Zad 4.8

Zmienna losowa X ma gęstość $f(x) = ax(x-3)\, 1_{(0,3)}(x)$.

Z własności ~~funkcji~~ gęstości wiemy że jest funkcją nieujemną oraz całkowalną.

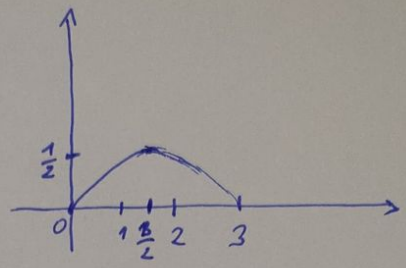
Z nieujemności wynika że musi być spełnione nierówność $ax(x-3) \geq 0 \overset{x\in[0;3]}{\Rightarrow} a<0$.

Skorzystamy teraz z warunku całkowalności do 1.

$$\int_{-\infty}^{+\infty} f(x) = a\int_0^3 (x^2-3x)dx = 1$$

$$a\left(\frac{x^3}{3} - 3\frac{x^2}{2}\right)\Big|_0^3 = 1$$

$$a \cdot -\frac{9}{2} = 1 \Rightarrow \underline{\underline{-\frac{2}{9} = a}}$$

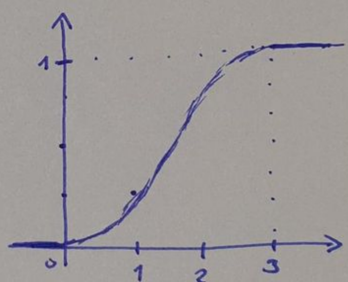


b) Skorzystamy ze wzoru na dystrybuantę $F(x) = \int_{-\infty}^{x} f(t)dt \Leftrightarrow F(t) = \int_{-\infty}^{t} f(x)dx$

$$F(t) = \int_{-\infty}^{t} f(x)dx = \begin{cases} 0 & \text{gdy } t \leq 0 \\ -\frac{2}{3}\left(\frac{t^3}{3} - 3\frac{t^2}{2}\right) & \text{gdy } 0 < x < 3 \\ 1 & \text{gdy } t \geq 3 \end{cases}$$

(Zauważ że dla $t=0 \Rightarrow F(0)=0$ oraz $t=3 \Rightarrow F(3)=1$)

$$\int_{-\infty}^{t} f(x)dx = -\frac{2}{3}\int_0^t (x^2-3x)dx = -\frac{2}{3}\left(\frac{t^3}{3} - \frac{3t^2}{2}\right)$$



c) Z definicji $E(X) = \int_{-\infty}^{+\infty} x f(x)$.

$$E(4X-1) = 4E(X) - 1 = 4\cdot\frac{3}{2} - 1 = \underline{\underline{5}}$$

$$E(X) = \int_{-\infty}^{+\infty} t f(t) = -\frac{2}{9}\int_0^3 (x^3-3x^2)dx = -\frac{2}{9}\cdot\left(\frac{x^4}{4} - x^3\right)\Big|_0^3 = \frac{3}{2}$$

Korzystając z dystrybuanty otrzymujemy.

$$P(X>1) = 1-P(X<1) = 1-P(X\leq 1) = 1 - F(1) = 1 - \frac{7}{27} = \underline{\underline{\frac{20}{27}}}$$

$$F(1) = -\frac{2}{9}\left(\frac{1}{3} - \frac{3}{2}\right) = \frac{7}{27}$$

Zad 4.

Zmienna X ma gęstość $p(x) = \alpha(x^2-1)\mathbb{1}_{(-1,1)}(x)$

a) $\int_{-1}^{1} p(x) = \int_{-1}^{1} \alpha(x^2-1)dx = \alpha\left(\int_{-1}^{1} x^2 dx - \int_{-1}^{1} 1\, dx\right) = \alpha\left(\left[\frac{1}{3}x^3\right]_{-1}^{1} - [x]_{-1}^{1}\right) = \alpha\left(\frac{2}{3}-2\right) = -\frac{4}{3}\alpha$

$-\frac{4}{3}\alpha = 1 \Rightarrow \alpha = -\frac{3}{4}$

b) $F(t) = \int_{-\infty}^{t} p(x)dx = \begin{cases} 0 & t<-1 \\ -\frac{3}{4}\left(\frac{t^3}{3}-t-\frac{2}{3}\right) & \text{gdy } -1 \le t \le 1 \\ 1 & t>1 \end{cases}$

$\int_{-1}^{t} -\frac{3}{4}(x^2-1)dx = -\frac{3}{4}\left(\left[\frac{x^3}{3}\right]_{-1}^{t} - [x]_{-1}^{t}\right) = -\frac{3}{4}\left(\frac{t^3}{3}+\frac{1}{3}-(t+1)\right) = -\frac{3}{4}\left(\frac{t^3}{3}-t-\frac{2}{3}\right)$

c) $\mathbb{P}(X>0) = 1-P(X=0) = 1-F(0) = 1-\left(-\frac{3}{4}\right)\left(-\frac{2}{3}\right) = \frac{1}{2}$

$E(1-X)^2 = E(x^2-2x+1) = E(x^2) - E(2X) - E(1) = E(X^2) - 2E(X) + \overset{1}{\overset{\shortparallel}{E(1)}} = \frac{1}{5}+1 = \frac{6}{5}$

$E(X) = \int_{-\infty}^{+\infty} x\,p(x)dx = \int_{-1}^{1} x\left(-\frac{3}{4}\right)(x^2-1)dx = -\frac{3}{4}\left(\int_{-1}^{1} x^3 dx - \int_{-1}^{1} x\,dx\right) = -\frac{3}{4}\left(\left[\frac{x^4}{4}\right]_{-1}^{1} - \left[\frac{x^2}{2}\right]_{-1}^{1}\right) =$

$-\frac{3}{4}(0-0) = 0$

$E(X^2) = \int_{-\infty}^{+\infty} x^2 p(x)dx = -\frac{3}{4}\int_{-1}^{1} x^2(x^2-1)dx = -\frac{3}{4}\left(\int_{-1}^{1} x^4 dx - \int_{-1}^{1} x^2 dx\right) = -\frac{3}{4}\cdot\left(\left[\frac{x^5}{5}\right]_{-1}^{1} - \left[\frac{x^3}{3}\right]\right.$

$= \left(-\frac{3}{4}\right)\cdot\left(-\frac{4}{15}\right) = \frac{1}{5}$

Za Zad 4.11.

a) $X \sim N(1, 4)$

$P(X>-1) = P\left(\frac{X-1}{4} > \frac{-1-1}{4}\right) = 1 - F_{(0,1)}\left(-\frac{2}{4}\right) = 1 - F_{(0,1)}\left(-\frac{1}{2}\right)$

$P(|X|<6) = P(X>-6 \wedge X<6) = P(X<6) - P(X>-6) =$

b) $= P\left(\frac{X-1}{16} < \frac{6-1}{16}\right) - P\left(\frac{X-1}{16} > \frac{-6-1}{16}\right) = F_{(0,1)}\left(\frac{5}{16}\right) - 1 + F_{(0,1)}\left(\frac{-7}{16}\right)$

Zad 4.12

$X \sim N(3, 3^2)$

a) $P(X>-2) = P\left(\frac{X-3}{3} > \frac{-2-3}{3}\right) = 1 - F_{(0,1)}\left(-\frac{5}{3}\right)$

c) $P(|X|<5) = P(X>-5 \wedge X<5) = P(X<5) - P(X>-5) = P\left(\frac{X-3}{3} < \frac{5-3}{3}\right) - P\left(\frac{X-3}{3} > \frac{-5-3}{3}\right) =$

$= F_{(0,1)}\left(\frac{2}{3}\right) - 1 + F_{(0,1)}\left(-\frac{8}{3}\right)$

Zad 4.14

$X = U(1,10)$

$2X + 2Y = 20 \Rightarrow X + Y = 10 \Rightarrow Y = 10 - X$

$E(X \cdot Y) = E(X \cdot (10-X)) = 10E(X) - E(X^2) = 10 \cdot \frac{11}{2} - 37 = 18$

$E(X) = \frac{10+1}{2} = \frac{11}{2} \qquad Var(X) = \frac{(10-1)^2}{12} = \frac{27}{4}$

$E(X^2) = Var(X) + (E(X))^2 = \frac{27}{4} + \left(\frac{11}{2}\right)^2 = 37$

Zad 4.15.

$X = U(1,4)$

$P = \frac{X \cdot \frac{1}{2}X}{2} = \frac{1}{4}X^2$

$E\left(\frac{1}{4}X^2\right) = \frac{1}{4}E(X^2) = \frac{1}{4} \cdot \frac{28}{4} = \frac{7}{4}$

$E(X) = \frac{1+4}{2} = \frac{5}{2}$

$Var(X) = \frac{(4-1)^2}{12} = \frac{9}{12} = \frac{3}{4}$

$E(X^2) = Var(X) + (E(X))^2 = \frac{3}{4} + \left(\frac{5}{2}\right)^2 = \frac{28}{4}$

Zad 4.9

a) $\int_{-1}^{2} \alpha(-x^2+x+2)\,dx = \alpha\left(-\int_{-1}^{2} x^2 dx + \int_{-1}^{2} x\,dx + \int_{-1}^{2} 2\,dx\right) = \alpha\left(\left[-\frac{x^3}{3}+\frac{x^2}{2}+2x\right]_{-1}^{2}\right) =$

$= \alpha\left(-\frac{8}{3}+\frac{4}{2}+4-\frac{1}{3}-\frac{1}{2}+2\right) = \frac{9}{2}\alpha$

$\frac{9}{2}\alpha = 1 \Rightarrow \alpha = \frac{2}{9}$

b) $F(t) = \int_{-\infty}^{t} f(x)\,dx = \begin{cases} 0 & t<-1 \\ 1 & t>2 \\ \frac{2}{9}\left(\frac{-t^3}{3}+\frac{t^2}{2}+2t-\frac{7}{6}\right) & -1\le t\le 2 \end{cases}$

$\int_{-1}^{t} \frac{2}{9}(-x^2+x+2)\,dx = \frac{2}{9}\left[\frac{-x^3}{3}+\frac{x^2}{2}+2x\right]_{-1}^{t} = \frac{2}{9}\left(-\frac{t^3}{3}+\frac{t^2}{2}+2t-\frac{1}{3}-\frac{1}{2}+2\right)$

$= \frac{2}{9}\left(-\frac{t^3}{3}+\frac{t^2}{2}+2t+\frac{7}{6}\right)$

c) $P(X>1) = 1-F(1) = 1-\frac{20}{27} = \frac{7}{27}$

$E(2X-1) = 2E(X)-E(1) = 2E(X)-1 = 2\cdot\frac{1}{2}-1 = 0$

$E(X) = \int_{-\infty}^{+\infty} x f(x)\,dx = \frac{2}{9}\int_{-1}^{2} x(-x^2+x+2)\,dx = \frac{2}{9}\int_{-1}^{2}(-x^3+x^2+2x)\,dx = \frac{2}{9}\left[-\frac{x^4}{4}+\frac{x^3}{3}+x^2\right]_{-1}^{2}$

$= \frac{2}{9}\cdot\frac{9}{4} = \frac{1}{2}$

Zad 4.10

a) $\int_{0}^{1} \alpha x^{\alpha-1} = \alpha\int_{0}^{1} x^{\alpha-1} = \alpha\cdot\left[\frac{x^\alpha}{\alpha}\right]_0^1 = \alpha\cdot\frac{1}{\alpha} = 1 \Rightarrow \alpha\in\mathbb{R},\ \alpha>0$

b) $F(t) = \int_{-\infty}^{t} f(x)\,dx = \begin{cases} 0 & t<0 \\ 1 & t>1 \\ t^\alpha & 0\le t\le 1 \end{cases}$

$\int_{0}^{t} \alpha x^{\alpha-1}\,dx = \alpha\left[\frac{x^\alpha}{\alpha}\right]_0^t = t^\alpha$

c) $P(X>\frac{1}{2}) = 1-F(\frac{1}{2}) = 1-\left(\frac{1}{2}\right)^\alpha$

$E(1-X)^2 =$ ~~$E(1)-E(X)\cdot 2\cdot E(X)+1$~~ $E(X^2)-2E(X)+1 = \frac{\alpha}{\alpha+2}-\frac{2\alpha}{\alpha+1}+1$

$E(X) = \int_{-\infty}^{+\infty} x f(x) = \alpha\int_{0}^{1} x^\alpha\,dx = \alpha\left[\frac{x^{\alpha+1}}{\alpha+1}\right]_0^1 = \alpha\frac{1}{\alpha+1} = \frac{\alpha}{\alpha+1}$

$E(X^2) = \int_{-\infty}^{+\infty} x^2 f(x) = \alpha\int_{0}^{1} x^{\alpha+1}\,dx = \alpha\cdot\left[\frac{x^{\alpha+2}}{\alpha+2}\right]_0^1 = \frac{\alpha}{\alpha+2}$

$\mathrm{Var}\,X = E(X^2)-(E(X))^2 = \frac{\alpha}{\alpha+2}-\left(\frac{\alpha}{\alpha+1}\right)^2$