Matrices & Determinants
Game Theory
Implicit Differentiation & Integral Calculus

WAEC General Mathematics Comprehensive Lecture Notes

Academic Lecture Series

1 Matrices and Determinants

1.1 Introduction to Matrices

Definition 1.1. A **matrix** is a rectangular array of numbers, symbols, or expressions arranged in rows and columns.

A matrix with m rows and n columns is called an $m \times n$ matrix (read "m by n").

General form:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

where a_{ij} is the element in row i, column j.

Definition 1.2. Types of Matrices:

- **Row Matrix:** $1 \times n$ matrix (single row)
- **Column Matrix:** $m \times 1$ matrix (single column)
- **Square Matrix:** $n \times n$ matrix (equal rows and columns)
- **Zero/Null Matrix:** All elements are zero
- **Identity Matrix** I_n: Square matrix with 1's on main diagonal, 0's elsewhere

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

1.2 Matrix Operations

Definition 1.3. Matrix Addition: Two matrices can be added if they have the same dimensions. Add corresponding elements.

If $A = (a_{ij})$ and $B = (b_{ij})$, then $A + B = (a_{ij} + b_{ij})$

Worked Example

Example 1.1: Find $A + B$ where:

$$A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 5 & 1 \\ 2 & 3 \end{pmatrix}$$

Solution:

$$A + B = \begin{pmatrix} 2+5 & 3+1 \\ 1+2 & 4+3 \end{pmatrix} = \begin{pmatrix} 7 & 4 \\ 3 & 7 \end{pmatrix}$$

Definition 1.4. Scalar Multiplication: Multiply each element of a matrix by a scalar (number).

If k is a scalar and $A = (a_{ij})$, then $kA = (ka_{ij})$

Worked Example

Example 1.2: Find $3A$ where $A = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$

Solution:

$$3A = \begin{pmatrix} 3(2) & 3(-1) \\ 3(3) & 3(4) \end{pmatrix} = \begin{pmatrix} 6 & -3 \\ 9 & 12 \end{pmatrix}$$

Definition 1.5. Matrix Multiplication: Matrix A (size $m \times n$) can be multiplied by matrix B (size $n \times p$) only if the number of columns in A equals the number of rows in B. The result is an $m \times p$ matrix.

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

(Multiply row i of A by column j of B)

Worked Example

Example 1.3: Find AB where:

$$A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 5 & 1 \\ 2 & 3 \end{pmatrix}$$

Solution:

$$\begin{aligned} AB &= \begin{pmatrix} 2(5)+3(2) & 2(1)+3(3) \\ 1(5)+4(2) & 1(1)+4(3) \end{pmatrix} \\ &= \begin{pmatrix} 10+6 & 2+9 \\ 5+8 & 1+12 \end{pmatrix} \\ &= \begin{pmatrix} 16 & 11 \\ 13 & 13 \end{pmatrix} \end{aligned}$$

Common Mistake

Important: Matrix multiplication is NOT commutative!
In general, $AB \neq BA$

1.3 Determinants

Definition 1.6. The **determinant** is a scalar value that can be computed from a square matrix. It has important properties related to matrix inverses and solving systems of equations.

For 2×2 matrix:

$$\det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

For 3×3 matrix:

$$\begin{aligned} \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} &= a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ &= a(ei - fh) - b(di - fg) + c(dh - eg) \end{aligned}$$

Worked Example

Example 1.4: Find the determinant of $A = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$

Solution:

$$\begin{aligned} |A| &= \begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix} \\ &= 3(4) - 2(1) \\ &= 12 - 2 \\ &= 10 \end{aligned}$$

Worked Example

Example 1.5: Find the determinant of:

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 0 & 4 & 1 \\ 1 & 2 & 0 \end{pmatrix}$$

Solution: Expanding along first row:

$$\begin{aligned} |A| &= 2\begin{vmatrix} 4 & 1 \\ 2 & 0 \end{vmatrix} - 1\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + 3\begin{vmatrix} 0 & 4 \\ 1 & 2 \end{vmatrix} \\ &= 2(4 \cdot 0 - 1 \cdot 2) - 1(0 \cdot 0 - 1 \cdot 1) + 3(0 \cdot 2 - 4 \cdot 1) \\ &= 2(-2) - 1(-1) + 3(-4) \\ &= -4 + 1 - 12 \\ &= -15 \end{aligned}$$

1.4 Inverse of a Matrix

Definition 1.7. For a square matrix A, the **inverse matrix** A^{-1} satisfies:

$$AA^{-1} = A^{-1}A = I$$

A matrix is **invertible** (or non-singular) if $|A| \neq 0$.

For 2×2 matrix: If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then:

$$A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Worked Example

Example 1.6: Find the inverse of $A = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$
Solution: First, find determinant:

$$|A| = 3(4) - 2(1) = 10$$

Since $|A| \neq 0$, the inverse exists:

$$A^{-1} = \frac{1}{10}\begin{pmatrix} 4 & -2 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 0.4 & -0.2 \\ -0.1 & 0.3 \end{pmatrix}$$

Verify: $AA^{-1} = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 0.4 & -0.2 \\ -0.1 & 0.3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (correct)

1.5 Solving Systems of Equations

Key Point

A system of linear equations can be written in matrix form:

$$AX = B$$

If A is invertible, the solution is:

$$X = A^{-1}B$$

Worked Example

Example 1.7: Solve the system using matrices:

$$3x + 2y = 7$$
$$x + 4y = 11$$

Solution: Write in matrix form:

$$\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 11 \end{pmatrix}$$

From Example 1.6, we know:

$$A^{-1} = \begin{pmatrix} 0.4 & -0.2 \\ -0.1 & 0.3 \end{pmatrix}$$

Therefore:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0.4 & -0.2 \\ -0.1 & 0.3 \end{pmatrix}\begin{pmatrix} 7 \\ 11 \end{pmatrix} = \begin{pmatrix} 0.4(7) - 0.2(11) \\ -0.1(7) + 0.3(11) \end{pmatrix} = \begin{pmatrix} 2.8 - 2.2 \\ -0.7 + 3.3 \end{pmatrix} = \begin{pmatrix} 0.6 \\ 2.6 \end{pmatrix}$$

Wait, let me recalculate properly:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0.4 & -0.2 \\ -0.1 & 0.3 \end{pmatrix}\begin{pmatrix} 7 \\ 11 \end{pmatrix} = \begin{pmatrix} 2.8 - 2.2 \\ -0.7 + 3.3 \end{pmatrix} = \begin{pmatrix} 0.6 \\ 2.6 \end{pmatrix}$$

Actually, this doesn't give integer values. Let me verify the inverse calculation again. From $A = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$:
$|A| = 12 - 2 = 10$
$A^{-1} = \frac{1}{10}\begin{pmatrix} 4 & -2 \\ -1 & 3 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 4 & -2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 7 \\ 11 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 28 - 22 \\ -7 + 33 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 6 \\ 26 \end{pmatrix} = \begin{pmatrix} 0.6 \\ 2.6 \end{pmatrix}$$

Hmm, non-integer. Let me verify: $3(0.6)+2(2.6) = 1.8+5.2 = 7$ (correct)
$0.6 + 4(2.6) = 0.6 + 10.4 = 11$ (correct)
So $x = 0.6, y = 2.6$ or in fractions: $x = \frac{3}{5}, y = \frac{13}{5}$

2 Game Theory

2.1 Introduction to Game Theory

Definition 2.1. Game Theory is the study of mathematical models of strategic interaction among rational decision-makers.

A **two-player zero-sum game** is a game where:

- Two players make decisions
- One player's gain is the other's loss
- Total payoff is zero

2.2 Payoff Matrix

Definition 2.2. A **payoff matrix** (or game matrix) shows the outcomes for each combination of strategies.

Notation:

- Rows represent Player A's strategies
- Columns represent Player B's strategies
- Entries show payoff to Player A (positive = gain, negative = loss)
- Player B's payoff is opposite of Player A's

Worked Example

Example 2.1: Consider the game with payoff matrix:

	B1	B2
A1	3	-2
A2	-1	4

Interpretation:

- If A plays A1 and B plays B1: A gains 3, B loses 3
- If A plays A1 and B plays B2: A loses 2, B gains 2
- If A plays A2 and B plays B1: A loses 1, B gains 1
- If A plays A2 and B plays B2: A gains 4, B loses 4

2.3 Pure Strategies and Saddle Points

Definition 2.3. A **saddle point** is an entry in the payoff matrix that is:

- The minimum in its row, AND
- The maximum in its column

If a saddle point exists, it represents the optimal pure strategy for both players, and its value is the **value of the game**.

Worked Example

Example 2.2: Find the saddle point, if any:

	B1	B2	B3
A1	2	4	3
A2	5	3	6
A3	4	2	5

Solution:
Find row minimums (worst for A in each row):

- Row A1: min = 2
- Row A2: min = 3
- Row A3: min = 2

Find column maximums (worst for B in each column):

- Column B1: max = 5
- Column B2: max = 4
- Column B3: max = 6

Check if any entry is both row min and column max:

- Entry (A2, B2) = 3 is row min for A2
- Entry (A2, B2) = 3 is NOT column max for B2 (max is 4)

Check all systematically:

	B1	B2	B3	Row Min
A1	2	4	3	2
A2	5	3	6	3
A3	4	2	5	2
Col Max	5	4	6	

No entry is both row min and column max, so there is NO saddle point.

Worked Example

Example 2.3: Find the saddle point:

	B1	B2	B3
A1	1	3	2
A2	4	2	5
A3	3	2	4

Solution:

	B1	B2	B3	Row Min
A1	1	3	2	1
A2	4	2	5	2
A3	3	2	4	2
Col Max	4	3	5	

Entry (A2, B2) = 2 is row minimum for A2.
Entry (A2, B2) = 2 is NOT column maximum for B2 (max is 3).
Entry (A3, B2) = 2 is row minimum for A3.
Entry (A3, B2) = 2 is NOT column maximum for B2 (max is 3).
No saddle point exists.

2.4 Dominance

Definition 2.4. Dominance: A strategy dominates another if it is at least as good in all situations and strictly better in at least one.

For Player A (row player):

- Strategy A_i dominates A_j if every entry in row i is $\geq$ corresponding entry in row j
- Remove dominated rows

For Player B (column player):

- Strategy B_i dominates B_j if every entry in column i is $\leq$ corresponding entry in column j
- Remove dominated columns

Worked Example

Example 2.4: Use dominance to simplify:

	B1	B2	B3
A1	2	1	4
A2	3	2	5
A3	1	0	3

Solution:
For Player A: Compare rows

- A2 vs A1: (3,2,5) vs (2,1,4) - A2 dominates A1 (all entries larger)
- A2 vs A3: (3,2,5) vs (1,0,3) - A2 dominates A3 (all entries larger)

Remove A1 and A3:

	B1	B2	B3
A2	3	2	5

For Player B: Compare columns (remember B wants to minimize A's payoff)

- B2 vs B1: (2) vs (3) - B2 dominates B1 (smaller for A)
- B2 vs B3: (2) vs (5) - B2 dominates B3 (smaller for A)

Remove B1 and B3:

	B2
A2	2

Optimal strategy: A plays A2, B plays B2, value of game = 2

3 Implicit Differentiation

3.1 Introduction

Definition 3.1. Implicit differentiation is used when y is not explicitly expressed as a function of x, but rather x and y are related by an equation like:

$$F(x, y) = 0$$

Examples: $x^2 + y^2 = 25$, $x^3 + y^3 = 6xy$

3.2 Method of Implicit Differentiation

Important Formula

Steps:

1. Differentiate both sides with respect to x
2. Treat y as a function of x (use chain rule)
3. When differentiating y, multiply by $\frac{dy}{dx}$
4. Collect all terms with $\frac{dy}{dx}$ on one side
5. Solve for $\frac{dy}{dx}$

Key Rules:

$$\frac{d}{dx}(y) = \frac{dy}{dx}$$
$$\frac{d}{dx}(y^n) = ny^{n-1}\frac{dy}{dx}$$
$$\frac{d}{dx}(xy) = x\frac{dy}{dx} + y$$

Worked Example

Example 3.1: Find $\frac{dy}{dx}$ if $x^2 + y^2 = 25$.
Solution: Differentiate both sides:

$$\begin{aligned} \frac{d}{dx}(x^2 + y^2) &= \frac{d}{dx}(25) \\ 2x + 2y\frac{dy}{dx} &= 0 \\ 2y\frac{dy}{dx} &= -2x \\ \frac{dy}{dx} &= -\frac{x}{y} \end{aligned}$$

Worked Example

Example 3.2: Find $\frac{dy}{dx}$ if $x^3 + y^3 = 6xy$.
Solution: Differentiate both sides:

$$\begin{aligned} \frac{d}{dx}(x^3 + y^3) &= \frac{d}{dx}(6xy) \\ 3x^2 + 3y^2\frac{dy}{dx} &= 6\left(x\frac{dy}{dx} + y\right) \\ 3x^2 + 3y^2\frac{dy}{dx} &= 6x\frac{dy}{dx} + 6y \\ 3y^2\frac{dy}{dx} - 6x\frac{dy}{dx} &= 6y - 3x^2 \\ \frac{dy}{dx}(3y^2 - 6x) &= 6y - 3x^2 \\ \frac{dy}{dx} &= \frac{6y - 3x^2}{3y^2 - 6x} \\ &= \frac{3(2y - x^2)}{3(y^2 - 2x)} \\ &= \frac{2y - x^2}{y^2 - 2x} \end{aligned}$$

Worked Example

Example 3.3: Find $\frac{dy}{dx}$ at the point $(1, 2)$ if $x^2y + xy^2 = 6$.
Solution: First, verify the point is on the curve:

$$1^2(2) + 1(2)^2 = 2 + 4 = 6$$

(correct)
Differentiate:

$$\begin{aligned}\frac{d}{dx}(x^2y + xy^2) &= \frac{d}{dx}(6)\\ x^2\frac{dy}{dx} + 2xy + x(2y\frac{dy}{dx}) + y^2 &= 0\\ x^2\frac{dy}{dx} + 2xy + 2xy\frac{dy}{dx} + y^2 &= 0\\ \frac{dy}{dx}(x^2 + 2xy) &= -2xy - y^2\\ \frac{dy}{dx} &= \frac{-2xy - y^2}{x^2 + 2xy}\end{aligned}$$

At $(1, 2)$:

$$\begin{aligned}\frac{dy}{dx} &= \frac{-2(1)(2) - 2^2}{1^2 + 2(1)(2)}\\ &= \frac{-4 - 4}{1 + 4}\\ &= \frac{-8}{5}\end{aligned}$$

4 Integral Calculus

4.1 Introduction to Integration

Definition 4.1. Integration is the reverse process of differentiation (anti-differentiation).

If $\frac{d}{dx}[F(x)] = f(x)$, then $\int f(x)\,dx = F(x) + C$

where C is the constant of integration.

4.2 Basic Integration Rules

Important Formula

Power Rule:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$

Basic Integrals:

$$\int k\,dx = kx + C$$

$$\int x^{-1}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

Sum Rule:

$$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

Constant Multiple:

$$\int kf(x)\,dx = k\int f(x)\,dx$$

Worked Example

Example 4.1: Find:

(a) $\int 5x^4\,dx$

(b) $\int (3x^2 - 2x + 5)\,dx$

(c) $\int \frac{1}{x^3}\,dx$

Solution:
(a) $\int 5x^4\,dx = 5 \cdot \frac{x^5}{5} + C = x^5 + C$
(b)

$$\begin{aligned}\int (3x^2 - 2x + 5)\,dx &= \int 3x^2\,dx - \int 2x\,dx + \int 5\,dx\\ &= 3 \cdot \frac{x^3}{3} - 2 \cdot \frac{x^2}{2} + 5x + C\\ &= x^3 - x^2 + 5x + C\end{aligned}$$

(c)

$$\begin{aligned}\int \frac{1}{x^3}\,dx &= \int x^{-3}\,dx\\ &= \frac{x^{-2}}{-2} + C\\ &= -\frac{1}{2x^2} + C\end{aligned}$$

4.3 Definite Integration

Important Formula

Definite Integral:

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$$

where $F(x)$ is an antiderivative of $f(x)$.
Properties:

$$\begin{aligned}\int_a^b f(x)\,dx &= -\int_b^a f(x)\,dx\\ \int_a^a f(x)\,dx &= 0\\ \int_a^b [f(x) + g(x)]\,dx &= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx\end{aligned}$$

Worked Example

Example 4.2: Evaluate $\int_1^3 (2x + 1)\,dx$
Solution:

$$\begin{aligned}\int_1^3 (2x + 1)\,dx &= [x^2 + x]_1^3\\ &= (3^2 + 3) - (1^2 + 1)\\ &= (9 + 3) - (1 + 1)\\ &= 12 - 2\\ &= 10\end{aligned}$$

Worked Example

Example 4.3: Evaluate $\int_0^2 (x^2 - 3x + 2)\,dx$
Solution:

$$\begin{aligned}\int_0^2 (x^2 - 3x + 2)\,dx &= \left[\frac{x^3}{3} - \frac{3x^2}{2} + 2x\right]_0^2\\ &= \left(\frac{8}{3} - \frac{12}{2} + 4\right) - (0)\\ &= \frac{8}{3} - 6 + 4\\ &= \frac{8}{3} - 2\\ &= \frac{8 - 6}{3}\\ &= \frac{2}{3}\end{aligned}$$

4.4 Area Under a Curve

Key Point

The definite integral $\int_a^b f(x)\,dx$ represents the area between the curve $y = f(x)$ and the x-axis from $x = a$ to $x = b$ (assuming $f(x) \geq 0$).

Worked Example

Example 4.4: Find the area bounded by $y = x^2$, the x-axis, and the lines $x = 1$ and $x = 3$.
Solution:

$$\begin{aligned}\text{Area} &= \int_1^3 x^2\,dx \\ &= \left[\frac{x^3}{3}\right]_1^3 \\ &= \frac{27}{3} - \frac{1}{3} \\ &= \frac{26}{3} \\ &= 8\frac{2}{3} \text{ square units}\end{aligned}$$

5 Practice Exercises

5.1 Matrices and Determinants

Exercise 5.1. Given $A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$, find:

(a) $A + B$

(b) $2A - B$

(c) AB

(d) $|A|$

Exercise 5.2. Find the inverse of $\begin{pmatrix} 4 & 3 \\ 5 & 4 \end{pmatrix}$

Exercise 5.3. Solve using matrices:

$$\begin{aligned} 2x + y &= 5 \\ x + 3y &= 8 \end{aligned}$$

5.2 Game Theory

Exercise 5.4. Find the saddle point, if any:

	B1	B2	B3
A1	3	2	4
A2	1	3	2
A3	2	1	3

5.3 Implicit Differentiation

Exercise 5.5. Find $\frac{dy}{dx}$ for:

(a) $x^2 + y^2 = 16$

(b) $xy = 12$

(c) $x^2 - y^2 = 9$

5.4 Integral Calculus

Exercise 5.6. Evaluate:

(a) $\int (6x^2 - 4x + 3)\,dx$

(b) $\int_0^1 (2x + 3)\,dx$

(c) $\int_1^2 x^3\,dx$

6 Solutions to Selected Exercises

Solution to Exercise 1:

(a) $A + B = \begin{pmatrix} 3 & 3 \\ 3 & 7 \end{pmatrix}$

(b) $2A - B = \begin{pmatrix} 4 & 2 \\ 6 & 8 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 6 & 5 \end{pmatrix}$

(c) $AB = \begin{pmatrix} 2(1)+1(0) & 2(2)+1(3) \\ 3(1)+4(0) & 3(2)+4(3) \end{pmatrix} = \begin{pmatrix} 2 & 7 \\ 3 & 18 \end{pmatrix}$

(d) $|A| = 2(4) - 1(3) = 8 - 3 = 5$

Solution to Exercise 2:

$$\begin{aligned} |A| &= 4(4) - 3(5) = 16 - 15 = 1 \\ A^{-1} &= \frac{1}{1}\begin{pmatrix} 4 & -3 \\ -5 & 4 \end{pmatrix} = \begin{pmatrix} 4 & -3 \\ -5 & 4 \end{pmatrix} \end{aligned}$$

Solution to Exercise 5(a):

$$\begin{aligned} 2x + 2y\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{x}{y} \end{aligned}$$

Solution to Exercise 6(a):

$$\int (6x^2 - 4x + 3)\,dx = 2x^3 - 2x^2 + 3x + C$$

Solution to Exercise 6(b):

$$\begin{aligned} \int_0^1 (2x + 3)\,dx &= [x^2 + 3x]_0^1 \\ &= (1 + 3) - 0 \\ &= 4 \end{aligned}$$

7 Summary

Important Formula

Matrices:

- Addition: Same size, add corresponding elements
- Multiplication: $(m \times n)(n \times p) = (m \times p)$
- Determinant (2×2): $ad - bc$
- Inverse (2×2): $A^{-1} = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Game Theory:

- Saddle point = row min AND column max
- Use dominance to simplify

Implicit Differentiation:

- Differentiate both sides with respect to x
- Use $\frac{d}{dx}(y^n) = ny^{n-1}\frac{dy}{dx}$
- Solve for $\frac{dy}{dx}$

Integration:

- $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$
- $\int_a^b f(x)\,dx = F(b) - F(a)$

End of Lecture Notes

Practice systematically - these topics build mathematical maturity!