

Tutorial 6

14.11

1. Reachability for VASS with negative semantic belongs to NP.

Brief theoretical introduction (hint) :

Linear programming (LP)

a minimization / maximization problem for a linear function of n arguments : x_1, x_2, \dots, x_n , where the arguments have to satisfy some conditions (linear equations or inequalities of x_i s)

Example:

$$\max x_1 + 2x_2$$

$$\text{s.t. } x_2 \leq x_1 + 2$$

$$2x_1 + x_2 \leq 4$$



solvable in
polynomial
time

Integer programming (IP)

for each $i \in \{1, 2, \dots, n\}$ we add a condition $x_i \in \mathbb{Z}$
this problem is NP-complete

Given: VASS (Q, T)

Question: $(p, u) \dashrightarrow^* (q, v)$?

↑
does there exist a run that can drop below zero

Attempt 1

- * denote transition vectors by v_1, v_2, \dots, v_n
- * solve a set of linear equations

$$\begin{bmatrix} : & : & : \\ \vdots & \vdots & \vdots \\ v_1 & v_2 & \cdots & v_n \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = v - u$$

where x_i is the number of times transition t_i is to be fired (used)

Problem: we can obtain $x_i \notin \mathbb{Z}$ and $x_i < 0$

Attempt 2

and the goal of IP: $\min \sum_{i=1}^n x_i$

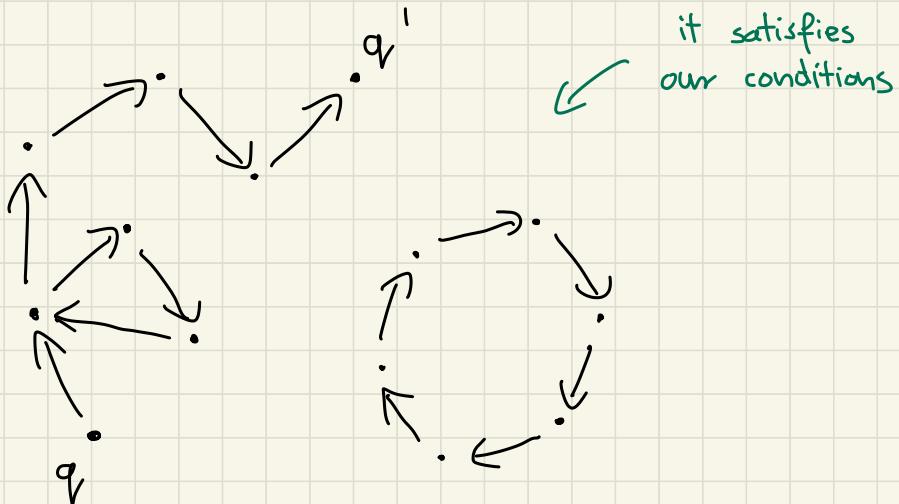
- * add conditions: $x_i \in \mathbb{Z}, x_i \geq 0$ for each i

Problem: firing each transition t_i x_i number of times gives $v - u$ but the transitions may not form a run in VASS

Attempt 3

- * add Kirchhoff's law conditions — for each state except q, q' the number of incoming transitions (sum of x_i) equals the number of outgoing ones
- * for q there is one more outgoing transition and for q' — one more incoming

Problem: x_i can describe a path from q to q' and a disjoint cycle...

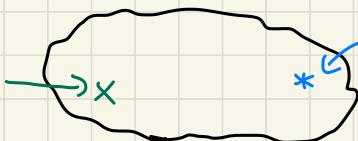


Attempt 4

- * after obtaining x_i 's, define $S = \{t_i \in T \mid x_i > 0\}$
and check whether each $t_i \in S$ can be
reached from q using transitions in S
- * the condition above ensures that we can
create the run given x_i 's (go from q using
any available transition until reaching q' with
the last transition incoming to q' ; then, if we
still have some spare transitions, we can create
some cycles and glue them to the run $q \rightarrow^* q'$)
- * it seems like it should be enough for S to be
connected in order to create such a run

Problem: Solution to our IP program may not
satisfy the condition from attempt 4, but
there can be some other run that does

received solution,
unable to create
a run based on it



another feasible
solution that corresponds
to a run but doesn't
minimize the goal function

Attempt 5 - final approach

- * to ensure that every possible run is taken into account we can guess the transition set S' that we allow the use of and then solve IP with conditions

$$\forall i \in S' \quad x_i > 0$$

$$\forall i \in T \setminus S' \quad x_i = 0$$

2. Using the insights from the previous problem present a way to check condition Θ_1

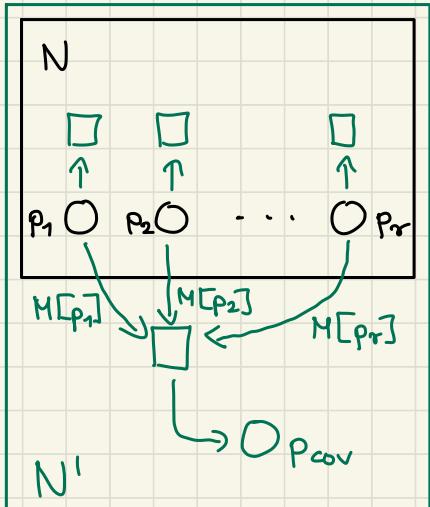
Θ_1 : for every $m \geq 1$ $(q, v) \xrightarrow{*} (q', v')$
using every transition $\geq m$ times

- * if there exists a run r' for some selected $m' > 0$ using every transition $t_i \quad x_i \geq m'$ number of times, then there also exists a run r'' that uses each transition at least $m'' = \max(x_i) + 1$ times (use notation $t_i \rightarrow y_i$)

- * both r' and r'' satisfy the Kirchhoff's law,
hence so does $r'' - r'$ which uses each
transition t_i at least $y_i - x_i > 1$ times
- * thus, $r'' - r'$ is a cycle that has no side
effect on a given configuration
- * the solution idea:
 - define two sets of variables: x_1, x_2, \dots, x_n
and $\Delta_1, \Delta_2, \dots, \Delta_n$ such that both sets
satisfy Kirchhoff's law (first with a
twist for q and q')
 - $x_i > 0$ (for a subset of transitions - as
discussed before), fixing all t_i 's x_i number
of times results in $v' - v$ change on
the configuration
 - $\Delta_i > 0$ for all transitions and fixing
 t_i 's don't change the configuration

4. How to reduce coverability to reachability in general Petri nets?

First solution



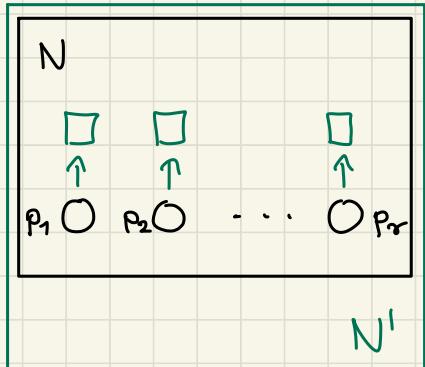
from now on we assume that
each initial net has r places

(N, M_0) : can we cover M ?

$$\xrightarrow{P_{cov}} \frac{(M_0, O)}{=}$$

(N', M'_0) : can we reach $(0, 0, \dots, 0, 1)^T$?

Second solution



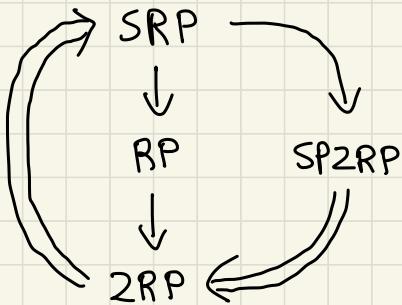
(N, M_0) : can we cover M ?

(N', M_0) : can we reach M ?

5. Show reductions between the following problems:

- 1) reachability problem: $M \in N^r$, is M reachable? (RP)
- 2) submarking reachability problem: \tilde{P} - subset of places, $M_{\tilde{P}}$ - configuration over \tilde{P} , does there exist a reachable configuration M such that over the places of \tilde{P} it agrees with $M_{\tilde{P}}$? (SRP)
- 3) zero reachability problem: is 0 reachable? (2RP)
- 4) single-place zero reachability problem: given a place p , does there exist a reachable configuration M , s.t. $M(p) = 0$? (SP2RP)

The idea is to show the following reductions:



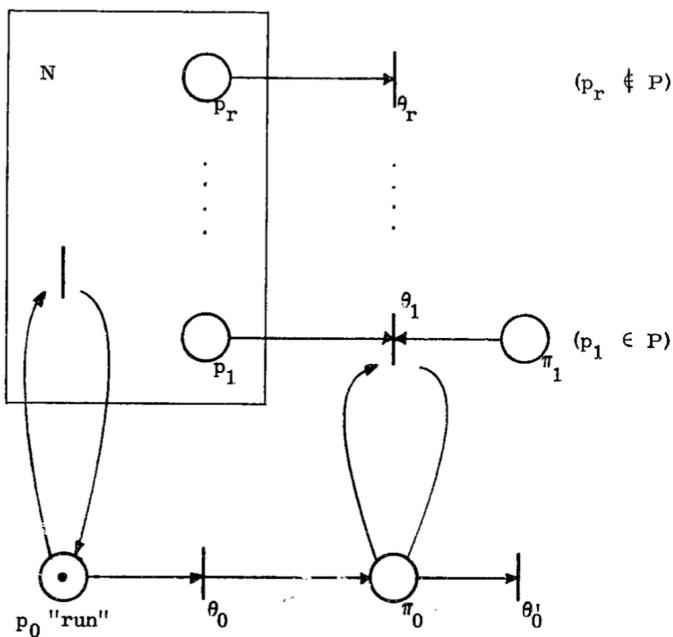
" Y is reducible to X "
notation: $\begin{array}{c} X \\ \longrightarrow \\ Y \end{array}$
know, how
to solve this
can solve
this
 \rightarrow easy reduction
 \Rightarrow reduction to present

$ZRP \Rightarrow SRP:$

$\leftarrow (P, T, F)$

- * given a net (N, M_0) and a submarking $M_{\tilde{P}}$ over a subset $\tilde{P} \subseteq P$, is there a reachable configuration M that agrees with $M_{\tilde{P}}$ over \tilde{P} ?
- * we will answer this question by asking for zero configuration reachability in another net (obtained by modifying (N, M_0))
 - * first, we add a "run" place p_0 that contains one token and is connected with each transaction of the original net in both ways, i.e., for each $t \in T$: $p_0 \in {}^{\circ}t$ and $p_0 \in t^{\bullet}$ (arc weight = 1)
 - * next, we add a transition θ_0 that takes one token from p_0 , i.e., stops the normal run of N and puts a token on the "check" place Π_0 — this token allows the firing of new transitions $\theta_1, \theta_2, \dots, \theta_r$
 - * for $i \in P \setminus \tilde{P}$ transition θ_i takes a token from p_i

- * for $i \in \tilde{P}$ transition θ_i takes one token from p_i and one from π_i , π_i initially having $M_{\tilde{P}}(p_i)$ tokens
- * finally, we have a transition θ'_0 that takes one token from π_0

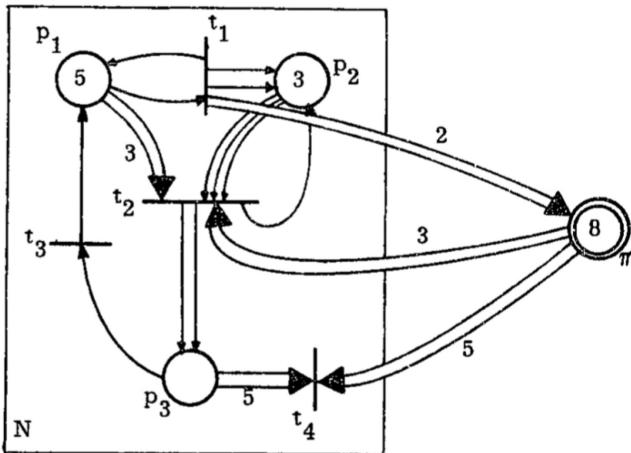


- * it is obvious that zero configuration is reachable iff configuration $M_{\tilde{P}}$ over \tilde{P} is reachable in N

SP2RP \Rightarrow 2RP:

- * now, we can use the blackbox for answering single-place zero reachability problem and have to answer the zero reachability problem in N
- * the idea is to have a new place π s.t. at all times π contains as many tokens as there are in all places of N , i.e., at every configuration M in the new net N' : $M(\pi) = \sum_{i=1}^r M(p_i)$
- * to obtain it, we connect π with all transitions of N with incoming and outgoing arcs - the weight of incoming arc is the sum of incoming arcs' weights in N and analogously for outgoing
- * initially, π contains the total number of tokens in configuration M_0 for N
- * obviously, $t_j \in T$ is fireable in N iff t_j is fireable in N'
- * answer for 2RP in N = answer for SP2RP in N'

* obtaining N' from N - example



Comment on the reachability problems

For some of the problems, we would like to investigate whether at least one configuration in a given set of configurations is reachable. If this set is finite, it involves just a finite number of using a blackbox for RP. If the set is infinite, we have to use a different approach. However, we have already considered such examples, for instance, submarking reachability problem or single-place zero reachability problem.

6. Structural unboundedness for general Petri nets
belongs to NP.

homework (not obligatory)