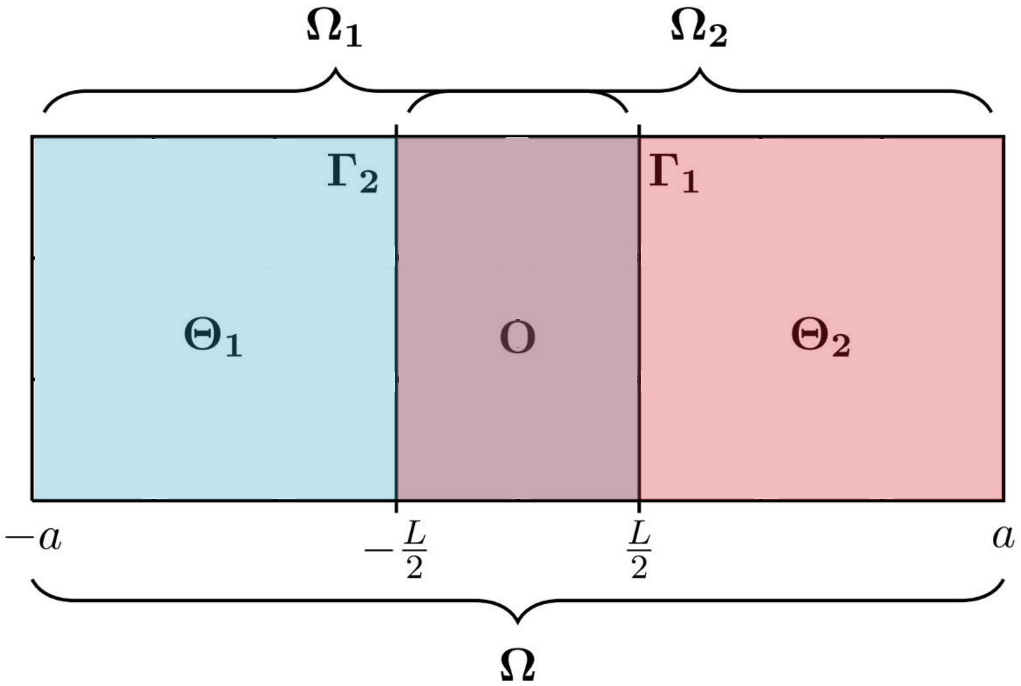
Schwarz methods, Schur complements, preconditioning and numerical linear algebra

Michal Outrata
supervised by Martin J. Gander

Outline

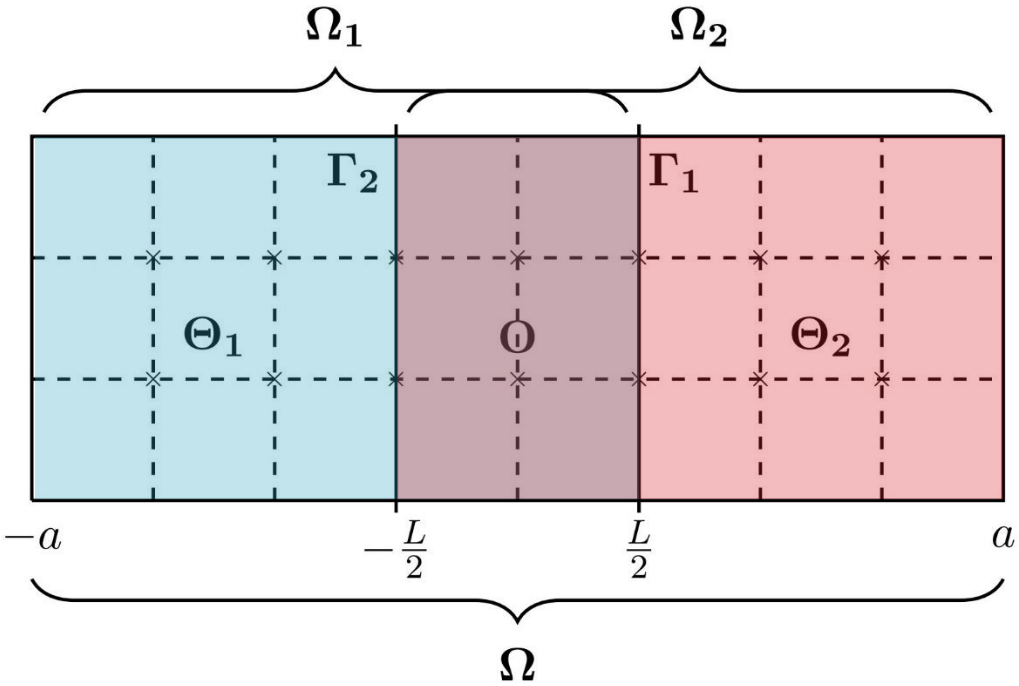
- Model problem and set-up
- Schwarz methods and DS
- Schwarz methods and ABC
- IRK preconditioners

Model problem



$$\begin{aligned} \Delta u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega \end{aligned}$$

Model problem

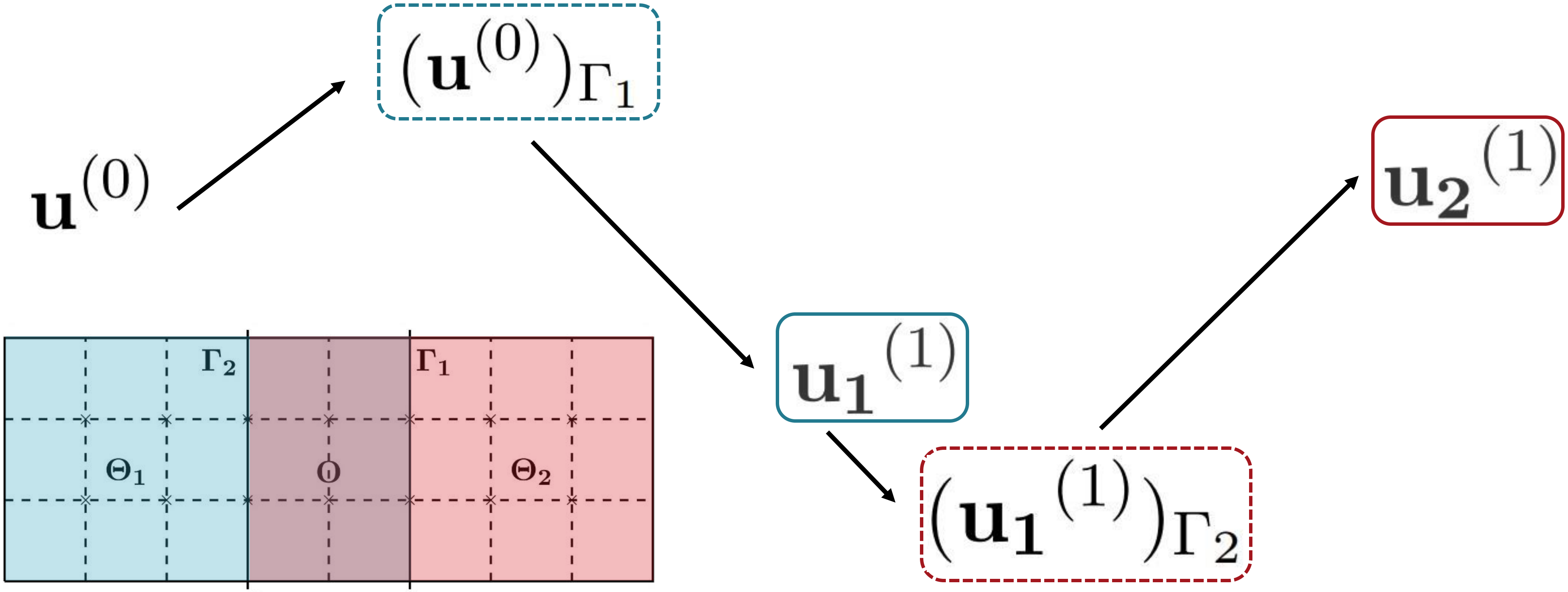


$$L\mathbf{u} = \mathbf{f}$$

blackboard (diag D)

Schwarz methods

Schwarz methods



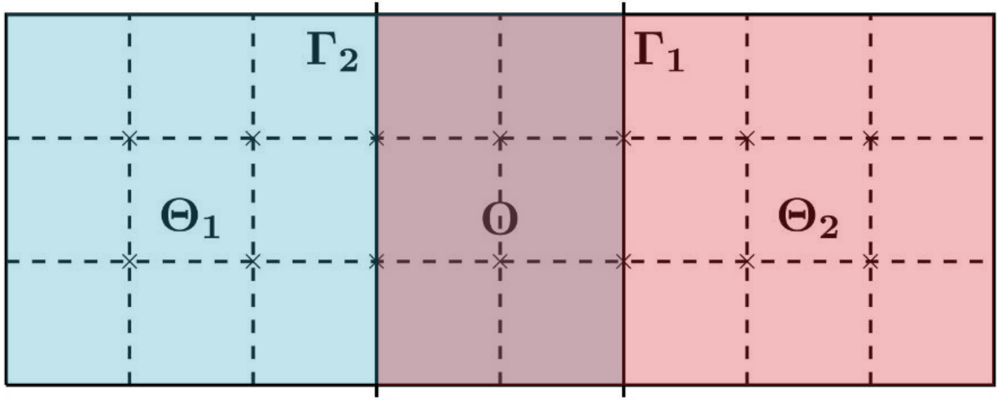
Schwarz methods

$$\frac{1}{h^2}\begin{bmatrix} D & I & & \\ I & \ddots & \ddots & \\ & \ddots & \ddots & I \\ & & I & D \end{bmatrix}\mathbf{u}_1^{(n)} = \mathbf{b}_1^{(n)}$$

$$\frac{1}{h^2}\begin{bmatrix} D & I & & \\ I & \ddots & \ddots & \\ & \ddots & \ddots & I \\ & & I & D \end{bmatrix}\mathbf{u}_2^{(n)} = \mathbf{b}_2^{(n)}$$

Optimal Schwarz methods

Optimal Schwarz methods



Optimal Schwarz methods

$$\frac{1}{h^2}\begin{bmatrix} D & I & & \\ I & \ddots & \ddots & \\ & \ddots & D & I \\ & & I & D - S_1^\star \end{bmatrix} \mathbf{u}_1^{(n)} = \mathbf{b}_1^{(n)}$$

$$\frac{1}{h^2}\begin{bmatrix} D - S_2^\star & I & & \\ I & \ddots & \ddots & \\ & \ddots & D & I \\ & & I & D \end{bmatrix} \mathbf{u}_2^{(n)} = \mathbf{b}_2^{(n)}$$

Optimal Schwarz methods

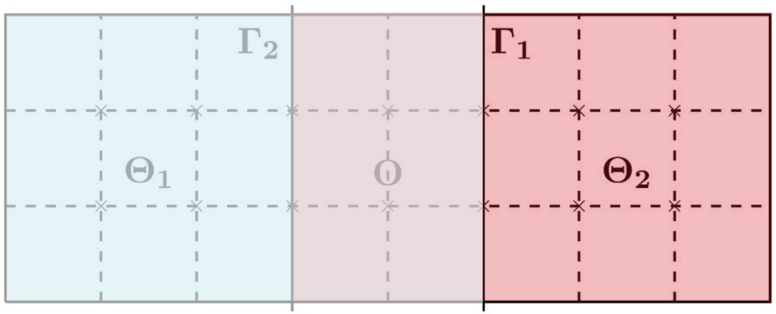
$$\frac{1}{h^2}\begin{bmatrix} D & I & & \\ I & \ddots & \ddots & \\ & \ddots & D & I \\ & & I & D - S_1^\star \end{bmatrix} \mathbf{u}_1^{(n)} = \mathbf{b}_1^{(n)}$$

$$\frac{1}{h^2}\begin{bmatrix} D - S_2^\star & I & & \\ I & \ddots & \ddots & \\ & \ddots & D & I \\ & & I & D \end{bmatrix} \mathbf{u}_2^{(n)} = \mathbf{b}_2^{(n)}$$

$$S_1^\star := E_{\Gamma_1}^T L_{\Theta_2}^{-1} E_{\Gamma_1}$$

Optimal Schwarz methods

$$S^\star := E_{\Gamma_1}^T L_{\Theta_2}^{-1} E_{\Gamma_1}$$



Optimized Schwarz methods

Optimized Schwarz methods

$$\frac{1}{h^2}\begin{bmatrix} D & I & & \\ I & \ddots & \ddots & \\ & \ddots & D & I \\ & & I & D - S_1 \end{bmatrix}\mathbf{u}_1^{(n)} = \mathbf{b}_1^{(n)}$$

$$\frac{1}{h^2}\begin{bmatrix} D - S_2 & I & & \\ I & \ddots & \ddots & \\ & \ddots & D & I \\ & & I & D \end{bmatrix}\mathbf{u}_2^{(n)} = \mathbf{b}_2^{(n)}$$

$$S^\star \to S$$

Optimized Schwarz methods

$$\frac{1}{h^2}\begin{bmatrix} D & I & & \\ I & \ddots & \ddots & \\ & \ddots & D & I \\ & & I & D - S_1 \end{bmatrix}\mathbf{u}_1^{(n)} = \mathbf{b}_1^{(n)}$$

$$\frac{1}{h^2}\begin{bmatrix} D - S_2 & I & & \\ I & \ddots & \ddots & \\ & \ddots & D & I \\ & & I & D \end{bmatrix}\mathbf{u}_2^{(n)} = \mathbf{b}_2^{(n)}$$

$$S^\star \to S \qquad \|S - S^\star\|$$

Optimized Schwarz methods

$$\frac{1}{h^2}\begin{bmatrix} D & I & & \\ I & \ddots & \ddots & \\ & \ddots & D & I \\ & & I & D - S_1 \end{bmatrix}\mathbf{u}_1^{(n)} = \mathbf{b}_1^{(n)}$$

$$\frac{1}{h^2}\begin{bmatrix} D - S_2 & I & & \\ I & \ddots & \ddots & \\ & \ddots & D & I \\ & & I & D \end{bmatrix}\mathbf{u}_2^{(n)} = \mathbf{b}_2^{(n)}$$

$$S^\star \to S \qquad \|S - S^\star\| \qquad \rho^{\mathrm{discr}}(S)$$

Schwarz methods & DS

Schwarz methods & DS

DS = data-sparse formats

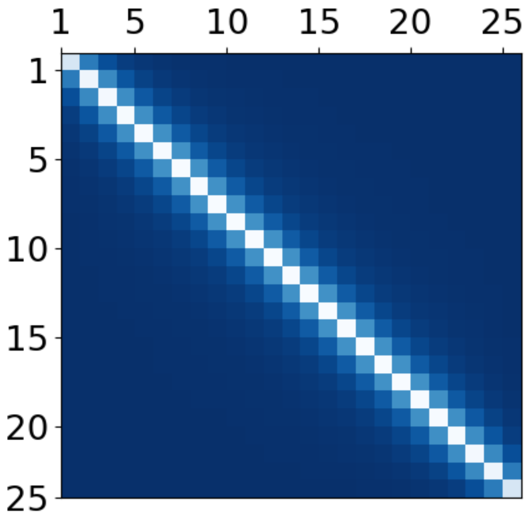
low-rank & HODLR

Schwarz methods & DS

$$S_1^\star := E_{\Gamma_1}^T L_{\Theta_2}^{-1} E_{\Gamma_1}$$

Schwarz methods & DS

$$S_1^\star := E_{\Gamma_1}^T L_{\Theta_2}^{-1} E_{\Gamma_1}$$

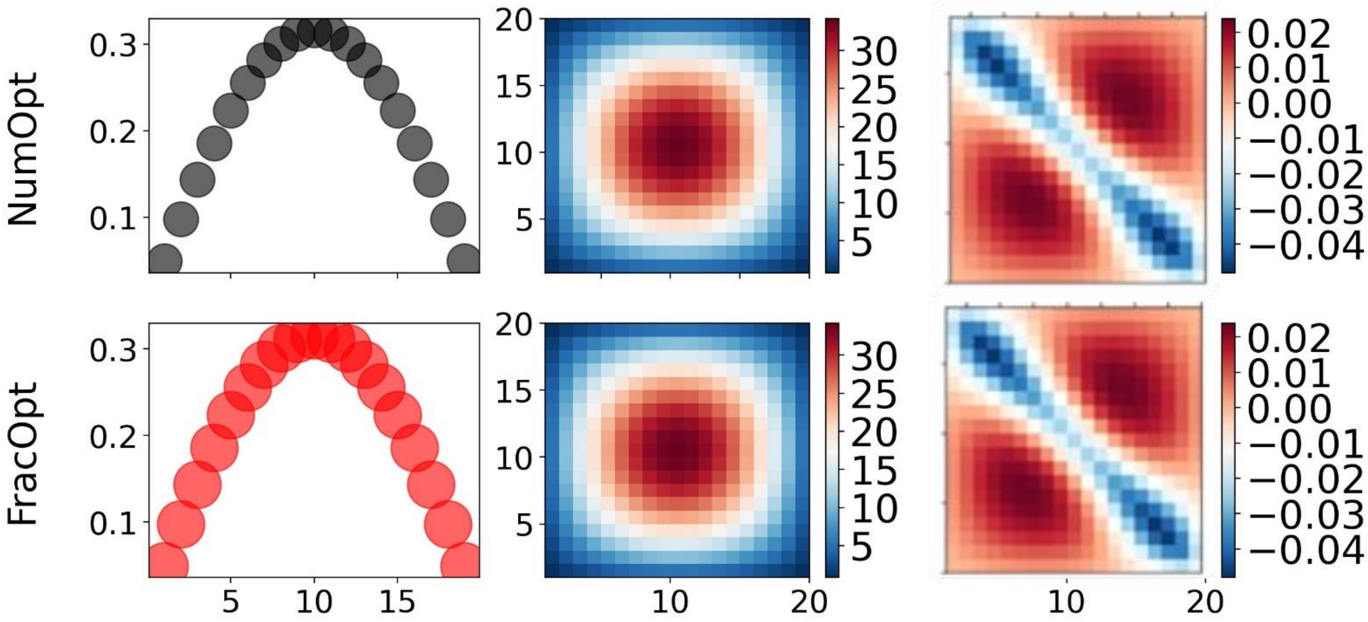


Schwarz methods & DS

data-sparse formats: low-rank

Schwarz methods & DS

data-sparse formats: low-rank

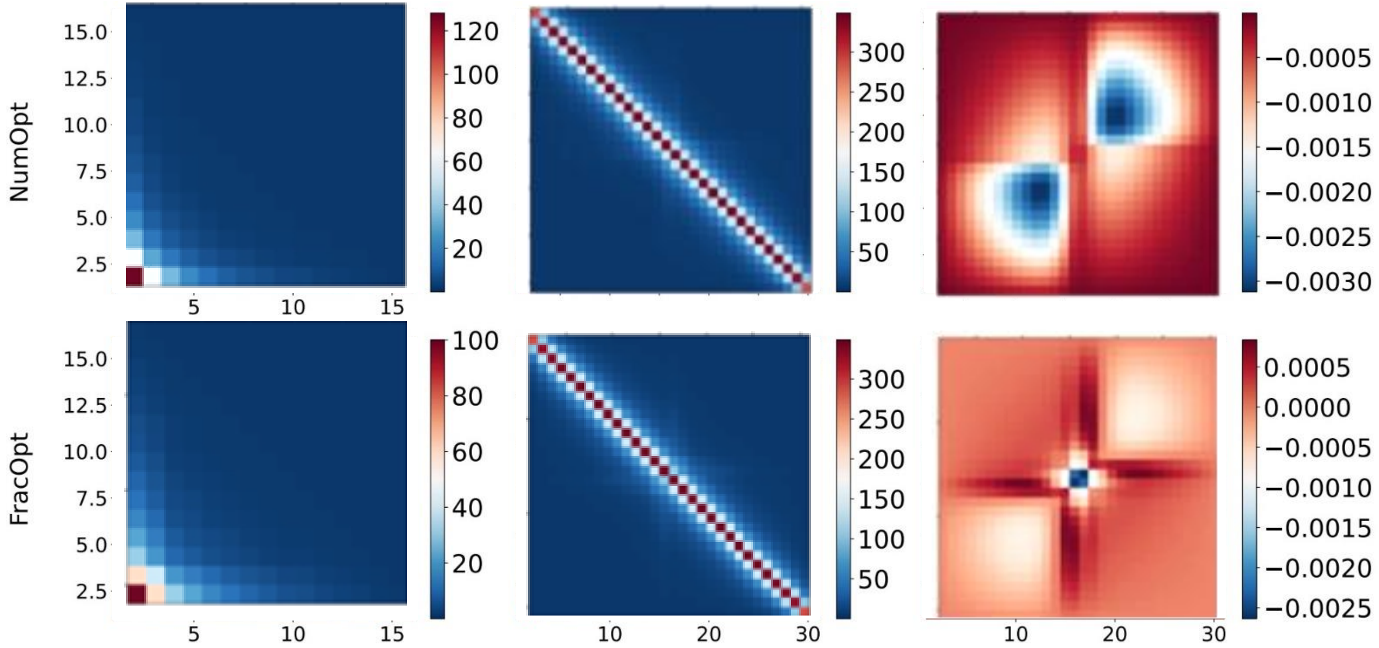


Schwarz methods & DS

data-sparse formats: HODLR

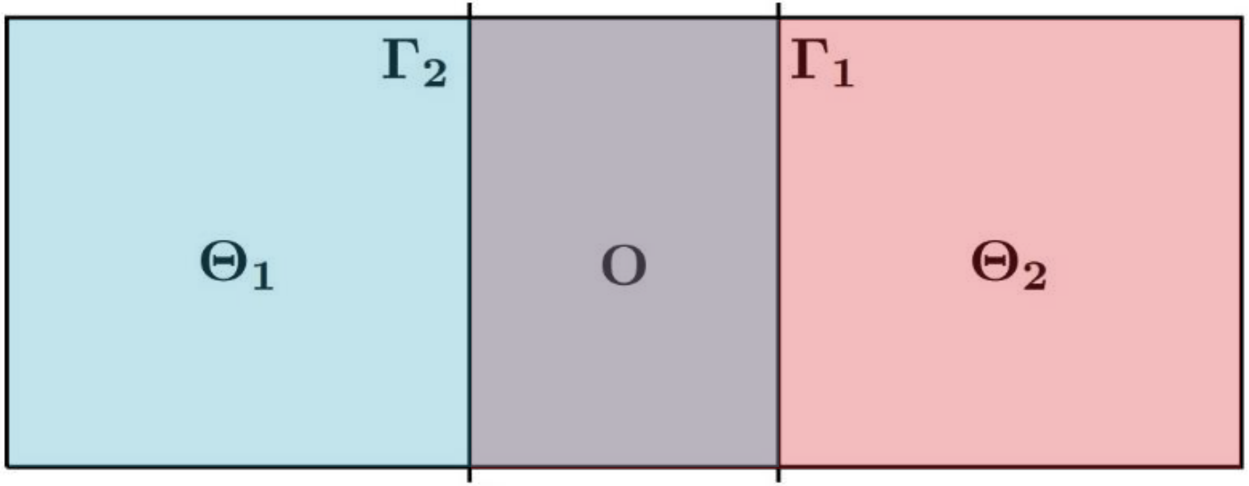
Schwarz methods & DS

data-sparse formats: HODLR

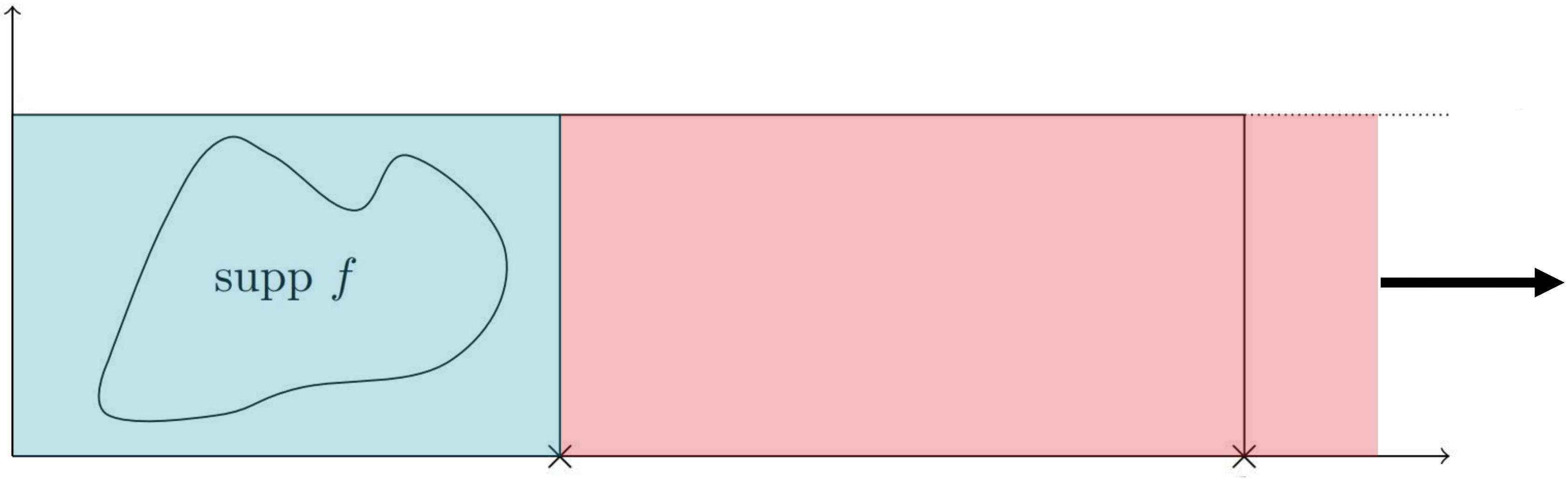


Schwarz methods & ABC

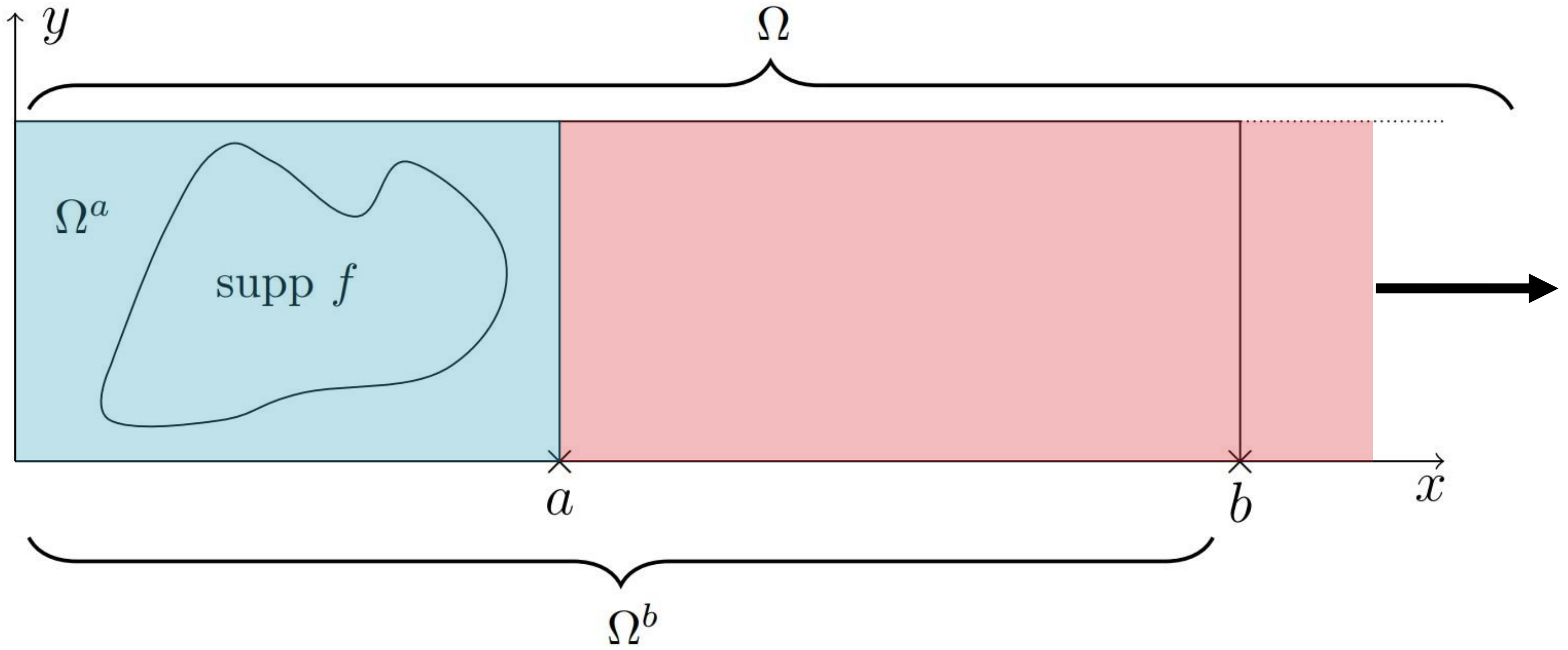
Schwarz methods & ABC



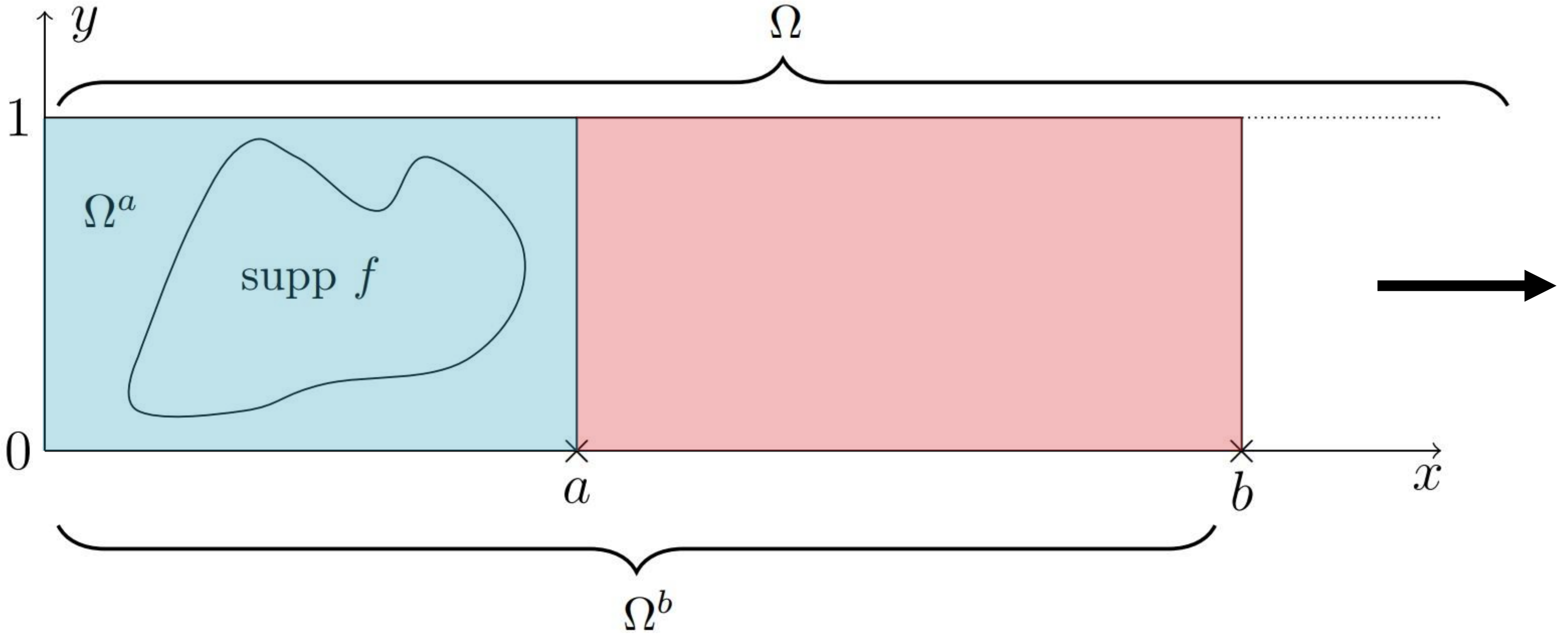
Schwarz methods & ABC



Schwarz methods & ABC



Schwarz methods & ABC



$-\Delta u$ & blackboard

Schwarz methods & ABC

$$L^a \mathbf{u}^a = \mathbf{f}^a \qquad L^b \mathbf{u}^b = \mathbf{f}^b \qquad L\mathbf{u} = \mathbf{f}$$

Schwarz methods & ABC

$$L^a \mathbf{u}^a = \mathbf{f}^a \qquad L^b \mathbf{u}^b = \mathbf{f}^b \qquad L\mathbf{u} = \mathbf{f}$$

$$\begin{pmatrix} D_1 & -I & & \\ -I & \ddots & \ddots & \\ & \ddots & D_{N^a-1} & -I \\ & & -I & D_{N^a} \end{pmatrix} \quad \begin{pmatrix} D_1 & -I & & \\ -I & \ddots & \ddots & \\ & \ddots & D_{N^b-1} & -I \\ & & -I & D_{N^b} \end{pmatrix}$$

where $D_i = D$ (blackboard)

Schwarz methods & CF

$$L^a \mathbf{u}^a = \mathbf{f}^a \qquad L^b \mathbf{u}^b = \mathbf{f}^b \qquad L\mathbf{u} = \mathbf{f}$$

$$\begin{pmatrix} D_1 & -I & & \\ -I & \ddots & \ddots & \\ & \ddots & D_{N^a-1} & -I \\ & & -I & D_{N^a} \end{pmatrix} \begin{pmatrix} D_1 & -I & & \\ -I & \ddots & \ddots & \\ & \ddots & D_{N^b-1} & -I \\ & & -I & D_{N^b} \end{pmatrix} \begin{pmatrix} h^2 L^b & & \\ & -I & \\ -I & D_{N^b+1} & \ddots \\ & \ddots & \ddots \end{pmatrix}$$

where $D_i = D$ (blackboard)

Schwarz methods & ABC

$$L^a \mathbf{u}^a = \mathbf{f}^a \qquad L^b \mathbf{u}^b = \mathbf{f}^b \qquad L\mathbf{u} = \mathbf{f}$$

$$\begin{pmatrix} D_1 & -I & & \\ -I & \ddots & \ddots & \\ & \ddots & D_{N^a-1} & -I \\ & & -I & D_{N^a} \end{pmatrix} \begin{pmatrix} D_1 & -I & & \\ -I & \ddots & \ddots & \\ & \ddots & D_{N^a-1} & -I \\ & & -I & \boxed{T^b_{N^a}} \end{pmatrix} \begin{pmatrix} h^2 A^b & & \\ & -I & \\ -I & D_{N^b+1} & \ddots \\ & \ddots & \ddots \end{pmatrix}$$

$T^b_{N^a}$ (blackboard)

Schwarz methods & ABC

$$L^a \mathbf{u}^a = \mathbf{f}^a \qquad L^b \mathbf{u}^b = \mathbf{f}^b \qquad L\mathbf{u} = \mathbf{f}$$

$$\begin{pmatrix} D_1 & -I & & \\ -I & \ddots & \ddots & \\ & \ddots & D_{N^a-1} & -I \\ & & -I & D_{N^a} \end{pmatrix} \begin{pmatrix} D_1 & -I & & \\ -I & \ddots & \ddots & \\ & \ddots & D_{N^a-1} & -I \\ & & -I & \boxed{T^b_{N^a}} \end{pmatrix} \begin{pmatrix} D_1 & -I & & \\ -I & \ddots & \ddots & \\ & \ddots & D_{N^a-1} & -I \\ & & -I & \boxed{T^\infty_{N^a}} \end{pmatrix}$$

$T^b_{N^a}, T^\infty_{N^a}$ (blackboard)

Schwarz methods & ABC

$$L^a\mathbf{u}^a = \mathbf{f}^a \qquad L^b\mathbf{u}^b = \mathbf{f}^b \qquad L\mathbf{u} = \mathbf{f}$$

$$\begin{pmatrix} D_1 & -I & & \\ -I & \ddots & \ddots & \\ & \ddots & D_{N^a-1} & -I \\ & & -I & D_{N^a} \end{pmatrix} \begin{pmatrix} D_1 & -I & & \\ -I & \ddots & \ddots & \\ & \ddots & D_{N^a-1} & -I \\ & & -I & \boxed{T^b_{N^a}} \end{pmatrix} \begin{pmatrix} D_1 & -I & & \\ -I & \ddots & \ddots & \\ & \ddots & D_{N^a-1} & -I \\ & & -I & T^\infty_{N^a} \end{pmatrix}$$

What is the effect of increasing b ?

Schwarz methods & ABC

$$\boxed{T^b_{N^a} :}$$

Schwarz methods & ABC

$$\hat{T}_i^b = Q\frac{D}{h^2}Q^T - Q\frac{\left(T_{i+1}^b\right)^{-1}}{h^4}Q^T = \frac{\Lambda}{h^2} - \frac{\left(\hat{T}_{i+1}^b\right)^{-1}}{h^4}$$

$\boxed{T_{N^a}^b :}$

Schwarz methods & ABC

$$\hat{T}_i^b = Q\frac{D}{h^2}Q^T - Q\frac{(T_{i+1}^b)^{-1}}{h^4}Q^T = \frac{\Lambda}{h^2} - \frac{(\hat{T}_{i+1}^b)^{-1}}{h^4}$$

$\boxed{T_{N^a}^b :}$

$$\hat{t}_i^b(\lambda) = \frac{1}{h^2}\left(\lambda - \frac{1}{h^2\hat{t}_{i+1}^b(\lambda)}\right)$$

Schwarz methods & ABC

$$\hat{T}_i^b = Q\frac{D}{h^2}Q^T - Q\frac{(T_{i+1}^b)^{-1}}{h^4}Q^T = \frac{\Lambda}{h^2} - \frac{(\hat{T}_{i+1}^b)^{-1}}{h^4}$$

$\boxed{T_{N^a}^b :}$

$$\hat{t}_i^b(\lambda) = \frac{1}{h^2}\left(\lambda - \frac{1}{h^2\hat{t}_{i+1}^b(\lambda)}\right)$$

$$\hat{t}_i^b(\lambda) = \frac{1}{h^2}\left(\lambda - \frac{1}{\lambda - \frac{1}{h^2\hat{t}_{i+2}^b}}\right)$$

Schwarz methods & ABC

$\boxed{T^b_{N^a}:}$

$$\hat{t}^b_{N^a}(\lambda) = \frac{1}{h^2}\left(\lambda - \cfrac{1}{\lambda - \cfrac{\ddots}{\lambda - \frac{1}{\lambda}}}\right)$$

$N^b - N^a$ levels; blackboard

Schwarz methods & ABC

$\boxed{T^{\infty}_{Na}:}$

Schwarz methods & ABC

$$\hat{T}^{\infty}_{N^a}(\lambda) = \lim_{b \to +\infty} \hat{T}^{b}_{N^a}(\lambda)$$

$$T^{\infty}_{N^a}:$$

Schwarz methods & ABC

$$\hat{T}^{\infty}_{N^a}(\lambda) = \lim_{b\to+\infty} \hat{T}^{b}_{N^a}(\lambda) \qquad \hat{t}^{\infty}_{N^a}(\lambda) = \lim_{b\to+\infty} \hat{t}^{b}_{N^a}(\lambda)$$

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Schwarz methods & ABC

$$\hat{T}^{\infty}_{N^a}(\lambda) = \lim_{b \to +\infty} \hat{T}^{b}_{N^a}(\lambda) \qquad \hat{t}^{\infty}_{N^a}(\lambda) = \lim_{b \to +\infty} \hat{t}^{b}_{N^a}(\lambda)$$

$\boxed{T^{\infty}_{N^a}:}$

$$\hat{t}^{b}_{i}(\lambda) = \frac{1}{h^2}\left(\lambda - \frac{1}{h^2 \hat{t}^{b}_{i+1}(\lambda)}\right)$$

Schwarz methods & ABC

$$\hat{T}^{\infty}_{N^a}(\lambda) = \lim_{b \to +\infty} \hat{T}^{b}_{N^a}(\lambda) \qquad \hat{t}^{\infty}_{N^a}(\lambda) = \lim_{b \to +\infty} \hat{t}^{b}_{N^a}(\lambda)$$

$$\boxed{T^{\infty}_{N^a}:}$$

$$\hat{t}^{b}_{i}(\lambda) = \frac{1}{h^2}\left(\lambda - \frac{1}{h^2 \hat{t}^{b}_{i+1}(\lambda)}\right)$$

$$\hat{t}^{\infty}_{N^a}(\lambda) = \frac{1}{h^2}\left(\lambda - \frac{1}{h^2 \hat{t}^{\infty}_{N^a}(\lambda)}\right)$$

Schwarz methods & ABC

$$\hat{T}^{\infty}_{N^a}(\lambda) = \lim_{b \to +\infty} \hat{T}^{b}_{N^a}(\lambda) \qquad \hat{t}^{\infty}_{N^a}(\lambda) = \lim_{b \to +\infty} \hat{t}^{b}_{N^a}(\lambda)$$

$$\boxed{T^{\infty}_{N^a}:}$$

$$\hat{t}^{b}_{i}(\lambda) = \frac{1}{h^2}\left(\lambda - \frac{1}{h^2 \hat{t}^{b}_{i+1}(\lambda)}\right)$$

$$\hat{t}^{\infty}_{N^a}(\lambda) = \frac{1}{h^2}\left(\lambda - \frac{1}{h^2 \hat{t}^{\infty}_{N^a}(\lambda)}\right)$$

$$\hat{t}^{\infty}_{N^a}(\lambda) = \frac{1}{h^2}\left(\lambda - \frac{1}{\lambda - \frac{1}{h^2 \hat{t}^{\infty}_{N^a}(\lambda)}}\right)$$

Schwarz methods & ABC

$\boxed{T^{\infty}_{N^a}:}$

$$\hat{t}^{\infty}_{N^a}(\lambda) = \frac{1}{h^2}\left(\lambda - \cfrac{1}{\lambda - \cfrac{\ddots}{\lambda - \frac{1}{\ddots}}}\right)$$

blackboard

Schwarz methods & ABC

$$\hat{t}^{\infty}_{N^a}(\lambda) = \frac{1}{h^2}\left(\lambda - \frac{1}{h^2 \hat{t}^{\infty}_{N^a}(\lambda)}\right)$$

$\boxed{T^{\infty}_{N^a} :}$

Schwarz methods & ABC

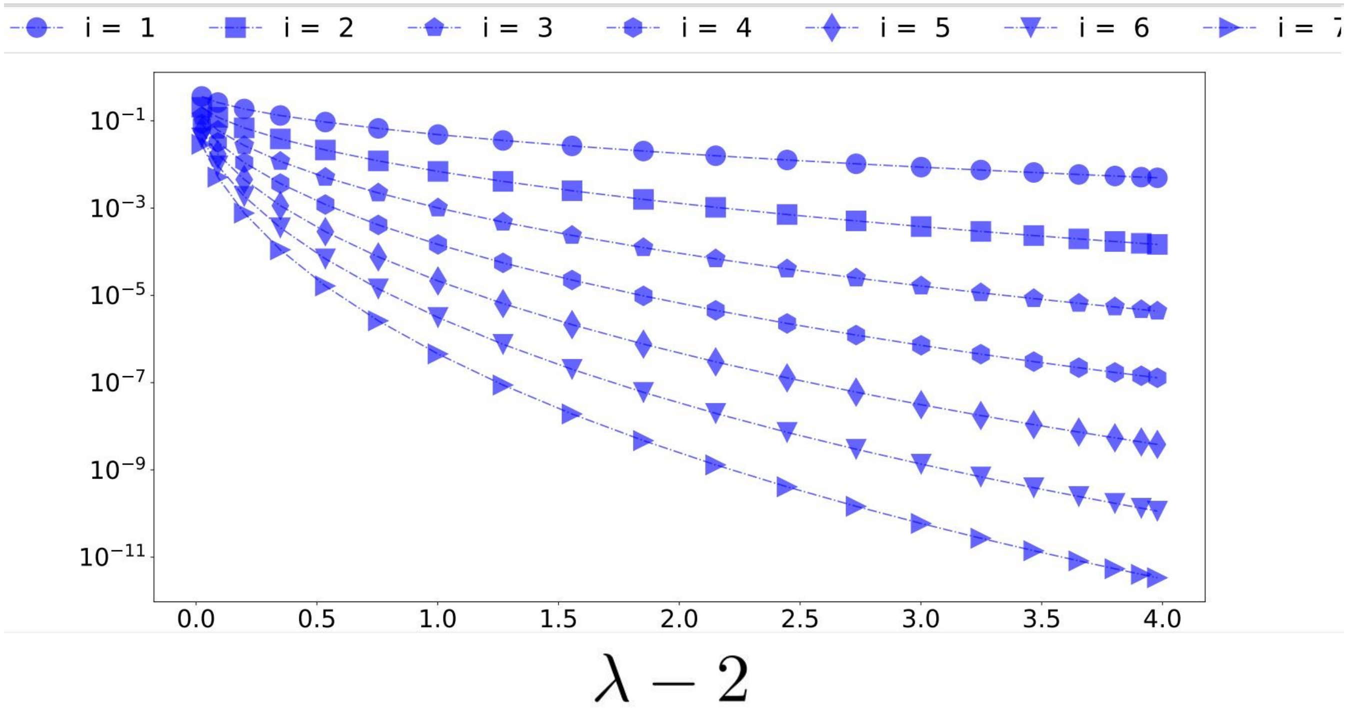
$$\hat{t}^{\infty}_{N^a}(\lambda) = \frac{1}{h^2}\left(\lambda - \frac{1}{h^2\hat{t}^{\infty}_{N^a}(\lambda)}\right)$$

$\boxed{T^{\infty}_{N^a}:}$

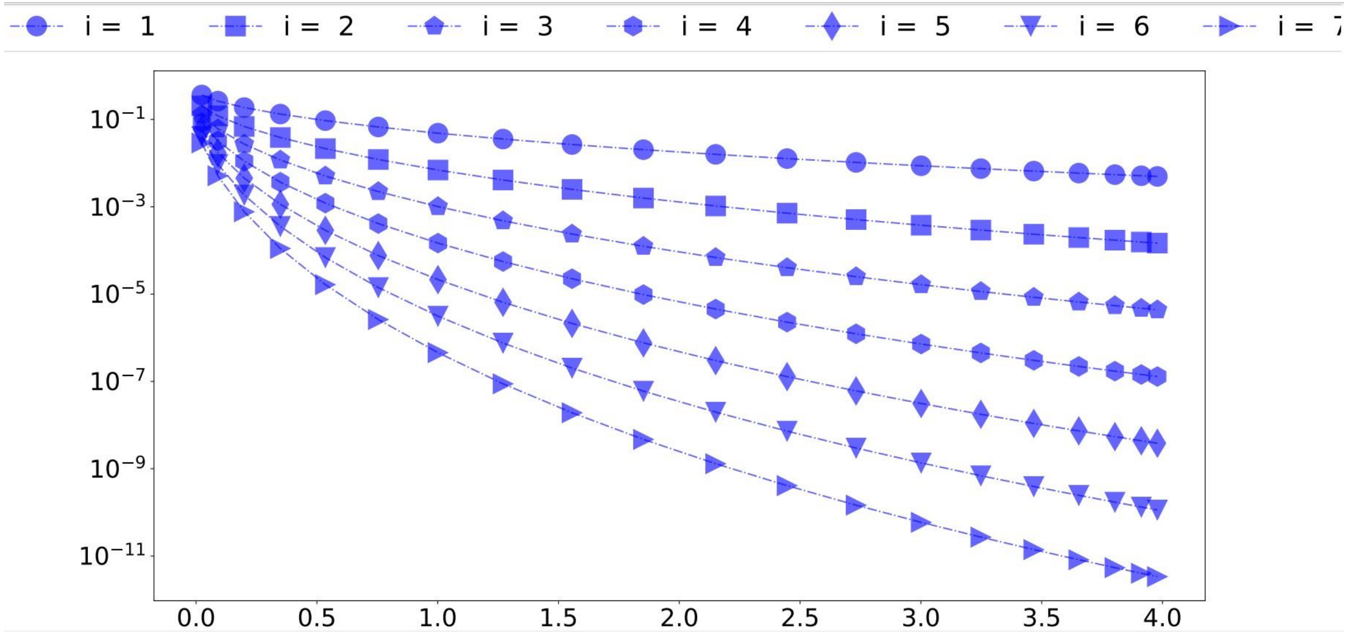
$$\hat{t}^{\infty}_{N^a}(\lambda) = \frac{\lambda + \sqrt{\lambda^2 - 4}}{2h^2}$$

blackboard

Schwarz methods & ABC

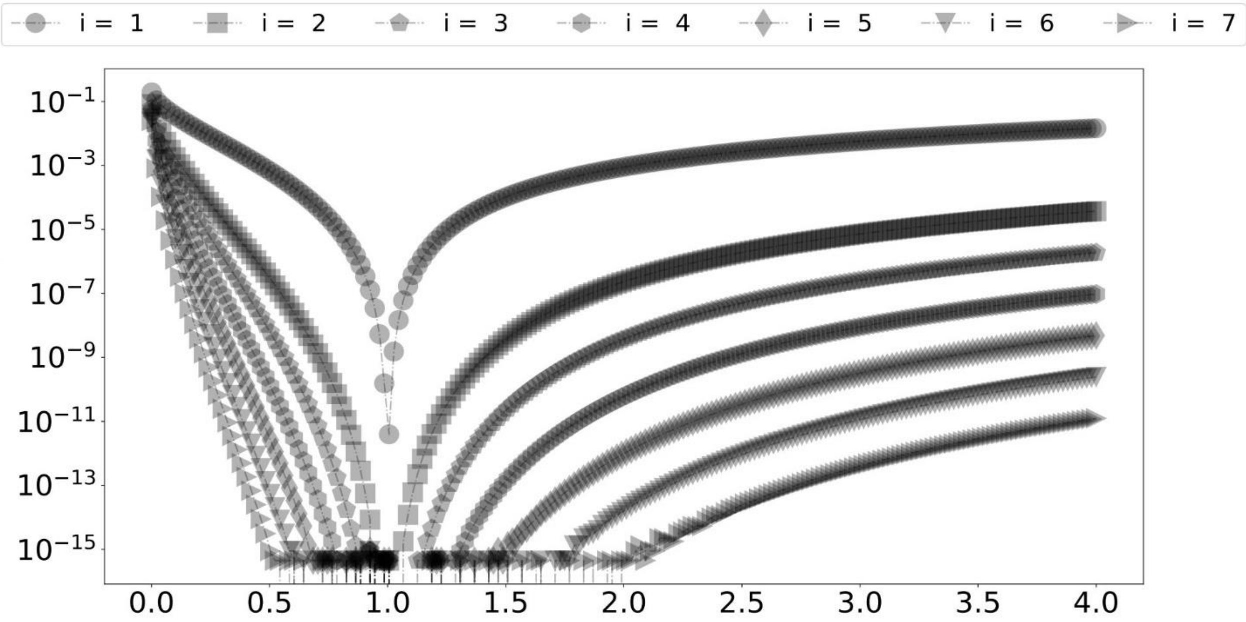


Schwarz methods & ABC



Padé approximation about endpoint

Schwarz methods & ABC



Shifted Padé approximation

IRK preconditioners

IRK preconditioners

$$\frac{\partial}{\partial t}u = \Delta u \quad \text{in } \Omega \times (0, T)$$

$$u = g \quad \text{on } \partial\Omega \times (0, T)$$

$$u = u_0 \;\; \text{at } \partial\Omega \times \{0\}$$

IRK preconditioners

$$\mathbf{u}^m = \mathbf{u}^{m-1} + \tau \sum_{i=1}^{s} b_i \mathbf{k}_i^m$$

$$\underbrace{\boxed{\left(I_s \otimes I_n - \frac{\tau}{h^2}(A \otimes L) \right)}}_{\boxed{M}} \mathbf{k}^m = \frac{1}{h^2}(I_s \otimes L)\mathbf{u}^{m-1}$$

IRK preconditioners

M. M. Rana, V. E. Howle, K. Long, A. Meek, and W. Milestone. A New Block Preconditioner for Implicit Runge-Kutta Methods for Parabolic PDE Problems, 2021.

IRK preconditioners

$$\mathrm{factor}\left(I_s \otimes I_n - \frac{\tau}{h^2} A \otimes L\right)$$

IRK preconditioners

$$\boxed{\operatorname{factor}\left(I_s \otimes I_n - \frac{\tau}{h^2} A \otimes L\right)} \approx I_s \otimes I_n - \frac{\tau}{h^2} \boxed{\operatorname{factor}(A)} \otimes L$$

IRK preconditioners

$$\text{factor}\left(I_s \otimes I_n - \frac{\tau}{h^2} A \otimes L\right) \approx I_s \otimes I_n - \frac{\tau}{h^2}\,\text{factor}\,(A) \otimes L$$

$$\boxed{I_s \otimes I_n - \frac{\tau}{h^2} D_A \otimes L =: P^{\text{diag}}}$$

IRK preconditioners

$$I_s \otimes I_n - \frac{\tau}{h^2} U_A \otimes L =: P^{\mathrm{diag}}$$

$$M \left(P^{\mathrm{diag}} \right)^{-1}$$

$$\texttt{sp.linalg.gmres}\,(M, \mathrm{rhs}, \, P^{\mathrm{diag}})$$

IRK preconditioners

`sp.linalg.gmres`

$$\frac{\|r_k\|}{\|r_0\|} \leq \min_{\substack{\varphi(0)=1 \\ \deg(\varphi)\leq k}} \|\varphi(M\left(P^{\text{diag}}\right)^{-1})\|$$

$$\frac{\|r_k\|}{\|r_0\|} \leq \kappa(S) \min_{\substack{\varphi(0)=1 \\ \deg(\varphi)\leq k}} \max_{\zeta_i \in \text{sp}(M(P^{\text{triang}})^{-1})} |\varphi(\zeta_i)|$$

$$\frac{\|r_k\|}{\|r_0\|} \leq \boxed{\kappa(S)} \boxed{\min_{\substack{\varphi(0)=1 \\ \deg(\varphi)\leq k}} \max_{\zeta \in \text{co}(sp(\cdots))} |\varphi(\zeta)|}$$

IRK preconditioners

$$s = 2$$

IRK preconditioners

Proposition. *Let* $s = 2$ *and* $a_{ij} \neq 0$. *Adopting the above notation and setting* $\mathrm{sp}(L) = \{\lambda_k\}_k$ *and* $\theta_k = \frac{\tau}{h^2}\lambda_k$ *we have* $\mathrm{sp}(M\left(P^{\mathrm{diag}}\right)^{-1}) = \zeta_{1,2}^{(k)}$ *with*

$$\boxed{\zeta_{1,2}^{(k)} = 1 \pm \sqrt{\frac{a_{12}a_{21}}{(|\theta_k^{-1}| + a_{11})(|\theta_k^{-1}| + a_{22})}},}$$

and the condition number of the eigenbasis is given by

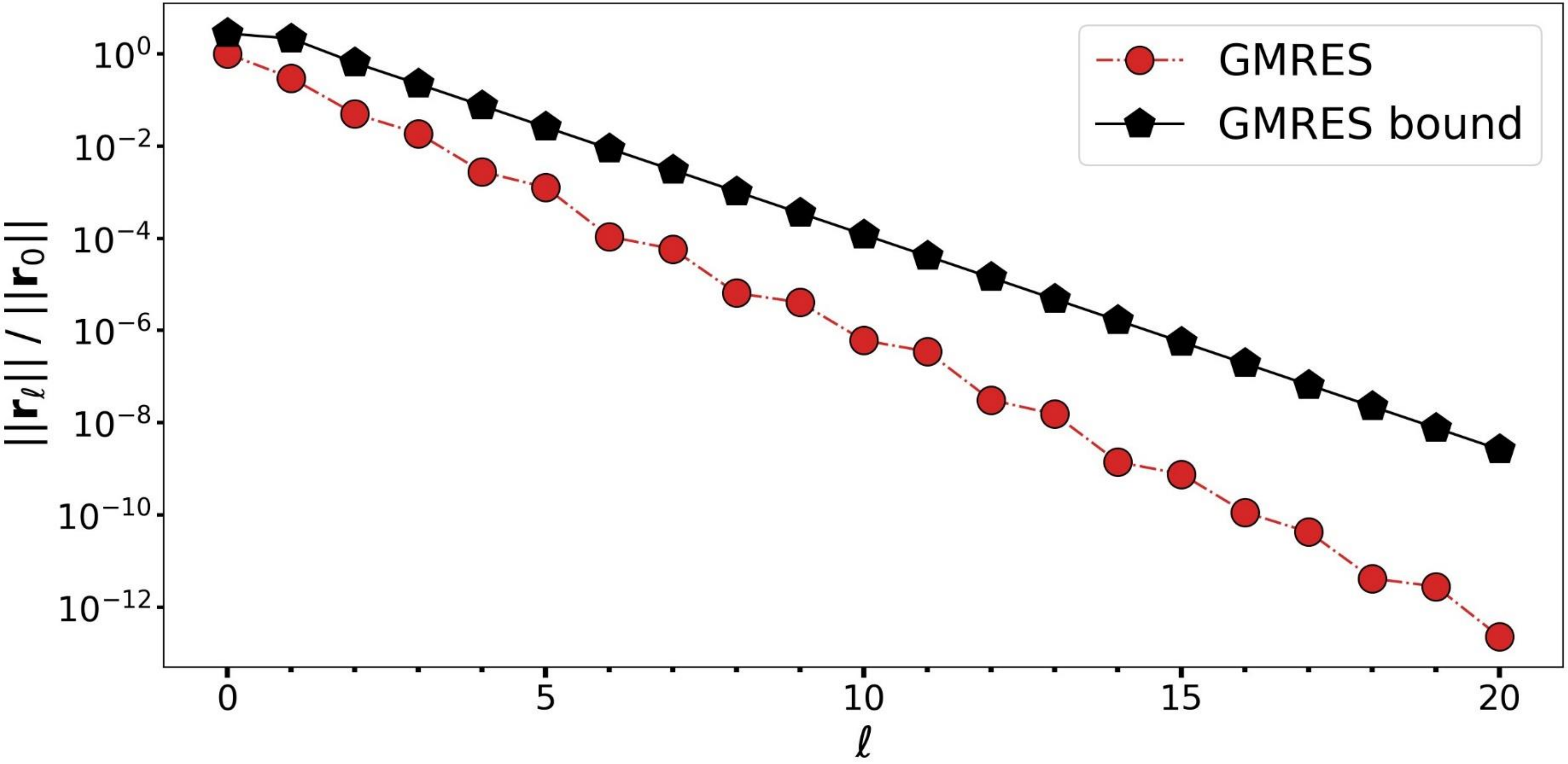
$$\boxed{\kappa(S) = \max_{k=1,\ldots,n} \sqrt{\frac{1 + \left|\frac{a_{21}(1+a_{22}|\theta_k|)}{a_{12}(1+a_{11}|\theta_k|)}\right| + \left|1 - \left|\frac{a_{21}(1+a_{22}|\theta_k|)}{a_{12}(1+a_{11}|\theta_k|)}\right|\right|}{1 + \left|\frac{a_{21}(1+a_{22}|\theta_k|)}{a_{12}(1+a_{11}|\theta_k|)}\right| - \left|1 - \left|\frac{a_{21}(1+a_{22}|\theta_k|)}{a_{12}(1+a_{11}|\theta_k|)}\right|\right|}}.}$$

IRK preconditioners

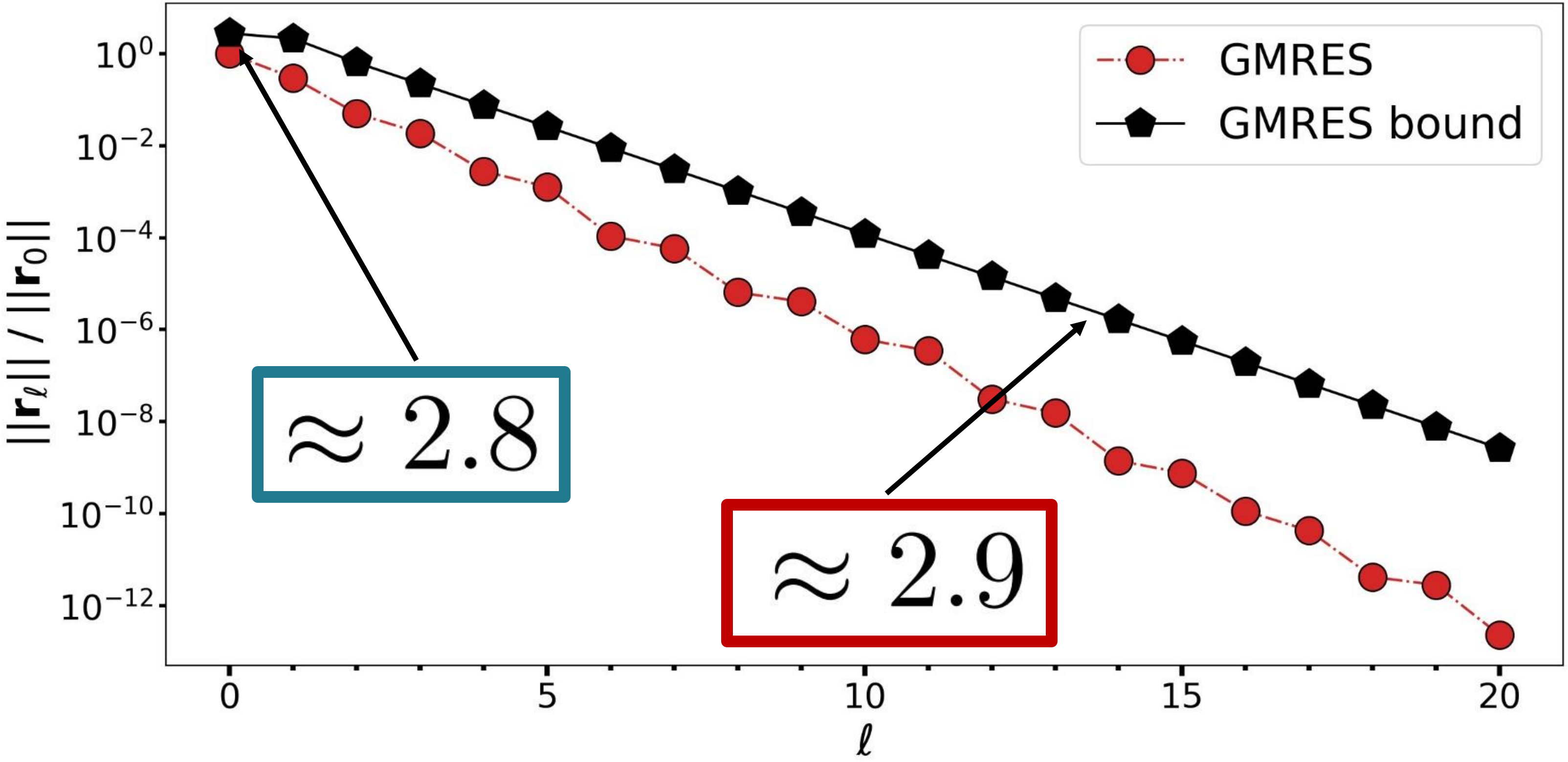
preconditioned GMRES
What can we predict ?

$$s = 2$$

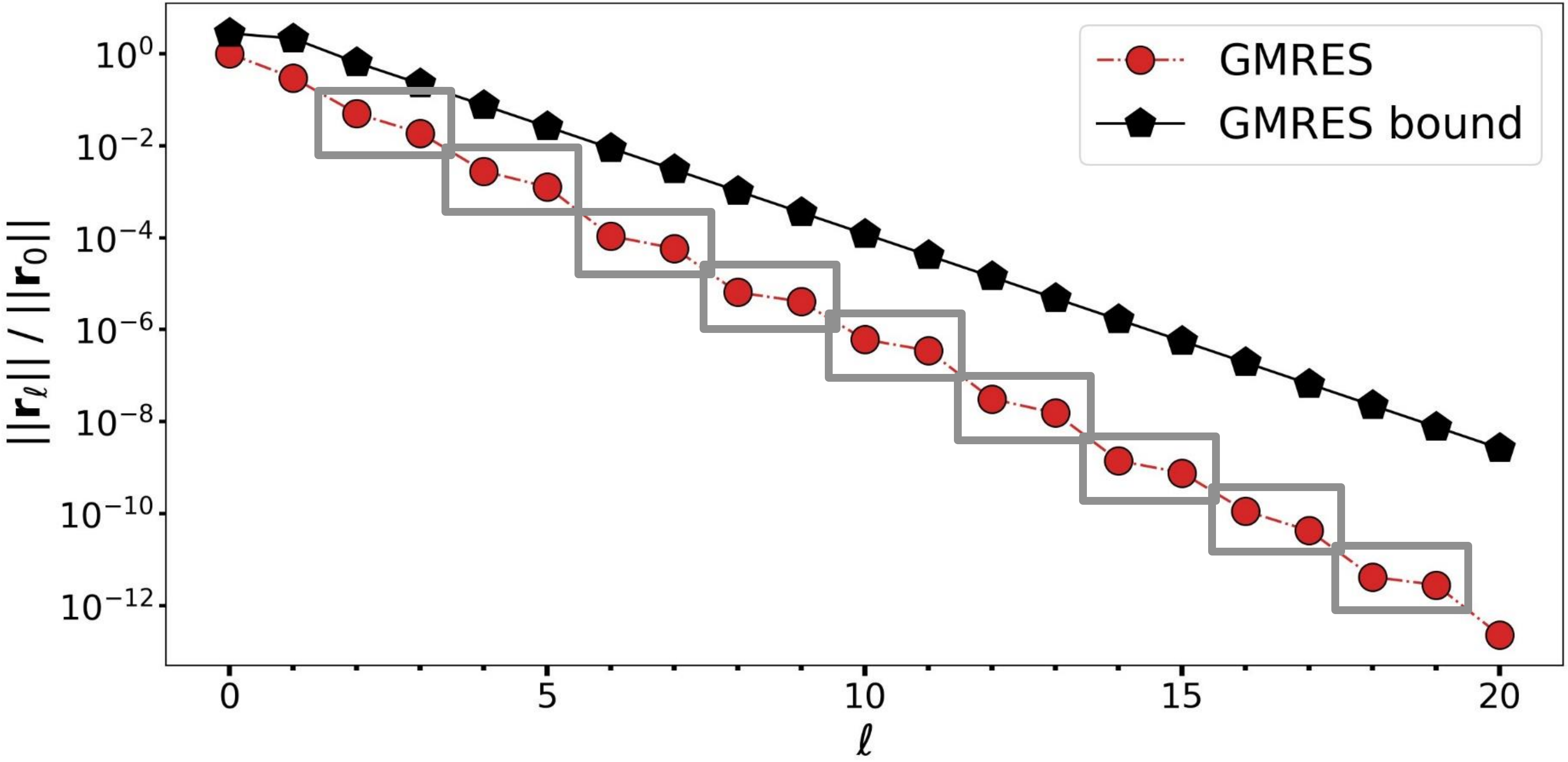
IRK preconditioners



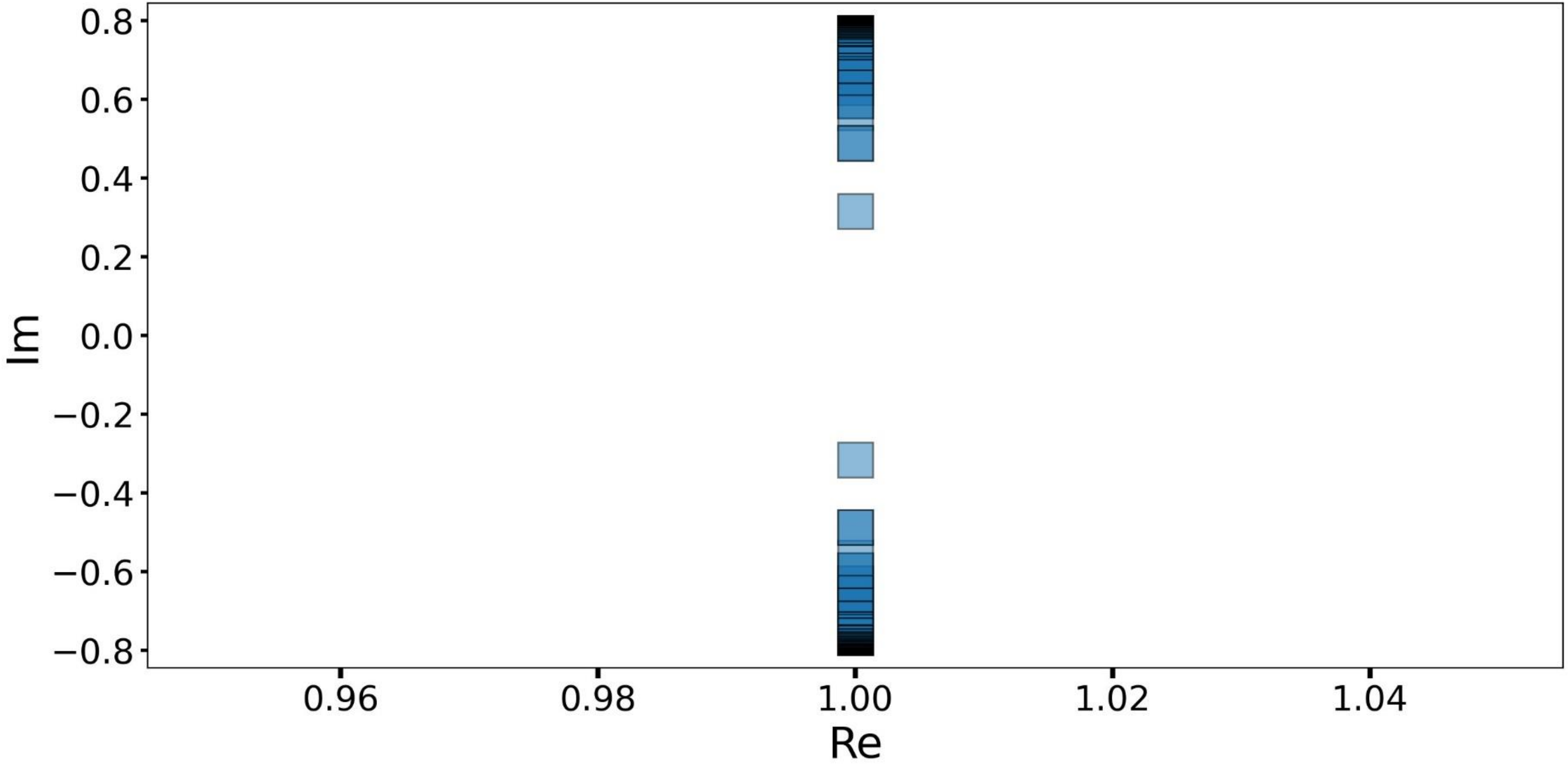
IRK preconditioners



IRK preconditioners



IRK preconditioners



IRK preconditioners

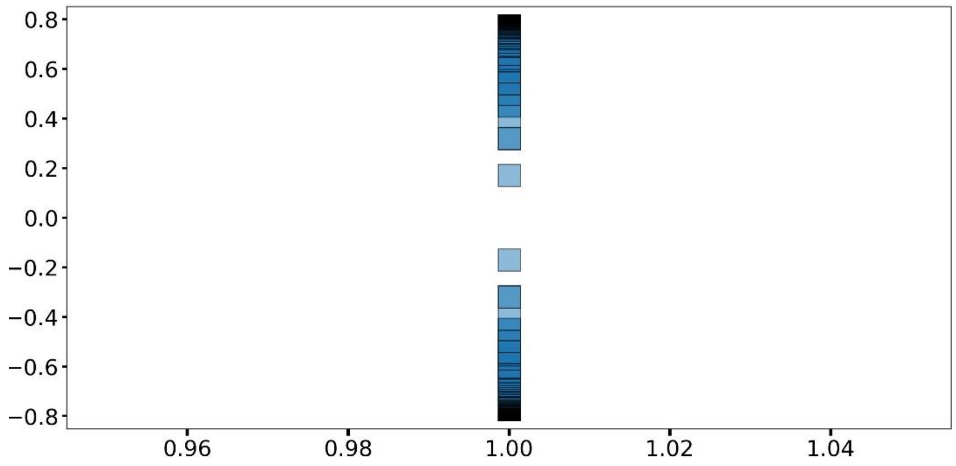
sp.linalg.gmres

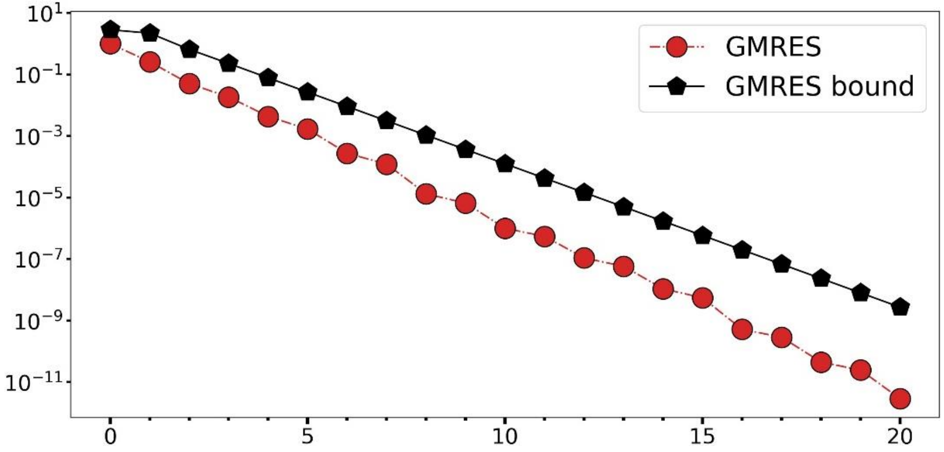
$$\frac{\|r_k\|}{\|r_0\|} \leq \min_{\substack{\varphi(0)=1 \\ \deg(\varphi)\leq k}} \|\varphi(M\left(P^{\mathrm{diag}}\right)^{-1})\|$$

$$\frac{\|r_k\|}{\|r_0\|} \leq \kappa(S) \min_{\substack{\varphi(0)=1 \\ \deg(\varphi)\leq k}} \max_{\zeta_i \in \mathrm{sp}(M(P^{\mathrm{triang}})^{-1})} |\varphi(\zeta_i)|$$

$$\frac{\|r_k\|}{\|r_0\|} \leq \boxed{\kappa(S)} \boxed{\min_{\substack{\varphi(0)=1 \\ \deg(\varphi)\leq k}} \max_{\zeta \in \mathrm{co}(sp(\cdots))} |\varphi(\zeta)|}$$

IRK preconditioners

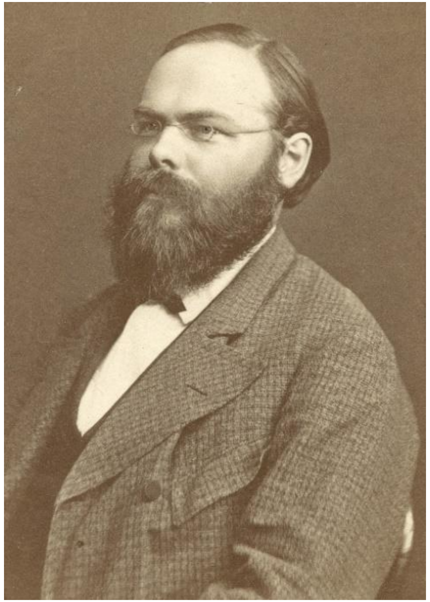




Conclusion

Thank you for your attention

Schwarz methods



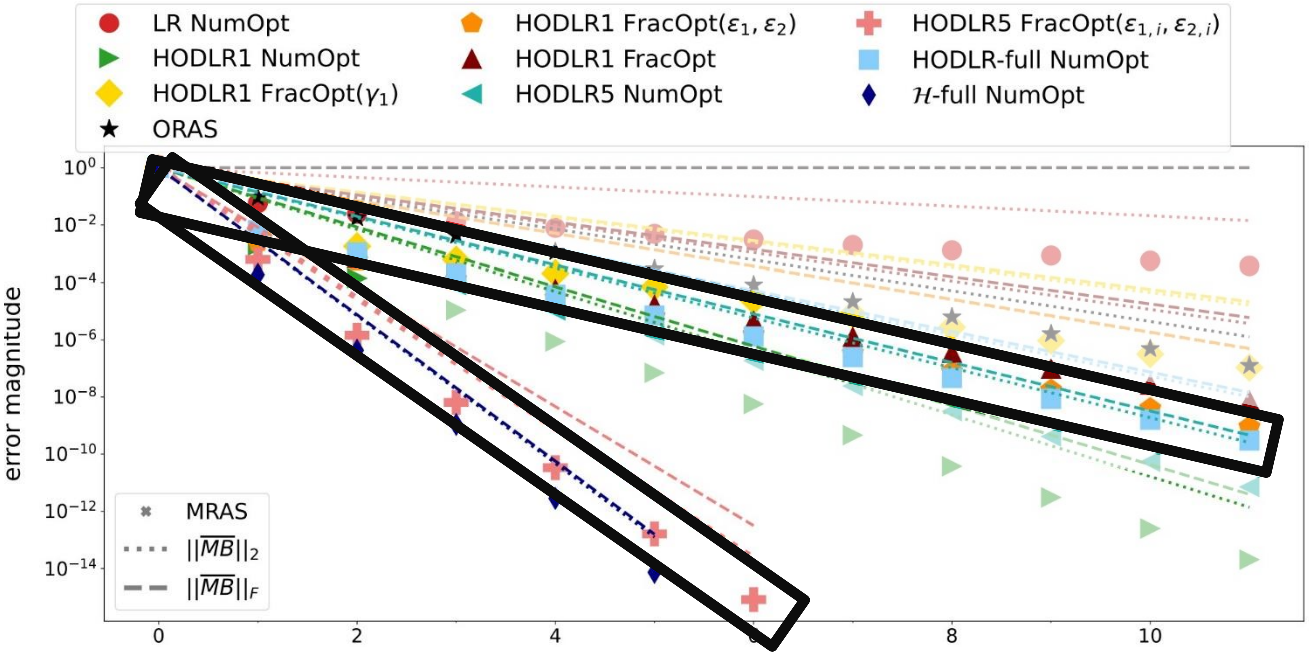
Karl Hermann Amandus Schwarz

Schwarz methods & DS

data-sparse formats: convergence

Schwarz methods & DS

data-sparse formats: convergence



Schwarz methods & ABC

Henri Eugène Padé

References

Continued fractions and Padé

Theorem. *For any* $\alpha \in (-1, +\infty)$ *we have*

$$\sqrt{1+\alpha} = 1 + \cfrac{\frac{\alpha}{2}}{1 + \cfrac{\frac{\alpha}{2}}{2 + \cfrac{\frac{\alpha}{2}}{1 + \cfrac{\frac{\alpha}{2}}{\ddots}}}}$$

Moreover, for any n *the* $[n+1, n]$*-Padé approximant of* $\sqrt{1+\alpha}$ *expanded about* $\alpha = 0$ *is the* $(2n+1)$*-st truncation of the continued fraction above.*

Continued fractions and Padé

To combine

$$\sqrt{1+\alpha} = 1 + \cfrac{\frac{\alpha}{2}}{1 + \cfrac{\frac{\alpha}{2}}{2 + \cfrac{\frac{\alpha}{2}}{1 + \cfrac{\frac{\alpha}{2}}{\ddots}}}}$$

Continued fractions and Padé

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and

$$\hat{t}^{\infty}_{N^a}(\lambda) = \frac{\lambda + \sqrt{\lambda^2 - 4}}{2h^2}$$

Continued fractions and Padé

To combine

$$\boxed{\sqrt{1+\alpha}} = 1 + \cfrac{\frac{\alpha}{2}}{1 + \cfrac{\frac{\alpha}{2}}{2 + \cfrac{\frac{\alpha}{2}}{1 + \cfrac{\frac{\alpha}{2}}{\ddots}}}}$$

and

$$\hat{t}^{\infty}_{N^a}(\lambda) = \frac{\lambda + \boxed{\sqrt{\lambda^2 - 4}}}{2h^2}$$

things get a little *calculaty*

Continued fractions and Padé

Having $\lambda = 2 + z$ we get

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$$\hat{t}^b_{N^a}(z) = \frac{1}{h^2}\left(2 + z - \cfrac{1}{2 + z - \cfrac{1}{\ddots_{\,2+z-\frac{1}{2+z-\frac{1}{2+z}}}}}\right)$$

Continued fractions and Padé

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$$\hat{t}^{\infty}_{N^a}(z) = \frac{1}{h^2}\left(2 + z - \cfrac{1}{2 + z - \cfrac{1}{2+z-\cfrac{1}{2+z-\cfrac{1}{\cdots}}}}\right) = \frac{1}{h^2}\left(1 + \frac{z}{2} + \frac{z}{2}\sqrt{1 + \frac{4}{z}}\right)$$

Continued fractions and Padé

Combining

$$\sqrt{1+\alpha} = 1 + \cfrac{\frac{\alpha}{2}}{1 + \cfrac{\frac{\alpha}{2}}{2 + \cfrac{\frac{\alpha}{2}}{1 + \cfrac{\frac{\alpha}{2}}{\ddots}}}}$$

Continued fractions and Padé

Combining

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$$\hat{t}^{\infty}_{N^a}(z) = \frac{1}{h^2}\left(1 + \frac{z}{2} + \frac{z}{2}\sqrt{1+\frac{4}{z}}\right)$$

Continued fractions and Padé

Combining

$$\boxed{\sqrt{1+\alpha}} = 1 + \cfrac{\frac{\alpha}{2}}{1 + \cfrac{\frac{\alpha}{2}}{2 + \cfrac{\frac{\alpha}{2}}{1 + \cfrac{\frac{\alpha}{2}}{\ddots}}}}$$

and

$$\hat{t}^{\infty}_{N^a}(z) = \frac{1}{h^2}\left(1 + \frac{z}{2} + \frac{z}{2}\boxed{\sqrt{1+\frac{4}{z}}}\right)$$

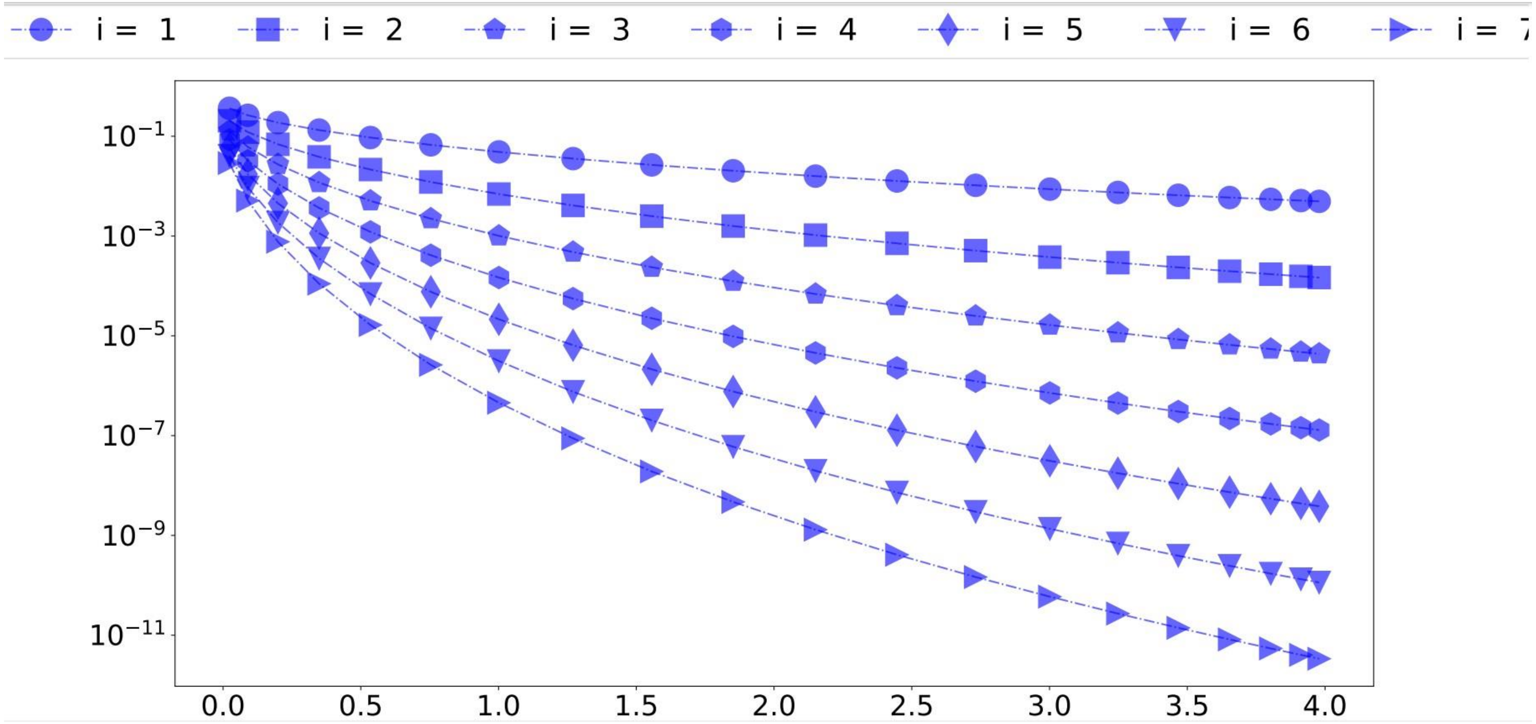
becomes a little easier

Continued fractions and Padé

Having $\lambda = 2 + z$ we get

Theorem. *The function* $\hat{t}^{b}_{N^a}(z)$ *is the* $[n, n]$*-Padé approximation about the expansion point* $z = +\infty$ *of* $\hat{t}^{\infty}_{N^a}(z)$, *where* $n = N^b - N^a$.

Continued fractions and Padé



IRK preconditioners

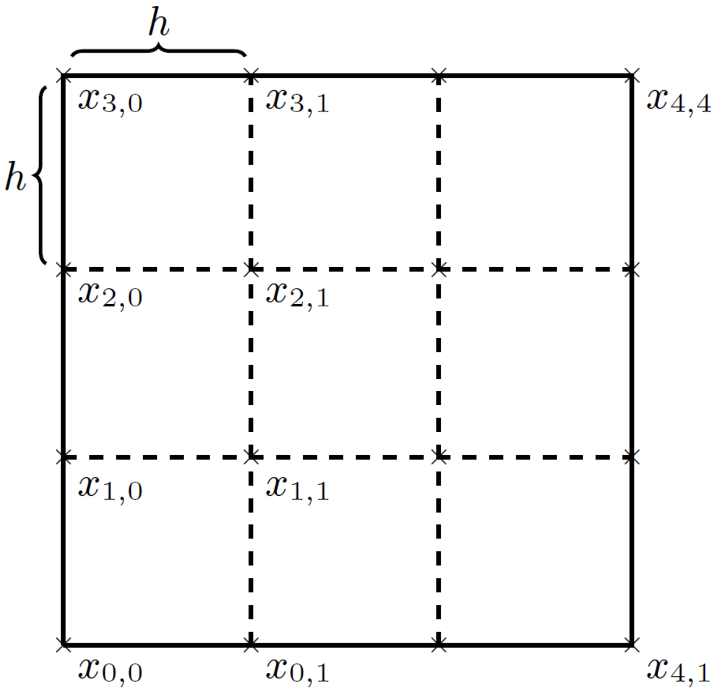
IRK preconditioners

Step I :

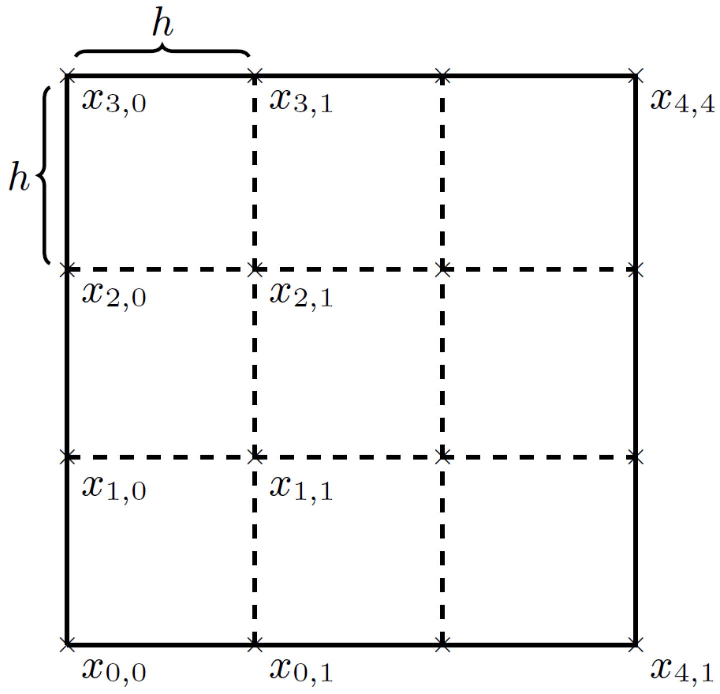
IRK preconditioners

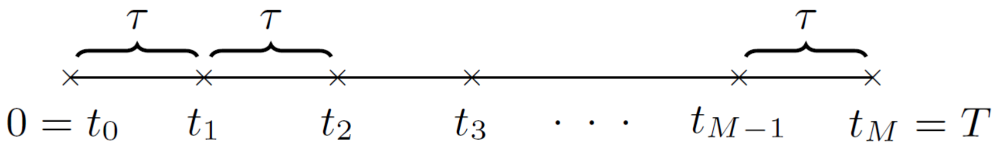
$$\Omega = (0,1) \times (0,1)$$

IRK preconditioners

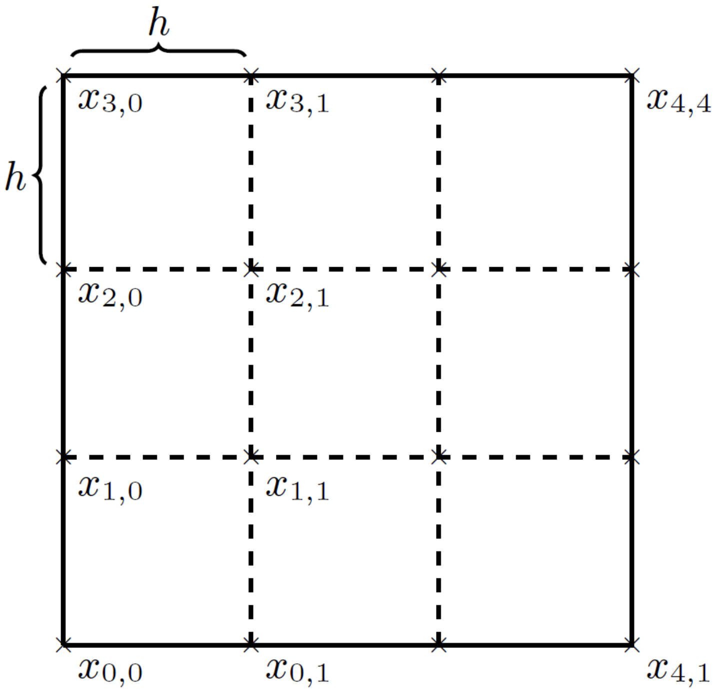


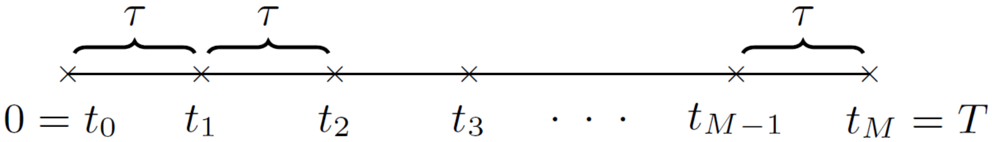
IRK preconditioners





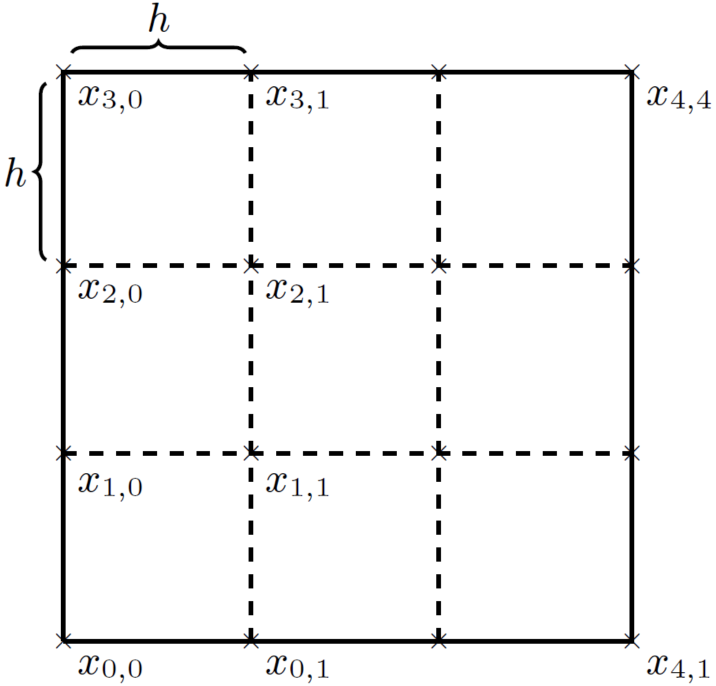
IRK preconditioners

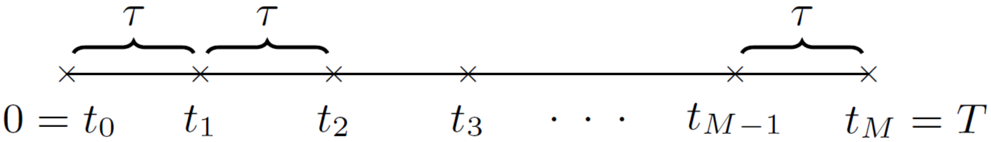




$$\mathbf{u}^m \approx u(t_m, x_{ij})$$

IRK preconditioners





$$\mathbf{u}^m \approx u(t_m, x_{ij})$$

$$\Delta \approx L$$

IRK preconditioners

IRK preconditioners

$$\mathbf{u}^m = \mathbf{u}^{m-1} + \tau \sum_{i=1}^{s} b_i \mathbf{k}_i^m$$

IRK preconditioners

Step I :

$$M\left(P^{\text{diag}}\right)^{-1} \sim \begin{bmatrix} X_{11} & \cdots & X_{1s} \\ \vdots & \ddots & \vdots \\ X_{s1} & \cdots & X_{ss} \end{bmatrix}$$

IRK preconditioners

Step I :

$$M\left(P^{\text{diag}}\right)^{-1} \sim \begin{bmatrix} X_{11} & \cdots & X_{1s} \\ \vdots & \ddots & \vdots \\ X_{s1} & \cdots & X_{ss} \end{bmatrix}$$

$$\text{with} \quad X_{ij} = \text{diag}\left(\xi_1^{(ij)}, \ldots, \xi_n^{(ij)}\right) \qquad \forall ij$$

Preconditioner analysis

Step II :

IRK preconditioners

Step II :

$$X = \begin{bmatrix} X_{11} & \dots & X_{1s} \\ \vdots & \ddots & \vdots \\ X_{s1} & \dots & X_{ss} \end{bmatrix} \quad \sim$$

$$\text{with} \quad X_{ij} = \operatorname{diag}\left(\xi_1^{(ij)}, \dots, \xi_n^{(ij)}\right)$$

$$X \in \mathbb{R}^{ns \times ns}$$

IRK preconditioners

Step II :

$$X = \begin{bmatrix} X_{11} & \dots & X_{1s} \\ \vdots & \ddots & \vdots \\ X_{s1} & \dots & X_{ss} \end{bmatrix} \quad \sim \quad X_k = \begin{bmatrix} \xi_k^{(11)} & \dots & \xi_k^{(1s)} \\ \vdots & \ddots & \vdots \\ \xi_k^{(s1)} & \dots & \xi_k^{(ss)} \end{bmatrix}$$

$$\text{with} \quad X_{ij} = \operatorname{diag}\left(\xi_1^{(ij)}, \dots, \xi_n^{(ij)}\right)$$

$$X_k \in \mathbb{R}^{s \times s}$$

$$X \in \mathbb{R}^{ns \times ns}$$

IRK preconditioners

Lemma. *Let $X \in \mathbb{R}^{ns \times ns}$ and $X_k \in \mathbb{R}^{s \times s}$ be as above and set*

$$\text{eigenpair}\,(X_k) = \left(\boxed{\mu_\ell^{(k)}}, \boxed{\boldsymbol{s}_\ell^{(k)}}\right).$$

Then the eigenpairs of X are equal to $\left(\boxed{\mu_\ell^{(k)}}, \boxed{\boldsymbol{s}_\ell^{(k)} \otimes \mathbf{e}_k}\right)$.