Esercizio 2 p. 24

Definire la funzione *append*, che concatena due liste, in modo tale che:

- $append(x, y) \in List(A)[x \in List(A),\ y \in List(A)]$;
- $append(x, \text{nil}) = x \in List(A)[x \in List(A)]$;
- $append(x, cons(y, a)) = cons(append(x, y), a) \in List(A)[x \in List(A),\ y \in List(A)]$.

Definizione:

$$append(x, y) \stackrel{\text{def}}{=} \text{El}_{List}(y, x, (y', a, acc).cons(acc, a))$$

Esempi:

$$\begin{aligned}
&append(cons(cons(\text{nil}, 1), 2), \text{nil}) = \\
&\quad = \text{El}_{List}(\text{nil}, cons(cons(\text{nil}, 1), 2), (y', a, acc).cons(acc, a)) = \\
&\quad = cons(cons(\text{nil}, 1), 2)
\end{aligned}$$

Usiamo la notazione $\{n_1, n_2, \ldots, n_k\}$ per le liste per facilitare la lettura.

$$\begin{aligned}
&append(\{1, 2\}, \{3, 4\}) = \\
&\quad = \text{El}_{List}(\{3, 4\}, \{1, 2\}, (y', a, acc).cons(acc, a)) = \\
&\quad = cons(\text{El}_{List}(\{3\}, \{1, 2\}, (y', a, acc).cons(acc, a)), 4) = \\
&\quad = cons(cons(\text{El}_{List}(\text{nil}, \{1, 2\}, (y', a, acc).cons(acc, a)), 3), 4) = \\
&\quad = cons(cons(\{1, 2\}, 3), 4) \equiv \{1, 2, 3, 4\}
\end{aligned}$$

Esercizio 3 p. 24

Specificare il tipo e definire le seguenti operazioni:

- *back*, che restituisce la lista meno l'ultimo elemento;
- $front$, che restituisce la lista meno il primo elemento;
- *last*, che restituisce l'ultimo elemento della lista;
- $first$, che restituisce il primo elemento della lista.

Siccome si tratta di funzioni che restituiscono un risultato solo se la lista è non vuota, allora definiamo un tipo opzionale $Opt(O) \stackrel{\text{def}}{=} O + \text{N}_1$, con cui modellare l'assenza o la presenza di un risultato:

- $inl(o) \in O + \text{N}_1\ [a \in O]$ modella la presenza di un risultato (o);
- $inr(*) \in O + \text{N}_1\ []$ modella l'assenza di un risultato.

Funzione *back*:

- $back(x) \in List(A) + N_1\ [x \in List(A)]$;

- $back(nil) = inr(*) \in List(A) + N_1$ [];
- $back(cons(s, a)) = s \in List(A) + N_1$ $[s \in List(A), a \in A]$.

Definizione:

$$back(x) \stackrel{\text{def}}{=} \text{El}_{List}(s, inr(*), (s, a, c).inl(\text{El}_{List}(s, \text{nil}, (s', a', acc).cons(acc, a'))))$$

Esempi:

$$\begin{aligned}
&back(cons(\text{nil}, 2)) = \\
&\quad = \text{El}_{List}(cons(\text{nil}, 2), inr(*), (s, a, c).inl(\text{El}_{List}(s, \text{nil}, (s', a', acc).cons(acc, a'))))\\
&\quad = inl(\text{El}_{List}(\text{nil}, \text{nil}, (s', a', acc).cons(acc, a'))) =\\
&\quad = inl(\text{nil})
\end{aligned}$$

$$\begin{aligned}
&back(\{1, 2\}) = \\
&\quad = \text{El}_{List}(\{1, 2\}, inr(*), (s, a, c).inl(\text{El}_{List}(s, \text{nil}, (s', a', acc).cons(acc, a'))))\\
&\quad = inl(\text{El}_{List}(\{1\}, \text{nil}, (s', a', acc).cons(acc, a'))) =\\
&\quad = inl(cons(\text{El}_{List}(\text{nil}, \text{nil}, (s', a', acc).cons(acc, a')), 1)) =\\
&\quad = inl(cons(\text{nil}, 1)) = inl(\{1\})
\end{aligned}$$

Funzione $front$:

- $front(x) \in List(A) + \text{N}_1$ $[x \in List(A)]$;
- $front(nil) = inr(*) \in List(A) + \text{N}_1$ [];
- $front(cons(s, a)) = inl(\text{El}_+(front(s), (s').cons(front(s'), a), (u).nil)) \in List(A) + \text{N}_1$ $[s \in List(A), a \in A]$

Definizione:

$front(x) = \text{El}_{List}(x, inr(*), (s, a, c).inl(\text{El}_{List}(cons(s, a), \text{nil}, e)))$
dove $e(s', a', acc) = \text{El}_{List}(s', \text{nil}, (x, y, z).cons(acc, a'))$

Esempi:

$$\begin{aligned}
&front(cons(\text{nil}, 2)) = \\
&\quad = \text{El}_{List}(cons(\text{nil}, 2), inr(*), (s, a, c).inl(\text{El}_{List}(cons(s, a), \text{nil}, e)))\\
&\quad = inl(\text{El}_{List}(cons(\text{nil}, 2), \text{nil}, e)) =\\
&\quad = inl(e(\text{nil}, 2, \text{El}_{List}(\text{nil}, \text{nil}, e))) =\\
&\quad = inl(e(\text{nil}, 2, \text{nil})) =\\
&\quad = inl(\text{El}_{List}(\text{nil}, \text{nil}, (x, y, z).cons(\text{nil}, 2))) = inl(\text{nil})
\end{aligned}$$

$front(\{1,2\}) =$
$$= \mathrm{El}_{List}(\{1,2\}, inr(*), (s,a,c).inl(\mathrm{El}_{List}(cons(s,a), \mathrm{nil}, e)))$$
$$= inl(\mathrm{El}_{List}(\{1,2\}, \mathrm{nil}, e)) =$$
$$= inl(e(\{1\}, 2, \mathrm{El}_{List}(\{1\}, \mathrm{nil}, e))) =$$
$$= inl(e(\{1\}, 2, e(\mathrm{nil}, 1, \mathrm{El}_{List}(\mathrm{nil}, \mathrm{nil}, e)))) =$$
$$= inl(e(\{1\}, 2, e(\mathrm{nil}, 1, \mathrm{nil}))) =$$
$$= inl(e(\{1\}, 2, \mathrm{El}_{List}(\mathrm{nil}, \mathrm{nil}, (x,y,z).cons(\mathrm{nil}, 1)))) =$$
$$= inl(e(\{1\}, 2, \mathrm{nil})) =$$
$$= inl(\mathrm{El}_{List}(\{1\}, \mathrm{nil}, (x,y,z).cons(\mathrm{nil}, 2))) =$$
$$= inl(cons(\mathrm{nil}, 2)) = inl(\{2\})$$

Funzione $last$:

- $last(x) \in A + \mathrm{N}_1\ [x \in List(A)]$;
- $last(nil) = inr(*) \in A + \mathrm{N}_1\ []$;
- $last(cons(s,a)) = inl(a) \in A + \mathrm{N}_1\ [s \in List(A), a \in A]$.

Definizione:

$$last(x) = \mathrm{El}_{List}(x, inr(*), (s', a', acc).inl(a))$$

Funzione $first$:

- $first(x) \in A + \mathrm{N}_1\ [x \in List(A)]$;
- $first(nil) = inr(*) \in A + \mathrm{N}_1\ []$;
- $first(cons(s,a)) = inl(\mathrm{El}_+(first(s), (a').a', (u).a)) \in A + \mathrm{N}_1$ $[s \in List(A), a \in A]$

Definizione:

$$first(x) = \mathrm{El}_{List}(x, inr(*), (s,a,acc).inl(\mathrm{El}_{List}(s,a,e)))$$
dove $e(s', a', acc) = \mathrm{El}_{List}(s', a', (x,y,z).acc)$

Esempi:

$first(\{1\}) =$
$$= \mathrm{El}_{List}(\{1\}, inr(*), (s,a,acc).inl(\mathrm{El}_{List}(s,a,e)))$$
$$= inl(\mathrm{El}_{List}(\mathrm{nil}, 1, e)) = inl(1)$$

$first(\{1,2,3\}) =$

$$
\begin{aligned}
&= \mathrm{El}_{List}(\{1,2,3\}, inr(*), (s,a,acc).inl(\mathrm{El}_{List}(s,a,e))) \\
&= inl(\mathrm{El}_{List}(\{1,2\},3,e)) = \\
&= inl(e(\{1\},2,\mathrm{El}_{List}(\{1\},3,e))) = \\
&= inl(e(\{1\},2,e(\mathrm{nil},1,\mathrm{El}_{List}(\mathrm{nil},3,e)))) = \\
&= inl(e(\{1\},2,e(\mathrm{nil},1,3))) = \\
&= inl(e(\{1\},2,\mathrm{El}_{List}(\mathrm{nil},1,(x,y,z).3))) = \\
&= inl(e(\{1\},2,1)) = \\
&= inl(\mathrm{El}_{List}(\{1\},2,(x,y,z).1)) = inl(1)
\end{aligned}
$$

Esercizio 1 p. 27 Scrivere le regole del tipo *Bool* che rappresenta valori booleani e provare che il *Bool* è rappresentabile da $N_1 + N_1$.

Formazione:

$$\frac{\Gamma\ cont}{\mathrm{Bool}\ type\ [\Gamma]}\ \text{F-Bool}$$

Sia $Bool \stackrel{\text{def}}{=} N_1 + N_1$. Allora la regola F-Bool è derivabile assumendo $\Gamma\ cont$:

$$\frac{\dfrac{\Gamma\ cont}{\mathrm{N}_1\ type\ [\Gamma]}\ \text{F-S} \qquad \dfrac{\Gamma\ cont}{\mathrm{N}_1\ type\ [\Gamma]}\ \text{F-S}}{\mathrm{N}_1 + \mathrm{N}_1\ type\ [\Gamma]}\ \text{F-+}$$

Introduzione (*false*):

$$\frac{\Gamma\ cont}{0 \in \mathrm{Bool}\ [\Gamma]}\ \text{I-0-Bool}$$

Sia $Bool \stackrel{\text{def}}{=} N_1 + N_1$ e $0 \stackrel{\text{def}}{=} inl(*)$. Allora la regola I-0-Bool è derivabile assumendo $\Gamma\ cont$:

$$\frac{\dfrac{\Gamma\ cont}{* \in \mathrm{N}_1\ [\Gamma]}\ \text{I-S} \qquad \dfrac{\Gamma\ cont}{\mathrm{N}_1\ type\ [\Gamma]}\ \text{F-S} \qquad \dfrac{\Gamma\ cont}{\mathrm{N}_1\ type\ [\Gamma]}\ \text{F-S}}{inl(*) \in \mathrm{N}_1 + \mathrm{N}_1\ [\Gamma]}\ \text{I-1-+}$$

Introduzione (*true*):

$$\frac{\Gamma\ cont}{1 \in \mathrm{Bool}\ [\Gamma]}\ \text{I-1-Bool}$$

Sia $Bool \stackrel{\text{def}}{=} N_1 + N_1$ e $1 \stackrel{\text{def}}{=} inr(*)$. Allora la regola I-1-Bool è derivabile assumendo $\Gamma\ cont$:

$$\cfrac{\cfrac{\Gamma\ cont}{* \in \mathrm{N}_1\ [\Gamma]}\ \text{I-S} \qquad \cfrac{\Gamma\ cont}{\mathrm{N}_1\ type\ [\Gamma]}\ \text{F-S} \qquad \cfrac{\Gamma\ cont}{\mathrm{N}_1\ type\ [\Gamma]}\ \text{F-S}}{inr(*) \in \mathrm{N}_1 + \mathrm{N}_1\ [\Gamma]}\ \text{I-2-}+$$

Eliminazione:

$$\frac{M(z)type\ [\Gamma, z \in Bool] \quad b \in Bool\ [\Gamma] \quad m_0 \in M(0)\ [\Gamma] \quad m_1 \in M(1)\ [\Gamma]}{\mathrm{El}_{Bool}(b, m_0, m_1) \in M(b)\ [\Gamma]}\ \text{E-Bool}$$

Sia $Bool \overset{\text{def}}{=} N_1 + N_1$ e $\mathrm{El}_{Bool}(b, m_0, m_1) \overset{\text{def}}{=} \mathrm{El}_+(b, e_0, e_1)$, con $e_0(*) \overset{\text{def}}{=} m_0$, $e_1(*) \overset{\text{def}}{=} m_1$. Allora la regola E-Bool è derivabile:

- da $M(z)type\ [\Gamma, z \in Bool]$ assumiamo $M(z)type\ [\Gamma, z \in \mathrm{N}_1 + \mathrm{N}_1]$
- da $b \in Bool\ [\Gamma]$ assumiamo $b \in \mathrm{N}_1 + \mathrm{N}_1\ [\Gamma]$
- da $m_0 \in M(0)\ [\Gamma]$ assumiamo $e_0(*) \in M(inl(*))\ [\Gamma]$
- da $m_1 \in M(1)\ [\Gamma]$ assumiamo $e_1(*) \in M(inr(*))\ [\Gamma]$

$$\frac{M(z)type\ [\Gamma, z \in \mathrm{N}_1 + \mathrm{N}_1] \quad b \in \mathrm{N}_1 + \mathrm{N}_1\ [\Gamma] \quad e_0(x_0) \in M(inl(x_0))\ [\Gamma, x_0 \in \mathrm{N}_1] \quad e_1(x_1) \in M(inl(x_1))\ [\Gamma, x_1 \in \mathrm{N}_1]}{\mathrm{El}_+(b, e_0, e_1) \in M(b)\ [\Gamma]}\ \text{E-}+$$

Che coincide con le premesse, siccome $x \in \mathrm{N}_1 \Longrightarrow x = *$.

Conversione (false):

$$\frac{M(z)type\ [\Gamma, z \in Bool] \quad m_0 \in M(0)\ [\Gamma] \quad m_1 \in M(1)\ [\Gamma]}{\mathrm{El}_{Bool}(0, m_0, m_1) = m_0 \in M(0)\ [\Gamma]}\ \text{C}_1\text{-Bool}$$

Sia $Bool \overset{\text{def}}{=} N_1 + N_1$ e $\mathrm{El}_{Bool}(b, m_0, m_1) \overset{\text{def}}{=} \mathrm{El}_+(b, e_0, e_1)$, con $e_0(*) \overset{\text{def}}{=} m_0$, $e_1(*) \overset{\text{def}}{=} m_1$. Allora la regola C-0-Bool è derivabile:

- da $M(z)type\ [\Gamma, z \in Bool]$ assumiamo $M(z)type\ [\Gamma, z \in \mathrm{N}_1 + \mathrm{N}_1]$
- da $m_0 \in M(0)\ [\Gamma]$ assumiamo $e_0(*) \in M(inl(*))\ [\Gamma]$
- da $m_1 \in M(1)\ [\Gamma]$ assumiamo $e_1(*) \in M(inr(*))\ [\Gamma]$

$$\cfrac{M(z)type\ [\Gamma, z \in N_1 + N_1] \quad \cfrac{\Gamma\ cont}{* \in \mathrm{N}_1\ [\Gamma]}\ \text{I-S} \quad e_0(*) \in M(inl(*))\ [\Gamma] \quad e_1(*) \in M(inr(*))\ [\Gamma]}{\mathrm{El}_+(inl(*), e_0, e_1) = e_0(*) \in M(inl(*))\ [\Gamma]}\ \text{C}_1\text{-}+$$

Conversione (true):

$$\frac{M(z)type\ [\Gamma, z \in Bool] \quad m_0 \in M(0)\ [\Gamma] \quad m_1 \in M(1)\ [\Gamma]}{\mathrm{El}_{Bool}(1, m_0, m_1) = m_1 \in M(1)\ [\Gamma]}\ \text{C}_2\text{-Bool}$$

Sia $Bool \overset{\text{def}}{=} N_1 + N_1$ e $\mathrm{El}_{Bool}(b, m_0, m_1) \overset{\text{def}}{=} \mathrm{El}_+(b, e_0, e_1)$, con $e_0(*) \overset{\text{def}}{=} m_0$, $e_1(*) \overset{\text{def}}{=} m_1$. Allora la regola C-1-Bool è derivabile:

- da $M(z)type\ [\Gamma, z \in Bool]$ assumiamo $M(z)type\ [\Gamma, z \in \mathrm{N}_1 + \mathrm{N}_1]$
- da $m_0 \in M(0)\ [\Gamma]$ assumiamo $e_0(*) \in M(inl(*))\ [\Gamma]$

- da $m_1 \in M(1)\ [\Gamma]$ assumiamo $e_1(*) \in M(inr(*))\ [\Gamma]$

$$\dfrac{M(z)type\ [\Gamma, z \in N_1 + N_1] \quad \dfrac{\Gamma\ cont}{* \in \mathrm{N}_1\ [\Gamma]}\text{ I-S} \quad e_0(*) \in M(inl(*))\ [\Gamma] \quad e_1(*) \in M(inr(*))\ [\Gamma]}{\mathrm{El}_+(inr(*), e_0, e_1) = e_1(*) \in M(inr(*))\ [\Gamma]}\ \mathrm{C}_2\text{-}+$$

Esercizio 4 p. 53

Supponendo il tipo $Bool \stackrel{\text{def}}{=} \mathrm{N}_1 + \mathrm{N}_1$, con $0 \stackrel{\text{def}}{=} inl(*)$ e $1 \stackrel{\text{def}}{=} inr(*)$ si vuole mostrare che esiste un proof-term q tale che:

$$q \in \forall_{x \in Bool}\ ((x = 0) \vee (x = 1))$$

Sia $\varphi = \forall_{x \in Bool}\ ((x = 0) \vee (x = 1))$. Allora $q \in \varphi$ sse esiste un t tale che $t \in (\varphi)^I$ sia derivabile nella teoria dei tipi.

Il tipo è:

$$(\varphi)^I = \Pi_{x \in Bool}\ (\mathrm{Id}(Bool, x, 0) + \mathrm{Id}(Bool, x, 1))$$

Il termine è $\lambda x^{Bool}.\mathrm{El}_{Bool}(x, inl(id(0)), inr(id(1)))$ (usiamo come regola di derivazione di El_{Bool} quella dell'esercizio 1 di p. 27).

$$\dfrac{\mathrm{Id}(Bool,z,0)+\mathrm{Id}(Bool,z,1)\ type\ [z\in Bool,\ x\in Bool] \qquad \dfrac{\dfrac{[]\ cont}{Bool\ type\ []}\,\text{F-Bool}}{x\in Bool\ [x\in Bool]}\,\text{var} \qquad inl(id(0))\in \mathrm{Id}(Bool,0,0)+\mathrm{Id}(Bool,0,1)\ [x\in Bool] \qquad inr(id(1))\in \mathrm{Id}(Bool,1,0)+\mathrm{Id}(Bool,1,1)\ [x\in Bool]}{\dfrac{\mathrm{El}_{Bool}(x,inl(id(0)),inr(id(1)))\in \mathrm{Id}(Bool,x,0)+\mathrm{Id}(Bool,x,1)\ [x\in Bool]}{\lambda x^{Bool}.\mathrm{El}_{Bool}(x,inl(id(0)),inr(id(1)))\in \Pi_{x\in Bool}\ (\mathrm{Id}(Bool,x,0)+\mathrm{Id}(Bool,x,1))\ []}\,\text{I-}\Pi}\,\text{E-Bool}$$

Per semplicità di lettura la derivazione dell'albero segue spezzata.

D'ora in poi la derivazione del giudizio $Bool\ type\ [z\in Bool,\ x\in Bool]$ e dei giudizi derivati nel seguente albero verranno omessi:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[]\ cont}{Bool\ type\ []}\,\text{F-Bool}}{x\in Bool\ cont}\,\text{F-c}}{Bool\ type\ [x\in Bool]}\,\text{F-Bool}}{z\in Bool,\ x\in Bool\ cont}\,\text{F-c}}{Bool\ type\ [z\in Bool,\ x\in Bool]}\,\text{F-Bool}$$

Il giudizio $\mathrm{Id}(Bool,z,0)+\mathrm{Id}(Bool,z,1)\ type\ [z\in Bool,\ x\in Bool]$ è derivabile:

$$\dfrac{\dfrac{\dfrac{[]\ cont}{Bool\ type\ [z\in Bool,\ x\in Bool]}\,[...] \qquad \dfrac{\dfrac{[]\ cont}{Bool\ type\ [x\in Bool]}\,[...]}{z\in Bool\ [z\in Bool,\ x\in Bool]}\,\text{var} \qquad \dfrac{\dfrac{[]\ cont}{z\in Bool,\ x\in Bool\ cont}\,[...]}{0\in Bool\ [z\in Bool,\ x\in Bool]}\,\text{I}_1\text{-Bool}}{\mathrm{Id}(Bool,z,0) type\ [z\in Bool,\ x\in Bool]}\,\text{F-Id} \qquad \dfrac{\dfrac{[]\ cont}{Bool\ type\ [z\in Bool,\ x\in Bool]}\,[...] \qquad \dfrac{\dfrac{[]\ cont}{Bool\ type\ [x\in Bool]}\,[...]}{z\in Bool\ [z\in Bool,\ x\in Bool]}\,\text{var} \qquad \dfrac{\dfrac{[]\ cont}{z\in Bool,\ x\in Bool\ cont}\,[...]}{1\in Bool\ [z\in Bool,\ x\in Bool]}\,\text{I}_2\text{-Bool}}{\mathrm{Id}(Bool,z,1) type\ [z\in Bool,\ x\in Bool]}\,\text{F-Id}}{\mathrm{Id}(Bool,z,0)+\mathrm{Id}(Bool,z,1)\ type\ [z\in Bool,\ x\in Bool]}\,\text{F-+}$$

Il giuduzio $inl(id(0))\in \mathrm{Id}(Bool,0,0)+\mathrm{Id}(Bool,0,1)\ [x\in Bool]$ è derivabile:

$$\dfrac{\dfrac{\dfrac{\dfrac{[]\ cont}{0\in Bool\ []}\,\text{I}_1\text{-Bool}}{id(0)\in \mathrm{Id}(Bool,0,0)\ []}\,\text{I-Id} \qquad \dfrac{\dfrac{[]\ cont}{Bool\ type\ []}\,\text{F-Bool} \qquad \dfrac{[]\ cont}{0\in Bool\ []}\,\text{I}_1\text{-Bool} \qquad \dfrac{[]\ cont}{0\in Bool\ []}\,\text{I}_2\text{-Bool}}{\mathrm{Id}(Bool,0,0)\ type\ []} \qquad \dfrac{\dfrac{[]\ cont}{Bool\ type\ []}\,\text{F-Bool} \qquad \dfrac{[]\ cont}{0\in Bool\ []}\,\text{I}_1\text{-Bool} \qquad \dfrac{[]\ cont}{1\in Bool\ []}\,\text{I}_2\text{-Bool}}{\mathrm{Id}(Bool,0,1)\ type\ []}}{inl(id(0))\in \mathrm{Id}(Bool,0,0)+\mathrm{Id}(Bool,0,1)\ []}\,\text{I}_1\text{-+} \qquad \dfrac{[]\ cont}{x\in Bool\ cont}\,[...]}{inl(id(0))\in \mathrm{Id}(Bool,0,0)+\mathrm{Id}(Bool,0,1)\ [x\in Bool]}\,\text{weak-te}$$

Il giudizio $inr(id(1)) \in \mathrm{Id}(Bool, 1, 0) + \mathrm{Id}(Bool, 1, 1)\ [x \in Bool]$ è derivabile:

$$
\dfrac{
\dfrac{
\dfrac{\dfrac{[]\ cont}{1 \in Bool\ []}\,\text{I}_2\text{-Bool}}{id(1) \in \mathrm{Id}(Bool,1,1)\ []}\,\text{I-Id}
\qquad
\dfrac{\dfrac{[]\ cont}{Bool\ type\ []}\,\text{F-Bool} \quad \dfrac{[]\ cont}{1 \in Bool\ []}\,\text{I}_2\text{-Bool} \quad \dfrac{[]\ cont}{0 \in Bool\ []}\,\text{I}_1\text{-Bool}}{\mathrm{Id}(Bool,1,0)\ type\ []}
\qquad
\dfrac{\dfrac{[]\ cont}{Bool\ type\ []}\,\text{F-Bool} \quad \dfrac{[]\ cont}{1 \in Bool\ []}\,\text{I}_1\text{-Bool} \quad \dfrac{[]\ cont}{1 \in Bool\ []}\,\text{I}_2\text{-Bool}}{\mathrm{Id}(Bool,1,1)\ type\ []}
}{inr(id(1)) \in \mathrm{Id}(Bool,1,0) + \mathrm{Id}(Bool,1,1)\ []}\,\text{I}_1\text{-}+
\qquad
\dfrac{[]\ cont}{x \in Bool\ cont}\,[...]
}{inr(id(1)) \in \mathrm{Id}(Bool,1,0) + \mathrm{Id}(Bool,1,1)\ [x \in Bool]}\,\text{weak-te}
$$

Pertanto, siccome il giudizio $\lambda x^{Bool}.\mathrm{El}_{Bool}(x, inl(id(0)), inr(id(1))) \in \Pi_{x \in Bool}\ (\mathrm{Id}(Bool, x, 0) + \mathrm{Id}(Bool, x, 1))\ []$ è derivabile, e dunque il tipo è abitato, esiste un proof term $p \in \forall_{x \in Bool}\ ((x = 0) \vee (x = 1))$.

∞

Esercizio 2 p. 53

1. Dimostrare che $\mathrm{N}_1 \to \mathrm{N}_0$ e N_0 sono isomorfi.
2. Fornire la traduzione in logica proposizionale *as-sets* secondo Curry-Howard-Martin-Löf.

1. $\mathrm{N}_1 \to \mathrm{N}_0$ e N_0 sono isomorfi sse esistono due termini $f(x)$ e $h(y)$ tali che:

- $f(x) \in \mathrm{N}_0\ [x \in \mathrm{N}_1 \to \mathrm{N}_0]$
- $h(y) \in \mathrm{N}_1 \to \mathrm{N}_0\ [y \in \mathrm{N}_0]$

siano derivabili nella teoria dei tipi; inoltre tali termini devono essere tali per cui esistano $\mathbf{pf_1}$ e $\mathbf{pf_2}$ per le proposizioni:

- $\mathbf{pf_1} \in x = h(f(x))\ [x \in \mathrm{N}_1 \to \mathrm{N}_0]$;
- $\mathbf{pf_2} \in y = f(h(y))\ [y \in \mathrm{N}_0]$.

I termini sono:

$f(x) = \mathrm{Ap}(x, *)$

$$\dfrac{\dfrac{\dfrac{\dfrac{[]\ \mathit{cont}}{\mathrm{N}_1 \to \mathrm{N}_0\ \mathit{type}\ []}\ \text{F-}\to}{x \in \mathrm{N}_1 \to \mathrm{N}_0\ \mathit{cont}}\ \text{F-c}}{* \in \mathrm{N}_1\ [x \in \mathrm{N}_1 \to \mathrm{N}_0]}\ \text{I-S} \qquad \dfrac{\dfrac{[]\ \mathit{cont}}{\mathrm{N}_1 \to \mathrm{N}_0\ \mathit{type}\ []}\ \text{F-}\to}{x \in \mathrm{N}_1 \to \mathrm{N}_0\ [x \in \mathrm{N}_1 \to \mathrm{N}_0]}\ \text{var}}{\mathrm{Ap}(x, *) \in \mathrm{N}_0\ [x \in \mathrm{N}_1 \to \mathrm{N}_0]}\ \text{E-}P$$

$h(y) = \lambda s^{\mathrm{N}_1}.y$

$$\dfrac{\dfrac{\dfrac{\dfrac{[]\ \mathit{cont}}{\mathrm{N}_1\ \mathit{type}\ []}\ \text{F-S}}{s \in \mathrm{N}_1\ \mathit{cont}}\ \text{F-c}}{y \in \mathrm{N}_0\ [y \in \mathrm{N}_0\ s \in \mathrm{N}_1]}\ \text{var}}{\lambda s^{\mathrm{N}_1}.y \in \mathrm{N}_1 \to \mathrm{N}_0\ [y \in \mathrm{N}_0]}\ \text{I-}\to$$

Pertanto siccome

- $h(f(x)) = s^{\mathrm{N}_1}.\mathrm{Ap}(x, *)$;
- $f(h(y)) = \mathrm{Ap}(\lambda s^{\mathrm{N}_1}.y, *)$.

la traduzione in teoria dei tipi delle proposizioni è:

- $x = h(f(x))\ \mathit{prop}\ [x \in \mathrm{N}_1 \to \mathrm{N}_0] = \mathrm{Id}(\mathrm{N}_1 \to \mathrm{N}_0, x, \lambda s^{\mathrm{N}_1}.\mathrm{Ap}(x, *))\ [x \in \mathrm{N}_1 \to \mathrm{N}_0]$;
- $y = f(h(y))\ \mathit{prop}\ [y \in \mathrm{N}_0] = \mathrm{Id}(\mathrm{N}_0, y, \mathrm{Ap}(\lambda s^{\mathrm{N}_1}.y, *))\ [y \in \mathrm{N}_0]$.

I *proof term* sono:

$\mathbf{pf_1} = \mathrm{El}_{\mathrm{N}_0}(Ap(x,*))$

$$\dfrac{\dfrac{[]\ \mathit{cont}}{\mathrm{Ap}(x,*) \in \mathrm{N}_0\ [x \in \mathrm{N}_1 \to \mathrm{N}_0]}\ [...] \qquad \dfrac{\dfrac{\dfrac{[]\ \mathit{cont}}{\mathrm{N}_1 \to \mathrm{N}_0\ \mathit{type}\ [x \in \mathrm{N}_1 \to \mathrm{N}_0]}\ [...] \quad \dfrac{[]\ \mathit{cont}}{x \in \mathrm{N}_1 \to \mathrm{N}_0\ [x \in \mathrm{N}_1 \to \mathrm{N}_0]}\ [...] \quad \dfrac{\dfrac{[]\ \mathit{cont}}{\mathrm{Ap}(x,*) \in \mathrm{N}_0\ [x \in \mathrm{N}_1 \to \mathrm{N}_0,\ s \in \mathrm{N}_1]}\ [...]}{\lambda s^{\mathrm{N}_1}.\mathrm{Ap}(x,*) \in \mathrm{N}_1 \to \mathrm{N}_0\ [x \in \mathrm{N}_1 \to \mathrm{N}_0]}\ \text{I-}\to}{\mathrm{Id}(\mathrm{N}_1 \to \mathrm{N}_0, x, \lambda s^{\mathrm{N}_1}.\mathrm{Ap}(x,*))\ \mathit{type}\ [x \in \mathrm{N}_1 \to \mathrm{N}_0]}\ \text{F-Id} \quad \dfrac{[]\ \mathit{cont}}{x \in \mathrm{N}_1 \to \mathrm{N}_0,\ \mathit{cont}}\ ...}{\mathrm{Id}(\mathrm{N}_1 \to \mathrm{N}_0, x, \lambda s^{\mathrm{N}_1}.\mathrm{Ap}(x,*))\ \mathit{type}\ [x \in \mathrm{N}_1 \to \mathrm{N}_0,\ z \in \mathrm{N}_0]}\ \text{weak-ty}}{\mathrm{El}_{\mathrm{N}_0}(Ap(x,*)) \in \mathrm{Id}(\mathrm{N}_1 \to \mathrm{N}_0, x, \lambda s^{\mathrm{N}_1}.\mathrm{Ap}(x,*))\ [x \in \mathrm{N}_1 \to \mathrm{N}_0]}\ \text{E-N}_0$$

$\mathbf{pf_2} = \mathrm{El}_{\mathrm{N}_0}(y)$

$$\dfrac{\dfrac{[]\ \mathit{cont}}{y \in \mathrm{N}_0\ [y \in \mathrm{N}_0]}\ [...] \qquad \dfrac{\dfrac{\dfrac{[]\ \mathit{cont}}{\mathrm{N}_0\ \mathit{type}\ [y \in \mathrm{N}_0]}\ [...] \quad \dfrac{[]\ \mathit{cont}}{y \in \mathrm{N}_0\ [y \in \mathrm{N}_0]}\ [...] \quad \dfrac{\dfrac{[]\ \mathit{cont}}{* \in \mathrm{N}_1\ [y \in \mathrm{N}_0]}\ [...] \quad \dfrac{\dfrac{[]\ \mathit{cont}}{y \in \mathrm{N}_0\ [y \in \mathrm{N}_0]}\ [...]}{\lambda s^{\mathrm{N}_1}.y \in \mathrm{N}_1 \to \mathrm{N}_0\ [y \in \mathrm{N}_0]}\ \text{I-}\to}{\mathrm{Ap}(\lambda s^{\mathrm{N}_1}.y,*) \in \mathrm{N}_0\ [y \in \mathrm{N}_0]}\ \text{E-}\to}{\mathrm{Id}(\mathrm{N}_0, y, \mathrm{Ap}(\lambda s^{\mathrm{N}_1}.y,*))\ \mathit{type}\ [y \in \mathrm{N}_0]}\ \text{F-Id} \quad \dfrac{[]\ \mathit{cont}}{y \in \mathrm{N}_0,\ z \in \mathrm{N}_0\ \mathit{cont}}\ [...]}{\mathrm{Id}(\mathrm{N}_0, y, \mathrm{Ap}(\lambda s^{\mathrm{N}_1}.y,*))\ \mathit{type}\ [y \in \mathrm{N}_0,\ z \in \mathrm{N}_0]}\ \text{weak-ty}}{\mathrm{El}_{\mathrm{N}_0}(y) \in \mathrm{Id}(\mathrm{N}_0, y, \mathrm{Ap}(\lambda s^{\mathrm{N}_1}.y,*))\ [y \in \mathrm{N}_0]}\ \text{E-N}_0$$

2. Siccome il tipo $\mathrm{N}_1 \to \mathrm{N}_0$ corrisponde a $\Pi_{x \in \mathrm{N}_1} \mathrm{N}_0$, allora possiamo dare le seguenti traduzioni, dato il contesto Γ:

- $\mathrm{tt} \to \perp \in Form(\Gamma)$;
- $\neg \mathrm{tt} \in Form(\Gamma)$;
- $(\forall_{x \in \mathrm{N}_1} \perp) \in Form(\Gamma)$.

Notare che tutte queste formule equivalgono a $\perp \in Form(\Gamma)$

Esercizio 5 p. 59

Sia $Bool = \mathrm{N}_1 + \mathrm{N}_1$ e siano $a, b \in Un_0$. Codificare il tipo $T(a) \times T(b)$ utilizzando solamente Π, $Bool$ e l'universo Un_0.

Utilizziamo di seguito le regole di $Bool$ come presentate in precedenza, nello svolgimento dell'esercizio 1 di pagina 27.

Definiamo:

- il tipo $T(a) \times T(b) \stackrel{\text{def}}{=} \Pi_{q \in Bool} T(\mathrm{El}_{Bool}(q, a, b))$;
- il costruttore $\langle x, y \rangle \stackrel{\text{def}}{=} \lambda p.\mathrm{El}_{Bool}(p, x, y)$;
- l'eliminatore $\pi_1(d) \stackrel{\text{def}}{=} Ap(d, 0)$;
- l'eliminatore $\pi_2(d) \stackrel{\text{def}}{=} Ap(d, 1)$.

Viene mostrato di seguito come le regole coincidano con quelle di $T(a) \times T(b)$.

Formazione:

$$\dfrac{a \in Un_0\ [\Gamma] \qquad b \in Un_0\ [\Gamma]}{T(a) \times T(b)\ type\ [\Gamma]}\ \text{F-}\times$$

La regola F-$\times$ è derivabile assumendo $a \in Un_0\ [\Gamma]$, $b \in Un_0\ [\Gamma]$ e Γ *cont*:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma\ cont}{Un_0\ type\ [\Gamma,\ q \in Bool, z \in Bool]}\ [...] \qquad \dfrac{\Gamma\ cont}{q \in Bool\ [\Gamma,\ q \in Bool]}\ [...] \qquad a \in Un_0\ [\Gamma,\ q \in Bool] \qquad b \in Un_0\ [\Gamma,\ q \in Bool]}{\mathrm{El}_{Bool}(q, a, b) \in Un_0\ [\Gamma,\ q \in Bool]}\ \text{E-Bool}}{T(\mathrm{El}_{Bool}(q, a, b))\ type\ [\Gamma,\ q \in Bool]}\ \text{E-}Un_0}{\Pi_{q \in Bool} T(\mathrm{El}_{Bool}(q, a, b))\ type\ [\Gamma]}\ \text{F-}\Pi$$

Introduzione:

Definiamo $\langle x, y \rangle \stackrel{\text{def}}{=} \lambda p.\mathrm{El}_{Bool}(p, x, y)$.

$$\dfrac{x \in T(a)\ [\Gamma] \qquad y \in T(b)\ [\Gamma]}{\langle x, y \rangle \in T(a) \times T(b)\ [\Gamma]}\ \text{I-}\times$$

La regola I-$\times$ è derivabile assumendo $x \in T(a)\ [\Gamma]$, $y \in T(b)\ [\Gamma]$ e Γ *cont* (segue nella pagina successiva):

L'albero è spezzato in rami per facilitare la lettura:

$$\dfrac{\dfrac{\dfrac{\Gamma\ \mathit{cont}}{T(\mathrm{El}_{Bool}(p,a,b))\ \mathit{type}\ [\Gamma,\ p\in Bool]}\,[...] \qquad \dfrac{\Gamma\ \mathit{cont}}{p\in Bool\ [\Gamma,\ p\in Bool]}\,[...] \qquad x\in T(\mathrm{El}_{Bool}(0,a,b))\ [\Gamma,\ p\in Bool] \qquad y\in T(\mathrm{El}_{Bool}(1,a,b))\ [\Gamma,\ p\in Bool]}{\mathrm{El}_{Bool}(p,x,y)\in T(\mathrm{El}_{Bool}(p,a,b))\ [\Gamma,\ p\in Bool]}\ \text{E-Bool}}{\lambda p.\mathrm{El}_{Bool}(p,x,y)\in \Pi_{q\in Bool}T(\mathrm{El}_{Bool}(q,a,b))\ [\Gamma]}\ \text{I-}\Pi$$

Ramo $x\in T(\mathrm{El}_{Bool}(0,a,b))\ [\Gamma,\ p\in Bool]$:

$$\dfrac{x\in T(a)\ [\Gamma,\ p\in Bool] \qquad \dfrac{\dfrac{\dfrac{[]\ \mathit{cont}}{Un_0\ \mathit{type}\ [a\in Un_0,\ b\in Un_0,\ z\in Bool]}\,[...] \qquad \dfrac{[]\ \mathit{cont}}{a\in Un_0[a\in Un_0,\ b\in Un_0]}\,[...] \qquad \dfrac{[]\ \mathit{cont}}{b\in Un_0\ [a\in Un_0,\ b\in Un_0]}\,[...]}{a=\mathrm{El}_{Bool}(0,a,b)\in Un_0\ [a\in Un_0,\ b\in Un_0]}\ \text{C}_2\text{-Bool}}{T(a)=T(\mathrm{El}_{Bool}(0,a,b))\ \mathit{type}\ [a\in Un_0,\ b\in Un_0]}\ \text{eq-E-}Un_0}{x\in T(\mathrm{El}_{Bool}(0,a,b))\ [\Gamma,\ p\in Bool]}\ \text{conv}$$

Ramo $y\in T(\mathrm{El}_{Bool}(1,a,b))\ [\Gamma,\ p\in Bool]$:

$$\dfrac{y\in T(b)\ [\Gamma,\ p\in Bool] \qquad \dfrac{\dfrac{\dfrac{[]\ \mathit{cont}}{Un_0\ \mathit{type}\ [a\in Un_0,\ b\in Un_0,\ z\in Bool]}\,[...] \qquad \dfrac{[]\ \mathit{cont}}{a\in Un_0[a\in Un_0,\ b\in Un_0]}\,[...] \qquad \dfrac{[]\ \mathit{cont}}{b\in Un_0\ [a\in Un_0,\ b\in Un_0]}\,[...]}{b=\mathrm{El}_{Bool}(1,a,b)\in Un_0\ [a\in Un_0,\ b\in Un_0]}\ \text{C}_1\text{-Bool}}{T(b)=T(\mathrm{El}_{Bool}(1,a,b))\ \mathit{type}\ [a\in Un_0,\ b\in Un_0]}\ \text{eq-E-}Un_0}{y\in T(\mathrm{El}_{Bool}(1,a,b))\ [\Gamma,\ p\in Bool]}\ \text{conv}$$

Eliminazione:

Definiamo $\pi_1(d) = Ap(d,0)$ e $\pi_2(d) = Ap(d,1)$.

$$\frac{d \in T(a) \times T(b)}{\pi_1(d) \in T(a)}\ \mathrm{E}_1\text{-}\times \qquad\qquad \frac{d \in T(a) \times T(b)}{\pi_2(d) \in T(b)}\ \mathrm{E}_2\text{-}\times$$

Le regole sono derivabili assumendo $d \in \Pi_{q \in Bool} T(\mathrm{El}_{Bool}(q,a,b))\ [\Gamma]$ e Γ *cont* (le dimostrazioni sono nella pagina seguente):

$\pi_1(d) = Ap(d,0)$

$$
\dfrac{
\dfrac{\dfrac{\Gamma\ cont}{0 \in Bool}\ [...] \qquad d \in \Pi_{q \in Bool} T(\mathrm{El}_{Bool}(q,a,b))\ [\Gamma]}{Ap(d,0) \in T(\mathrm{El}_{Bool}(0,a,b))\ [\Gamma]}\ \text{E-}\Pi
\qquad
\dfrac{\dfrac{\dfrac{[]\ cont}{Un_0\ type\ [a \in Un_0,\ b \in Un_0,\ z \in Bool]}\ [...] \quad \dfrac{[]\ cont}{a \in Un_0 [a \in Un_0,\ b \in Un_0]}\ [...] \quad \dfrac{[]\ cont}{b \in Un_0\ [a \in Un_0,\ b \in Un_0]}\ [...]}{a = \mathrm{El}_{Bool}(0,a,b) \in Un_0\ [a \in Un_0,\ b \in Un_0]}\ \text{C}_2\text{-Bool}}{T(a) = T(\mathrm{El}_{Bool}(0,a,b))\ type\ [a \in Un_0,\ b \in Un_0]}\ \text{eq-E-}Un_0
}{Ap(d,0) \in T(a)\ [\Gamma]}\ \text{conv}
$$

$\pi_2(d) = Ap(d,1)$:

$$
\dfrac{
\dfrac{\dfrac{\Gamma\ cont}{1 \in Bool}\ [...] \qquad d \in \Pi_{q \in Bool} T(\mathrm{El}_{Bool}(q,a,b))\ [\Gamma]}{Ap(d,1) \in T(\mathrm{El}_{Bool}(1,a,b))\ [\Gamma]}\ \text{E-}\Pi
\qquad
\dfrac{\dfrac{\dfrac{[]\ cont}{Un_0\ type\ [a \in Un_0,\ b \in Un_0,\ z \in Bool]}\ [...] \quad \dfrac{[]\ cont}{a \in Un_0 [a \in Un_0,\ b \in Un_0]}\ [...] \quad \dfrac{[]\ cont}{b \in Un_0\ [a \in Un_0,\ b \in Un_0]}\ [...]}{b = \mathrm{El}_{Bool}(1,a,b) \in Un_0\ [a \in Un_0,\ b \in Un_0]}\ \text{C}_1\text{-Bool}}{T(b) = T(\mathrm{El}_{Bool}(1,a,b))\ type\ [a \in Un_0,\ b \in Un_0]}\ \text{eq-E-}Un_0
}{Ap(d,1) \in T(b)\ [\Gamma]}\ \text{conv}
$$

Conversione:

$$\frac{x \in T(a)\ [\Gamma] \qquad y \in T(b)\ [\Gamma]}{\pi_1(\langle x, y\rangle) = x \in T(a)\ [\Gamma]}\ \beta_1\text{-}\times \qquad\qquad \frac{x \in T(a)\ [\Gamma] \qquad y \in T(b)\ [\Gamma]}{\pi_2(\langle x, y\rangle) = y \in T(b)\ [\Gamma]}\ \beta_2\text{-}\times$$

Le regole sono derivabili assumendo:

- $x \in T(a)\ [\Gamma]$;
- $a \in Un_0\ [\Gamma]$;
- $y \in T(b)\ [\Gamma]$;
- $b \in Un_0\ [\Gamma]$;
- $\Gamma\ cont$.

$d \in \Pi_{q \in Bool} T(\mathrm{El}_{Bool}(q, a, b))\ [\Gamma]$ e $\Gamma\ cont$ (le dimostrazioni sono nella pagina seguente):

Per facilitare la lettura, un ramo in comune a entrambi gli alberi è derivato in coda:

$\pi_1(\langle x, y\rangle) = x \in T(a)\ [\Gamma]$:

$$\dfrac{\dfrac{\dfrac{\Gamma\ cont}{0 \in Bool}\,[...] \qquad \mathrm{El}_{Bool}(p,x,y) \in T(\mathrm{El}_{Bool}(p,a,b))\ [\Gamma,\ p \in Bool]}{Ap(\lambda p.\mathrm{El}_{Bool}(p,x,y),0) = \mathrm{El}_{Bool}(0,x,y) \in T(\mathrm{El}_{Bool}(0,a,b))\ [\Gamma]}\,\beta\text{-}\Pi \qquad \dfrac{\dfrac{\dfrac{[]\ cont}{Un_0\ type\ [a \in Un_0,\ b \in Un_0,\ z \in Bool]}\,[...] \quad \dfrac{[]\ cont}{a \in Un_0[a \in Un_0,\ b \in Un_0]}\,[...] \quad \dfrac{[]\ cont}{b \in Un_0\ [a \in Un_0,\ b \in Un_0]}\,[...]}{a = \mathrm{El}_{Bool}(0,a,b) \in Un_0\ [a \in Un_0,\ b \in Un_0]}\,\text{C}_2\text{-Bool}}{T(a) = T(\mathrm{El}_{Bool}(0,a,b))\ type\ [a \in Un_0,\ b \in Un_0]}\,\text{eq-E-}Un_0}{Ap(\lambda p.\mathrm{El}_{Bool}(p,x,y),0) = \mathrm{El}_{Bool}(0,x,y) \in T(a)\ [\Gamma]}\,\text{conv}$$

$\pi_2(\langle x, y\rangle) = y \in T(b)\ [\Gamma]$:

$$\dfrac{\dfrac{\dfrac{\Gamma\ cont}{1 \in Bool}\,[...] \qquad \mathrm{El}_{Bool}(p,x,y) \in T(\mathrm{El}_{Bool}(p,a,b))\ [\Gamma,\ p \in Bool]}{Ap(\lambda p.\mathrm{El}_{Bool}(p,x,y),1) = \mathrm{El}_{Bool}(1,x,y) \in T(\mathrm{El}_{Bool}(1,a,b))\ [\Gamma]}\,\beta\text{-}\Pi \qquad \dfrac{\dfrac{\dfrac{[]\ cont}{Un_0\ type\ [a \in Un_0,\ b \in Un_0,\ z \in Bool]}\,[...] \quad \dfrac{[]\ cont}{a \in Un_0[a \in Un_0,\ b \in Un_0]}\,[...] \quad \dfrac{[]\ cont}{b \in Un_0\ [a \in Un_0,\ b \in Un_0]}\,[...]}{b = \mathrm{El}_{Bool}(1,a,b) \in Un_0\ [a \in Un_0,\ b \in Un_0]}\,\text{C}_1\text{-Bool}}{T(b) = T(\mathrm{El}_{Bool}(1,a,b))\ type\ [a \in Un_0,\ b \in Un_0]}\,\text{eq-E-}Un_0}{Ap(\lambda p.\mathrm{El}_{Bool}(p,x,y),1) = \mathrm{El}_{Bool}(1,x,y) \in T(b)\ [\Gamma]}\,\text{conv}$$

Entrambi gli alberi presentano il seguente ramo:

$$\dfrac{\dfrac{a \in Un_0\ [\Gamma,\ p \in Bool,\ q \in Bool] \quad b \in Un_0\ [\Gamma,\ p \in Bool,\ q \in Bool]}{T(\mathrm{El}_{Bool}(q,a,b))\ type\ [\Gamma,\ p \in Bool,\ q \in Bool]}\,\text{[dimostrato in precedenza]} \quad \dfrac{\Gamma\ cont}{p \in Bool\ [\Gamma,\ p \in Bool]}\,[...] \quad \dfrac{x \in T(a)\ [\Gamma,\ p \in Bool] \quad \dfrac{\Gamma\ cont}{T(a) = T(\mathrm{El}_{Bool}(0,a,b))\ [\Gamma,\ p \in Bool]}\,[...]}{x \in T(\mathrm{El}_{Bool}(0,a,b))\ [\Gamma,\ p \in Bool]}\,\text{conv} \quad \dfrac{y \in T(b)\ [\Gamma,\ p \in Bool] \quad \dfrac{\Gamma\ cont}{T(b) = T(\mathrm{El}_{Bool}(1,a,b))\ [\Gamma,\ p \in Bool]}\,[...]}{y \in T(\mathrm{El}_{Bool}(1,a,b))\ [\Gamma,\ p \in Bool]}\,\text{conv}}{\mathrm{El}_{Bool}(p,x,y) \in T(\mathrm{El}_{Bool}(p,a,b))\ [\Gamma,\ p \in Bool]}$$