Let $F \to X$ be a vector bundle. Denote by $\mathrm{Fr}(F)$ it's frame bundle. Then if $E \to X$ is a subbundle of F, and F has a metric, we can consider the decomposition

$$F = E \oplus E^{\perp}.$$

Hence we have an injection

$$i : \mathrm{Fr}(E) \times \mathrm{Fr}\left(E^{\perp}\right) \hookrightarrow \mathrm{Fr}(F).$$

Definition 1. Let A be a connection on $\mathrm{Fr}(F)$. Let $\pi : \mathfrak{g}_F \to \mathfrak{g}_E$ denote the projection of the lie algebras. Then the induced connection on Fr E is the one form given by

$$\pi i^* A \tag{1}$$

Lemma 2. Let A be a connection on $\mathrm{Fr}(F)$. Then the curvature of the induced connection (1) is given by

$$F_{\pi i^* A} = \pi i^* F_A + \frac{1}{2}\left([\pi i^* A, \pi i^* A] - \pi i^* [A, A]\right)$$

Remark 3. This is a principal bundle version of the Gauss equation.

Proof. The curvature of the induced connection is given by

$$d\left(\pi i^* A\right) + \frac{1}{2}[\pi i^* A, \pi i^* A].$$

d commutes with pullbacks, so we can rewite this as

$$\pi i^* dA + \frac{1}{2}\pi i^* [A, A] + \frac{1}{2}\left([\pi i^* A, \pi i^* A] - \pi i^* [A, A]\right).$$

which gives the result. $\square$

Lemma 4. If we let

$$A^{\perp} = A - \pi A$$

then we can rewrite the above as

$$F_{\pi i^* A} = \pi i^* F_A + \pi i^* [\pi A, A^{\perp}] + \frac{1}{2}\pi i^* [A^{\perp}, A^{\perp}]$$

$\square$

Hence to calculate the curvature of the induced connection, we just need to calculate the algebraic objects $[\pi A, A^{\perp}]$ and $[A^{\perp}, A^{\perp}]$. It might be tempting to think that $\pi[A^{\perp}, A^{\perp}]$ will be 0, but this won't usually be the case (eg $\mathfrak{g}_F \neq \mathfrak{g}_E \oplus \mathfrak{g}_{E^{\perp}}$ in general).