds values of k up to 1000. Here are the results:

```
[1]   -4.440892e-15+ 5.329071e-15i   1.154632e-14+ 8.881784e-15i   1.776357e-15- 4.440892e-15i   1.776357e-15- 3.108624e-15i
[5]   -4.615118e+09- 3.291681e+09i  -4.615118e+09- 3.291681e+09i  -4.615118e+09- 3.291681e+09i  -4.615118e+09- 3.291681e+09i
[9]   -4.615118e+09- 3.291681e+09i  -4.615118e+09- 3.291681e+09i  -4.615118e+09- 3.291681e+09i  -4.615118e+09- 3.291681e+09i
[13]  -4.615118e+09- 3.291681e+09i  -4.615118e+09- 3.291681e+09i  -4.615118e+09- 3.291681e+09i  -4.112331e+18- 4.546411e+18i
```

Figure 6: First 16 zeroes starting at k=0 with increasing values of k by 1

```
[349] -8.140845e+306-7.462607e+307i -8.140845e+306-7.462607e+307i -8.140845e+306-7.462607e+307i          NaN-          Infi
```

Figure 7: 349th to 352nd zeroes starting at k=0 with increasing vaues of k by 1

The first 4 values of k yield reasonable approximation, however the error starts to grow exponentially. With 5th example it is in millions and then it starts to grow to infinity. The programme treats error above k=352 as infinity as seen on the same example below.

So why is the error so huge? Is there a mistake in calculations, the code or the conditions for a zero to exist? There is an explanation. The exponential growth of it provides a clue where to look for expaination. In the solution there appears an expression $e^{(-\frac{-d}{c^2+d^2})\theta}$. As $2k\pi$ is a part of θ, therefore this expression will grow really fast, which may cause many inaccuracies in machine approximation. Below is the graph of the function $f(k) = e^{2k\pi}$.

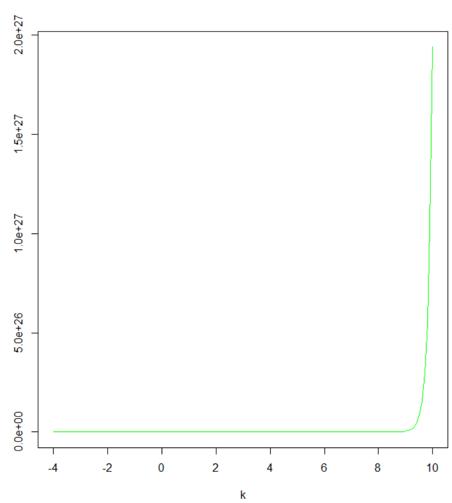

Figure 8: Graph of $f(k) = e^{2k\pi}, k \in [-4, 10]$

This expains the margin of error, as computers operate on approximations of non-rational numbers. The numbers themselves become too big. In the graph it can be seen that already for $k = 10$ the function has a value of about 2×10^{27}. This combined with the operations of exponentiation and multiplication in checking the solutions provides such a huge error. Maybe this is the reason why WolframAlpha could not find the zeroes in the first place, bringing my attention to this problem.

I have checked what happens when the k increases, now I will check what error does the approximation produce while it decreases. Below are the result.

```
 [1]  -2.664535e-15+ 3.019807e-14i   7.105427e-15- 8.881784e-15i   8.881784e-16- 1.776357e-15i   9.769963e-15- 3.552714e-15i
 [5]   1.065814e-14- 1.776357e-15i   1.953993e-14+ 7.460699e-14i   1.421085e-14+ 1.776357e-15i   4.973799e-14+ 6.128431e-14i
 [9]   1.598721e-14+ 5.329071e-15i   2.309264e-14+ 8.171241e-14i   1.776357e-14+ 8.881784e-15i  -6.144231e+27+ 7.983133e+27i
[13]  -6.144231e+27+ 7.983133e+27i  -6.144231e+27+ 7.983133e+27i  -6.144231e+27+ 7.983133e+27i  -6.144231e+27+ 7.983133e+27i
[17]  -6.144231e+27+ 7.983133e+27i  -6.144231e+27+ 7.983133e+27i  -6.144231e+27+ 7.983133e+27i  -6.144231e+27+ 7.983133e+27i
```

Figure 9: Zeroes of the fucntion starting at k=0 with decreasing values of k by 1

The approximation holds up to k=-11, but then again starts increasing up to infinity, probably for the same reason as with positive values of k, instead just now the number becames so small.

5 Conclusion

Throughout this paper I have explored the function and its zeroes that WolframAlpha is yet to fully comprehend. While considering other functions with periodic solutions, Wolphram is able to find periodic anser to those solutions. Why does it fail in this case? I have found a probable reason for that, which is how quickly the values of infinitely many zeroes like these grow. It implies a surprising statement, that computers and computational technology may not always be able to produce as accurate results as we would want to. It points out to its limitations while operating on irrational numbers. I myself was able only to find a couple of approximations using my programming skills. At the same time the exact solution is highly complicated, making calculations impractical. There are many problems in mathematics, for example Navier-Stokes problem, where the exact solution is not known, however approximations for certain conditions may be calculated. But the problem in this paper shows, that sometimes even the approximation may be too difficult to compute. We may find problems in mathematics, which are nearly impossible to solve, and extremely difficult to approximate. It sparks a question about the extent our technology is able to effectively help our creativity and problem solving skills.

This paper has made me realise couple of thing about mathematics. The complexity of the issues I dealt with have made me realise the importance of tidiness and uniformity of mathematical notation. I have experienced the difficulty of expressing complex mathematical ideas accurately. However I am glad that I undertook this challenge. It let me combine my computer science skills and passions for different branches of mathematics: complex numbers and statistics. While I in the future I will rather use applied mathematics it was important to me to experience it in its purer form. I also hope, after I finish school I can contact WolframAlpha team, and maybe I will interest them in my findings, so they can improve their platform.

6 Appendix

```
  [1]  -4.440892e-15+ 5.329071e-15i   1.154632e-14+ 8.881784e-15i   1.776357e-15- 4.440892e-15i   1.776357e-15- 3.108624e-15i
  [5]  -4.615118e+09- 3.291681e+09i  -4.615118e+09- 3.291681e+09i  -4.615118e+09- 3.291681e+09i  -4.615118e+09- 3.291681e+09i
  [9]  -4.615118e+09- 3.291681e+09i  -4.615118e+09- 3.291681e+09i  -4.615118e+09- 3.291681e+09i  -4.615118e+09- 3.291681e+09i
 [13]  -4.615118e+09- 3.291681e+09i  -4.615118e+09- 3.291681e+09i  -4.615118e+09- 3.291681e+09i  -4.112331e+18- 4.546411e+18i
 [17]  -4.112331e+18- 4.546411e+18i  -4.112331e+18- 4.546411e+18i  -4.112331e+18- 4.546411e+18i  -4.112331e+18- 4.546411e+18i
 [21]  -4.112331e+18- 4.546411e+18i  -4.112331e+18- 4.546411e+18i  -4.112331e+18- 4.546411e+18i  -4.112331e+18- 4.546411e+18i
 [25]  -4.112331e+18- 4.546411e+18i  -3.290500e+27- 5.755305e+27i  -3.290500e+27- 5.755305e+27i  -3.290500e+27- 5.755305e+27i
 [29]  -3.290500e+27- 5.755305e+27i  -3.290500e+27- 5.755305e+27i  -3.290500e+27- 5.755305e+27i  -3.290500e+27- 5.755305e+27i
 [33]  -3.290500e+27- 5.755305e+27i  -3.290500e+27- 5.755305e+27i  -3.290500e+27- 5.755305e+27i  -2.142277e+36- 6.841860e+36i
 [37]  -2.142277e+36- 6.841860e+36i  -2.142277e+36- 6.841860e+36i  -2.142277e+36- 6.841860e+36i  -2.142277e+36- 6.841860e+36i
 [41]  -2.142277e+36- 6.841860e+36i  -2.142277e+36- 6.841860e+36i  -2.142277e+36- 6.841860e+36i  -2.142277e+36- 6.841860e+36i
 [45]  -2.142277e+36- 6.841860e+36i  -6.776279e+44- 7.723557e+45i  -6.776279e+44- 7.723557e+45i  -6.776279e+44- 7.723557e+45i
 [49]  -6.776279e+44- 7.723557e+45i  -6.776279e+44- 7.723557e+45i  -6.776279e+44- 7.723557e+45i  -6.776279e+44- 7.723557e+45i
 [53]  -6.776279e+44- 7.723557e+45i  -6.776279e+44- 7.723557e+45i  -6.776279e+44- 7.723557e+45i   1.073801e+54- 8.315542e+54i
 [57]   1.073801e+54- 8.315542e+54i   1.073801e+54- 8.315542e+54i   1.073801e+54- 8.315542e+54i   1.073801e+54- 8.315542e+54i
 [61]   1.073801e+54- 8.315542e+54i   1.073801e+54- 8.315542e+54i   1.073801e+54- 8.315542e+54i   1.073801e+54- 8.315542e+54i
 [65]   1.073801e+54- 8.315542e+54i   3.061034e+63- 8.535051e+63i   3.061034e+63- 8.535051e+63i   3.061034e+63- 8.535051e+63i
 [69]   3.061034e+63- 8.535051e+63i   3.061034e+63- 8.535051e+63i   3.061034e+63- 8.535051e+63i   3.061034e+63- 8.535051e+63i
 [73]   3.061034e+63- 8.535051e+63i   3.061034e+63- 8.535051e+63i   3.061034e+63- 8.535051e+63i   3.061034e+63- 8.535051e+63i
 [77]   5.211045e+72- 8.306468e+72i   5.211045e+72- 8.306468e+72i   5.211045e+72- 8.306468e+72i   5.211045e+72- 8.306468e+72i
 [81]   5.211045e+72- 8.306468e+72i   5.211045e+72- 8.306468e+72i   5.211045e+72- 8.306468e+72i   5.211045e+72- 8.306468e+72i
 [85]   5.211045e+72- 8.306468e+72i   5.211045e+72- 8.306468e+72i   7.429176e+81- 7.566843e+81i   7.429176e+81- 7.566843e+81i
 [89]   7.429176e+81- 7.566843e+81i   7.429176e+81- 7.566843e+81i   7.429176e+81- 7.566843e+81i   7.429176e+81- 7.566843e+81i
 [93]   7.429176e+81- 7.566843e+81i   7.429176e+81- 7.566843e+81i   7.429176e+81- 7.566843e+81i   7.429176e+81- 7.566843e+81i
 [97]   9.600852e+90- 6.271610e+90i   9.600852e+90- 6.271610e+90i   9.600852e+90- 6.271610e+90i   9.600852e+90- 6.271610e+90i
[101]   9.600852e+90- 6.271610e+90i   9.600852e+90- 6.271610e+90i   9.600852e+90- 6.271610e+90i   9.600852e+90- 6.271610e+90i
[105]   9.600852e+90- 6.271610e+90i   9.600852e+90- 6.271610e+90i  1.159472e+100- 4.400242e+99i  1.159472e+100- 4.400242e+99i
[109]  1.159472e+100- 4.400242e+99i  1.159472e+100- 4.400242e+99i  1.159472e+100- 4.400242e+99i  1.159472e+100- 4.400242e+99i
[113]  1.159472e+100- 4.400242e+99i  1.159472e+100- 4.400242e+99i  1.159472e+100- 4.400242e+99i  1.159472e+100- 4.400242e+99i
[117]  1.326727e+109-1.961492e+108i  1.326727e+109-1.961492e+108i  1.326727e+109-1.961492e+108i  1.326727e+109-1.961492e+108i
[121]  1.326727e+109-1.961492e+108i  1.326727e+109-1.961492e+108i  1.326727e+109-1.961492e+108i  1.326727e+109-1.961492e+108i
[125]  1.326727e+109-1.961492e+108i  1.326727e+109-1.961492e+108i  1.326727e+109-1.961492e+108i  1.446895e+118+1.002143e+117i
[129]  1.446895e+118+1.002143e+117i  1.446895e+118+1.002143e+117i  1.446895e+118+1.002143e+117i  1.446895e+118+1.002143e+117i
[133]  1.446895e+118+1.002143e+117i  1.446895e+118+1.002143e+117i  1.446895e+118+1.002143e+117i  1.446895e+118+1.002143e+117i
[137]  1.446895e+118+1.002143e+117i  1.505161e+127+4.411118e+126i  1.505161e+127+4.411118e+126i  1.505161e+127+4.411118e+126i
[141]  1.505161e+127+4.411118e+126i  1.505161e+127+4.411118e+126i  1.505161e+127+4.411118e+126i  1.505161e+127+4.411118e+126i
[145]  1.505161e+127+4.411118e+126i  1.505161e+127+4.411118e+126i  1.505161e+127+4.411118e+126i  1.487720e+136+8.147086e+135i
[149]  1.487720e+136+8.147086e+135i  1.487720e+136+8.147086e+135i  1.487720e+136+8.147086e+135i  1.487720e+136+8.147086e+135i
[153]  1.487720e+136+8.147086e+135i  1.487720e+136+8.147086e+135i  1.487720e+136+8.147086e+135i  1.487720e+136+8.147086e+135i
[157]  1.487720e+136+8.147086e+135i  1.382733e+145+1.205304e+145i  1.382733e+145+1.205304e+145i  1.382733e+145+1.205304e+145i
[161]  1.382733e+145+1.205304e+145i  1.382733e+145+1.205304e+145i  1.382733e+145+1.205304e+145i  1.382733e+145+1.205304e+145i
[165]  1.382733e+145+1.205304e+145i  1.382733e+145+1.205304e+145i  1.382733e+145+1.205304e+145i  1.181329e+154+1.593572e+154i
[169]  1.181329e+154+1.593572e+154i  1.181329e+154+1.593572e+154i  1.181329e+154+1.593572e+154i  1.181329e+154+1.593572e+154i
[173]  1.181329e+154+1.593572e+154i  1.181329e+154+1.593572e+154i  1.181329e+154+1.593572e+154i  1.181329e+154+1.593572e+154i
[177]  1.181329e+154+1.593572e+154i  1.181329e+154+1.593572e+154i  8.786199e+162+1.957039e+163i  8.786199e+162+1.957039e+163i
[181]  8.786199e+162+1.957039e+163i  8.786199e+162+1.957039e+163i  8.786199e+162+1.957039e+163i  8.786199e+162+1.957039e+163i
```

Figure 10

```
[349] -8.140845e+306-7.462607e+307i -8.140845e+306-7.462607e+307i -8.140845e+306-7.462607e+307i           NaN-         Infi
[353]           NaN-         Infi           NaN-         Infi           NaN-         Infi           NaN-         Infi
[357]           NaN-         Infi           NaN-         Infi           NaN-         Infi           NaN-         Infi
[361]           NaN-         Infi           Inf+         NaNi           Inf+         NaNi           Inf+         NaNi
[365]           Inf+         NaNi           Inf+         NaNi           Inf+         NaNi           Inf+         NaNi
[369]           Inf+         NaNi           Inf+         NaNi           Inf+         NaNi           Inf+         NaNi
[373]           Inf+         NaNi           Inf+         NaNi           Inf+         NaNi           Inf+         NaNi
[377]           Inf+         NaNi           Inf+         NaNi           Inf+         NaNi           Inf+         NaNi
[381]           Inf+         NaNi           Inf+         NaNi           Inf+         NaNi           Inf+         NaNi
[385]           Inf+         NaNi           Inf+         NaNi           Inf+         NaNi           Inf+         NaNi
[389]           Inf+         NaNi           Inf+         NaNi           Inf+         NaNi           Inf+         NaNi
[393]           Inf+         NaNi           Inf+         NaNi           Inf+         NaNi           Inf+         NaNi
[397]           Inf+         NaNi           Inf+         NaNi           Inf+         NaNi           Inf+         NaNi
[401]           Inf+         NaNi           Inf+         NaNi           Inf+         NaNi           Inf+         NaNi
[405]           Inf+         NaNi           Inf+         NaNi           Inf+         NaNi           Inf+         NaNi
[409]           Inf+         NaNi           Inf+         NaNi           Inf+         NaNi           Inf+         NaNi
[413]           Inf+         NaNi           Inf+         NaNi           Inf+         NaNi           Inf+         NaNi
[417]           Inf+         NaNi           Inf+         NaNi           Inf+         NaNi           Inf+         NaNi
[421]           Inf+         NaNi           Inf+         NaNi           Inf+         NaNi           Inf+         NaNi
[425]           Inf+         NaNi           Inf+         NaNi           Inf+         NaNi           Inf+         NaNi
[429]           Inf+         NaNi           Inf+         NaNi           Inf+         NaNi           Inf+         NaNi
[433]           Inf+         NaNi           Inf+         NaNi           Inf+         NaNi           Inf+         NaNi
[437]           Inf+         NaNi           Inf+         NaNi           Inf+         NaNi           Inf+         NaNi
[441]           Inf+         NaNi           Inf+         NaNi           NaN+         Infi           NaN+         Infi
[445]           NaN+         Infi           NaN+         Infi           NaN+         Infi           NaN+         Infi
```

Figure 11

```
 [1]  -2.664535e-15+ 3.019807e-14i   7.105427e-15- 8.881784e-15i   8.881784e-16- 1.776357e-15i   9.769963e-15- 3.552714e-15i
 [5]   1.065814e-14- 1.776357e-15i   1.953993e-14+ 7.460699e-14i   1.421085e-14+ 1.776357e-15i   4.973799e-14+ 6.128431e-14i
 [9]   1.598721e-14+ 5.329071e-15i   2.309264e-14+ 8.171241e-14i   1.776357e-14+ 8.881784e-15i  -6.144231e+27+ 7.983133e+27i
[13]  -6.144231e+27+ 7.983133e+27i  -6.144231e+27+ 7.983133e+27i  -6.144231e+27+ 7.983133e+27i  -6.144231e+27+ 7.983133e+27i
[17]  -6.144231e+27+ 7.983133e+27i  -6.144231e+27+ 7.983133e+27i  -6.144231e+27+ 7.983133e+27i  -6.144231e+27+ 7.983133e+27i
[21]  -6.144231e+27+ 7.983133e+27i  -6.144231e+27+ 7.983133e+27i  -6.144231e+27+ 7.983133e+27i  -6.144231e+27+ 7.983133e+27i
[25]  -6.144231e+27+ 7.983133e+27i  -6.144231e+27+ 7.983133e+27i  -6.144231e+27+ 7.983133e+27i  -6.144231e+27+ 7.983133e+27i
[29]  -6.144231e+27+ 7.983133e+27i  -6.144231e+27+ 7.983133e+27i  -6.144231e+27+ 7.983133e+27i   3.098402e+54- 1.038768e+55i
[33]   3.098402e+54- 1.038768e+55i   3.098402e+54- 1.038768e+55i   3.098402e+54- 1.038768e+55i   3.098402e+54- 1.038768e+55i
[37]   3.098402e+54- 1.038768e+55i   3.098402e+54- 1.038768e+55i   3.098402e+54- 1.038768e+55i   3.098402e+54- 1.038768e+55i
[41]   3.098402e+54- 1.038768e+55i   3.098402e+54- 1.038768e+55i   3.098402e+54- 1.038768e+55i   3.098402e+54- 1.038768e+55i
[45]   3.098402e+54- 1.038768e+55i   3.098402e+54- 1.038768e+55i   3.098402e+54- 1.038768e+55i   3.098402e+54- 1.038768e+55i
[49]   3.098402e+54- 1.038768e+55i   3.098402e+54- 1.038768e+55i   3.098402e+54- 1.038768e+55i   3.098402e+54- 1.038768e+55i
[53]   8.880746e+80+ 1.163043e+82i   8.880746e+80+ 1.163043e+82i   8.880746e+80+ 1.163043e+82i   8.880746e+80+ 1.163043e+82i
[57]   8.880746e+80+ 1.163043e+82i   8.880746e+80+ 1.163043e+82i   8.880746e+80+ 1.163043e+82i   8.880746e+80+ 1.163043e+82i
[61]   8.880746e+80+ 1.163043e+82i   8.880746e+80+ 1.163043e+82i   8.880746e+80+ 1.163043e+82i   8.880746e+80+ 1.163043e+82i
[65]   8.880746e+80+ 1.163043e+82i   8.880746e+80+ 1.163043e+82i   8.880746e+80+ 1.163043e+82i   8.880746e+80+ 1.163043e+82i
[69]   8.880746e+80+ 1.163043e+82i   8.880746e+80+ 1.163043e+82i   8.880746e+80+ 1.163043e+82i   8.880746e+80+ 1.163043e+82i
[73]   8.880746e+80+ 1.163043e+82i -5.372156e+108-1.134354e+109i -5.372156e+108-1.134354e+109i -5.372156e+108-1.134354e+109i
```

Figure 12