Notes on `AsPy`

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Abstract: In these brief notes we discuss details of the numerical method to compute the aspherical determinant implemented in code `AsPy` written in `Python`.

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Contents

The notations in these notes are different from the ones used in Ref.[1]. In particular, we use r_{in} to denote Lagrangian coordinates, r-Eulerian coordinates.

1 Finite difference equations

Let us first rewrite partial differential equations (5.7) from Ref.[1] using finite differences. In this section we will omit the subscript ℓ denoting linear aspherical perturbations. Working in Lagrangian coordinates r_{in}, we rewrite Eqs. (5.7) as

$$\begin{aligned}
\dot{\Theta} &= A(\eta, r_{in})\Theta + \frac{3}{2}\delta + B(\eta, r_{in})\partial_{r_{in}}\Theta + C(\eta, r_{in})\partial_{r_{in}}\Psi + \ell(\ell+1)D(\eta, r_{in})\psi\,, \\
\dot{\delta} &= E(\eta, r_{in})\delta + B(\eta, r_{in})\partial_{r_{in}}\delta + F(\eta, r_{in})\partial_{r_{in}}\psi + G(\eta, r_{in})\Theta\,,
\end{aligned} \tag{1.1}$$

with we defined the following background functions:

$$A = -\frac{1}{2} + 2\left(\hat{\Theta} - \frac{2}{r}\partial_r\hat{\Psi}\right)\,, \qquad E = \hat{\Theta}\,, \tag{1.2a}$$

$$B = \frac{1}{\frac{\partial r}{\partial r_{in}}}\partial_r\hat{\Psi}\,, \qquad F = \frac{1}{\left(\frac{\partial r}{\partial r_{in}}\right)^2}\partial_{r_{in}}\hat{\delta}\,, \tag{1.2b}$$

$$C = \frac{1}{\frac{\partial r}{\partial r_{in}}}\left(\frac{1}{\frac{\partial r}{\partial r_{in}}}\partial_{r_{in}}\hat{\Theta} - \frac{4}{r}\left(\hat{\Theta} - 3\frac{\partial_r\hat{\Psi}}{r}\right)\right)\,, \qquad G = 1 + \hat{\delta}\,, \tag{1.2c}$$

$$D = 2\left(\hat{\Theta} - 3\frac{\partial_r\hat{\Psi}}{r}\right)\frac{1}{r^2}\,. \tag{1.2d}$$

We switch to the coordinates comoving with the background flow, which allows to absorb the shift terms into a time derivative,

$$\frac{\partial}{\partial\eta} - \partial_r\hat{\Psi}\frac{\partial}{\partial r} \equiv \left.\frac{d}{d\eta}\right|_{\text{flow}}\,. \tag{1.3}$$

We do not have to impose boundary conditions for the velocity and density fields as the their differential equations contain only time derivatives in the comoving frame.

The Euler and continuity equations are supplemented by the Poisson equation

$$\left(\partial_r^2 + \frac{2\partial_r}{r} - \frac{\ell(\ell+1)}{r^2}\right)\Psi(\eta, r_{in}) = \Theta(\eta, r_{in})\,. \tag{1.4}$$

The boundary conditions for the velocity potential are given by,

$$\begin{aligned} &\Psi(\eta, r_{in}) \propto r^\ell(r_{in})\,, \quad \text{at} \quad r \to 0 \\ &\Psi(\eta, r_{in,max}) = \mathrm{e}^{(\eta-\eta_{min})}\Psi(\eta_{min}, r_{in,max})\,. \end{aligned} \tag{1.5}$$

The boundary condition at the origin is dictated by the structure of the Poisson equation (1.4). The second condition comes from the assumption that at spatial infinity the velocity potential follows the linear evolution. Indeed, the relevant background profile is close to a top-hat and hence has support only within a region close to the size of the window function.

We work on a rectangle $[r_{in,min}, r_{in,max}] \times [\eta_{min}, 0]$ and use a uniformly spaced grid with nodes

$$\begin{aligned} &x_j = r_{in,min} + h \cdot j\,, \quad j = 0, ..., N_x\,, \\ &\eta_m = \eta_{min} + \tau \cdot m\,, \quad m = 0, ..., N_t\,, \end{aligned} \tag{1.6}$$

with

$$h = \frac{r_{in,max} - r_{in,min}}{N_x}\,, \quad \tau = \frac{-\eta_{min}}{N_t}\,. \tag{1.7}$$

We use an implicit second-order Runge-Kutta scheme (RK2) for the Euler and continuity equations. Since the Poisson equation does not contain time derivatives, let us consider it separately.

Poisson equation

We use the spatial second order finite difference method for the Poisson equation. However, we have to go through two additional steps here. First, we need to regularize the wavefunctions at $r \to 0$ to ensure the boundary condition $\Psi \propto r^l$ and reduce the numerical error at the origin. To this end we introduce new wavefunctions $\tilde{\Psi}, \tilde{\Theta}, \tilde{\delta}$ such that

$$\Psi = \tilde{\Psi}\frac{r^\ell}{r^\ell + r_0^\ell} \equiv \tilde{\Psi}\cdot\Lambda(r, r_0)\,, \quad \Theta = \tilde{\Theta}\cdot\Lambda(r, r_0)\,, \quad \delta = \tilde{\delta}\cdot\Lambda(r, r_0)\,, \tag{1.8}$$

where r_0 is some arbitrary regularization scale. The Poisson equation for the regularized functions takes the form:

$$\begin{aligned} &\frac{\partial^2\tilde{\Psi}}{\partial r^2} + \frac{2}{r}\left(1 + \frac{\ell r_0^l}{r^\ell + r_0^\ell}\right)\frac{\partial\tilde{\Psi}}{\partial r} \\ &\quad + \left(\frac{\ell r_0^\ell((\ell-1)r_0^\ell - (\ell+1)r^\ell)}{(r^\ell + r_0^\ell)^2 r^2} + \frac{2\ell r_0^\ell}{(r^\ell + r_0^\ell)r^2} - \frac{\ell(\ell+1)}{r^2}\right)\tilde{\Psi} = \tilde{\Theta}\,. \end{aligned} \tag{1.9}$$

Second, we rewrite the above equation in the Lagrangian coordinates r_{in}:

$$\frac{1}{\left(\frac{\partial r}{\partial r_{in}}\right)^2}\frac{\partial^2\tilde{\Psi}}{\partial r_{in}^2}+\frac{\partial\tilde{\Psi}}{\partial r_{in}}\left(\frac{-\frac{\partial^2 r}{\partial r_{in}^2}}{\left(\frac{\partial r}{\partial r_{in}}\right)^3}+\frac{2}{r}\frac{1}{\frac{\partial r}{\partial r_{in}}}\left(1+\frac{\ell r_0^\ell}{r^\ell+r_0^\ell}\right)\right)$$
$$+\left(\frac{\ell r_0^\ell((\ell-1)r_0^\ell-(\ell+1)r^\ell)}{(r^\ell+r_0^\ell)^2 r^2}+\frac{2\ell r_0^\ell}{(r^\ell+r_0^\ell)r^2}-\frac{\ell(\ell+1)}{r^2}\right)\tilde{\Psi}=\tilde{\Theta}\,. \tag{1.10}$$

We are going to use the following notation for grid functions:

$$\psi_j^{m+1}=\tilde{\Psi}(\eta_{m+1},x_j)\,,\quad \delta_j^{m+1}=\tilde{\delta}(\eta_{m+1},x_j)\,,\quad u_j^{m+1}=\tilde{\Theta}(\eta_{m+1},x_j)\,. \tag{1.11}$$

We will employ a second order central finite difference method, i.e.

$$\partial^2_{r_{in}}\Psi(\eta_{m+1},x_j)=\frac{\psi_{j+1}^{m+1}-2\psi_j^{m+1}+\psi_{j-1}^{m+1}}{h^2}\,,\quad \partial_{r_{in}}\Psi(\eta_{m+1},x_j)=\frac{\psi_{j+1}^{m+1}-\psi_{j-1}^{m+1}}{2h}\,. \tag{1.12}$$

Finally, we obtain the following discrete Poisson equation:

$$\psi_j^{m+1}\left[-\frac{2}{h^2}\frac{1}{\left(\frac{\partial r}{\partial r_{in}}\right)^2}\Bigg|_j^{m+1}+\frac{\ell r_0^\ell((\ell-1)r_0^\ell-(\ell+1)(r_j^{m+1})^\ell)}{((r_j^{m+1})^\ell+r_0^\ell)^2(r_j^{m+1})^2}+\frac{2\ell r_0^\ell}{((r_j^{m+1})^\ell+r_0^\ell)(r_j^{m+1})^2}-\frac{\ell(\ell+1)}{(r_j^{m+1})^2}\right]$$
$$+\psi_{j+1}^{m+1}\left[\frac{1}{\left(\frac{\partial r}{\partial r_{in}}\right)^2}\Bigg|_j^{m+1}\frac{1}{h^2}+\frac{1}{2h}\left(\frac{-\frac{\partial^2 r}{\partial r_{in}^2}}{\left(\frac{\partial r}{\partial r_{in}}\right)^3}\Bigg|_j^{m+1}+\frac{2}{r_j^{m+1}}\frac{1}{\left(\frac{\partial r}{\partial r_{in}}\right)}\Bigg|_j^{m+1}\left(1+\frac{\ell r_0^\ell}{(r_j^{m+1})^\ell+r_0^\ell}\right)\right)\right]$$
$$+\psi_{j-1}^{m+1}\left[\frac{1}{\left(\frac{\partial r}{\partial r_{in}}\right)^2}\Bigg|_j^{m+1}\frac{1}{h^2}-\frac{1}{2h}\left(\frac{-\frac{\partial^2 r}{\partial r_{in}^2}}{\left(\frac{\partial r}{\partial r_{in}}\right)^3}\Bigg|_j^{m+1}+\frac{2}{r_j^{m+1}}\frac{1}{\left(\frac{\partial r}{\partial r_{in}}\right)}\Bigg|_j^{m+1}\left(1+\frac{\ell r_0^\ell}{(r_j^{m+1})^\ell+r_0^\ell}\right)\right)\right]$$
$$=u_j^{m+1}\,. \tag{1.13}$$

The discrete boundary conditions for a mode with wavenumber k are:

$$\psi_0^{m+1}-\psi_2^{m+1}=0\,,$$
$$\psi_{N_x}^{m+1}=-\frac{4\pi}{k^2}\mathrm{e}^{\eta_{m+1}}j_\ell(kr_{in,max})\frac{1}{\Lambda(r_{N_x}^{m+1},r_0)}\,. \tag{1.14}$$

Euler and continuity equations

In the implicit RK2 scheme we replace each term in Eqs. (1.1) as

$$f(\eta,r_{in})\Big|_{\eta=\eta_m,\,r_{in}=x_j}\to\frac{1}{2}f(\eta_m,x_j)+\frac{1}{2}f(\eta_{m+1},x_j) \tag{1.15}$$

and use the following forward time derivative

$$\dot{u}(\eta_m, x_j) = \frac{u_j^{m+1} - u_j^m}{\tau}\,.$$

In the Euler and continuity equations we use the following first order lattice derivative for the velocity potential:

$$\partial_{r_{in}} \tilde{\Psi}(\eta_m, x_j) = \frac{\psi_{j+1}^m - \psi_j^m}{h}\,. \tag{1.16}$$

At $(m+1)$ step in time we solve the following system of linear equations on the coefficients $u_j^{m+1}, \delta_j^{m+1}$:

$$\begin{aligned}
&\text{for} \quad j = 0, ..., N_x - 1: \\
&u_j^{m+1}\left(\frac{1}{\tau} - \frac{A_j^{m+1}}{2} + \ell\frac{\dot{r}_j^{k+1}}{2r_j^{k+1}}\left(1 - \frac{(r_j^{k+1})^\ell}{(r_j^{k+1})^\ell + r_0^\ell}\right)\right) - \frac{3}{4}\delta_j^{m+1} - \frac{C_j^{m+1}}{2h}\psi_{j+1}^{m+1} \\
&+ \frac{\psi_j^{m+1}}{2}\left(\frac{C_j^{m+1}}{h} - C_j^{m+1}\frac{\ell r_0^l}{(r_j^{m+1})^\ell + r_0^\ell}\frac{\psi_j^{m+1}}{(r_j^{m+1})}\frac{\partial r}{\partial r_{in}}\bigg|_j^{m+1} - \ell(\ell+1)D_j^{m+1}\right) \\
&= \left(\frac{1}{\tau} + \frac{A_j^m}{2} - \ell\frac{\dot{r}_j^m}{2r_j^m}\left(1 - \frac{(r_j^m)^\ell}{(r_j^m)^\ell + r_0^\ell}\right)\right)u_j^m + \frac{3}{4}\delta_j^m + \frac{C_j^m}{2h}\psi_{j+1}^m \\
&- \frac{\psi_j^m}{2}\left(\frac{C_j^m}{h} - C_j^m\frac{\ell r_0^\ell}{(r_j^m)^l + r_0^\ell}\frac{\psi_j^m}{(r_j^m)}\frac{\partial r}{\partial r_{in}}\bigg|_j^m - \ell(\ell+1)D_j^m\right)\,,
\end{aligned} \tag{1.17}$$

$$\begin{aligned}
&\text{for} \quad j = 0, ..., N_x - 1: \\
&\delta_j^{m+1}\left(\frac{1}{\tau} - \frac{E_j^{m+1}}{2} - \ell\frac{\dot{r}_j^{m+1}}{2r_j^{m+1}}\left(1 - \frac{(r_j^{m+1})^\ell}{(r_j^{m+1})^\ell + r_0^\ell}\right)\right) - \frac{F_j^{m+1}}{2h}\psi_{j+1}^{m+1} \\
&+ \frac{\psi_j^{m+1}}{2}\left(\frac{F_j^{m+1}}{h} - F_j^{m+1}\frac{\ell r_0^\ell}{(r_j^{m+1})^\ell + r_0^\ell}\frac{\psi_j^{m+1}}{(r_j^{m+1})}\frac{\partial r}{\partial r_{in}}\bigg|_j^{m+1}\right) - \frac{1}{2}G_j^{m+1}u_j^{m+1} \\
&= \delta_j^m\left(\frac{1}{\tau} + \frac{E_j^m}{2} - \ell\frac{\dot{r}_j^m}{2r_j^m}\left(1 - \frac{(r_j^m)^\ell}{(r_j^m)^\ell + r_0^\ell}\right)\right) + \frac{F_j^m}{2h}\psi_{j+1}^m \\
&- \frac{\psi_j^m}{2}\left(\frac{F_j^m}{h} - F_j^m\frac{\ell r_0^\ell}{(r_j^m)^\ell + r_0^\ell}\frac{\psi_j^m}{(r_j^m)}\frac{\partial r}{\partial r_{in}}\bigg|_j^m\right) + \frac{1}{2}G_j^m u_j^m\,,
\end{aligned} \tag{1.18}$$

along with the boundary condition at infinity

$$\delta_{N_x}^{m+1} = u_{N_x}^{m+1} = 4\pi e^{\eta_{m+1}} j_\ell(kx_{N_x})\frac{1}{\Lambda(r_{N_x}^{m+1}, r_0)}\,, \tag{1.19}$$

and the initial conditions

$$\text{for} \quad j=0,...,N_x: \quad u_j^0=\delta_j^0=\mathrm{e}^{\eta_{min}}(4\pi)j_\ell(kx_j)\frac{1}{\Lambda(r_j^0,r_0)}\,, \quad \psi_j^0=-\frac{u_j^0}{k^2}\,, \tag{1.20}$$

Combined with the Poisson equation and corresponding boundary conditions (1.13) and (1.14) they complete a system of equations to be solved.

Discrete equations for $\mu^{(2)}$ and $r_\eta^{(2)}$

At a next step we discretize the ODE for $\mu_2=\mu^{(2)}$ and $r_2=r_\eta^{(2)}$,

$$\begin{aligned}
&\frac{d\mu_2}{d\eta}+\frac{dr_2}{d\eta}L(\eta)+r_2\frac{dL(\eta)}{d\eta}=\hat{R}^2\Upsilon_\delta\,,\\
&\frac{d^2r_2}{d\eta^2}+\frac{1}{2}\frac{dr_2}{d\eta}-\frac{r_2}{2}+P(\eta)r_2+\frac{3}{2\hat{R}^2}=-\Upsilon_\Theta\,,
\end{aligned} \tag{1.21}$$

where we defined

$$L\equiv\hat{R}^2(1+\hat{\delta})\,, \quad P\equiv\frac{3}{2}(1+\hat{\delta}(\hat{R}_{in}))-\frac{\hat{R}_{in}^3}{\hat{R}^3}\,. \tag{1.22}$$

The initial and final conditions for these two equations are given by:

$$\begin{aligned}
&\mu_2(\eta_{min})=\hat{R}_{in}^2\frac{5\Upsilon_\delta+2\Upsilon_\Theta}{7}\Big|_{\eta=\eta_{min}}\,,\\
&\dot{r_2}L(\eta)+r_2\dot{L}(\eta)\Big|_{\eta=\eta_{min}}=-2\hat{R}_{in}^2\frac{1}{14}(3\Upsilon_\delta+4\Upsilon_\Theta)\Big|_{\eta=\eta_{min}}\,,\\
&r_2(0)=0\,,
\end{aligned} \tag{1.23}$$

where we have imposed the initial condition not on r_2, but on the variable r_2L to ensure numerical stability. The second-order finite difference scheme for Eqs. (1.21) is given by the following system of linear equations on coefficients $\{r_2^0,...,r_2^{N_t},\mu_2^0,...,\mu_2^{N_t}\}$:

$$\begin{aligned}
&\mu_2^0=R_{in}^2\frac{5\Upsilon_\delta^0+2\Upsilon_\Theta^0}{7}\,, \quad r_2^{N_t}=0\,, \quad \frac{r_2^2-r_2^0}{2\tau}L^1+r_2^1\frac{L^2-L^0}{2\tau}=-\frac{R_{in}^2}{7}(3\Upsilon_\delta^0+4\Upsilon_\Theta^0)\,,\\
&\text{for} \quad m=1,...,N_t-1\ :\\
&\quad \frac{r_2^{m+1}-2r_2^m+r_2^{m-1}}{\tau^2}+\frac{1}{2}\frac{r_2^{m+1}-r_2^{m-1}}{2\tau}-\frac{r_2^m}{2}+P^mr_2^m+\frac{3}{2(R_{R_{in}}^m)^2}\mu_2^k=-\Upsilon_\Theta^m\,,\\
&\quad \frac{\mu_2^{m+1}-\mu_2^m}{\tau}+\frac{r_2^{m+1}-r_2^m}{\tau}L^m+r_2^m\frac{L^{m+1}-L^m}{\tau}=\frac{(R_{R_{in}}^m)^2\Upsilon_\delta^m+(R_{R_{in}}^{m+1})^2\Upsilon_\delta^{m+1}}{2}\,.
\end{aligned} \tag{1.24}$$

2 Equations for ΛCDM

In the ΛCDM-universe the relevant equations are modified as follows:

$$
\begin{aligned}
&\dot{u}(\eta,r_{in}) = A(\eta,r_{in})u(\eta,r_{in}) + \frac{3}{2}\frac{\Omega_{m,\eta}}{f^2}\delta + B(\eta,r_{in})\partial_{r_{in}}u(\eta,r_{in}) + C(\eta,r_{in})\partial_{r_{in}}\psi(\eta,r_{in}) + \ell(\ell+1)D(\eta,\\
&\dot{\delta}(\eta,r_{in}) = E(\eta,r_{in})\delta(\eta,r_{in}) + B(\eta,r_{in})\partial_{r_{in}}\delta(\eta,r_{in}) + F(\eta,r_{in})\partial_{r_{in}}\psi(\eta,r_{in}) + G(\eta,r_{in})u(\eta,r_{in})\,,\\
&\left(\partial_r^2 + \frac{2\partial_r}{r} - \frac{\ell(\ell+1)}{r^2}\right)\psi(\eta,r_{in}) = u(\eta,r_{in})\,,
\end{aligned}
\tag{2.1}
$$

with the functions defined as

$$
A = -\frac{1}{2}\left(3\frac{\Omega_{m,\eta}}{f^2} - 2\right) + 2\left(\hat{\Theta} - \frac{2}{r}\partial_r\hat{\Psi}\right)\,, \qquad E = \hat{\Theta}\,, \tag{2.2}
$$

$$
B = \frac{1}{\frac{\partial r}{\partial r_{in}}}\partial_r\hat{\Psi}\,, \tag{2.3}
$$

$$
C = \frac{1}{\frac{\partial r}{\partial r_{in}}}\left(\frac{1}{\frac{\partial r}{\partial r_{in}}}\partial_{r_{in}}\hat{\Theta} - \frac{4}{r}\left(\hat{\Theta} - 3\frac{\partial_r\hat{\Psi}}{r}\right)\right)\,, \qquad F = \frac{1}{\left(\frac{\partial r}{\partial r_{in}}\right)^2}\partial_{r_{in}}\hat{\delta}\,, \tag{2.4}
$$

$$
D = 2\left(\hat{\Theta} - 3\frac{\partial_r\hat{\Psi}}{r}\right)\frac{1}{r^2}\,, \qquad G = 1 + \hat{\delta}\,. \tag{2.5}
$$

The other equations are

$$
\begin{aligned}
&\frac{d\mu_2}{d\eta} + \frac{dr_2}{d\eta}L(\eta) + r_2\frac{dL(\eta)}{d\eta} = R^2\Upsilon_\delta\,,\\
&\frac{d^2r_2}{d\eta^2} + \frac{\alpha}{2}\frac{dr_2}{d\eta} - \beta\frac{r_2}{2} + P(\eta)r_2 + \beta\frac{3\mu^2}{2R^2} = -\Upsilon_\Theta\,,
\end{aligned}
\tag{2.6}
$$

where

$$
\begin{aligned}
&L = \hat{R}^2(1+\hat{\delta})\,,\\
&P = \beta\left(\frac{3}{2}(1+\hat{\delta}) - \frac{R_{in}^3}{\hat{R}^3}\right)\,.
\end{aligned}
\tag{2.7}
$$

and we defined

$$
\alpha = 3\frac{\Omega_{m,\eta}}{f^2} - 2\,, \quad \beta = \frac{\Omega_{m,\eta}}{f^2}\,. \tag{2.8}
$$

References

[1] M. M. Ivanov, A. A. Kaurov and S. Sibiryakov, arXiv:1811.07913 [astro-ph.CO].