Foundations of Mathematics (MATH 5000)
HANDOUT 1 (October 3–6)

Proofs and the Completeness Theorem

A proof is a finite procedure, based on some logical rules, that is used to establish that a statement is a consequence of other statements (or assumptions).

In first order logic 'statements' are formalized by sentences of a given language L, and 'consequence' is captured by the notion of entailment (logical consequence), $\Sigma \models \varphi$, where Σ is a set of L-sentences and φ is an L-sentence. It will be useful to extend the concept of entailment to arbitrary formulas. Let $\bar{v} = (v_0, v_1, \ldots)$ be the sequence of variables in L, so $\Gamma = \Gamma(\bar{v})$ and $\varphi = \varphi(\bar{v})$. We say that Γ *entails* φ, and write $\Gamma \models \varphi$, if for each nonempty L-structure $\mathcal{M}$ and each tuple $\bar{a}$ in M (matching $\bar{v}$) such that $\mathcal{M} \models \gamma(\bar{a})$ for all $\gamma \in \Gamma$, we have that $\mathcal{M} \models \varphi(\bar{a})$.

How can we formalize a 'proof'? Proofs have two essential ingredients:

- logical axioms: statements that are true by their 'logical structure'; for example, "S or not S" for arbitrary statement S.
- rules of inference: rules saying that from statements of a certain form one can deduce another statement; for example, *modus ponens* is the rule of inference saying that from statements "R" and "if R then S" one can deduce "S".

To formalize the concept of a proof in first order logic it will be more convenient to use $\neg$, $\to$, $\forall$, and $=$ as our primary logical symbols (so $\wedge$, $\vee$, $\exists$ will be abbreviations). We adopt this convention throughout this handout. The choice of logical axioms and rules of inference for a deductive system is far from unique. The system we will discuss here will have only one rule of inference, modus ponens, and a set Λ of valid (logically true) L-formulas (not necessarily sentences) as logical axioms, which we will specify later.

Definition. Let $\Gamma \subseteq L$ and $\varphi \in L$. A *proof* of φ from Γ is a finite sequence $(\alpha_1, \alpha_2, \ldots, \alpha_m)$ of L-formulas such that $\alpha_m = \varphi$ and for each i ($1 \leq i \leq m$) one of the following holds:

- $\alpha_i \in \Lambda$, that is, α_i is a logical axiom;
- $\alpha_i \in \Gamma$;
- there exist $j, k < i$ ($j, k \geq 1$) such that $\alpha_k = \alpha_j \to \alpha_i$; that is, α_i is obtained from two earlier formulas α_j and $\alpha_k = \alpha_j \to \alpha_i$ in the sequence by *modus ponens*.

We say that φ is *provable* from Γ, and write $\Gamma \vdash \varphi$, if there exists a proof of φ from Γ.

Since we want to use proofs to establish logical consequence, a primary requirement on a deductive system is

> $\boxed{\text{soundness:}}$ a formula φ is provable from a set Γ of formulas only if φ is a consequence of Γ.

Modus ponens is a sound rule of inference, because $\{\alpha, \alpha \to \beta\} \models \beta$ for all L-formulas α and β. Since the logical axioms are valid formulas, one can easily show that any deductive system described above is sound.

Soundness Theorem: *If* $\Gamma \vdash \varphi$, *then* $\Gamma \models \varphi$.

Examples. Let's look at the two extreme choices for Λ.

1. If Λ is the set of all valid L-formulas, then every valid formula φ has a one-step proof from $\Gamma = \emptyset$, but to check whether or not this is a proof is nothing else than checking whether or not φ is valid. Such a deductive system does not help establishing logical consequence.
2. If $\Lambda = \emptyset$, then no formula is provable from $\Gamma = \emptyset$. Such a deductive system is useless in establishing nontrivial instances of logical consequence.

We want the set of logical axioms of our deductive system to be small enough so that

> proofs can be effectively checked: if Γ is finite, an algorithm can decide whether or not a given sequence of formulas is a proof,

and large enough so that we have

> completeness: φ is provable from Γ whenever φ is a consequence of Γ.

Our goal is to show that such a deductive system exists. First we will describe the set of axioms we will use. We call a formula of the form $\forall x_1 \cdots \forall x_n\, \psi$ where $x_1, \ldots, x_n$ are variables of L and ψ is an L-formula a *generalization of ψ.*

Axioms. The set Λ of axioms consists of all generalizations of the formulas listed in (i)–(viii) below where x, y are arbitrary variables in L and α, β, γ are arbitrary L-formulas:

(i) $\alpha \to (\beta \to \alpha)$,
(ii) $(\gamma \to (\alpha \to \beta)) \to ((\gamma \to \alpha) \to (\gamma \to \beta))$,
(iii) $(\neg\beta \to \neg\alpha) \to (\alpha \to \beta)$,
(iv) $\forall x\, \alpha \to \alpha_x(t)$
for each L-term t that is substitutable for x in α,
(v) $\forall x(\alpha \to \beta) \to (\forall x\, \alpha \to \forall x\, \beta)$,
(vi) $\alpha \to \forall x\, \alpha$
if x does not occur free in α,
(vii) $x = x$,
(viii) $x = y \to (\alpha \to \alpha')$
where α is atomic, and α' is obtained from α by replacing several (zero or more) occurences of x by y.

Since there is an algorithm for parsing formulas[1], it follows easily that there exists an algorithm that decides for each formula (in fact, for each string of symbols) whether or not it is one of the axioms. Hence the deductive system described above satisfies the requirement that proofs can be checked effectively.

The proof of completeness requires some preparation.

A formula of propositional logic (with logical connectives $\neg$ and $\to$) is a well-formed expression built up from statement symbols, using the connectives $\neg$ and $\to$.[2] A formula of propositional logic is called a tautology if it is true for all truth assignments of the statement symbols. An L-formula is called a *tautology* if it is obtained from a tautology of propositional logic by replacing each statement symbol by an L-formula.

For example, the formulas in (i)-(iii) above are tautologies.

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[1]This is one way of proving unique readability.

[2]The precise definition proceeds by recursion.

The next five theorems state important properties of our deductive system which are needed for the proof of completeness. In the theorems Γ always denotes an arbitrary set of L-formulas, and φ, ψ, $\varphi_1, \ldots, \varphi_n$ are arbitrary L-formulas.

Theorem on Tautologies.

(1) *Every tautology is provable from $\emptyset$, using only axioms in (i)–(iii).*
(2) *If $\Gamma \vdash \varphi_1, \ldots, \varphi_n$ and $\varphi_1 \to (\varphi_2 \to (\ldots(\varphi_n \to \varphi)\ldots))$ is a tautology, then $\Gamma \vdash \varphi$.*

Part (1) is essentially a completenes theorem for propositional logic. The proof is rather technical. We could have avoided using a completeness theorem for propositional logic by choosing a larger set Λ' of logical axioms, which is the same as Λ, except that it contains the generalizations of all tautologies (rather than the generalizations of the tautologies (i)–(iii) only). For the deductive system with logical axioms Λ' the statement in (1) becomes trivial. This deductive system also satisfies the requirement that proofs can be checked effectively, because truth tables provide an effective method for checking whether or not a propositional formula is a tautology.

Part (2) is an easy consequence of part (1), using modus ponens. Note that the formula $\varphi_1 \to (\varphi_2 \to (\ldots(\varphi_n \to \varphi)\ldots))$ in (2) is logically equivalent to $(\varphi_1 \wedge \varphi_2 \wedge \ldots \wedge \varphi_n) \to \varphi$.

Generalization Theorem. *If $\Gamma \vdash \varphi$ and x does not occur free in Γ, then $\Gamma \vdash \forall x\, \varphi$.*

This theorem can be proved by induction on the length of a proof of φ from Γ, using axioms of types (v) and (vi) (i.e., generalizations of formulas in (v) and (vi)).

Deduction Theorem. *If $\Gamma \cup \{\varphi\} \vdash \psi$ then $\Gamma \vdash \varphi \to \psi$.*

This can also be proved by induction on the length of a proof of φ from Γ, using axioms of types (i) and (ii). The converse of the Deduction Theorem is also true; the proof is an easy application of modus ponens.

Theorem on Equality. *We have*

- $\vdash \forall x\,(x = x)$,
- $\vdash \forall x\, \forall y\,(x = y \to y = x)$,
- $\vdash \forall x\, \forall y\, \forall z\,(x = y \to (y = z \to x = z))$.

Furthermore, for arbitrary function symbol f (say n-ary)

- $\vdash \forall \bar{x}\, \forall \bar{y}\,(x_1 = y_1 \to (x_2 = y_2 \to (\ldots(x_n = y_n \to f(\bar{x}) = f(\bar{y}))\ldots)))$,

and for arbitrary relation symbol R (say n-ary) in L,

- $\vdash \forall \bar{x}\, \forall \bar{y}\,(x_1 = y_1 \to (x_2 = y_2 \to (\ldots(x_n = y_n \to (R(\bar{x}) \to R(\bar{y})))\ldots)))$.

The proof uses the axioms of types (vii) and (viii) and earlier theorems.

Generalization on Constants.

(1) *If $\Gamma \vdash \varphi$ and c is a constant symbol that does not occur in Γ, then there is a variable y (not occurring in φ) such that the formula $\forall y\, \varphi_c(y)$ has a proof from Γ in which c does not occur.*
(2) *In particular, if $\Gamma \vdash \varphi_x(c)$ where the constant symbol c does not occur in any formula in Γ, then the formula $\forall x\, \varphi$ has a proof from Γ in which c does not occur.*

In part (1) $\varphi_c(y)$ denotes the formula obtained from φ by replacing each occurrence of c in φ by y. The idea of the proof is to find first an appropriate proof of $\varphi_c(y)$ from a finite subset of Γ (using a proof of φ from Γ), and then use the generalization theorem.

The proof of part (2) relies on part (1) and axioms of type (iv).

We will sketch the proof of the completeness theorem for sentences only. With slight modifications the proof works also for arbitrary formulas.

So from now on Σ will be a set of L-sentences and φ an L-sentence.

Completeness Theorem: If $\Sigma \models \varphi$, then $\Sigma \vdash \varphi$.

We need a few definitions.

Definitions. For a set Σ of L-sentences let $\Sigma^{\vdash}$ denote the set of all L-sentences that are provable from Σ, and call it the *deductive closure* of Σ. We say that Σ is *deductively closed* if $\Sigma = \Sigma^{\vdash}$. If $\Sigma^{\vdash}$ contains a contradiction (i.e., a sentence of the form $\neg(\beta \to \beta)$ for some $\beta \in L_0$), then Σ is called *inconsistent*; otherwise Σ is *consistent*.[3] An *L-theory* is a consistent, deductively closed set of L-sentences.

Let Σ be an arbitrary set of L-sentences. The following statements are easy to show:

- $\Sigma^{\vdash}$ is deductively closed for all $\Sigma \subseteq L_0$.
- Σ is inconsistent if and only if $\Sigma^{\vdash}$ is the set of all L-sentences.
  [By the Theorem on Tautologies.]
- (Lindenbaum's Theorem) Every consistent set $\Sigma \subseteq L_0$ is contained in a maximal consistent set $\Delta \subseteq L_0$.
  [By Zorn's Lemma.]
- If $\varphi \notin \Sigma^{\vdash}$, then $\Sigma \cup \{\neg\varphi\}$ is consistent.
  [By the Deduction Theorem and the Theorem on Tautologies.]
- Every maximal consistent set $\Delta \subseteq L_0$ is deductively closed, and has the property that $\varphi \in \Delta$ or $\neg\varphi \in \Delta$ for all $\varphi \in L_0$;
  [By the previous two items.]
- (Conservation Lemma) If Σ is a consistent set of L-sentences, then Σ is also consistent as a set of $L(C)$-sentences for arbitrary set C of new constants.
  [By Generalization on Constants.]

Definition. Let T be an L-theory. We say that T *has witnesses in L* if for every L-formula $\psi = \psi(x)$ with at most one free variable there exists a constant symbol c in L such that T contains the formula $\neg\forall x\, \psi(x) \to \neg\psi(c)$.

Sketch of proof of the Completeness Theorem. We prove first that the Completeness Theorem is equivalent to the following statement:

$(*)$ Every consistent set of sentences has a model.

(CT) $\Rightarrow$ $(*)$: If Σ does not have a model, then $\Sigma \models \neg(\beta \to \beta)$ for $\beta \in L_0$, so by the Completeness Theorem $\Sigma \vdash \neg(\beta \to \beta)$, that is, Σ is inconsistent.

$(*)$ $\Rightarrow$ (CT): If $\Sigma \models \varphi$, then $\Sigma \cup \{\neg\varphi\}$ does not have a model, so by $(*)$ $\Sigma \cup \{\neg\varphi\}$ is inconsistent. Now the Deduction Theorem and the Theorem on Tautologies imply that $\Sigma \vdash \neg\varphi \to \neg(\beta \to \beta) \vdash \varphi$.

[3] These are the usual definitions of 'deductive closure' and '(in)consistent', not the ones in Rothmaler's text.

Therefore it suffices to prove $(*)$. Let Σ be a consistent set of L-sentences.

Step 1. *There exist a language $L(C)$ obtained from L by adding new constant symbols and an $L(C)$-theory T such that $\Sigma \subseteq T$ and T has witnesses in $L(C)$.*

Let $\lambda = |L|$ be the cardinality of L. Let C be a set of new constant symbols such that $|C| = \lambda$, and fix a well-ordering of C isomorphic to the natural ordering of λ. By the Conservation Lemma Σ is consistent as a set of $L(C)$-sentences. The set of all $L(C)$-formulas $\psi = \psi(x)$ with at most one free variable also has cardinality λ, therefore we can enumerate these formulas by ordinals less than λ: $\psi_\gamma = \psi_\gamma(x_\gamma)$, $\gamma < \lambda$. For each $\gamma < \lambda$ let θ_γ be the $L(C)$-sentence

$$\neg\forall x_\gamma\, \psi_\gamma(x_\gamma) \to \neg\psi(c_\gamma)$$

where c_γ is the first symbol in C that does not occur in ψ_γ or any θ_δ, $\delta < \gamma$.

We let $\Theta = \{\theta_\gamma : \gamma < \lambda\}$, and define T to be the deductive closure of $\Sigma \cup \Theta$. We claim that T is an $L(C)$-theory that has witnesses in $L(C)$. The only thing that requires proof is that T, or equivalently, $\Sigma \cup \Theta$ is consistent.

Suppose $\Sigma \cup \Theta$ is inconsistent. Since Σ is consistent, there exist $m \geq 0$ and $\theta'_1, \ldots, \theta'_{m+1} \in \Theta$ such that $\Sigma \cup \{\theta'_1, \ldots, \theta'_{m+1}\}$ is inconsistent. We assume here that $\theta'_i = \theta_{\gamma_i}$ with $\gamma_1 < \cdots < \gamma_m < \gamma_{m+1}$. Choose such a set with m as small as possible. Since θ'_{m+1} is of the form $\neg\forall x\, \psi(x) \to \neg\psi(c)$, the Deduction Theorem and the Theorem on Tautologies imply that

$$\Sigma \cup \{\theta'_1, \ldots, \theta'_m\} \vdash \neg\theta'_{m+1} \vdash \neg\forall x\, \psi(x),\ \psi(c).$$

The constant symbol $c = c_{\gamma_{m+1}}$ does not occur in $\Sigma \cup \{\theta'_1, \ldots, \theta'_m\}$, therefore by Generalization on Constants,

$$\Sigma \cup \{\theta'_1, \ldots, \theta'_m\} \vdash \forall x\, \psi(x).$$

But then $\Sigma \cup \{\theta'_1, \ldots, \theta'_m\}$ is inconsistent, which contradicts the minimality of m.

Step 2. *Let Δ be a maximal consistent set of $L(C)$-sentences that contains T, and let B be the set of all constant $L(C)$-terms. There exists an equivalence relation $\equiv$ on B and an $L(C)$-structure $\mathcal{A}$ on the set $A = B/{\equiv}$ of $\equiv$-classes such that for every $L(C)$-sentence φ*

$$(\dagger) \qquad \mathcal{A} \models \varphi \quad \Leftrightarrow \quad \varphi \in \Delta.$$

In particular, it follows that the reduct of $\mathcal{A}$ to the language L is a model of Σ.

We define the relation $\equiv$ on B as follows:

$$b \equiv b' \quad \Leftrightarrow \quad b = b' \in \Delta.$$

The Theorem on Equality (combined with applications of axioms of type (iv)) implies that $\equiv$ is an equivalence relation, and that an $L(C)$-structure can be defined on the set $A = B/{\equiv}$ of equivalence classes as follows:

- for each constant symbol c in $L(C)$, $c^{\mathcal{A}} = c/{\equiv}$,
- for each n-ary function symbol f in $L(C)$, $f^{\mathcal{A}}(b_1/{\equiv}, \ldots, b_n/{\equiv}) = f(b_1, \ldots, b_n)/{\equiv}$,
- for each n-ary relation symbol R in $L(C)$, $R^{\mathcal{A}}(b_1/{\equiv}, \ldots, b_n/{\equiv}) \;\Leftrightarrow\; R(b_1, \ldots, b_n) \in \Delta$.

This definition ensures that $(\dagger)$ is true for atomic sentences.

Exercise. Show (without using the Completeness Theorem) that for every constant term b in the language $L(C)$,

$$\vdash \neg\forall x\, \neg(b = x).$$

Deduce from this that every element of the structure $\mathcal{A}$ is of the form $c/{\equiv}$ for some new constant symbol $c \in C$.

The proof of (†) for arbitrary $L(C)$-sentences proceeds by induction on the complexity of φ.[4] For example, if φ is of the form $\forall x\, \psi(x)$, then (†) can be proved as follows. Assume first that $\mathcal{A} \models \forall x\, \psi(x)$. Using the fact that T has witnesses in $L(C)$, choose $c \in C$ such that the formula $\neg\forall x\, \psi(x) \to \neg\psi(c)$ is in T, and hence in Δ. By the Theorem on Tautologies, we get that $\Delta \vdash \psi(c) \to \forall x\, \psi(x)$, and from the assumption $\mathcal{A} \models \forall x\, \psi(x)$ we get that $\mathcal{A} \models \psi(c/\!\equiv)$, that is, $\mathcal{A} \models \psi(c)$. The induction hypothesis for $\psi(c)$ implies that $\psi(c) \in \Delta$. Thus $\Delta \vdash \psi(c),\ \psi(c) \to \forall x\, \psi(x) \vdash \forall x\, \psi(x)$. Since Δ is deductively closed, we get that $\forall x\, \psi(x) \in \Delta$. Now assume that $\mathcal{A} \not\models \forall x\, \psi(x)$. Then some element of $\mathcal{A}$ does not satisfy ψ, so by the Exercise, there exists $c \in C$ such that $\mathcal{A} \not\models \psi(c/\!\equiv)$, that is, $\mathcal{A} \not\models \psi(c)$. The induction hypothesis for $\psi(c)$ implies that $\psi(c) \notin \Delta$. Hence $\neg\psi(c) \in \Delta$. Since the axiom $\forall x\, \psi(x) \to \psi(c)$ is in Δ we get have that $\Delta \vdash \neg\psi(c),\ \neg\psi(c) \to \neg\forall x\, \psi(x) \vdash \neg\forall x\, \psi(x)$. Hence $\neg\forall x\, \psi(x) \in \Delta$ and therefore $\forall x\, \psi(x) \notin \Delta$.

This proof has the following important consequence.

Corollary. *Every consistent set Σ of L-sentences has a model of cardinality $\le |L|$.*

Remark. The following more general version of the Completeness Theorem is also true: for arbitrary set Γ of L-formulas and for arbitrary L-formula φ,

$$\text{if } \Gamma \models \varphi \text{ then } \Gamma \vdash \varphi.$$

Equivalently, if no contradiction is provable from a set Γ' of L-formulas, then there exists a nonempty L-structure $\mathcal{A}'$ and a tuple $\bar{a} = (a_0, a_1, \ldots)$ of elements in $\mathcal{A}'$ corresponding to the variables in L such that $\mathcal{A}' \models \sigma(\bar{a})$ for all $\sigma \in \Gamma'$.

The construction of $\mathcal{A}'$ is similar to the construction of $\mathcal{A}$, except that one uses all $L(C)$-terms instead of the constant $L(C)$-terms. To deal with the issue of substitutability another preparatory theorem is needed, which states that the bound variables of a formula φ can be suitably changed to obtain a 'deductively equivalent' formula φ' (i.e., a φ' such that $\varphi \vdash \varphi'$ and $\varphi' \vdash \varphi$) into which a given term is substitutable.

Compactness Theorem. *For arbitrary set Σ of L-sentences, Σ has a model if and only if every finite subset of Σ has a model.*

Proof. Let $\beta \in L_0$. For any set Σ' of L-sentences,

$$\begin{aligned} \Sigma' \text{ has no model} \quad &\Leftrightarrow \quad \Sigma' \models \neg(\beta \to \beta) \\ &\Leftrightarrow \quad \Sigma' \vdash \neg(\beta \to \beta) \quad \text{(by Soundness and Completeness).} \end{aligned}$$

Since proofs are finite, $\Sigma \vdash \neg(\beta \to \beta)$ if and only if $\Sigma_0 \vdash \neg(\beta \to \beta)$ for some finite $\Sigma_0 \subseteq \Sigma$. Thus Σ has no model if and only if some finite subset of Σ has no model.

[4] Since every element of $\mathcal{A}$ is named by a constant symbol, it is possible to do induction on sentences.