The volume of an n-dimensional hypersphere

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Abstract

An explicit calculation of the volume of a hypersphere in Euclidean space using only elementary methods is given.

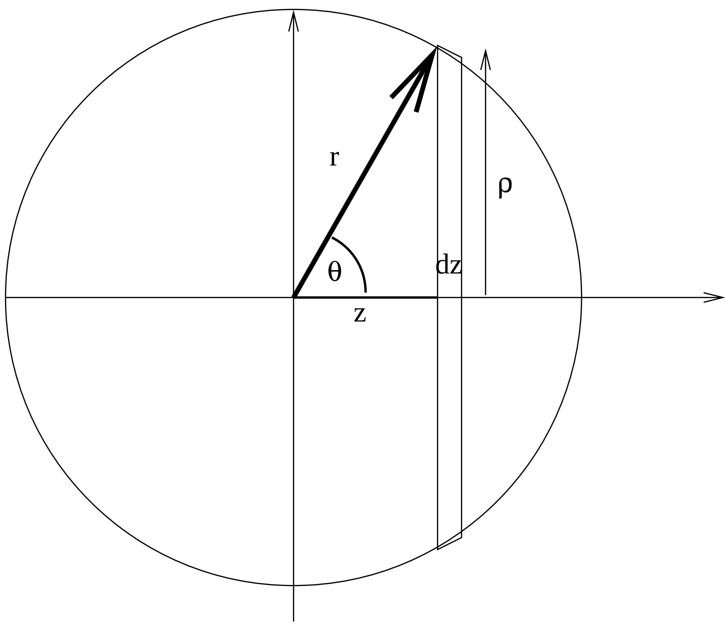
1 Introduction

The volume of an n-dimensional hypersphere is often needed in information theory among other places.

Write $V(r)$ for the volume of an n-dimensional hypersphere S_r of radius r. That is the volume of the set of points

$$S_r = \{(x_1, \ldots, x_n) \mid \sum_{i=1}^{n} x_i^2 = r^2, x_i \in \mathbb{R}\}$$

The volume here is integrated from that of elementary n-cuboid elements with volume as the product of their sides. This corresponds to all the usual measures including Riemann and Lebesgue.



It is clear that

$$V_n(r) = \int_{-r}^{r} V_{n-1}(\rho) dz$$

because when the n^{th} coordinate has a value z, the other coordinates range over a $(n-1)$-sphere of radius $\rho = r\sin\theta$. The n-volume element matching an increment dz is thus $V_{n-1}(\rho)dz$[1].

So we have

$$V_n(r) = r\int_0^\pi \sin\theta\, V_{n-1}(r\sin\theta)\,d\theta$$

But we can write $V_n(r) = r^n\alpha(n)$, so

$$\alpha(n) = \int_0^\pi \sin^n\theta\alpha(n-1)\,d\theta = \alpha(n-1)\int_0^\pi \sin^n\theta\,d\theta$$

or

$$\alpha(n) = I(n)\alpha(n-1) \quad \text{with} \quad I(n) = \int_0^\pi \sin^n\theta\,d\theta$$

So

$$\alpha(n) = I(n)I(n-1)\ldots I(2)\alpha(1) \tag{1}$$

Although $I(n)$ is related to standard integrals, we can evaluate it with elementary methods.

2 Properties of $I(n)$

$$I(0) = \int_0^\pi d\theta = \pi, \quad I(1) = \int_0^\pi \sin\theta\,d\theta = [\cos\theta]_\pi^0 = 2$$

For $n \geq 2$, we can integrate by parts:

$$\begin{aligned} I(n) &= \int_0^\pi \sin^n\theta\,d\theta = \int_0^\pi \sin\theta\sin^{n-1}\theta\,d\theta \\ &= \left[(-\cos\theta)\sin^{n-1}\theta\right]_0^\pi + \int_0^\pi \cos^2\theta(n-1)\sin^{n-2}\theta\,d\theta \\ &= (n-1)\int_0^\pi (1-\sin^2\theta)\sin^{n-2}\theta\,d\theta \\ &= (n-1)\left[I(n-2) - I(n)\right] \end{aligned}$$

So

$$I(n) = \frac{n-1}{n}I(n-2) \quad (n \geq 2) \tag{2}$$

Notice that this gives $I(2) = \frac{1}{2}I(0) = \frac{\pi}{2}$.

From equation 1, we see that it will be useful to calculate $\Pi(n) = I(n)I(n-1)I(n-2)\ldots I(1)I(0)$. Equation 2 gives

$$I(n)I(n-1) = \frac{n-2}{n}\,I(n-2)I(n-3) \quad (n \geq 3) \tag{3}$$

[1]Readers familiar with non-standard analysis will realise that this is reasonable as it stands. Those familiar with measure theory know how to express this properly if they are in a pedantic mood. And every one else will wonder what this note is about.

We use this repeatedly below:

$$\begin{aligned} I(2k+1)I(2k)\ldots &= \frac{2k-1}{2k+1}\left[I(2k-1)I(2k-2)\right]^2\ldots \\ &= \frac{2k-1}{2k+1}\left(\frac{2k-3}{2k-1}\right)^2\left[I(2k-3)I(2k-4)\right]^3\ldots \\ &= \frac{1}{2k+1}\,\frac{1}{2k-1}\,\ldots\frac{1}{3}\left[I(1)I(0)\right]^{k+1} \end{aligned}$$

which gives

$$\Pi(2k+1) = \frac{k!}{(2k+1)!}2^{2k+1}\pi^{k+1} \tag{4}$$

which can be confirmed by induction.

Similarly

$$\begin{aligned} I(2k)I(2k-1)\ldots I(0) &= \frac{2k-2}{2k}\left[I(2k-2)I(2k-3)\right]^2\ldots \\ &= \frac{2k-2}{2k}\left(\frac{2k-4}{2k-2}\right)^2\left[I(2k-4)I(2k-5)\right]^5\ldots \\ &= \frac{1}{2k}\,\frac{1}{2k-2}\,\ldots\frac{1}{4}2^{k-1}\left[I(2)I(1)\right]^k I(0) \end{aligned}$$

which gives

$$\Pi(2k) = \frac{\pi^{k+1}}{k!} \tag{5}$$

3 Results

Since $\alpha(n) = \frac{\Pi(n)}{I(1)I(0)}\ \alpha(1)$, we have $\alpha(n) = \frac{\Pi(n)}{2\pi}\ \alpha(1) = \frac{\Pi(n)}{\pi}$ since $\alpha(1) = 2$. So using equations 5 and 4:

$$\alpha(2k) = \tfrac{\pi^k}{k!} \quad \text{and} \quad V_{2k}(r) = \tfrac{\pi^k}{k!}r^{2k}$$

and

$$\alpha(2k+1) = \tfrac{k!}{(2k+1)!}2^{2k+1}\pi^k \quad \text{and} \quad V_{2k+1}(r) = \tfrac{k!}{(2k+1)!}2^{2k+1}\pi^k r^{2k+1}$$

4 Summary

The volume of an $2k$ or $2k+1$ dimensional hypersphere of radius r is given by

$$V_{2k}(r) = \frac{\pi^k}{k!}r^{2k} \tag{6}$$

$$V_{2k+1}(r) = \frac{k!}{(2k+1)!}2^{2k+1}\pi^k r^{2k+1} \tag{7}$$