See [DK97, Section 5.4].

**Proposition 1.** Let $X$ be a oriented four manifold, with a principal $SU(2)$ bundle $P \to X$. Then, after picking an orientation of $H_+(X)$, the moduli space of $ASD$ instantons has a natural orientation.

*Proof.* We there are two steps to do prove this. The first is proving that the moduli space is orientable, and then we have to pick an orientation. □

**Definition 2.** Let $A$ be an ASD connection. Then we have a complex

$$\Omega^0(\mathrm{Ad}P) \xrightarrow{d_A} \Omega^1(\mathrm{Ad}P) \xrightarrow{d_A^+} \Omega^2(\mathrm{Ad}P)$$

We define $H_A^i$, $i = 1, 2$, to be the cohomology of this complex. So

$$H_A^1 = \frac{\ker\, d_A^+}{\mathrm{Im}\, d_A}, H_A^2 = \ker d_A = \mathrm{coker}\, d_A^+$$

**Remark 3.** Using transversality arguments, for a generic metric $H_A^2 = 0$ at all the irreducible connections.

**Proposition 4.** Let $M^s$ be the subset of $M$ consisting of the irreducible ASD connections where $H_A^2 = 0$. The moduli space of ASD instantons on $P$ is orientable.

*Proof.* Recall that the tangent space to the moduli space is the kernel of the elliptic operator

$$\delta_A = (d_A^* \oplus d_A^+) : \Omega^1(\mathrm{Ad}P) \to \Omega^1(\mathrm{Ad}P) \oplus \Omega_+^2(\mathrm{Ad}P).$$

Hence, $M^s$ is orientable if the bundle $\Lambda^{\max}\ker(\delta_A)$ is trivial. Note that

$$\mathrm{coker}\, \delta_A = \mathrm{coker}\, d_A^* \oplus \mathrm{coker}\, d_A^+ = 0,$$

since, by elliptic theory we have that

$$\Omega^0(\mathrm{Ad}P) = \ker\, d_A \oplus \mathrm{Im}\, d_A^*.$$

The operator $\delta_A : M^s \to \Omega^1(\mathrm{Ad}P) \oplus \Omega_+^2(\mathrm{Ad}P)$ extends to an operator on $\mathscr{B}^*$.

$$\det \mathrm{ind}\, \delta = \Lambda^{\max}\ker\, \delta \oplus \Lambda^{\max}\mathrm{coker}\, \delta$$

Hence, if we can prove that $\det \mathrm{ind}\, \delta \to \mathscr{B}^*$ is trivial, then we will have that $M^s$ is orientable. □

We will now prove the following

**Proposition 5.** The bundle $\det \mathrm{ind}\, \delta \to \mathscr{B}^*$ is trivial.

*Proof.* Recall that we can consider the framed instantons as in `FramedConnections/`Definition 5
. □

## References

[DK97] S. K. Donaldson and P. B. Kronheimer. *The Geometry of Four-Manifolds.* Oxford Mathematical Monographs. Oxford, New York: Oxford University Press, Dec. 4, 1997. 456 pp. ISBN: 978-0-19-850269-2.