Tutorial 7 21.11

1. How to find all P-invariants of a given net?

* P-invariant I has to satisfy equation $I \cdot N = \vec{0}$ where $N = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ t_1 & t_2 & \cdots & t_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$ is a matrix consisting of all available transitions

* hence, to find all P-invariants, one has to solve the set of linear equations

2. Define a polynomial algorithm checking whether a given set of linear equations has a solution satisfying a given set of implications of form $x \geq 0 \Rightarrow y \geq 0$, where x, y are variables.

* a detailed solution is presented in the following video (by Piotr Hofman)

https://drive.google.com/file/d/11U54tA4LXrJoqlJswaMTf7a-bQXlmv1q/view?usp=sharing

3. Structural unboundedness for general Petri nets belongs to NP.

What condition should be satisfied by a net to be structurally unbounded?

Intuition: there exists a sequence of transitions that has a non-negative effect on all places and positive effect on at least one of them

To justify it, we can prove the following fact: the conditiones listed below are equivalent:

$$N = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ t_1 & t_2 & \cdots & t_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

1) place p is structurally bounded,

2) there exists $y \geqslant \mathbb{1}_p$ s.t. $y^T N \leqslant \mathbf{0}$, where $\mathbb{1}_p \in \mathbb{Z}^{|P|}$, $\mathbb{1}_p[q] = 1$ if $q = p$ and 0 otherwise,

3) no $x \geqslant \mathbf{0}$ satisfies $Nx \geqslant \mathbb{1}_p$. $x \in \mathbb{N}^d$, $y \in \mathbb{N}^n$

Why can't we use coverability tree?

structurally
||
$\forall$ configuration

1) $\Rightarrow$ 3)

* we will prove $\neg 3) \Rightarrow \neg 1)$ (which is equivalent)

* let $x \in \mathbb{N}^d$ be a vector such that $Nx \geqslant \mathbb{1}_p$

* we can interpret x as a multiset of transitions and say that we fire x in a given net, meaning that we fire each transition the number of times it occures in x

* let M be the initial configuration big enough to fire x, then from M we obtain $M_1 > M$ on at least place p

* of course we can fire x in M_1 and iterate this process further, proving that place p is unbounded

3) $\Rightarrow$ 2)

* to prove this implication, we use the following fact from the theory of dual programs

Farkas' lemma

Theorem. Exactly one of the following equation systems has a solution:

1) $Ax \geq b$

2) $\bar{y} \geq \mathbf{0}$
$\bar{y}^T A = \mathbf{0}$
$\bar{y}^T b > 0$

* we have to prove:

no $x \geq \mathbf{0}$ s.t. $Nx \geq \mathbb{1}_p$ $\Rightarrow$ $\exists\, y \geq \mathbb{1}_p$ s.t. $y^T N \leq \mathbf{0}$

* we observe the following

$$\begin{bmatrix} N \\ I \end{bmatrix} \cdot x \geq \underbrace{\begin{bmatrix} \mathbb{1}_p \\ \mathbf{0} \end{bmatrix}}_{=:\, b}$$

no solution

$\Rightarrow$

$$\bar{y} \geq \mathbf{0}$$
$$\bar{y}^T \cdot \begin{bmatrix} N \\ I \end{bmatrix} = \mathbf{0}$$
$$\bar{y}^T \cdot b > 0$$

has solution

* green box represents $Nx \geq \mathbb{1}_p$ and $x \geq \mathbf{0}$
* to analyze the blue box let us write

$$\bar{y}^T = [\, y^T \;\vdots\; z^T \,]$$

* it is easy to notice that $\overset{\text{solution}}{\bar{y}} \in \mathbb{Q}^{|P|+|T|}$

* first, it holds that

$$[y^T \,\vdots\, z^T] \cdot \begin{bmatrix} N \\ \hline I \end{bmatrix} = \mathbf{0} \iff y^T N + \underbrace{\underbrace{z^T}_{\geqslant \mathbf{0}} I}_{\geqslant \mathbf{0}} = \mathbf{0}$$

hence, $y^T N \leq \mathbf{0}$

* second, we have that $\bar{y} \geqslant \mathbf{0} \Rightarrow y \geqslant \mathbf{0}$ and

$$[y^T \,\vdots\, z^T] \cdot \begin{bmatrix} \mathbb{1}_p \\ \mathbf{0} \end{bmatrix} > 0 \iff y^T \cdot \mathbb{1}_p + z^T \cdot \mathbf{0} > 0$$

thus, $y^T \cdot \mathbb{1}_p > \mathbf{0} \Rightarrow y[p] > 0$

* however, $y \in \mathbb{Q}^{|P|}$ and hence we have to multiply all its coordinates by some number $K \in \mathbb{N}$ s.t. $Ky \in \mathbb{N}^{|P|}$ and $(Ky)[p] \geqslant 1$

2) $\Rightarrow$ 1)

* we assume that $\exists\, y \geqslant \mathbb{1}_p$ s.t. $y^T N \leqslant \mathbf{0}$
* look at initial conf. i and any reachable $z = i + Nx$
* we would like to prove $z[p] \leqslant \overset{\text{one for all } z}{C}$ for some $C \in \mathbb{N}$
* we can consider bounding $\overbrace{y[p]}^{\geqslant 1} \cdot z[p]$ instead
* $y[p] \cdot z[p] \leqslant \underset{\text{both non-negative}}{y^T z} = \underbrace{y^T i}_{:= m \geqslant 0} + \underbrace{y^T N}_{:= u \leqslant 0} x = m + \underset{\leqslant 0}{u}\,\underset{\geqslant 0}{x} \leqslant m$

* finally, we obtain $z[p] \leqslant \frac{m}{y[p]}$

Thus, to check structural unboundedness it is enough to check whether there exists a sequence of transitions that has a non-negative effect on all the places and positive effect for at least one of them.

* we use analoguous algorithm as the last time (integer programming)
* this time instead of equalities we have a set of inequalities
* we have to guess a place that increases its token count after firing the sequence of transitions

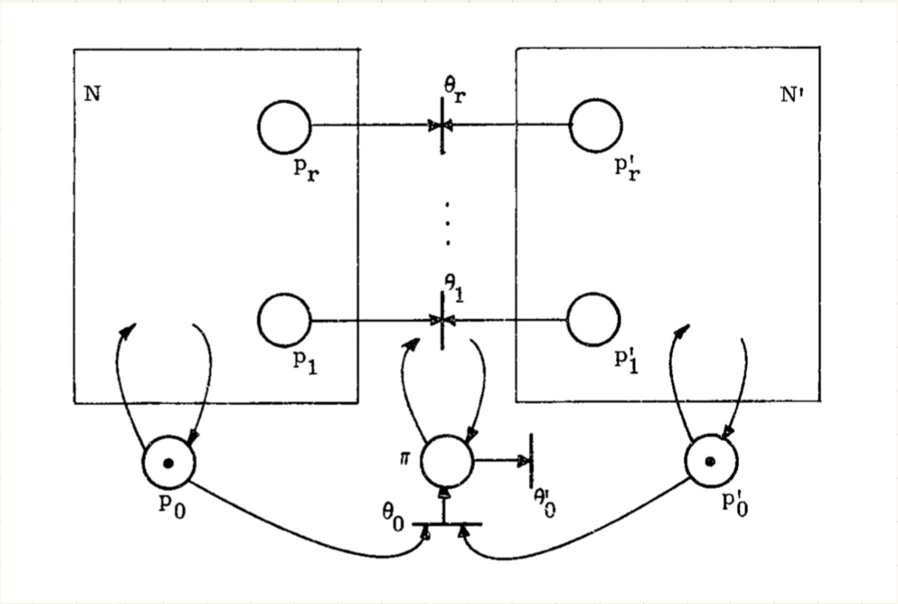
* you can work on the details of this algorithm on your own

A set $S \subseteq \mathbb{N}^r$ is RP-solvable iff the problem of deciding whether there exists a reachable configuration in S for a given net N with initial configuration M_0 is reducible to RP.

4. Every Reachability Set (a set of all configurations reachable in some net (N, M_0)) is RP-solvable.

   * let $R_N(M_0) \subseteq \mathbb{N}^r$ be the Reachability Set of (N, M_0)
   * our task is to show that for every other Petri net (N', M_0') of r places we can decide whether $R_N(M_0) \cap R_{N'}(M_0') \neq \emptyset$ using a blackbox for RP
   * instead of reducing to RP, we can reduce to 2RP
   * given N, N', we construct a new net N''
   * for each of the initial nets, we add a new „run" place – p_0 and p_0', respectively – as we've done previously

* to stop the usual runs of N and N', we add a new transition θ_0 that takes a tokens from p_0 and p_0', and puts a token on a new place π
* then, we check whether configurations of N and N' agree using transitions $\theta_1, \theta_2, \ldots, \theta_r$



* it is easy to see that N'' can reach the zero configuration iff some configuration can be reached in both N and N'

5. Prove that reachability is reducible to non-liveness in general Petri nets.

homework (not obligatory)