This is covered in [Hay19]. The main idea of this is to relate the different ways of viewing connections on principal bundles. In particular, the relationship between the connection one-form $A \in \Omega^1(P, \mathrm{Ad}P)$ and derivative operators ∇ on associated vector bundles. Let X be a manifold and $P \to X$ be a Principal G-bundle, where G is a Lie group with lie algebra $\mathfrak{g}$. Of particular importance is the bundle $\mathrm{ad}P$, the vector bundle associated to the adjoint action of G, that is

$$\mathrm{ad}P = P \times_{\mathrm{ad}} \mathfrak{g}.$$

Let V be a G representation, with the action given by $\rho : G \to \mathrm{Aut}(V)$. Then we can consider the associated bundle

$$E = P \times_\rho V,$$

and forms taking values in E, that is, the bundles $\Lambda^p T^* M \times E$. We denote the spaces of sections of such forms by

$$\Omega^p(E).$$

We will prove a correspondence between sections of $\Omega^p(E)$ and forms on P taking values in the trivial bundle $\underline{V} \to P$ with fibre V.

Definition 1. A form $s \in \Omega^q(P, \underline{V})$ is basic if $s_p(\phi(\xi_1), \dots \phi(\xi_q)) = 0$ for all $\xi \in \mathfrak{g}$, where $\phi : \mathfrak{g} \to T_p P$ is the map

$$\xi \mapsto \left.\frac{d}{dt}\right|_{t=0} p \exp(t\xi).$$

A zero-form is trivially basic.

Definition 2. A form $s \in \Omega^q(P, \underline{V})$ is G-equivariant, or just equivariant, if for all $g \in G$,

$$R_g^* s = \rho(g) s.$$

Proposition 3. There is a one to one correspondance between equivariant, basic forms in $\Omega^q(P, \underline{V})$ and forms in $\Omega^q(E)$ on X.

Proof. Given a form $s \in \Omega^q(P, \underline{V})$ that is equivariant and basic, we can define a form in $\Omega^q(E)$ by defining

$$\bar{s}(x)(\bar{X}_1, \dots, \bar{X}_p) = [p, s(p)(X_1, \dots, X_q)],$$

where $\pi(p) = x$, and $d\pi X_i = \bar{X}_i$. This is well defined because $\bar{s}$ is equivariant and basic. We want to prove that this is a bijection, so we can find an inverse to this map. Let φ_i be a partition of unity subordinate to a covering U_i of X over which P can be trivialised, that is on each U_i we have an isomorphism

$$f : P_{U_i} \to G \times U_i$$

In particular, this means that E is trivial over U_i via the map $[p, v] \mapsto [f(p), v] \in E \times U_i$.

Let $\bar{s} \in \Omega^q(E)$, we can define a one form $s_i \in \Omega^q(P_{U_i}, \underline{V})$ by setting

$$s_i((g,x))(X_1, \ldots, X_q) = \rho(g)\bar{s}(x)(\bar{X}_1, \ldots, \bar{X}_q),$$

where again an overline denots projection onto X. This form is clearly G equivariant and basic. Now take the sum $s = \sum_i \varphi_i s_i$. We want to show that this is the form that we want. What we need to show is for all $p \in U_i \cap U_j$, $s_i(p) = s_j(p)$. This is not obvious because changing the trivialisation could, a priori, change the value in V that the trivial sections s_i take. To prove that this is not the case, we can consider a gauge local transformation

$$\alpha : U_i \to G$$

So we have two trivialisations, one given by f and one given by $f(p)\alpha(\pi p)$, which in turn gives two trivialisations of E over U_i. One as before, and the other given by

$$[p, v] \mapsto [f(p), \rho(\alpha^{-1})v]$$

But then the definition of s_i in the second trivialisation is simply

$$s_i'((g', x)) = \rho(g')\bar{s}' = \rho(g)\rho(\alpha)\rho(\alpha^{-1})v = s_i(g, x)$$

hence the values in V that the s_i take do not depend on the choice of gauge. In particular, $s_i = s_j$ on $U_i \cap U_j$. This proves the correspondence. $\square$

References

[Hay19] Andriy Haydys. *Introduction to Gauge Theory*. Oct. 23, 2019. DOI: 10.48550/arXiv.1910.10436. arXiv: arXiv:1910.10436. URL: http://arxiv.org/abs/1910.10436 (visited on 01/28/2023). preprint.