Let X be a manifold and $P \to X$ a principal G bundle.

Definition 1. A connection on P, A, is irreducible if for some $x \in X$, the holonomy group of A, H_A, lies in a proper subgroup of G. The isotropy group of A is defined to be

$$\Gamma_A = \{u \in \mathscr{G} | u(A) = A\} \supseteq Z(G)$$

Lemma 2. If X is connected Γ_A is isomorphic to the centraliser of H_A in G.

Proof. Pick a point $x \in X$. Then an element $u \in \Gamma_A$ defines an element $u(x) \in G$ via evaluation, that is that the fibre of P at x is transformed by right multiplication by $u(x)$ under u. If $\gamma : S^1 \to X$ is a loop based at x, and $g_\gamma \in G$ is the holonomy around gamma, then we necessarily have $g_\gamma u(x) = u(x) g_\gamma$, since applying the gauge transformation and then calculating the holonomy must be the same as calculating the holonomy and then applying the gauge transformation. Now suppose $h \in C(H_A)$. We want to find a gauge transformation such that $u(A) = A$ and $u(x) = h$. h lies in the identity component of G, so we may assume for simplicity that $h = \exp(\xi)$, for an ξ in the lie algebra of H_A. Given a trivialisation of P about x, we can then define $u(y)$ to be given by $\exp(\xi(y))$ for y near x, where $d\xi = 0$. $\square$

Corollary 3. The isotropy group Γ_A is closed.

Proof. The centraliser of any set is closed. $\square$

Definition 4. By $\mathscr{A}$ we denote the affine space of all connections on P, and by $\mathscr{A}^*$ we denote the irreducible connections. The gauge group, $\mathscr{G}$, acts on $\mathscr{A}$, and we denote the quotients

$$\mathscr{B} = \mathscr{A}/\mathscr{G}, \ \mathscr{B}^* = \mathscr{A}^*/\mathscr{G}.$$

A problem with these spaces is that the action of $\mathscr{G}$ is not free. To get around this we can consider framed connections.

Definition 5. Let $x_0 \in X$ be a base point. Then a framed connection is a pair (A, φ), where A is a connection and $\varphi : G \to P_{x_0}$ is an isomorphism of G spaces.

Lemma 6. If A is a unitary connection, then a framed connection (A, φ) is equivalent to A with a choice of orthonormal basis at x_0.

Proof. A choice of framing at x_0 of the unitary vector bundle specifies a point in the fibre of P. If we define $\varphi(e)$ to be this point, then this uniquely defines φ.Going the other way, we can define the frame to be $\varphi(e)$. $\square$

Lemma 7. The group $\mathscr{G}$ acts freely on $\tilde{\mathscr{A}} = \mathscr{A} \times \mathrm{Hom}(G, P_{x_0})$

Proof. The isotropy subgroup Γ_A acts on $\mathrm{Hom}(G, P_{x_0})$ by right multiplication. This action is free. $\square$

Corollary 8. Under the Sobolev L^2_l completions of $\mathscr{A}$, and L^2_{l-1} of $\mathscr{G}$, the space

$$\tilde{\mathscr{B}} = \tilde{\mathscr{A}}/\mathscr{G}$$

is a Banach Manifold.

Alternatively, one can view $\tilde{\mathscr{B}}$ as the quotient of $\tilde{\mathscr{A}}$ by the based gauge group

$$\mathscr{G}_0 = \{u \in \mathscr{G} \mid u(x_0) = \mathrm{Id}\}\,.$$

Definition 9. Let $\tilde{\mathscr{B}}^*$ be the space of framed irreducible connections. Then we can consider $\tilde{\mathscr{B}}^*$ as a principal $G/Z(G)$ bundle

$$\tilde{\mathscr{B}}^* \to \mathscr{B}^*.$$