Chapter 1

Measure Theory

1.1 Sigma algebra

Example 1.1.1. *We define* $\mathcal{P}(X)$ *as the power set of set* X*. Assume that set* $X = \{a, b\}$*, the power set* $P(X)$ *would be* $\{\emptyset, X, \{a\}, \{b\}\}$

Definition 1.1.1 (Sigma algebra)**.** $\mathcal{A} \subseteq P(X)$ *is called a* $\sigma -$ algebra*:*

$$(a)\ \emptyset, X \in \mathcal{A} \tag{1.1}$$

$$(b)\ A \in \mathcal{A} \Longrightarrow A^c := X \setminus A \in \mathcal{A} \tag{1.2}$$

$$(c)\ A_i \in \mathcal{A},\ i \in \mathcal{N} \Longrightarrow \cup_{i=1}^{\infty} A_i \in \mathcal{A}. \tag{1.3}$$

Definition 1.1.2 (Measurable sets)**.** $A \in \mathcal{A}$ *is called a* $\mathcal{A}$*-measurable set.*

Example 1.1.2.

$$(1)\ \mathcal{A} = \{\emptyset, X\} \tag{1.4}$$

$$(2)\ \mathcal{A} = \{P(X)\}. \tag{1.5}$$

Lemma 1.1.1. *Assume* A_i *is* σ*-algebra on* X*,* $i \in I$ *(index set). Then, we have* $\cap_{i \in I} \mathcal{A}_i$ *is also a* σ*-algebra on* X*.*

Definition 1.1.3 (Sigma algebra generated by $\mathcal{M}$)**.** *For* $\mathcal{M} \subseteq P(X)$*, there is a smallest* σ*-algebra that contains* $\mathcal{M}$*:*

$$\sigma(\mathcal{M}) := \cap_{\mathcal{A} \supseteq \mathcal{M},\ A\ \sigma-algebra} \mathcal{A}. \tag{1.6}$$

Example 1.1.3. *We define* $X = \{a, b, c, d\}$ *and* $\mathcal{M} = \{\{a\}, \{b\}\}$*. Then we have*

$$\sigma(\mathcal{M}) = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c, d\}, \{a, c, d\}, \{c, d\}\}. \tag{1.7}$$

Definition 1.1.4 (Borel sigma algebra)**.** *Let* $(X, \mathcal{T})$ *be a topological space (Let* X *be a metric space/Let* X *be a subset of* $\mathbb{R}^n$*; We need "open sets".). We then define* $\mathcal{B}(X)$ *is the borel* σ*-algebra on* X *as*

$$\mathcal{B}(X) := \sigma(\mathcal{T}), \tag{1.8}$$

which is the σ*-algebra generated by the open sets* $\mathcal{T}$*.*

1.2 What is a measure?

Definition 1.2.1 (Measure)**.** *$(X, \mathcal{A})$ is called a measurable space, where X is a set and $\mathcal{A}$ is a σ-algebra on X. A map $\mu : A \to [0, \infty] := [0, \infty) + \{\infty\}$ is called a measure if it satisfies:*

$$(a)\ \mu(\emptyset) = 0 \tag{1.9}$$

$$(b)\ \mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)\ with\ A_i \cap A_j = \emptyset,\ i \neq j\ for\ all\ A_i \in \mathcal{A}.(\sigma - additive) \tag{1.10}$$

Definition 1.2.2. *$(X, \mathcal{A}, \mu)$ is called a measure space.*

Example 1.2.1. *Given X and $\mathcal{A} = \mathcal{P}(X)$.*

- *Counting measure ($A \in \mathcal{A}$) is defined as*

$$\mu(A) := \begin{cases} \#A, & A\ has\ finitely\ many\ elements \\ \infty & else \end{cases} \tag{1.11}$$

  where $\#A$ means the number of elements in A.

  Calculation rules in $[0, \infty]$:

$$x + \infty := \infty\ for\ all\ x \in [0, \infty] \tag{1.12}$$
$$x \cdot \infty := \infty\ for\ all\ x \in (0, \infty] \tag{1.13}$$
$$0 \cdot \infty := 0\ (only\ true\ in\ most\ cases\ in\ measure\ theory!) \tag{1.14}$$

- *Dirac measure for $p \in X$ is defined as*

$$\delta_p(A) := \begin{cases} 1, & p \in A \\ 0, & else \end{cases} \tag{1.15}$$

- *We search a measure on $X \in \mathcal{R}^n$ satisfying:*

$$(1)\ \mu([0, 1]^n) = 1 \tag{1.16}$$
$$(2)\ \mu(x + A) = \mu(A)\ for\ all\ x \in \mathcal{R}^n, \tag{1.17}$$

  which is known as Lebesgue measure where the σ-algebra is not equal to power set.

1.3 Not everything is lebesgue measurable

Measure problem: search measure μ on $\mathcal{P}(\mathbb{R})$ with:

- (1) $\mu([a, b]) = b - a,\ b > a,$
- (2) $\mu(x + A) = \mu(A),\ A \in \mathcal{P}(\mathbb{R}),\ x \in \mathbb{R}.$

$\implies \mu$ does not exist.

Claim: Let μ be a measure on $\mathcal{P}(\mathbb{R})$ with $\mu((0, 1]) < \infty$ and (2). $\implies \mu = 0$.

Proof. (a) Definitions: $I \in (0,1]$ with equivalence relation on I: $x\ y \iff x - y \in \mathbb{Q}$ i.e., $[x] := \{x + r | r \in \mathbb{Q},\ x + r \in I\}$. Following this definition, we have a disjoint decomposition of I into boxes, possibly uncontable many of them! We then pick one element a_n from each box $[x_n]$ and form a set $A \in I$, i.e., $\{a_1, a_2, \cdots\} = A$. We have $A \in I$ with prperty:

- (1) For each $[x]$, there is an $a \in A$ with $a \in [x]$.
- (2) For all $a, b \in A:\ a, b \in [x] \Longrightarrow a = b$.

In uncountable case, the existence of $A \in I$ with the above property is guaranted by the axiom of choice of set theory.

We define $A_n := r_n + A$, where $(r_n)_{n\in\mathbb{N}}$ enumeration of $\mathbb{Q}_n(-1,1]$.

(b) We then claim that $A_n \cap A_m = \emptyset \Longleftarrow n \neq m$. The proof is as follows: $x \in A_n \cap A_m \Longrightarrow x = r_n + a_n,\ a_n \in A$ and $x = r_m + a_m,\ a_m \in A. \Longrightarrow r_n + a_n = r_m + a_m \Longrightarrow a_n - a_m = r_n - r_m \in \mathbb{Q} \Longrightarrow a_n\ a_m \Longrightarrow a_m, a_n \in [a_m] \Longrightarrow a_n = a_m \Longrightarrow r_n = r_m \Longrightarrow n = m$.

(c) We claim that $(0,1] \subseteq \cup_{n\in\mathbb{N}} A_n \subseteq (-1,2]$. The proof is as follows:

Assume now: μ measure on $\mathcal{P}(\mathbb{R})$ with $\mu((0,1]) < \infty$ and (2).

By (2): $\mu(1 + A) = \mu(A)$ for all $n \in \mathbb{N}$.

By (c): we have

$$\mu((0,1]) \leq \mu(\cup_{n\in\mathbb{N}}) \leq \mu((-1,2]) \tag{1.18}$$

We know: $\mu((0,1]) =: C < \infty$. By using (2) and σ-additivity, we get $\mu((-1,2]) = \mu\left((-1,0] \cup (0,1] \cup (1,2] = 3C\right)$. $\Longrightarrow_{1.18,(b)} C \leq \sum_{n=1}^{\infty} \mu(A_n) \leq 3C \Longrightarrow C \leq \sum_{n=1}^{\infty} \mu(A) \leq 3C \Longrightarrow \mu(A) = 0 \Longrightarrow C = 0(\text{henceL } \mu\left((0,1]\right) = 0) \Longrightarrow \mu(\mathbb{R}) = \mu\left(\cup_{n\in\mathbb{Z}}(m, m+1]\right) = 0 \Longrightarrow \mu = 0$. □

1.4 Measurable maps

Definition 1.4.1 (Measurable maps)**.** *$(\Omega_1, \mathcal{A}_1)$ and $(\Omega_2, \mathcal{A}_2)$ are measurable spaces. $f : \Omega_1 \to \Omega_2$ is a measurable map w.r.t. $\mathcal{A}_1$ and $\mathcal{A}_2$ if $f^{-1}(A_2) \in \mathcal{A}_1$ for all $A_2 \in \mathcal{A}_2$.*

Example 1.4.1.

- *$(\Omega, \mathcal{A})$ and $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ are two measurable spaces. We define characteristic fucntion (aksi indicator function) as $\chi_A : \Omega \to \mathbb{R}$, where*

$$\chi_A(w) := \begin{cases} 1, & w \in A \\ 0, & w \notin A \end{cases} \tag{1.19}$$

For all measurable $A \in \mathcal{A}$, χ_A is a measurable map. We have

$$\chi_A^{-1}(\emptyset) = \emptyset \in \mathcal{A},\ \chi_A^{-1}(\mathbb{R}) = \Omega \in \mathcal{A} \tag{1.20}$$
$$\chi_A^{-1}(\{A\}) = A,\ \chi_A^{-1}(\{0\}) = A^c \in \mathcal{A}. \tag{1.21}$$

- *Composition of measurable maps.*

Lemma 1.4.1. $(\Omega_1, \mathcal{A}_\infty)$, $(\Omega_2, \mathcal{A}_\in)$, $(\Omega_3, \mathcal{A}_\ni)$ *are measurable space. We define* $\Omega_1 \xrightarrow{f} \Omega_2 \xrightarrow{g} \Omega_3$. *Then* f, g *are measurable implies* $g \circ f$ *is measurable.*

Proof.

$$
\begin{aligned}
(g \circ f)^{-1}(A_3) &= f^{-1}(g^{-1}(A_3)) && (1.22)\\
&\in \mathcal{A}_1 && (1.23)
\end{aligned}
$$

□

Important measurable maps

Lemma 1.4.2. $(\Omega, \mathcal{A})$ *and* $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ *are measurable spaces.* $f, g : \Omega \to \mathbb{R}$ *are measurable maps indicates that* $f + g$, $f - g$, $f \cdot g$, $|f|$ *are measurable maps.*

1.5 Lebesgue integral

Example 1.5.1. *Define Characteristic function* $\chi_A : X \to \mathbb{R}$, $A \in \mathcal{A}$. *We define* $I(A) := \mu(A)$. *Surprisingly,* $I(A)$ *is nothing but the integral of* χ_A *over* A.

Definition 1.5.1 (Simple/Step/Staircsae functions,...). *For* $A_1, A_2, \ldots, A_n \in \mathcal{A}$, *and* $c_1, c_2, \cdots, c_n \in \mathbb{R}$. *We define*

$$
f(x) := \sum_{i=1}^{n} c_i \cdot \chi_{A_i}(x). \qquad (1.24)
$$

We then have $f(x)$ *is measurable and the integraal of* f *is defined as* $I(f) := \sum_{i=1}^{n} c_i \mu(A_i)$.

Remark 1.5.1. *The problem of the integral* $I(f)$ *is that it is undefined when* $\mu(A_i) = \infty$. *The problem can be solved by exclude* ∞ *by defintion or the following way.*

Definition 1.5.2 (Lebesgue integral). *Define* $S^+ := \{f : X \to \mathbb{R} | f \text{ simple function}, \ f \geq 0\}$. $f \in S^+$ *and choose representation* $f(x) = \sum_{i=1}^{n} c_i \chi_{A_i}(x)$, $c_i \geq 0$. *The lebesgue integral of* f *w.r.t.* μ *is defined as*

$$
\begin{aligned}
\int_X f(x) \ \mathrm{d}\mu(x) &= \int_X f \ \mathrm{d}\mu && (1.25)\\
&= I(f) && (1.26)\\
&= \sum_{i=1}^{n} c_i \cdot \mu(A_i) && (1.27)\\
&= [0, \infty]. && (1.28)
\end{aligned}
$$

Property 1.5.1.

- $I(\alpha f + \beta g) = \alpha I(f) + \beta I(g)$, $\alpha, \beta \geq 0$.
- $f \leq g \Longrightarrow I(f) \leq I(g)$ *(monotomicity)*

Definition 1.5.3. *Define a measurable map* $f : X \to [0, \infty)$. $h = \sum_{i=1}^{n} c_i \cdot \chi_{A_i}$. *The lebesgue integral of* f *w.r.t.* μ *is defined as*

$$
\begin{aligned}
\int_X f \ \mathrm{d}\mu &:= \sup \left\{ I(h) | h \in S^+, \ h \leq f \right\} && (1.29)\\
&\in [0, \infty]. && (1.30)
\end{aligned}
$$

f *is called* μ*-integrable if* $\int_X f \ \mathrm{d}\mu < \infty$.

Property 1.5.2. *Define measurable maps $f, g : X \to [0, \infty)$, we have*

- *1. $f = g$ for μ-almost everywhere(a.e.), which satisfies $\mu(\{x \in X | f(x) \neq g(x)\}) = \implies \int_X f \, \mathrm{d}\mu = \int_X g \, \mathrm{d}\mu$.*
- *2. $f \leq g$ for μ a.e. $\implies \int_X f \, \mathrm{d}\mu \leq \int_X g \, \mathrm{d}\mu$*
- *3. $f = 0$ for μ-a.e. $\iff \int_X f \, \mathrm{d}\mu = 0$.*

Proof of 2.: monotonicity. Let $h := X \to [0, \infty)$ be a simple function, i.e.,

$$h(x) = \sum_{i=1}^{n} c_i \chi_{A_i}(x) \tag{1.31}$$

$$= \sum_{t \in h(X)} t \cdot \chi_{\{x \in X | h(x) = t\}}. \tag{1.32}$$

Let $X = \tilde{X}^c \cup \tilde{X}$ with $\mu(\tilde{X}^c) = 0$,

$$\tilde{h}(x) := \begin{cases} h(x), & x \in \tilde{X} \\ a, & x \in \tilde{X}^c \end{cases} \tag{1.33}$$

$$\tilde{h}(x) = \sum_{t \in h(X)} t \cdot \chi_{\left\{x \in \tilde{X} | h(x) = t\right\}} + a \cdot \chi_{\tilde{X}^c} \tag{1.34}$$

$$I(\tilde{h}) = \sum_{t \in h(X)} t \cdot \mu(\left\{x \in \tilde{X} | h(x) = t\right\}) + a \cdot \mu(\tilde{X}^c) \tag{1.35}$$

$$= \sum_{t \in h(X)} t \left[\mu\left(\left\{x \in \tilde{X} | h(x) = t\right\}\right) + \mu\left(\left\{x \in \tilde{X}^c | h(x) = t\right\}\right)\right] \tag{1.36}$$

$$= \sum_{t \in h(X)} t \left[\mu\left(\left\{x \in \tilde{X} | h(x) = t\right\} \cup \left\{x \in \tilde{X}^c | h(x) = t\right\}\right)\right] \tag{1.37}$$

$$I(h) = \sum_{t \in h(X) \setminus \{0\}} t \cdot \mu(\{x \in X | h(x) = t\}). \tag{1.38}$$

We define

$$\tilde{X} := \{x \in X | f(x) \leq g(x)\}, \tag{1.39}$$

$$\mu(\tilde{X}^c) = 0 \tag{1.40}$$

$$\int_X f \, \mathrm{d}\mu = \sup\left\{I(h) | h \in S^+, h \leq f\right\} \tag{1.41}$$

$$= \sup\{I(\tilde{h}) | \tilde{h} \in S^+, \tilde{h} \leq f \text{ on } \tilde{X}\} \tag{1.42}$$

$$\leq \sup\{I(\tilde{h}) | \tilde{h} \in S^+, h \leq g \text{ on } \tilde{X}\} \tag{1.43}$$

$$= \sup\{I(h) | h \in S^+, h \leq g\} \tag{1.44}$$

$$= \int_X g \, \mathrm{d}\mu. \tag{1.45}$$

□

Theorem 1 (Monotone convergence theorem)**.** *$(X, \mathcal{A}, \mu)$ measurable spaces, $f_n : X \to [0, \infty]$, $(f : X \to [0, \infty])$ measurable for all $n \in \mathbb{N}$ with*

$$f_1 \le f_2 \le f_3 \le \cdots \quad \mu - a.e. \tag{1.46}$$

$$(\lim_{n\to\infty} \int_X f_n \, \mathrm{d}\mu = \int_X f \, \mathrm{d}\mu \quad \mu - a.e.(x \in X)) \tag{1.47}$$

This implies that

$$\lim_{n\to\infty} \int_X f_n \, \mathrm{d}\mu = \int_X \lim_{n\to} f_n \, \mathrm{d}\mu. \tag{1.48}$$

Proof. $\int_X f_1 \, \mathrm{d}\mu \le \int_X f_2 \, \mathrm{d}\mu \le \cdots$ and $\int_X f_n \, \mathrm{d}\mu \le \int_X f \, \mathrm{d}\mu$ for $n \in \mathbb{N}$. Then we have

$$\lim_{n\to\infty} \int_X f_n \, \mathrm{d}\mu \le \int_X f \, \mathrm{d}\mu, \tag{1.49}$$

which is the first part of 1.48.

Let h be a simple function $0 \le f \le f$ and $\varepsilon > 0$. We define

$$X_n := \{x \in X | f_n(x) \ge (1 - \varepsilon)h(x)\} \tag{1.50}$$

with $\cup_{n=1}^{\infty} X_n = \tilde{X}$, and $\mu(\tilde{X}^c) = 0$. We have

$$\int_X f_n \, \mathrm{d}\mu \ge \int_{X_n} f_n \, \mathrm{d}\mu \ge \int_{X_n} (1 - \varepsilon)h \, \mathrm{d}\mu \tag{1.51}$$

$$\lim_{n\to\infty} \int_X f_n \, \mathrm{d}\mu \ge \lim_{n\to\infty} \int_{X_n} (1 - \varepsilon)h \, \mathrm{d}\mu \tag{1.52}$$

$$= \int_{\tilde{X}} (1 - \varepsilon)h \, \mathrm{d}\mu \tag{1.53}$$

$$= \int_X (1 - \varepsilon)h \, \mathrm{d}\mu. \tag{1.54}$$

This implies

$$\lim_{n\to\infty} \int_X f_n \, \mathrm{d}\mu \ge \int_X h \, \mathrm{d}\mu, \tag{1.55}$$

since $\varepsilon > 0$ arbitrarily. Then we have

$$\lim_{n\to\infty} \int_X f_n \, \mathrm{d}\mu \ge \int_X f \mathrm{d}\mu, \tag{1.56}$$

since h is arbitrary and $h \le f$, which is second part of 1.48. □

Applictions Given a series $(g_n)_{n\in\mathbb{N}}$, $g_n : X \to [0, \infty]$ measurable for all n. Then we have $\sum_{n=1}^{\infty} g_n : X \to [0, \infty]$ measurable and

$$\int_X \sum_{n=1}^{\infty} g_n \, \mathrm{d}\mu = \sum_{n=1}^{\infty} \int_X g_n \, \mathrm{d}\mu, \tag{1.57}$$

which means the integral and sum can exchange.

1.6 Fatou' lemma

Lemma 1.6.1 (Fatou' lemma)**.** *Given $(X, \mathcal{A}, \mu)$ measurable space, $f_n : X \to [0, \infty]$ measurable for all $n \in \mathbb{N}$. Then we have*

$$\int_X \liminf_{n\to\infty} f_n \ \mathrm{d}\mu \leq \liminf_{n\to\infty} \int_X f_n \ \mathrm{d}\mu. \tag{1.58}$$

Remark 1.6.1. $\liminf_{n\to\infty} f_n : X \to [0, \infty]$ *is a function. This is*

$$\begin{aligned} g(x) &:= \left(\liminf_{n\to\infty} f_n\right)(x) &(1.59)\\ &:= \lim_{n\to\infty}\left(\inf_{k\geq n} f_k(x)\right) &(1.60)\\ &\in [0,\infty] &(1.61)\\ g_n(x) &:= \inf_{k\geq n} f_k(x). &(1.62)\end{aligned}$$

We have

$$g_1 \leq g_2 \leq g_3 \leq \cdots, \tag{1.63}$$

which is monotonically increasing. All these functions are measurable.

Proof.

Since (1),

$$\begin{aligned} \int_X \lim_{n\to\infty} g_n \ \mathrm{d}\mu &= \lim_{n\to\infty}\int_X g_n \ \mathrm{d}\mu &(1.64)\\ &= \liminf_{n\to\infty}\int_X g_n \ \mathrm{d}\mu. &(1.65)\end{aligned}$$

We know that $g_n \leq f_n$ for all $n \in \mathbb{N}$. By (1.5.2), we have

$$\int_X g_n \ \mathrm{d}\mu \leq \int_X f_n \ \mathrm{d}\mu, \tag{1.66}$$

for all $n \in \mathbb{N}$. Then we have

$$\begin{aligned} \int_X \liminf_{n\to\infty} f_n \ \mathrm{d}\mu &= \liminf_{n\to\infty}\int_X g_n \ \mathrm{d}\mu &(1.67)\\ &\leq \liminf_{n\to\infty}\int_X f_n \ \mathrm{d}\mu. &(1.68)\end{aligned}$$

□

1.7 Lebesgue's dominated convergence theorem

$(X, \mathcal{A}, \mu)$, $\mathcal{L}^1 := \left\{f : X \to \mathbb{R} \text{ } \textit{measurable} \middle| \int_X |f|^1 \ \mathrm{d}\mu < \infty\right\}$. For $f \in \mathcal{L}^1(\mu)$, write $f = f^+ - f^-$, where $f^+, f^- \geq 0$. Define $\int_X f \ \mathrm{d}\mu := \int_X f^+ \ \mathrm{d}\mu - \int_X f^- \ \mathrm{d}\mu$.

Theorem 2 (Lebesgue's dominated convergence theorem)**.** *$f_n : X \to \mathbb{R}$ measurable for all $n \in \mathbb{N}$. $f : X \to \mathbb{R}$ with $\overset{n\to\infty}{f(x)}$ for $x \in X$ (μ-a.e.) and $|f_n| \leq g$ with $g \in \mathcal{L}^1(\mu)$ for all $n \in \mathbb{N}$, where g is called integral majorant. Then: we have $f_1, f_2, \dots \in \mathcal{L}^1(\mu)$, $f \in \mathcal{L}^1(\mu)$ and*

$$\lim_{n\to\infty} \int_X f_n \ \mathrm{d}\mu = \int_X f \ \mathrm{d}\mu. \tag{1.69}$$

Proof.

$$|f_n| \leq g \overset{monotonicity}{\Longrightarrow} \int_X g \ \mathrm{d}\mu < \infty \tag{1.70}$$

$$\Longrightarrow f_1, f_2, \dots \in \mathcal{L}^1(\mu) \tag{1.71}$$

$$|f| \leq g \ for \ \mu - \text{a.e.} \Longrightarrow f \in \mathcal{L}^1(\mu) \tag{1.72}$$

We will show $\int_X |f_n - f| \ \mathrm{d}\mu \overset{n\to\infty}{\Longrightarrow} 0$.

$$|f_n - f| \leq |f_n| + |f| \leq 2g \tag{1.73}$$

$$\Longrightarrow h_n := 2g - |f_n - f| \geq 0 \tag{1.74}$$

Hence: $h_n : X \to [0, \infty]$ measurable for all $n \in \mathbb{N}$. Then by (1.6.1),

$$\Longrightarrow \int_X \liminf_{n\to\infty} h_n \ \mathrm{d}\mu \leq \liminf_{n\to\infty} \int_X h_n \ \mathrm{d}\mu \tag{1.75}$$

$$\Longrightarrow \int_X 2g \ \mathrm{d}\mu \leq \int_X 2g \ \mathrm{d}\mu - \limsup_{n\to\infty} \int_X |f_n - f| \ \mathrm{d}\mu \tag{1.76}$$

$$\Longrightarrow 0 \leq \liminf_{n\to\infty} \int_X |f_n - f| \ \mathrm{d}\mu \leq \limsup_{n\to\infty} \int_X |f_n - f| \ \mathrm{d}\mu \leq 0 \tag{1.77}$$

$$\Longrightarrow \tag{1.78}$$

Limits exists and $\lim_{n\to\infty} |f_n - f| \ \mathrm{d}\mu = 0$. We conclude that

$$\tag{1.79}$$

$$0 \leq |\int_X f_n \ \mathrm{d}\mu - \int_X f \ \mathrm{d}\mu| = |\int_X (f_n - f) \ \mathrm{d}\mu| \leq \int_X |f_n - f| \ \mathrm{d}\mu \overset{n\to\infty}{\longrightarrow} 0, \tag{1.80}$$

where the third inequality is due to the integral's triangle inequality.

$$\tag{1.81}$$

$$\Longrightarrow \lim_{n\to\infty} \int_X f_n \ \mathrm{d}\mu = \int_X f \ \mathrm{d}\mu. \tag{1.82}$$

□