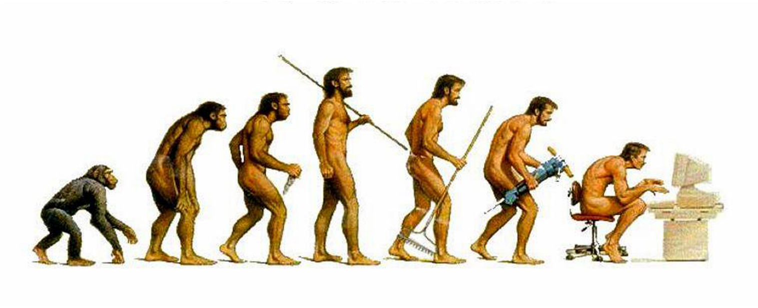
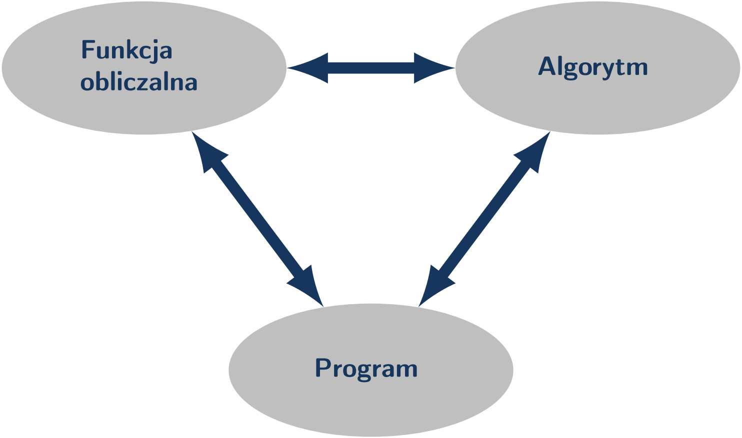


Teoria Obliczalności

Marcin Piątkowski

Wykład 2


Funkcja obliczalna
Algorytm
Program

Program **P** na maszynę licznikową nazywamy programem w postaci standardowej, jeśli dla każdej instrukcji `I(m,n,q)` zachodzi warunek $\mathtt{q} \leq \; k+1$, gdzie k jest liczbą instrukcji programu **P**.

```
0:  T(1,0)
1:  I(2,3,27)
2:  S(0)
3:  S(3)
4:  I(0,0,0)
5:
 ⋮
27:
```

```
0:  T(1,0)
1:  I(2,3,5)
2:  S(0)
3:  S(3)
4:  I(0,0,0)
5:
```

Program w postaci standardowej

Program **P** na maszynę licznikową nazywamy programem w postaci standardowej, jeśli dla każdej instrukcji `I(m,n,q)` zachodzi warunek $\mathtt{q} \leq k+1$, gdzie k jest liczbą instrukcji programu **P**.

```
0:  T(1,0)
1:  I(2,3,27)
2:  S(0)
3:  S(3)
4:  I
5
27:
```

```
0)
,3,5)
S(0)
3:  S(3)
4:  I(0,0,0)
5:
```

W dalszej części wykładu będziemy rozważać wyłącznie programy w postaci standardowej.

Składanie programów

$$P = \{\, I_1 \cdots I_j \cdots I_{k_1} \,\}$$

$$R = \{\, J_1 \cdots J_m \cdots J_n \cdots J_{k_2} \,\}$$

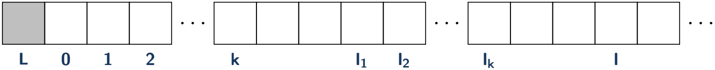
$$PR = \{\, I_1 \cdots I_j \cdots I_{k_1}\; J'_1 \cdots J'_m \cdots J'_n \cdots J'_{k_2}$$

!

$$J_i \in \{Z(k), S(k), T(m,n)\} \quad \Rightarrow \quad J'_i = J_i$$

$$J_i = I(m,n,q) \quad \Rightarrow \quad J'_i = I(m,n,q+k_1+1)$$

Podprogramy



⋮

$\mathtt{T(l_1,1)}$

⋮

$\mathtt{T(l_k,k)}$

$\mathtt{Z(0)}$

$\mathtt{Z(k+1)}$

⋮

$\mathtt{Z}(\rho(\mathtt{P}))$

P

$\mathtt{T(0,l)}$

⋮

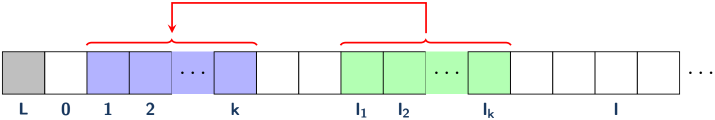
!

$\mathtt{P}$ – program na maszynę licznikową

$\mathtt{P}[\mathtt{l}_1, \ldots, \mathtt{l}_k \to \mathtt{l}]$ – wywołanie $\mathtt{P}$ jako podprogram

$\rho(\mathtt{P})$ – najmniejszy adres rejestru nieużywanego przez program $\mathtt{P}$

Podprogramy



$\vdots$

$T(l_1, 1)$

$\vdots$

$T(l_k, k)$

$Z(0)$

$Z(k+1)$

$\vdots$

$Z(\rho(P))$

P

$T(0, l)$

$\vdots$

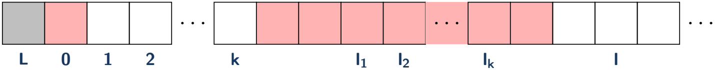
!

P – program na maszynę licznikową

$P[l_1, \ldots, l_k \to l]$ – wywołanie P jako podprogram

$\rho(P)$ – najmniejszy adres rejestru nieużywanego przez program P

Podprogramy



⋮

`T(l₁,1)`

⋮

`T(lₖ,k)`

`Z(0)`

`Z(k+1)`

⋮

$\mathtt{Z}(\rho(\mathtt{P}))$

P

`T(0,l)`

⋮

!

P – program na maszynę licznikową

$\mathtt{P}[\mathtt{l}_1, \ldots, \mathtt{l}_k \to \mathtt{l}]$ – wywołanie P jako podprogram

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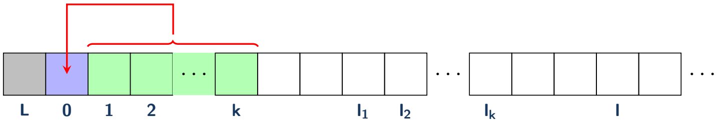
Podprogramy



⋮

T(l_1,1)

⋮

T(l_k,k)

Z(0)

Z(k+1)

⋮

Z(ρ(P))

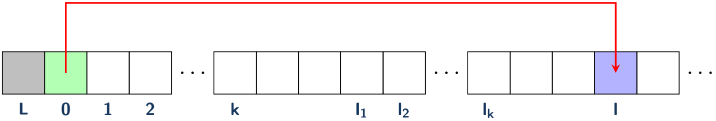
P

T(0,l)

⋮

> **!**
>
> P – program na maszynę licznikową
>
> $P[l_1, \ldots, l_k \to l]$ – wywołanie P jako podprogram
>
> $\rho(P)$ – najmniejszy adres rejestru nieużywanego przez program P

Podprogramy



$\vdots$

T(l_1,1)

$\vdots$

T(l_k,k)

Z(0)

Z(k+1)

$\vdots$

Z(ρ(P))

P

T(0,1)

$\vdots$

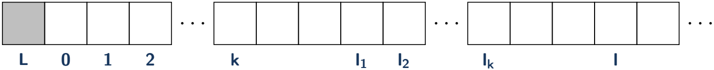
!

P – program na maszynę licznikową

P[$l_1, \ldots, l_k \to l$] – wywołanie P jako podprogram

ρ(P) – najmniejszy adres rejestru nieużywanego przez program P

Podprogramy



$\vdots$

`T(l₁,1)`

$\vdots$

`T(lₖ,k)`

`Z(0)`

`Z(k+1)`

$\vdots$

`Z(ρ(P))`

P

`T(0,l)`

$\vdots$

Kopia zapasowa pamięci programu

!

$\mathtt{P}$	–	program na maszynę licznikową
$\mathtt{P}[\mathtt{l}_1,\ldots,\mathtt{l}_\mathtt{k} \to \mathtt{l}]$	–	wywołanie $\mathtt{P}$ jako podprogram
$\rho(\mathtt{P})$	–	najmniejszy adres rejestru nieużywanego przez program $\mathtt{P}$

Podstawianie

$$f(x_1, x_2, x_3) = x_1 + x_2 + x_3$$

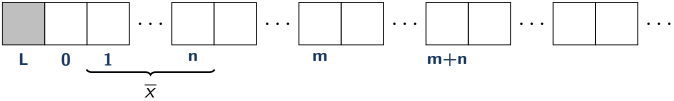
$$g_1(x_1, x_2) = x_1 \cdot x_2$$
$$g_2(x_1, x_2) = x_1^{x_2}$$
$$g_3(x_1, x_2) = x_1^2 + x_2^2$$

$$h(x_1, x_2) = f\Big(g_1(x_1, x_2), g_2(x_1, x_2), g_3(x_1, x_2)\Big) \;=\; (x_1 \cdot x_2) + x_1^{x_2} + (x_1^2 + x_2^2)$$

Twierdzenie

Niech $f : \boldsymbol{N}^k \to \boldsymbol{N}$ oraz $g_1, \ldots, g_k : \boldsymbol{N}^n \to \boldsymbol{N}$ będą obliczalne. Wówczas $h : \boldsymbol{N}^n \to \boldsymbol{N}$ określona $h(\overline{x}) = f\Big(g_1(\overline{x}), \ldots, g_k(\overline{x})\Big)$ również jest obliczalna.

Podstawianie



$$\mathtt{T(1,m+1)}$$
$$\mathtt{T(2,m+2)}$$
$$\vdots$$
$$\mathtt{T(n,m+n)}$$
$$\mathtt{G}_1[\mathtt{m}+1,\ldots,\mathtt{m}+\mathtt{n}\to\mathtt{m}+\mathtt{n}+1]$$
$$\mathtt{G}_2[\mathtt{m}+1,\ldots,\mathtt{m}+\mathtt{n}\to\mathtt{m}+\mathtt{n}+2]$$
$$\vdots$$
$$\mathtt{G}_\mathtt{k}[\mathtt{m}+1,\ldots,\mathtt{m}+\mathtt{n}\to\mathtt{m}+\mathtt{n}+\mathtt{k}]$$
$$\mathtt{F}[\mathtt{m}+\mathtt{n}+1,\ldots,\mathtt{m}+\mathtt{n}+\mathtt{k}\to 0]$$

$\mathtt{G}_1$ – program obliczający g_1

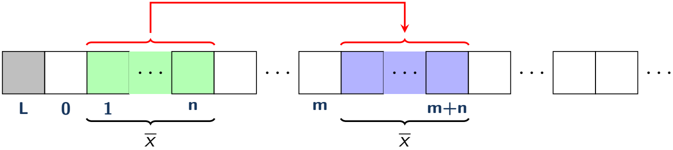
$\vdots$

$\mathtt{G}_\mathtt{k}$ – program obliczający g_k

$\mathtt{F}$ – program obliczający f

m $=$ $\max\{n, \rho(\mathtt{G}_1), \ldots, \rho(\mathtt{G}_\mathtt{k}), \rho(\mathtt{F})\}$

Podstawianie



$\texttt{T(1,m+1)}$

$\texttt{T(2,m+2)}$

$\vdots$

$\texttt{T(n,m+n)}$

$\mathtt{G}_1[\mathtt{m}+1,\ldots,\mathtt{m}+\mathtt{n}\to\mathtt{m}+\mathtt{n}+1]$

$\mathtt{G}_2[\mathtt{m}+1,\ldots,\mathtt{m}+\mathtt{n}\to\mathtt{m}+\mathtt{n}+2]$

$\vdots$

$\mathtt{G}_\mathtt{k}[\mathtt{m}+1,\ldots,\mathtt{m}+\mathtt{n}\to\mathtt{m}+\mathtt{n}+\mathtt{k}]$

$\mathtt{F}[\mathtt{m}+\mathtt{n}+1,\ldots,\mathtt{m}+\mathtt{n}+\mathtt{k}\to 0]$

!

$\mathtt{G}_1$ – program obliczający g_1

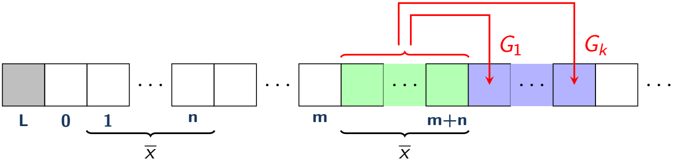
$\vdots$

$\mathtt{G}_\mathtt{k}$ – program obliczający g_k

$\mathtt{F}$ – program obliczający f

m $= \max\{n, \rho(\mathtt{G}_1), \ldots, \rho(\mathtt{G}_\mathtt{k}), \rho(\mathtt{F})\}$

Podstawianie



$\mathtt{T(1,m+1)}$

$\mathtt{T(2,m+2)}$

$\vdots$

$\mathtt{T(n,m+n)}$

$\mathtt{G_1[m+1,\ldots,m+n\to m+n+1]}$

$\mathtt{G_2[m+1,\ldots,m+n\to m+n+2]}$

$\vdots$

$\mathtt{G_k[m+1,\ldots,m+n\to m+n+k]}$

$\mathtt{F[m+n+1,\ldots,m+n+k\to 0]}$

!

$\mathtt{G_1}$ – program obliczający g_1

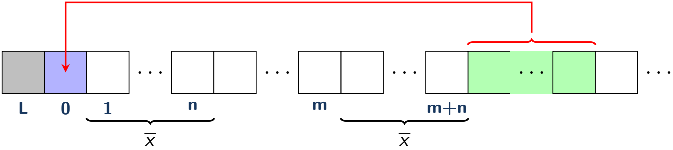
$\vdots$

$\mathtt{G_k}$ – program obliczający g_k

$\mathtt{F}$ – program obliczający f

m $= \max\{n, \rho(\mathtt{G_1}), \ldots, \rho(\mathtt{G_k}), \rho(\mathtt{F})\}$

Podstawianie



$\mathtt{T(1,m+1)}$

$\mathtt{T(2,m+2)}$

$\vdots$

$\mathtt{T(n,m+n)}$

$\mathtt{G}_1[\mathtt{m}+1,\ldots,\mathtt{m}+\mathtt{n} \to \mathtt{m}+\mathtt{n}+1]$

$\mathtt{G}_2[\mathtt{m}+1,\ldots,\mathtt{m}+\mathtt{n} \to \mathtt{m}+\mathtt{n}+2]$

$\vdots$

$\mathtt{G}_\mathtt{k}[\mathtt{m}+1,\ldots,\mathtt{m}+\mathtt{n} \to \mathtt{m}+\mathtt{n}+\mathtt{k}]$

$\mathtt{F}[\mathtt{m}+\mathtt{n}+1,\ldots,\mathtt{m}+\mathtt{n}+\mathtt{k} \to 0]$

!

$\mathtt{G}_1$ – program obliczający g_1

$\vdots$

$\mathtt{G}_\mathtt{k}$ – program obliczający g_k

$\mathtt{F}$ – program obliczający f

m $=$ $\max\{n, \rho(\mathtt{G}_1), \ldots, \rho(\mathtt{G}_\mathtt{k}), \rho(\mathtt{F})\}$

Rekursja

$$
\begin{aligned}
h(\overline{x}, 0) &= f(\overline{x}) \\
h(\overline{x}, y+1) &= g\Big(\overline{x}, y, h(\overline{x}, y)\Big)
\end{aligned}
$$

$$h(\overline{x}, \color{red}{2}) = g\left(\overline{x}, 2, \underbrace{h(\overline{x}, 1)}\right)$$

$$\downarrow$$

$$g\left(\overline{x}, 1, \underbrace{h(\overline{x}, 0)}\right)$$

$$\downarrow$$

$$f(\overline{x})$$

$$h(\overline{x}, 0) = f(\overline{x})$$
$$h(\overline{x}, y+1) = g\Big(\overline{x}, y, h(\overline{x}, y)\Big)$$

$$\boxed{h(\overline{x}, 2)} = \boxed{g\big(\overline{x}, 2, \underbrace{h(\overline{x}, 1)}\big)}$$

$$\boxed{g\big(\overline{x}, 1, \underbrace{h(\overline{x}, 0)}\big)}$$

$$\boxed{f(\overline{x})}$$

Rekursja

$$
\begin{aligned}
h(\overline{x}, 0) &= f(\overline{x}) \\
h(\overline{x}, y+1) &= g\Big(\overline{x}, y, h(\overline{x}, y)\Big)
\end{aligned}
$$

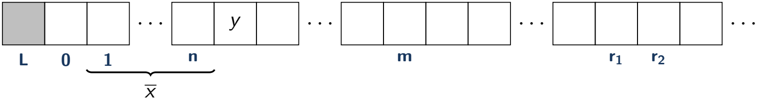
$$h(\overline{x}, \color{red}{2}) = g\left(\overline{x}, 2, \underbrace{h(\overline{x}, 1)}\right)$$

$$g\left(\overline{x}, 1, \underbrace{h(\overline{x}, 0)}\right)$$

$$f(\overline{x})$$

Twierdzenie o operatorze rekursji

Niech $f : \mathbb{N}^n \to \mathbb{N}$ oraz $g : \mathbb{N}^{n+2} \to \mathbb{N}$ będą obliczalne. Wówczas funkcja $h : \mathbb{N}^{n+1} \to \mathbb{N}$ określona za pomocą operatora rekursji również jest obliczalna.



$\mathtt{T(n+1,m+1)}$

$\mathtt{T(1,m+2)}$

$\vdots$

$\mathtt{T(n,m+n+1)}$

$\mathtt{F}[\mathtt{m}+2,\ldots,\mathtt{m}+\mathtt{n}+1 \to \mathtt{r}_2]$

$\mathtt{L}$: $\mathtt{I}(\mathtt{m}+1,\mathtt{r}_1,\mathtt{K})$

$\mathtt{G}[\mathtt{m}+2,\ldots,\mathtt{r}_1,\mathtt{r}_2 \to \mathtt{r}_2]$

$\mathtt{S}(\mathtt{r}_1)$

$\mathtt{I(0,0,L)}$

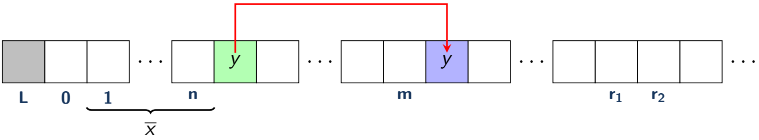
$\mathtt{K}$: $\mathtt{T}(\mathtt{r}_2,0)$

!

$\mathtt{G}$ – program obliczający g

$\mathtt{F}$ – program obliczający f

m $=$ max$\{n+2,\rho(\mathtt{G}),\rho(\mathtt{F})\}$



$\color{red}{\texttt{T(n+1,m+1)}}$

$\texttt{T(1,m+2)}$

$\vdots$

$\texttt{T(n,m+n+1)}$

$\texttt{F}[\texttt{m}+2,\ldots,\texttt{m}+\texttt{n}+1 \to \texttt{r}_2]$

$\texttt{L:}\ \texttt{I}(\texttt{m}+1,\texttt{r}_1,\texttt{K})$

$\texttt{G}[\texttt{m}+2,\ldots,\texttt{r}_1,\texttt{r}_2 \to \texttt{r}_2]$

$\texttt{S}(\texttt{r}_1)$

$\texttt{I(0,0,L)}$

$\texttt{K:}\ \texttt{T}(\texttt{r}_2,0)$

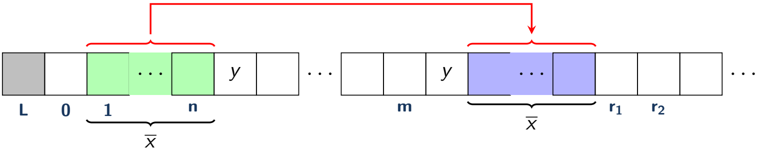
!

$\texttt{G}$ – program obliczający g

$\texttt{F}$ – program obliczający f

m $=$ max$\{n+2, \rho(\texttt{G}), \rho(\texttt{F})\}$

Rekursja



T(n+1,m+1)

T(1,m+2)

⋮

T(n,m+n+1)

$\mathtt{F}[\mathtt{m}+2,\ldots,\mathtt{m}+\mathtt{n}+1\to\mathtt{r}_2]$

L: $\mathtt{I}(\mathtt{m}+1,\mathtt{r}_1,\mathtt{K})$

$\mathtt{G}[\mathtt{m}+2,\ldots,\mathtt{r}_1,\mathtt{r}_2\to\mathtt{r}_2]$

$\mathtt{S}(\mathtt{r}_1)$

I(0,0,L)

K: $\mathtt{T}(\mathtt{r}_2,0)$

!

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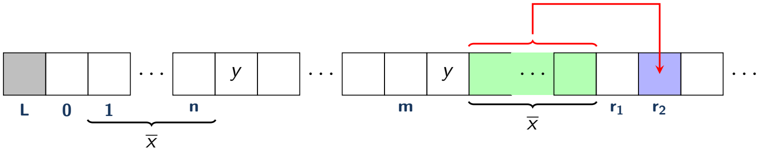
F – program obliczający f

m $=\max\{n+2,\rho(\mathtt{G}),\rho(\mathtt{F})\}$



$\mathtt{T(n+1,m+1)}$

$\mathtt{T(1,m+2)}$

$\vdots$

$\mathtt{T(n,m+n+1)}$

$\mathtt{F}[\mathtt{m}+2,\dots,\mathtt{m}+\mathtt{n}+1 \to \mathtt{r}_2]$

$\mathtt{L:}\ \mathtt{I}(\mathtt{m}+1,\mathtt{r}_1,\mathtt{K})$

$\mathtt{G}[\mathtt{m}+2,\dots,\mathtt{r}_1,\mathtt{r}_2 \to \mathtt{r}_2]$

$\mathtt{S}(\mathtt{r}_1)$

$\mathtt{I(0,0,L)}$

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!

$\mathtt{G}$ – program obliczający g

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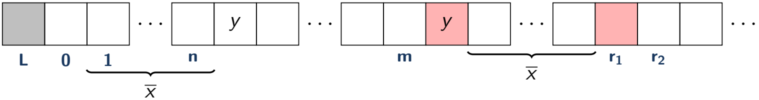
$\mathtt{m} = \max\{n+2, \rho(\mathtt{G}), \rho(\mathtt{F})\}$

Rekursja



$$
\begin{array}{ll}
 & \mathtt{T(n+1,m+1)} \\
 & \mathtt{T(1,m+2)} \\
 & \vdots \\
 & \mathtt{T(n,m+n+1)} \\
 & \mathtt{F}[\mathtt{m}+2,\ldots,\mathtt{m}+\mathtt{n}+1 \to \mathtt{r}_2] \\
\mathtt{L:} & \mathtt{I}(\mathtt{m}+1,\mathtt{r}_1,\mathtt{K}) \\
 & \mathtt{G}[\mathtt{m}+2,\ldots,\mathtt{r}_1,\mathtt{r}_2 \to \mathtt{r}_2] \\
 & \mathtt{S}(\mathtt{r}_1) \\
 & \mathtt{I(0,0,L)} \\
\mathtt{K:} & \mathtt{T}(\mathtt{r}_2,0)
\end{array}
$$

!

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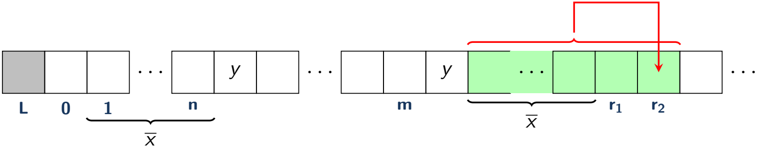
Rekursja



$\mathtt{T(n+1,m+1)}$

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$\vdots$

$\mathtt{T(n,m+n+1)}$

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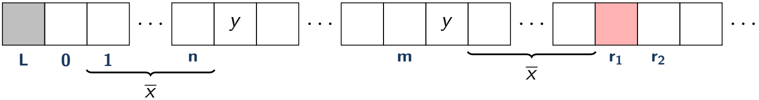
m $=$ $\max\{n+2, \rho(\mathtt{G}), \rho(\mathtt{F})\}$



$\mathtt{T(n+1,m+1)}$

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$\vdots$

$\mathtt{T(n,m+n+1)}$

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$\mathtt{G}[\mathtt{m}+2,\ldots,\mathtt{r}_1,\mathtt{r}_2 \to \mathtt{r}_2]$

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$\mathtt{I(0,0,L)}$

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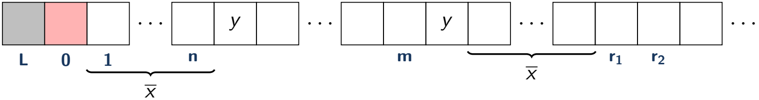
m $=$ $\max\{n+2, \rho(\mathtt{G}), \rho(\mathtt{F})\}$



T(n+1,m+1)

T(1,m+2)

⋮

T(n,m+n+1)

$\mathtt{F}[\mathtt{m}+2,\ldots,\mathtt{m}+\mathtt{n}+1 \to \mathtt{r}_2]$

L: $\mathtt{I}(\mathtt{m}+1,\mathtt{r}_1,\mathtt{K})$

$\mathtt{G}[\mathtt{m}+2,\ldots,\mathtt{r}_1,\mathtt{r}_2 \to \mathtt{r}_2]$

$\mathtt{S}(\mathtt{r}_1)$

I(0,0,L)

K: $\mathtt{T}(\mathtt{r}_2,0)$

!

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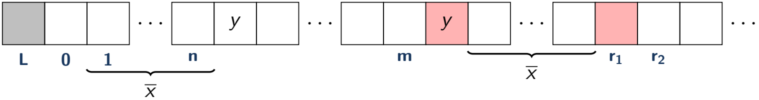
m $=\max\{n+2,\rho(\mathtt{G}),\rho(\mathtt{F})\}$



$\mathtt{T(n+1,m+1)}$

$\mathtt{T(1,m+2)}$

$\vdots$

$\mathtt{T(n,m+n+1)}$

$\mathtt{F}[\mathtt{m}+2,\ldots,\mathtt{m}+\mathtt{n}+1 \to \mathtt{r}_2]$

L: $\mathtt{I}(\mathtt{m}+1,\mathtt{r}_1,\mathtt{K})$

$\mathtt{G}[\mathtt{m}+2,\ldots,\mathtt{r}_1,\mathtt{r}_2 \to \mathtt{r}_2]$

$\mathtt{S}(\mathtt{r}_1)$

$\mathtt{I(0,0,L)}$

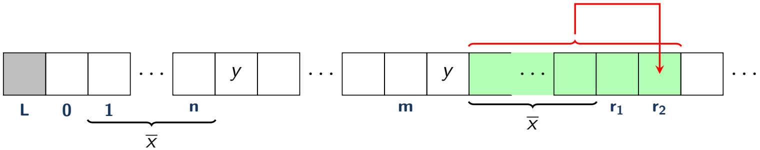
K: $\mathtt{T}(\mathtt{r}_2,0)$

!

$\mathtt{G}$ – program obliczający g

$\mathtt{F}$ – program obliczający f

m $=$ $\max\{n+2,\rho(\mathtt{G}),\rho(\mathtt{F})\}$



$\mathtt{T(n+1,m+1)}$

$\mathtt{T(1,m+2)}$

$\vdots$

$\mathtt{T(n,m+n+1)}$

$\mathtt{F}[\mathtt{m}+2,\ldots,\mathtt{m}+\mathtt{n}+1 \to \mathtt{r}_2]$

$\mathtt{L}\!:\ \mathtt{I}(\mathtt{m}+1,\mathtt{r}_1,\mathtt{K})$

$\mathtt{G}[\mathtt{m}+2,\ldots,\mathtt{r}_1,\mathtt{r}_2 \to \mathtt{r}_2]$

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$\mathtt{G}$ – program obliczający g

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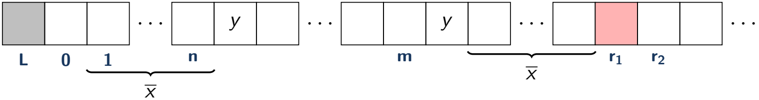
$\mathtt{m} = \max\{n+2, \rho(\mathtt{G}), \rho(\mathtt{F})\}$



$\mathtt{T(n+1,m+1)}$

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$\vdots$

$\mathtt{T(n,m+n+1)}$

$\mathtt{F}[\mathtt{m}+2,\ldots,\mathtt{m}+\mathtt{n}+1 \to \mathtt{r}_2]$

$\mathtt{L:}\ \mathtt{I}(\mathtt{m}+1,\mathtt{r}_1,\mathtt{K})$

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$\mathtt{G}$ – program obliczający g

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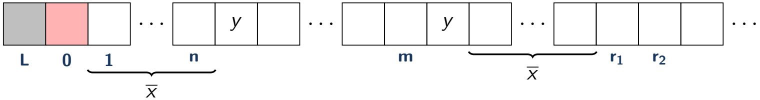
$\mathtt{m} = \max\{n+2, \rho(\mathtt{G}), \rho(\mathtt{F})\}$

Rekursja



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$\mathtt{F}[\mathtt{m}+2,\dots,\mathtt{m}+\mathtt{n}+1 \to \mathtt{r}_2]$

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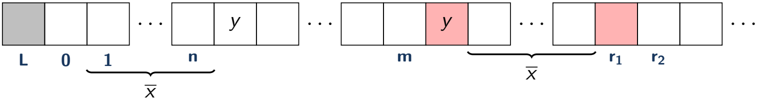
$\mathtt{G}$ – program obliczający g

$\mathtt{F}$ – program obliczający f

$\mathtt{m} = \max\{n+2, \rho(\mathtt{G}), \rho(\mathtt{F})\}$

Rekursja



$\mathtt{T(n+1,m+1)}$

$\mathtt{T(1,m+2)}$

$\vdots$

$\mathtt{T(n,m+n+1)}$

$\mathtt{F[m+2,\ldots,m+n+1 \to r_2]}$

$\mathtt{L:}$ $\mathtt{I(m+1,r_1,K)}$

$\mathtt{G[m+2,\ldots,r_1,r_2 \to r_2]}$

$\mathtt{S(r_1)}$

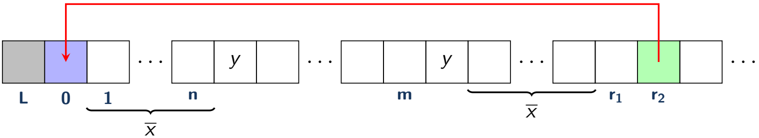
$\mathtt{I(0,0,L)}$

$\mathtt{K:}$ $\mathtt{T(r_2,0)}$

!

$\mathtt{G}$	–	program obliczający g
$\mathtt{F}$	–	program obliczający f
m	=	$\max\{n+2, \rho(\mathtt{G}), \rho(\mathtt{F})\}$

Rekursja



T(n+1,m+1)

T(1,m+2)

⋮

T(n,m+n+1)

$\mathtt{F}[\mathtt{m}+2,\ldots,\mathtt{m}+\mathtt{n}+1\to \mathtt{r}_2]$

L: $\mathtt{I}(\mathtt{m}+1,\mathtt{r}_1,\mathtt{K})$

$\mathtt{G}[\mathtt{m}+2,\ldots,\mathtt{r}_1,\mathtt{r}_2\to \mathtt{r}_2]$

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I(0,0,L)

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$\mathtt{G}$ – program obliczający g

$\mathtt{F}$ – program obliczający f

m $=$ $\max\{n+2,\rho(\mathtt{G}),\rho(\mathtt{F})\}$

Minimalizacja $f(\overline{x}, y)$

$\boxed{g(\overline{x}) = \mu y \Big(f(\overline{x}, y) = 0 \Big)}$ – najmniejsza wartość $y \in \boldsymbol{N}$ taka, że

1. $\forall_{z<y}\ f(\overline{x}, z)$ jest określona
2. $\forall_{z<y}\ f(\overline{x}, z) \neq 0$
3. $f(\overline{x}, y) = 0$

$f(x, y) = (x + y)\,\%\,7$

$$g(3) = \mu y \Big((3 + y)\,\%\,7 = 0 \Big) = 4$$

$$g(5) = \mu y \Big((5 + y)\,\%\,7 = 0 \Big) = 2$$

$f(x, y) = x + y$

$$g(0) = \mu y \Big((0 + y) = 0 \Big) = 0$$

$$g(3) = \mu y \Big((3 + y) = 0 \Big) = \infty$$

Minimalizacja $f(\overline{x}, y)$

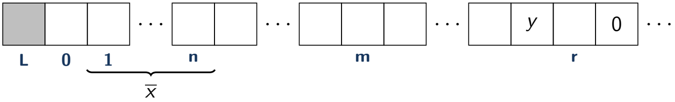
$\boxed{g(\overline{x}) = \mu y \Big(f(\overline{x}, y) = 0 \Big)}$ – najmniejsza wartość $y \in \boldsymbol{N}$ taka, że

1. $\forall_{z<y}\ f(\overline{x}, z)$ jest określona
2. $\forall_{z<y}\ f(\overline{x}, z) \neq 0$
3. $f(\overline{x}, y) = 0$

Twierdzenie o operatorze minimalizacji

Niech $f : \boldsymbol{N}^{n+1} \rightarrow \boldsymbol{N}$ będzie obliczalna. Wówczas funkcja $g : \boldsymbol{N}^n \rightarrow \boldsymbol{N}$ określona jako $g(\overline{x}) = \mu y \Big(f(\overline{x}, y) = 0 \Big)$ również jest obliczalna.

Minimalizacja



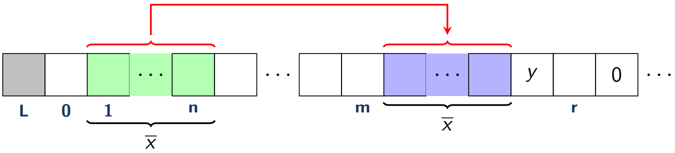

```
      T(1,m+1)
      T(2,m+2)
      ⋮
      T(n,m+n)
L:    F[m + 1, ..., m + n + 1 → r]
      I(r,m+n+3,K)
      S(m+n+1)
      I(0,0,L)
K:    T(m+n+1,0)
```

!

F – program obliczający f

m $=$ $\max\{n+1, \rho(\mathtt{F})\}$

Minimalizacja



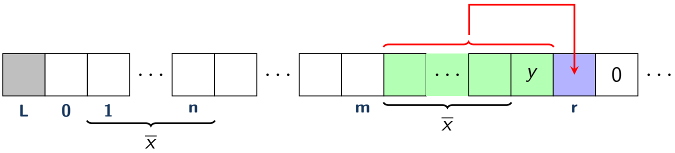

```
    T(1,m+1)
    T(2,m+2)
       ⋮
    T(n,m+n)
L:  F[m + 1, ..., m + n + 1 → r]
    I(r,m+n+3,K)
    S(m+n+1)
    I(0,0,L)
K:  T(m+n+1,0)
```

!

F – program obliczający f

m $=$ $\max\{n+1, \rho(\mathtt{F})\}$

Minimalizacja

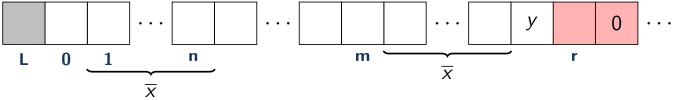



```
    T(1,m+1)
    T(2,m+2)
      ⋮
    T(n,m+n)
L:  F[m+1,...,m+n+1 → r]
    I(r,m+n+3,K)
    S(m+n+1)
    I(0,0,L)
K:  T(m+n+1,0)
```

!

F – program obliczający f

m $= \max\{n+1, \rho(\mathtt{F})\}$




```
    T(1,m+1)
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    ⋮
    T(n,m+n)
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    I(0,0,L)
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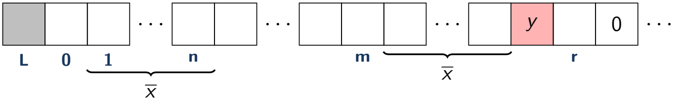
```

!

F – program obliczający f

m $=$ $\max\{n+1, \rho(\mathtt{F})\}$

Minimalizacja




```
        T(1,m+1)
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        T(n,m+n)
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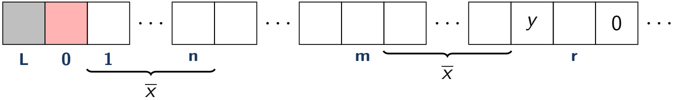
Minimalizacja

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F – program obliczający f

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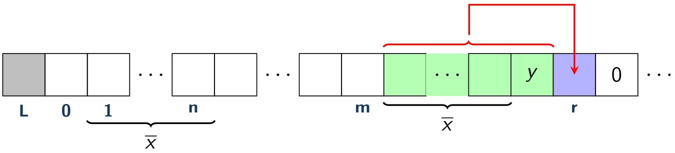
Minimalizacja

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!

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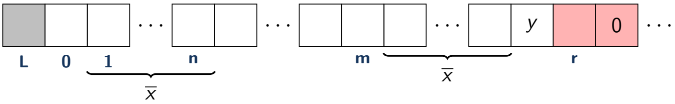
Minimalizacja

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       T(1,m+1)
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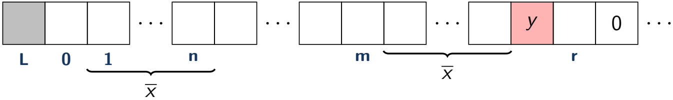
> **!**
> F – program obliczający f
>
> m $=$ $\max\{n+1, \rho(\mathtt{F})\}$

Minimalizacja

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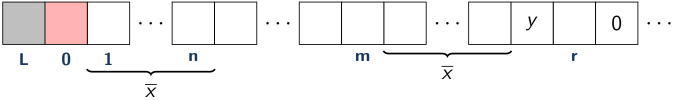


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!

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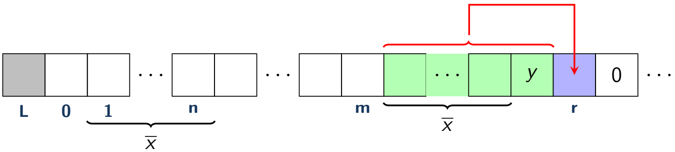
m $= \max\{n+1, \rho(\mathtt{F})\}$

Minimalizacja

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> **!**
> $\mathtt{F}$ – program obliczający f
>
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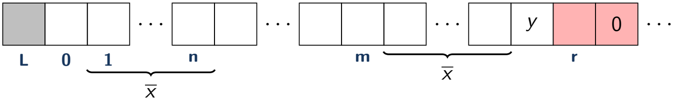
Minimalizacja

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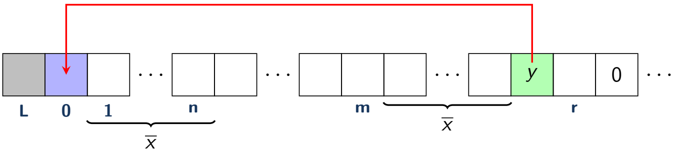
Minimalizacja



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$\vdots$

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$\mathtt{I(r,m+n+3,K)}$

$\mathtt{S(m+n+1)}$

$\mathtt{I(0,0,L)}$

$\mathtt{K}$: $\mathtt{T(m+n+1,0)}$

!

$\mathtt{F}$ – program obliczający f

$\mathtt{m} = \max\{n+1, \rho(\mathtt{F})\}$

Minimalizacja

! Operator minimalizacji zastosowany do funkcji totalnej nie musi dać w wyniku funkcji totalnej.

$$\mathsf{f(x,y) = (x+y)\,\%\,7}$$

$$g(3) = \mu y\Big((3+y)\,\%\,7 = 0\Big) = 4$$

$$g(5) = \mu y\Big((5+y)\,\%\,7 = 0\Big) = 2$$

$$\mathsf{f(x,y) = x+y}$$

$$g(0) = \mu y\Big((0+y) = 0\Big) = 0$$

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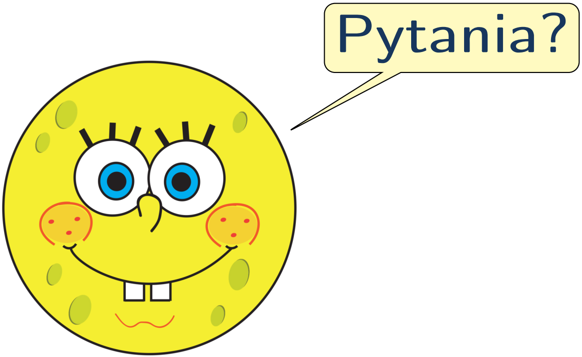
Pytania?