Chapter 1

Sensing

1.1 Joint measurement of TFE via SFG

1.2 Mathematical preliminaries

1.3 Physics preminaries

1.3.1 SFE

The SFG process in a $\chi^{(2)}$ nonlinear medium could be modeled as the following evolution operator:

$$V = I + \varepsilon \left(\int \mathrm{d}\omega_p \mathrm{d}\omega_s \mathrm{d}\omega_i a_p^\dagger(\omega_p) a_s(\omega_s) a_i(\omega_i) \delta(\omega_p - \omega_s - \omega_i) - H.C. \right), \tag{1.1}$$

where photons in the signal mode $a_s(\omega_s)$ and ideler mode $a_i(\omega_i)$ are annihilated to generate photons in the pump mode $a_p(\omega_p)$ and ε characterizes the interaction strength. The SPDC process is the time-reversal process of SFG, which can also be described by the same revolution operator.

1.3.2 SPDC

1.4 Introduction

1.5 Terminology

1.6 Problem formulation

The frequency sum and time difference of two photons could be simultaneously measured through the sum-frequency generation process.

1.7 Protocol

Given the close connection between the spontaneous parametric down-conversion (SPDC) process and time-frequency entanglement (TFE), it's natural to utilize the time-reversal of the SPDC process, i.e., sum frequency generation (SFG) to obtain a TFE joint measurement based protocol.

Definition 1.7.1 (Frequency sum (FS) operator). *The frequency sum operator $P_{\delta_\omega}(\omega)$ that selects states with the frequency sum $\omega_s+\omega_i$ of the signal and idler photon being around ω within uncertatinty δ_ω is defined as:*

$$P_{\delta_\omega}(\omega)=\int\int \mathrm{d}\omega_s \mathrm{d}\omega_i a_s^\dagger(\omega_s)a_i^\dagger(\omega_i)a_s(\omega_s)a_i(\omega_i)\mathrm{Gate}\left(\frac{\omega-\omega_s-\omega_i}{\delta_\omega}\right), \tag{1.2}$$

where $\mathrm{Gate}(x)=1$ *for* $|x|\leq 1/2$ *and* $\mathrm{Gate}(x)=0$ *otherwise.*

Lemma 1.7.1 (Frequency sum operator is a projection operator). *The frequency sum operator $P_{\delta_\omega}(\omega)$ is a projection operator satisfying*

$$P_{\delta_\omega}(\omega)^2=P_{\delta_\omega}(\omega). \tag{1.3}$$

Definition 1.7.2 (Time difference (TD) operator). *The time difference operator $P_{\delta_t}(t)$ that selects states with the time difference t_s-t_i of the signal and idler photon being around t within uncertatinty δ_t is defined as:*

$$P_{\delta_t}(t)=\int\int \mathrm{d}t_s \mathrm{d}t_i \tilde{a_s}^\dagger(t_s)\tilde{a_i}^\dagger(t_i)\tilde{a_s}(t_s)\tilde{a_i}(t_i)\mathrm{Gate}\left(\frac{t_s-t_i-t}{\delta_t}\right), \tag{1.4}$$

where $\mathrm{Gate}(x)=1$ *for* $|x|\leq 1/2$ *and* $\mathrm{Gate}(x)=0$ *otherwise and*

$$\tilde{a}_x=\frac{1}{\sqrt{2\pi}}\int \mathrm{d}\omega \exp\left(-i\omega t\right)a_x(\omega) \tag{1.5}$$

Lemma 1.7.2 (Time difference operator is a projection operator). *The time difference operator $P_{\delta_t}(t)$ is a projection operator satisfying*

$$P_{\delta_t}(t)^2=P_{\delta_t}(t). \tag{1.6}$$

Definition 1.7.3 (Joint projection operator). *The joint projection operator of the time difference and frequency sum is defined as*

$$P_{\delta_\omega,\delta_t}(\omega,t)=P_{\delta_\omega}(\omega)P_{\delta_t}(t), \tag{1.7}$$

which means selecting states of which the time difference between the signal and idler photon t_s-t_i is around t within uncertainty δ_t and frequency sum of the signal and idler photon being around ω within unicertainty δ_ω, simultaneously.

Lemma 1.7.3 (Commutation relationship between frequency-time operators). *We have the commutation relationship*

$$[P_{\delta_\omega}(\omega),P_{\delta_t}(t)]=0. \tag{1.8}$$

Definition 1.7.4 (TD an FS probability density operator (PDF)). *We define*

$$P(\omega,t)=\lim_{\delta_t\to 0,\delta_\omega\to 0}\frac{1}{\delta_\omega\delta_t}P_{\delta_\omega}(\omega)P_{\delta_t}(t). \tag{1.9}$$

Lemma 1.7.4. *We have*

$$P(\omega,t)=\frac{1}{2\pi}B_p^\dagger B_p, \tag{1.10}$$

where

$$B_p=\int\int \mathrm{d}\omega_s\mathrm{d}\omega_i\delta(\omega_s+\omega_i-\omega)\exp\left[i\omega_i t\right]a_s(\omega_s)a_i(\omega_i). \tag{1.11}$$

Lemma 1.7.5 (Connection between TD and FS PDF and SFG Process)**.** *We have*

$$B^\dagger B=2\pi P(\omega,0). \tag{1.12}$$

This lemma did not show the SFG process

Lemma 1.7.6 (Discrete sum of evolution operator of SFG process)**.** *The discrete sum of the evolution operator of SFG process can be obtained by a* ***two-step Schmidt decomposition*** *as:*

$$v=I+\varepsilon\sum_m\left(\sqrt{\lambda_m^{(1)}}A_m^\dagger B_m-H.C.\right), \tag{1.13}$$

where

$$B_m=\sum_n\sqrt{\lambda_{m,n}^{(2)}}F_{m,n}G_{m,n}, \tag{1.14}$$

$$A_m=\int \mathrm{d}\omega\psi_{A,m}(\omega)a_p(\omega), \tag{1.15}$$

$$B_m=\int \mathrm{d}\omega_s\mathrm{d}\omega_i\psi_{B,m}(\omega_s,\omega_i)a_s(\omega_s)a_i(\omega_i), \tag{1.16}$$

$$F_{m,n}=\int \mathrm{d}\psi_{F,m,n}(\omega)a_s(\omega), \tag{1.17}$$

$$G_{m,n}=\int \mathrm{d}\psi_{G,m,n}(\omega)a_i(\omega). \tag{1.18}$$

Lemma 1.7.7 (Non-uniqueness of the first step Schmidt Decomposition)**.** *If the function f_0 can be written in the following form:*

$$\delta(\omega_p-\omega_s-\omega_i)f_0(\omega_p-\omega_s-\omega_i)=\delta(\omega_p-\omega_s-\Omega_i)f(\frac{\omega_s-\Omega_i}{\sqrt{2}}), \tag{1.19}$$

then the first step Schmidt decomposition in the main text is not unique.

Lemma 1.7.8 (An useful communtation relationship)**.** *An useful communtation relationship:*

$$[B_{m'},B_{m''}^\dagger]=\delta_{m'm''}+\int \mathrm{d}\omega_s'\mathrm{d}\omega_i\mathrm{d}\omega_s''\psi_{B,m''}^*(\omega_s',\omega_i)a_s^\dagger(\omega_s'')a_s(\omega_s') \tag{1.20}$$

$$+\int \mathrm{d}\omega_s\mathrm{d}\omega_i'\mathrm{d}\omega_i''\psi_{B,m''}^*(\omega_s,\omega_i'')\psi_{B,m'}a_i^\dagger(\omega_i'')a_i(\omega_i'). \tag{1.21}$$

Corollary 1.7.1. *By (1.7.8), we have*

$$[B_{m'},B_{m''}^\dagger]=\delta_{m'm''}|0\rangle. \tag{1.22}$$

Lemma 1.7.9. *We have the commutation relation between time difference projection operator and frequency sum projection operator:*

$$[P_{\delta_\omega}(\omega), P_{\delta_t}(t)] = 0. \tag{1.23}$$

Lemma 1.7.10. *The frequency spectrum $S(\omega)$ of the generated pump photon is given by the expectation value of the spectral density operator $a_p^\dagger(\omega_p)a_p(\omega_p)$*

$$S(\omega_p) = \frac{\epsilon^2 \exp\left[\frac{1}{8}\left(-4\Delta t^2\sigma_-^2 - \frac{\Delta\omega^2}{\sigma_-^2} - \frac{4(\Delta\omega+\omega_0-\Omega_p)^2}{\sigma_+^2}\right)\right]}{2\sqrt{\pi}\sigma_+}. \tag{1.24}$$

Definition 1.7.5. *The discrete mode operator $F_{m,n}^(b)$ for the noise photons is defined as*

$$F_{m,n}^{(b)} = \int \mathrm{d}\omega \psi_{F_{m,n}}(\omega) a_s^{(b)}(\omega). \tag{1.25}$$

Definition 1.7.6. *The virtual beam-splitter is modeledd as the following unitary transform:*

$$U_{loss} = \Pi_n \exp\left[i\arccos(\eta)(F_{0,n}^\dagger F_{0,n}^{(b)} + H.C.)\right]. \tag{1.26}$$

Definition 1.7.7. *We use a density matrix ρ_b that satisfies the following conditions to describe the noise photons:*

$$\mathrm{Tr}[F_{0,n''}^{(b)\dagger} F_{0,n'}^{(b)} \rho_b] = \delta_{n',n''}\mu_b, \tag{1.27}$$

$$\mathrm{Tr}[F_{0,n'}^{(b)} \rho_b] = 0 \tag{1.28}$$

Definition 1.7.8. *The signal and idler photon pair source is described by the biphoton state $|pair\rangle$:*

$$|pair\rangle = B_0^\dagger|0\rangle \tag{1.29}$$

$$= \sum_n \sqrt{\lambda_{0,n}^{(2)}} F_{0,n}^\dagger G_{0,n}|0\rangle. \tag{1.30}$$

Definition 1.7.9. *The unitary trasform of the SFG process is given by:*

$$V = I + \epsilon \sum_m \left[\sqrt{\lambda_m^{(1)}} A_m^\dagger B_m - H.C.\right]. \tag{1.31}$$

Definition 1.7.10. *In the Heisenberg picture, the photon number operator of the generated pump photon in each pump mode A_m after the beam-splitter transform and the SFG process is given by:*

$$U_{loss}^\dagger V^\dagger A_m^\dagger V U_{loss}. \tag{1.32}$$

Proposition 1.7.1. *When the transmission of the signal photon is perfect ($\eta = 1$), the pump photon can only generate in mode $A_0 (m = 0)$.*

Lemma 1.7.11. *We have*

$$\langle U_{loss}^\dagger V^\dagger A_0^\dagger A_0 V | U_{loss}\rangle = \epsilon^2 \lambda_0^{(1)} (\eta + \mu_b \sum_n \lambda_{0,n}^2). \tag{1.33}$$

Definition 1.7.11. *The generated SPDC state is given by:*

$$V = |0\rangle - \epsilon\sqrt{\lambda_0^{(1)}}\alpha B_0^\dagger|0\rangle. \tag{1.34}$$

Lemma 1.7.12. *The joint density operator of the noise-idler state ρ_j is given by the tensor product of ρ_i and ρ_b:*

$$\begin{aligned}\rho_j &= \rho_i \otimes \rho_b &(1.35)\\ &= \mu_b \int\int \mathrm{d}\omega_s \mathrm{d}\omega_s' \int\int \mathrm{d}\omega_i' \mathrm{d}\omega_i'' \phi_0^*(\omega_s', \omega_i')\phi_0(\omega_s', \omega_i'')a_i^\dagger(\omega_i')a_s^\dagger(\omega_s)|0\rangle\langle 0|a_i(\omega_i'')a_s(\omega_s). &(1.36)\end{aligned}$$

Lemma 1.7.13. *The spectral density $S(\omega)$ of the upconverted photons is*

$$S(\omega) = \frac{\epsilon^2 \mu_b \exp\left[-\frac{(\omega-\omega_0)^2}{8\sigma_-^2 - 2\sigma_+^2}\right]}{\sqrt{\pi}\sqrt{4\sigma_-^2 + \sigma_+^2}}. \tag{1.37}$$

Theorem 1. *The error exponent of the classical Chernoff bound of the TFE QI protocol is given by C_{QI} [NS09]:*

$$C_{QI} = -\log \min_{s\in[0,1]} \left\{ \sum_{b\in 0,1} p_0(b)^s p_1(b)^{(1-s)} \right\}. \tag{1.38}$$

1.8 Performance evaluation

Lemma 1.8.1.

$$\tag{1.39}$$

Bibliography

[NS09] Michael Nussbaum and Arleta Szkoła. The chernoff lower bound for symmetric quantum hypothesis testing. 2009.