Exercises of Dimensional Analysis

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1 Exercise 1

1.1 Question 1

It is told that bigwhigs drive faster car than other people (maybe nowadays the size of shoes is more relevant than the size of whig). Try to prove using velocity v of the car (m/s) depends only on its fuel consumption $\dot{V}$ (m^3/day) and the length l of the driver's whig(m). What is your conclusion?

1.2 Solution 1

The basic quantities are L and T. The relevant quantities are v,V and l.The velocity v,$[v] = L^1T^{-1}$, consumption V, $[V] = L^3T^{-1}$ and length l, $[l] = L^1T^0$.The dimension matrix A is

A	L	T
v	1	-1
V	3	-1
l	1	0

or $A = \begin{pmatrix} 1 & -1 \\ 3 & -1 \\ 1 & 0 \end{pmatrix}$

The rank of A r(A) is 2. So there are $3 - 2 = 1$ π number π_1. Moreover, F$(\pi_1) = 0$ for some injective F : $\mathbb{R}_+ \to \mathbb{R}_+$. Hence $\pi_1 = C^2$ with some positive constant C. The exponents k_1, k_2, k_3 in

$$\pi_1 = \prod_{i=1}^{n} X_i^{k_i} = v^{k_1}V^{k_2}l^{k_3}, X_1 = v, X_2 = V, X_3 = l \tag{1}$$

can be solved from $kA = 0$ or $(kA)^t = 0$. Hence

$$\begin{cases} k_1 & +3k_2 & +k_3 & =0 \\ -k_1 & -k_2 & & =0 \end{cases} \tag{2}$$

whose all solutions are $k = (\lambda, -\lambda, 2\lambda)$, $\forall\lambda \in \mathbb{R}$. We choose $\lambda = 1$,$k = (1, -1, 2)$. Thus the velocity v of the car is

$$v = \frac{V}{l^2}C \tag{3}$$

2 Exercise 2

2.1 Question 2

Form the dimensional matrix of the location x of a body, its velocity v, its acceleration a, its mass m, the force $F = ma$ acting on it, and of its kinetic energy
(a)in the system $\{M,L,T\}$ of basic quantities,
(b)in the system $\{F, L, T\}$ of basic quantities.

2.2 Solution 2

2.2.1 Solution of (a)

The basic quantities are M, L, and T. The relevant quantities are x, v, a, m, F,E_k.
The location x, $[x]$=$M^0L^1T^0$, the velocity v,$[v]$=$M^0L^1T^{-1}$,
the acceleration a, $[a]$=$M^0L^1T^{-2}$, the mass m,$[m]$=$M^1L^0T^0$,
the force F, $[F]$=$M^1L^1T^{-2}$, the kinetic energy E_k=$[E_k]$=$M^1L^2T^{-2}$.
The dimension matrix A is

A	M	L	T
x	0	1	0
v	0	1	-1
a	0	1	-2
m	1	0	0
F	1	1	-2
E_k	1	2	-2

or $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & -2 \\ 1 & 0 & 0 \\ 1 & 1 & -2 \\ 1 & 2 & -2 \end{pmatrix}$

2.2.2 Solution of (b)

The basic quantities are F, L, and T. Then the location x,$[x]$=$F^0L^1T^0$, the velocity v=$F^0L^1T^{-1}$,
the acceleration a,$[a]$=$F^0L^1T^{-2}$, the mass m, $[m]$=$F^1L^{-1}T^2$,
the force F, $[F]$=$F^1L^0T^0$, the kinetic energy E_k, $[E_k]$=$F^1L^1T^0$.
The dimension matrix A is

A	F	L	T
x	0	1	0
v	0	1	-1
a	0	1	-2
m	1	-1	2
F	1	0	0
E_k	1	1	0

or $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & -2 \\ 1 & -1 & 2 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$

3 Exercise 3

3.1 Question 3

Apply Buckingham's π theorem and derive carefully the formula

$$h = r\phi(\frac{r^2\rho g}{\gamma}) \tag{4}$$

of the lectures for the capillarity phenomenon.

3.2 Solution 3

The dimension matrix in the system of basic quantities $\{M, L, T\}$. The dimension matrix A is

A	M	L	T
h	0	1	0
r	0	1	0
ρ	1	-3	0
γ	1	0	-2
g	0	1	-2

or $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & -3 & 0 \\ 1 & 0 & -2 \\ 0 & 1 & -2 \end{pmatrix}$

The rank of A $r(A)$ is 3, and we get $5 - 3 = 2$ π numbers. To solve $(kA)^t = 0$. Hence

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & -3 & 0 & 1 \\ 0 & 0 & 0 & -2 & -2 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_5 \end{pmatrix} = 0 \tag{5}$$

whose results are

$$\begin{cases} k_1 = (1, -1, 0, 0, 0) \\ k_2 = (0, 2, 1, -1, -1) \end{cases} \tag{6}$$

We succeed:

$$\pi_1 = \frac{h}{r}, \pi_2 = \frac{r^2 \rho g}{\gamma} \tag{7}$$

According to Buckingham's π theorem,

$$h = r\psi(\frac{r^2 \rho g}{\gamma}) \tag{8}$$

where ψ: $\mathbb{R}_+ \to \mathbb{R}_+$ is an unknown function.

4 Exercise 4

4.1 Question 4

We continue with the capillarity. Divide the dimension of length to two components R and Z. Derive carefully by Buckingham's π number theorem the formula

$$h = C\frac{\gamma}{g\rho r} \tag{9}$$

where $C > 0$ is constant. Is this formula credible? What was neglected (look the surface of water in glass near its edge)? Is this result still credible or useful? Is this result in contradiction with the result of the previous exercise?

4.2 Solution 4

The dimension matrix in the system of basic quantities $\{M, R, Z, T\}$ is matrix A,

$$\begin{array}{c|cccc} A & M & R & Z & T \\ \hline h & 0 & 0 & 1 & 0 \\ r & 0 & 1 & 0 & 0 \\ \rho & 1 & -2 & -1 & 0 \\ \gamma & 1 & -1 & 1 & -2 \\ g & 0 & 0 & 1 & -2 \end{array} \quad \text{or } A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -2 & -1 & 0 \\ 1 & -1 & 1 & -2 \\ 0 & 0 & 1 & -2 \end{pmatrix}$$

The rank of A $r(A)$ is 4, so there is only one π number. To solve $(kA)^t = 0$. Hence

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & -2 & -1 & 0 \\ 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & -2 & -2 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_5 \end{pmatrix} = 0 \tag{10}$$

whose all solutions are $k = (\lambda, \lambda, \lambda, -\lambda, \lambda)$, $\lambda \in \mathbb{R}$. We choose $\lambda = 1$, which means $k = (1, 1, 1, -1, 1)$.
By Buckingham's π theorem,

$$h = C \frac{\gamma}{g\rho r} \tag{11}$$

where $C > 0$ is some constant.

The formula is credible, because we deduce it by dimensional analysis. However, when looking the surface of water in glass near its edge, we notice that the liquid on the glass rising along the wall making it a concave type.

Thus the result is not useful in precise measure. But when it comes to the actual measurement, we can reduce the error as much as possible through error analysis. It is not in contradiction with the previous.

5 Exercise 5

5.1 Question 5

A round metal plate of a diametre d is fixed perpendicularly to an axis which rotates by the rotational velocity v. The plate is in oil whose density is ρ and coefficient of dynamic viscosity is μ. The torque τ needed to rotate the axis depends only on these quantities. Find by dimensional analysis a formula for τ, that is:
(a)Determine the dimension matrix A.
(b)Solve the equation $kA = 0$ by the Guass-Jordan method and write the π numbers.
(c)Apply Buckingham's π theorem.

5.2 Solution 5

5.2.1 Solution of (a)

The basis quantities are L, M, T. The relevant quantities are d, v, ρ, μ, τ. The dimension matrix A is

A	L	M	T
d	1	0	0
v	0	0	-1
ρ	-3	1	0
μ	-1	1	-1
τ	2	1	-2

$$\text{or } A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ -3 & 1 & 0 \\ -1 & 1 & -1 \\ 2 & 1 & -2 \end{pmatrix}$$

5.2.2 Solution of (b)

The equation $kA = 0$ and $(kA)^t = 0$ is equivalent. Hence

$$\begin{cases} \quad -k_2 \qquad -k_4 \quad -2k_5 \quad = 0 \\ \qquad +k_3 \quad +k_4 \quad +k_5 \quad = 0 \\ k_1 \quad -3k_3 \quad -k_4 \quad +2k_5 \quad = 0 \end{cases} \tag{12}$$

The rank of A is $r(A) = 3$. So there are $5 - 3 = 2$ π numbers π_1, π_2. And for each $\pi_i \in \{\pi_1, \pi_2\}$. There are equation

$$\pi = \prod_{j=1}^{5} X_i^{k_i} = d^{k_1} v^{k_2} \rho^{k_3} \mu^{k_4} \tau^{k_5} \tag{13}$$

X_i means relevant quantity which $i \in \mathbb{N}$ from 1 to 5. Using the Guass-Jordan method, equations (12) have to linearly independent vectors solutions.

$$\begin{cases} k_1 = (-1, 0, 1, -2, 1) \\ k_2 = (2, 1, 1, -1, 0) \end{cases} \tag{14}$$

According to equation (13), π numbers are

$$\begin{cases} \pi_1 = \dfrac{\tau\rho}{d\mu^2} \\ \pi_2 = \dfrac{d^2 v\rho}{\mu} \end{cases} \tag{15}$$

5.2.3 Solution of (c)

By Buckingham's π theorem,

$$\tau = \frac{d\mu^2}{\rho}\phi(d^2 v\rho\mu) \tag{16}$$

where ϕ: $\mathbb{R}_+ \to \mathbb{R}_+$ is an unknown function.

6 Exercise 6

6.1 Question 6

We continue with the previous exercise. For the full-sized device, $d_p = 0.5\ m$, $v_p = 200\ rpm$, $\rho_p = 750\ kg/m^3$, and $\mu_p = 0.20\ Ns/m^2$. The scaled model is made geometrically similarly in the scale $1:5$ (all lengths are on fifth part of the prototype). The miniature is filled by water, whose $\rho_m = 1000\ kg/m^3$ and $\mu_m = 0.0010\ Ns/m^2$. What its rotational velocity v_m should be? If the torque $\tau_m = 0.020\ Nm$ suffices for it, what is the torque τ_m need to run the full-sized device?

6.2 Solution 6

According to results form Exercise 5, we got he π numbers showing in equation (15).

$$(\frac{\tau\rho}{d\mu^2})_p = (\frac{\tau\rho}{d\mu^2})_m \tag{17}$$

and

$$(\frac{d^2v\rho}{\mu})_p = (\frac{d^2v\rho}{\mu})_m \tag{18}$$

where subsripts m and p refer to the model and the prototype, respectively. And we get v_m

$$v_m = \frac{d_p^2 v_p \rho_p \mu_m}{d_m^2 \rho_m \mu_p} = \frac{(0.5)^2 \cdot 200 \cdot 750 \cdot 0.0010}{(0.1)^2 \cdot 1000 \cdot 0.20} = 18.75\ rpm \tag{19}$$

Then the v_m of scaled model is $18.75\ rpm$. If the $\tau_m = 0.020\ Nm$, and we get torque τ_p

$$\tau_p = (\frac{\tau_m \rho_m}{d_m \mu_m^2}) \cdot (\frac{d_p \mu_p^2}{\rho_p}) = [\frac{0.020 \cdot 1000}{0.1 \cdot (0.0010)^2}] \cdot [\frac{0.5 \cdot (0.20)^2}{750}] \doteq 5333.33\ Nm \tag{20}$$

Then the τ_p of full-sized device need to run in $5333.33\ Nm$.

7 Exercise 7

7.1 Result from Dimensioanalyysikone

Use software Dimensioanalyysikone in Exercise $1-5$, we got results for each exercise showing in figure. As the result we have calculated k by Gauss-Jordan method, And $k_1 = (1,-1,2)$ is equal to $k_1^{'} = (1,-1,2)$, and $k_{1_{2a}}^{'} = (1,0,0,0,1,-1)$, $k_{2_{2a}}^{'} = (0,1,0,0.5,0,0.5)$, $k_{3_{2a}}^{'} = (0,0,1,1,-1,0)$. $k^{'}$ mean k is calculated by software.

$$\begin{cases} k_{1_3}^{'} = (1,0,0.5,-0.5,0.5) \\ k_{2_3}^{'} = (0,1,0.5,-0.5,0.5) \end{cases} \tag{21}$$

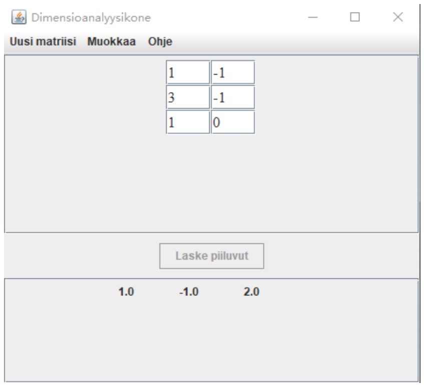

Figure 1: Result Exercise 1

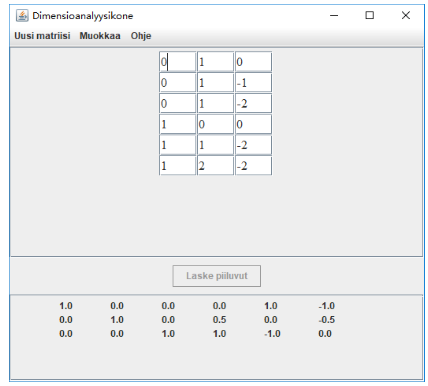

Figure 2: Result Exercise $2(a)$

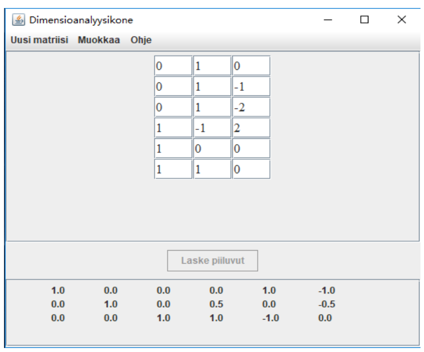

Figure 3: Result Exercise $2(b)$

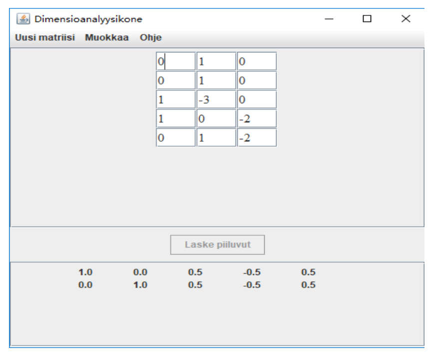

Figure 4: Result Exercise 3

But according to equation (6) we got in 3.2 Solution 3, Figure show the other results of equations (5).Which $k'_{1_3} \neq k_{1_3}$, $k'_{2_3} \neq k_{2_3}$. and also figure 6 show different looking results.

7.2 Explanation

According to the Advanced Algebra, in the $n-$dimensional linear space on the number field P, $\mathbb{P}^n$, all the solution vectors η of the homogeneous linear equations $(kA)^t = 0$, from a subspace of $\mathbb{P}^n$.

$$\begin{cases} a_{11}k_1 & +a_{12}k_2 & +\cdots & +a_{1n}k_n & =0 \\ a_{21}k_1 & +a_{22}k_2 & +\cdots & +a_{2n}k_n & =0 \\ & & \cdots & & \\ a_{n1}k_1 & +a_{n2}k_2 & +\cdots & +a_{nn}k_n & =0 \end{cases} \tag{22}$$

This subspace is called the solution space of homogeneous linear equations. The basis solution space is the basic solution system of the equations, dimension of

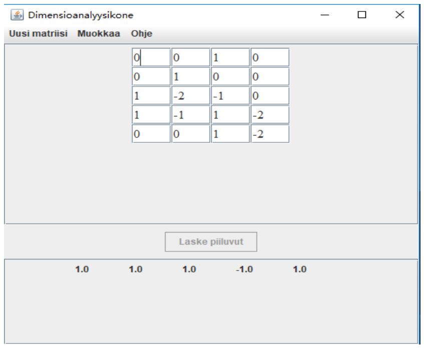

Figure 5: Result Exercise 4

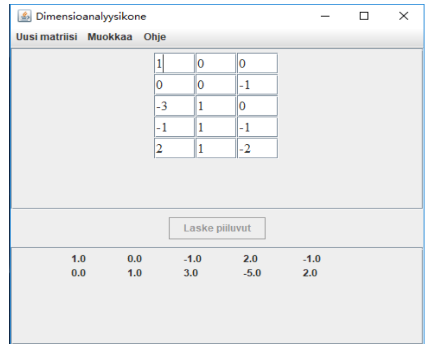

Figure 6: Result Exercise 5

basis solution space is $p = n - rank(A)$. Because there are infinite basis vectors for the basis in linear space, there are infinite basis vectors

$$(k_1, k_2, \cdots, k_p),\, i = 1, 2, \cdots n, \cdots$$

for the basic solution of the solution space for homogeneous linear systems. So, when you solve the same homogeneous linear system, you get different sets of vectors, and also the relation between $(k_1, k_2, \cdots, k_p)$ and $(k_1^{'}, k_2^{'}, \cdots, k_p^{'})$ $i = 1, 2, \cdots n, \cdots$ is for any η in solution space, equation

$$\eta = x_1 k_1 + k_2 + \cdots + x_p k_p = x_1^{'} k_1^{'} + x_2^{'} k_2^{'} + \cdots + x_p^{'} k_p^{'} \tag{23}$$

always established. Which $(x_1, x_2, \cdots, x_p)$ is coordinate under basic vectors array $(k_1, k_2, \cdots, k_p)$.