Note

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Chapter 1

Introduction to Quantum Optics

1.1 Introduction

- classicaal: classical atom and light
- semiclassical: quantized atom and classical light
- quantum mechanical: quantized atom and light

Light-Atom Interaction Hamiltonian

- classical dipole in eletric field: dipole moment $\overrightarrow{d} = q\overrightarrow{r}$, $U_I = -\overrightarrow{d} \cdot \overrightarrow{E}$. We have

$$\hat{H}_I = -\hat{d} \cdot \overrightarrow{E}(\overrightarrow{v_0}, t), \tag{1.1.1}$$

  where $\hat{d} = q\hat{v}$ is the dipole operator.

- induced atomic dipole

1.2 Light Atom Quantum Evolution

Time Evolution We have the Schrodinger equation (both sides) as

$$i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = (\hat{H}_0 + \hat{H}_I(t))|\Psi(t)\rangle, \tag{1.2.1}$$

where the general ansatz (assumption) is

$$|\Psi(t)\rangle = \sum_n c_n(t)e^{-iE_nt/\hbar}|n\rangle, \tag{1.2.2}$$

and

$$\hat{H}_0|n\rangle = E_n|n\rangle \tag{1.2.3}$$

is the atomic eigenstates. Inserting $|\Psi(t)\rangle$ and $\hat{H}_0|n\rangle$ into Schrodinger equation, we get

$$i\hbar \sum_n \left\{ \dot{c}_n e^{-iE_nt/\hbar}|n\rangle - \frac{iE_n}{\hbar} c_n e^{-iE_nt/\hbar}|n\rangle \right\} = \sum_n \left\{ c_n e^{-iE_nt/\hbar}|n\rangle + c_n e^{-iE_nt/\hbar}\hat{H}_I|n\rangle \right\} \tag{1.2.4}$$

$$\Longrightarrow i\hbar \sum_n \dot{c_n} e^{-iE_nt/\hbar}|n\rangle = \sum_n c_n e^{-iE_nt/\hbar}\hat{H}_I|n\rangle \tag{1.2.5}$$

$$\Longrightarrow i\hbar \dot{c_n} e^{-iE_kt/\hbar}| = \sum_n c_n(t) e^{-iE_nt/\hbar}\langle k\hat{H}_I(t)|n\rangle \tag{1.2.6}$$

$$\Longrightarrow i\hbar \dot{c}_k = \sum_n c_n(t) e^{-iE_{n,k}t/\hbar}\langle k|\hat{H}_I(t)|n\rangle, \tag{1.2.7}$$

where we use

$$\langle k|n\rangle = \delta_{kn}, \tag{1.2.8}$$

$$E_{n,k} = E_n - E_k, \tag{1.2.9}$$

$$\omega_{nk} = (E_n - E_k)/\hbar. \tag{1.2.10}$$

and $\langle k|\hat{H}_I(t)|n\rangle$ is the matrix element.

1.3 Time Dependent Perturbation Theory

Recall the time evolution:

$$i\hbar \dot{c}_k = \sum_n c_n(t) e^{-i\omega_{nk}t}\langle k|\hat{H}_I(t)|n\rangle, \tag{1.3.1}$$

and

$$\omega_{nk} = (E_n - E_k)/\hbar. \tag{1.3.2}$$

Consider the Simplification (Perturbation Theory)

- System only in state $|1\rangle$ at $t = 0 \Longrightarrow c_1|0\rangle = 1$ (only the ground state $|1\rangle$),
- Perturbative treatment of interaction term: weak perturbation $\forall |c_k(t)|^2 << 1$.

We then have

$$i\hbar \dot{c}_k = e^{i\omega_{1k}t}\langle k|\hat{H}_I(t)|1\rangle, \tag{1.3.3}$$

with $c_k(0) = 0$, we obtain:

$$c_k(t) = \frac{1}{i\hbar}\int_0^t e^{-i\omega_{1k}t}\langle k|\hat{H}_I(t')|1\rangle \mathrm{d}t'. \tag{1.3.4}$$

Example 1.3.1 (Sinusoidal perturbation)**.** *Define*

$$\hat{H}(t) = \hat{H}_I e^{-i\omega t}. \tag{1.3.5}$$

Given the figure in the video, we have

$$c_k(T) = \frac{1}{i\hbar}\int_0^T e^{i\Delta\omega t}\langle k|\hat{H}_I|1\rangle \mathrm{d}t \tag{1.3.6}$$

$$\Longrightarrow \textit{Transition probability } P_{k1}(T) = |c_k(T)|^2 = \frac{1}{\hbar^2}|\langle k|\hat{H}_I|1\rangle|^2 Y(\Delta\omega, T), \tag{1.3.7}$$

with

$$Y(\Delta\omega, T) = \frac{\sin^2(\Delta\omega T/2)}{(\Delta\omega/2)^2} \tag{1.3.8}$$

$$\sim \mathrm{sinc}^2 x, \tag{1.3.9}$$

where $\Delta\omega = \omega - \omega_{1k}$ *is the detwining.*

Let's take a look at the sinc function $Y(\Delta\omega, T) = \mathrm{sinc}^2 x$. Transition for $\Delta\omega \leq \frac{2\pi}{T}$, we have $\Delta\omega \cdot T \leq 2\pi$, which implies

$$\Delta E \cdot T \leq h, \tag{1.3.10}$$

which is the time-frequency uncertainty. (The expression in the video seems wrong, so I make corrections abrove.) We have the following case

$$\frac{1}{2\pi T} Y(\Delta\omega, T) \overset{T\to\infty}{\to} \delta(\Delta\omega), \tag{1.3.11}$$

then we have

$$P_{k1}(T \to \infty) = \frac{2\pi}{\hbar^2}|\langle k|\hat{H}_I|i\rangle|^2 \delta(\Delta\omega) T. \tag{1.3.12}$$

Fermi's Golden Rule $|k\rangle$ Quasi continuum of final states. We have the transition probability

$$P_{k1} = \Gamma_{k1} T, \tag{1.3.13}$$

where

$$\Gamma_{k1} = \frac{2\pi}{\hbar}|\langle k|\hat{H}_I|1\rangle|^2 \rho(E_k = E_1 + \hbar\omega) \tag{1.3.14}$$

is called the Femi's Golden Rule,

$$|\langle k|\hat{H}_I|1\rangle|^2 \tag{1.3.15}$$

is the coupling strength $\propto E_0^2$ and $\propto I$,

$$\rho(E_k = E_1 + \hbar\omega) \tag{1.3.16}$$

is the density states which is number of availble final states to the system,

$$\Gamma_{k1} \hat{=} Transition\ Rate = \frac{\mathrm{d}P_{k1}}{\mathrm{d}T}, \tag{1.3.17}$$

and density states

$$\rho(E) = \frac{\mathrm{d}N}{\mathrm{d}E}, \tag{1.3.18}$$

where ΔN is the number of states in an energy interval ΔE around energy E_k and we let ΔE approaches 0.

1.4 Two Level Atom (TLA)

Given by the figure, in state $|1\rangle$, we have $E_1 = \hbar\omega_1$ and in state $|2\rangle$, we have $E_2 = \hbar\omega_2$ and $E_2 - E_1 = \hbar(\omega_2 - \omega_1) = \omega_{21}$. We have the Hamiltonian

$$\hat{H} = \hat{H}_0 - \hat{d} \cdot E(t), \tag{1.4.1}$$

where

$$E(t) = \varepsilon E_0 \cos(\omega t), \tag{1.4.2}$$

where ε is the polarization vector, E_0 is the field amplitude, and ω is the frequency of the light field.

Ansatz for Solving TLA We have

$$|\Psi(t)\rangle = c_1(t)e^{-i\omega_1 t}|1\rangle + c_2(t)e^{-i\omega_2 t}|2\rangle. \tag{1.4.3}$$

Time Evolution Amplitude We have

$$\dot{c_1}(t) = i\frac{d_{12}^{\varepsilon} E_0}{\hbar} e^{-\omega_{21}} \cos(\omega t) c_2(t) \tag{1.4.4}$$

$$\dot{c_2}(t) = i\frac{d_{12}^{\varepsilon} E_0}{\hbar} e^{+\omega_{21}} \cos(\omega t) c_1(t), \tag{1.4.5}$$

where

$$\begin{aligned} d_{12}^{\varepsilon} &= \langle 1|\hat{d} \cdot \varepsilon|2\rangle && (1.4.6)\\ &= \langle 1|\hat{d}|2\rangle \cdot \varepsilon && (1.4.7)\\ &= \langle 1|\hat{d}_x|2\rangle \cdot \varepsilon_x + \langle 1|\hat{d}_y|2\rangle \cdot \varepsilon_y + \langle 1|\hat{d}_z|2\rangle \cdot \varepsilon_z. && (1.4.8) \end{aligned}$$

is the Dipole Matrix Element, which is the atomic property and we assume it's real. We also define

$$\Omega_0 = \frac{d_{12}^{\varepsilon} E_0}{\hbar} \tag{1.4.9}$$

as the Rubi frequency.

Time Evolution Using Euler' form, we have

$$\dot{c_1}(t) = i\frac{\Omega_0}{2} e^{-\omega_{21}}(e^{i\omega t} + e^{-i\omega t})c_2(t) \tag{1.4.10}$$

$$\dot{c_2}(t) = i\frac{\Omega_0}{2} e^{+\omega_{21}}(e^{i\omega t} + e^{-i\omega t})c_1(t) \tag{1.4.11}$$

by

$$\cos\alpha = \frac{1}{2}(e^{i\alpha} + e^{-i\alpha}) \tag{1.4.12}$$

and

$$e^{i\alpha} = \cos\alpha + i\sin\alpha. \tag{1.4.13}$$

Rotating Wave Approximation We have

$$\dot{c_1}(t) = i\frac{\Omega_0}{2}(e^{+i(\omega-\omega_{21})t} + e^{-i(\omega+\omega_{21})t})c_2(t) \tag{1.4.14}$$

$$\dot{c_2}(t) = i\frac{\Omega_0}{2}(e^{-i(\omega-\omega_{21})t} + e^{+i(\omega+\omega_{21})t})c_1(t), \tag{1.4.15}$$

and we ignore the sum frequency term and get

$$\dot{c_1}(t) = i\frac{\Omega_0}{2} e^{+i(\omega-\omega_{21})t}c_2(t) \tag{1.4.16}$$

$$\dot{c_2}(t) = i\frac{\Omega_0}{2} e^{-i(\omega-\omega_{21})t}c_1(t), \tag{1.4.17}$$

which is a good approcimation for detwining $\delta = \omega - \omega_{21} \approx 0$. We introduce

$$\tilde{c_1}(t) = c_1(t)e^{-i\frac{\delta}{2}t} \tag{1.4.18}$$

$$\tilde{c_2}(t) = c_2(t)e^{+i\frac{\delta}{2}t}. \tag{1.4.19}$$

$$\tag{1.4.20}$$

Ansatz Wavefunctions for TLA Whole time evolution in state amplitudes

$$|\Psi(t)\rangle = c_1'(t)|1\rangle + c_2'(t)|2\rangle. \tag{1.4.21}$$

Time evolution when field is off

$$|\Psi(t)\rangle = c_1'(0)e^{-i\omega_1 t}|1\rangle + c_2'(0)e^{-i\omega_2}|2\rangle. \tag{1.4.22}$$

However, this is boring. We chose different ansatz as

$$|\Psi(t)\rangle = c_1(t)e^{-i\omega_1 t}|1\rangle + c_2(t)e^{-i\omega_2 t}|2\rangle \tag{1.4.23}$$

$$\iff |\Psi(t)\rangle = c_1(t)|1\rangle + c_2(t)e^{-i\omega_{21} t}|2\rangle, \tag{1.4.24}$$

where $c_1(t)$ and $c_2(t)$ capture time evolution on top of eigenstate evolution! We now have

$$|\Psi(t)\rangle = c_1(t)|1\rangle + c_2(t)e^{-i\omega_{21} t}|2\rangle, \tag{1.4.25}$$

which is called the rotating frame of atom. We also have Rotating frame of light field as

$$|\Psi(t)\rangle = \tilde{c_1}(t)|1\rangle + \tilde{c_2}(t)e^{-i\omega t}|2\rangle, \tag{1.4.26}$$

where ω is the light frequency, $\tilde{c_1}$ and $\tilde{c_2}$ describe time evolution on top of fast light field oscilation.

Solving the TLA Dynamics We have the following equations:

$$\frac{d}{dt}\begin{pmatrix} \tilde{c_1}(t) \\ \tilde{c_2}(t) \end{pmatrix} = \frac{i}{2}\begin{pmatrix} -\delta & \Omega_0 \\ \Omega_0 & +\delta \end{pmatrix}\begin{pmatrix} \tilde{c_1}(t) \\ \tilde{c_2}(t) \end{pmatrix}. \tag{1.4.27}$$

Considering the simplest case $\delta = 0$

$$\frac{d}{dt}\tilde{c_1}(t) = \frac{i}{2}\Omega_0\tilde{c_2}(t) \tag{1.4.28}$$

$$\frac{d}{dt}\tilde{c_2}(t) = \frac{i}{2}\Omega_0\tilde{c_1}(t). \tag{1.4.29}$$

Take time dirivative of the first equation, then we have

$$\ddot{c_1}(t) = -\frac{\Omega_0^2}{4}\tilde{c_1}(t), \tag{1.4.30}$$

the solutions of which are

$$\tilde{c_1}(t) = \cos(\Omega_0 t/2) \tag{1.4.31}$$

$$\tilde{c_2}(t) = i\sin(\Omega_0 t/2) \tag{1.4.32}$$

for $\tilde{c_1}(0) = 1$ and $\tilde{c_2}(0) = 0$. Also we can obtain the excited state probability as

$$P_2(t) = |c_2(t)|^2 \tag{1.4.33}$$

$$= |\tilde{c_2}(t)|^2. \tag{1.4.34}$$

Rabi Oscillations (Resonant Case) Nonlinear Response can be seen from the figure.

General Rabi Oscillations (with detuning) Given the figurem.

$$|\tilde{c_2}(t)|^2 = \frac{\Omega_0^2}{\Omega} \sin^2\left(\frac{1}{2}\Omega t\right) \tag{1.4.35}$$

$$= \frac{\Omega_0^2}{2\Omega^2} \left\{1 - \cos(\Omega t)\right\}, \tag{1.4.36}$$

where $\Omega = \sqrt{\Omega_0^2 + \delta^2}$ is the effective Rabi frequency.

Interesting Special Cases a) Pi-Puls $\Omega_0 \tau = \pi$: swap population

$$|1\rangle \to i|2\rangle \tag{1.4.37}$$

$$|2\rangle \to i|1\rangle. \tag{1.4.38}$$

b) 2Pi-Puls $\Omega_0 \tau = 2\pi$: flip the sign

c) Pi/2-Puls $\Omega_0 \tau = \pi/2$: superposition state

1.5 Oscillating Dipoles

Atomic Eigenstates

$$|\Psi_{nlm}(t)\rangle = e^{-iE_{nlm}t/\hbar}|\Psi_{nlm}(0)\rangle, \tag{1.5.1}$$

$$\hat{H}_0|\Psi_{nlm}(0)\rangle = E_{nlm}|\Psi_{nlm}\rangle, \tag{1.5.2}$$

and the electron density is

$$\rho(r,\theta,\phi) = |\Psi(r,\theta,\phi,t=0)^2|. \tag{1.5.3}$$

Atomic Dipole Calculate (Oscillating) Dipole Moment for Atomic Eigenstate. We denote $|1\rangle = |\Psi_{nlm}\rangle$. We have

$$d(t) = \langle 1(t)|\hat{d}|1(t)\rangle \tag{1.5.4}$$

$$= \langle \hat{d}|1\rangle \tag{1.5.5}$$

$$= -e\langle 1|\hat{r}|1\rangle. \tag{1.5.6}$$

Then,

$$-e\langle 1|\hat{r}|1\rangle = -e\langle 1|\hat{P}\hat{P}^{-1}\hat{r}\hat{P}\hat{P}^{-1}|1\rangle \tag{1.5.7}$$

$$= +e\langle 1|\hat{r}|1\rangle, \tag{1.5.8}$$

which implies

$$\langle 1|\hat{r}|1\rangle = 0. \tag{1.5.9}$$

Atomic Dipole - Superposition States Calculate (Oscillating) Dipole Moment for Atomic Superposition State

$$|\Psi(0)\rangle = \frac{1}{\sqrt{2}}(|1\rangle + i|2\rangle). \tag{1.5.10}$$

Evolution

$$|\Psi(t)\rangle = \frac{1}{\sqrt{2}}(|1\rangle + ie^{-i\omega_{21}t}|2\rangle). \tag{1.5.11}$$

We have

$$d(t) = \langle\Psi(t)|\hat{d}|\Psi(t)\rangle \tag{1.5.12}$$
$$= \frac{1}{2}\left\{\langle 1|\hat{d}|1\rangle + \langle 2|\hat{d}|2\rangle + ie^{-i\omega_{21}t}\langle 1|\hat{d}|2\rangle - ie^{-i\omega_{21}t}\langle 2|\hat{d}|1\rangle\right\} \tag{1.5.13}$$
$$= d_{12} i\frac{1}{2}\left\{e^{-i\omega_{21}t} - e^{i\omega_{21}t}\right\} \tag{1.5.14}$$
$$= d_{12}\sin(\omega_{21}t), \tag{1.5.15}$$

where d_{12} is the dipole moment amplitude, ω_{21} is the natural resonance frequency.

Electron Density - Superposition States Calculate Electron Probability Density for Superposition State. The superposition state is

$$\Psi(r,t) = \frac{1}{\sqrt{2}}\left(\Psi_1(r) + ie^{-i\omega_{21}t}\Psi_2 D(t)\right). \tag{1.5.16}$$

The Electron Probability Density is

$$\rho(r,t) = |\Psi(r,t)|^2 \tag{1.5.17}$$
$$= \Psi^*\Psi \tag{1.5.18}$$
$$= \frac{1}{2}\left\{|\Psi_1(r)|^2 + |\Psi_2(r)|^2 + 2\mathrm{Re}\left(ie^{-i\omega_{21}t}\Psi_1^*(r)\Psi_2(r)\right)\right\}, \tag{1.5.19}$$

where $2\mathrm{Re}\left(ie^{-i\omega_{21}t}\Psi_1^*(r)\Psi_2(r)\right)$ is the interference term.

Examples This is shown by animation and figure in the video.

1.6 The Bloch Sphere

General Two-Level State

- General State Description

$$|\Psi\rangle = c_1'|1\rangle + c_2'|2\rangle \tag{1.6.1}$$
$$\text{Up to a global phase} \tag{1.6.2}$$
$$= |c_1'||1\rangle + e^{i\phi}|c_2'|2\rangle \tag{1.6.3}$$

satisfying $|c_1'|^2 + |c_2'|^2 = 1$.

- Alternative way

$$|\Psi\rangle = \cos(\theta/2)|1\rangle + e^{i\phi}\sin(\theta/2)|2\rangle, \tag{1.6.4}$$

since $\cos(\theta/2)^2 + \sin(\theta/2)^2 = 1$.

Geometric Description - Bloch Sphere We then have

$$|\Psi\rangle = \cos(\theta/2)|1\rangle + e^{i\phi}\sin(\theta/2)|2\rangle \tag{1.6.5}$$

with $0 \leq \theta \leq \pi$ as the latitude and $0 \leq \phi \leq 2\pi$ as the longitude. This is the Bloch Sphere representation. The definition of θ and θ and their ranges are different from my familiar coordinate system.

Special States on Bloch Sphere

Analogy to Spin -1/2 States Is is shown in the figure.

1.7 Density Operator and Density Matrix

The Problem How do we describe ”imperfect state preparation” in an experiment? For example, $50\%|1\rangle$ and $50\%|2\rangle$. We may think of

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle + |2\rangle\right).??? \tag{1.7.1}$$

This is $100\%|\Psi\rangle$ pure state. We need stable relative phase between the two states!

Optical Analogy - Controlled Phase The double slit problem is shown in the video.

Intensity on Detection Screen:

$$I \propto |E|^2 = |E_1 + e^{i\phi}E_2|^2 \tag{1.7.2}$$
$$= |E_1|^2 + |E_2|^2 + 2\mathrm{Re}\left(E_1E_2e^{i\phi}\right). \tag{1.7.3}$$

As ϕ varies, Interference pattern ”washed out”!

We need new formalism to describe mixed states!(imperfect state preparation, spontaneous emission,...)

Density Operator and Matrix The description of mixed states can be handled by the density operator (matrix) formalism!

- Density operator (hermitian)

$$\hat{\rho} = \sum p_i |\Psi_i\rangle\langle\Psi_i| \tag{1.7.4}$$

$$\hat{\rho} = I\hat{\rho}I \tag{1.7.5}$$

$$= \sum_{i,j} |i\rangle\langle i|\hat{\rho}|j\rangle\langle j| \tag{1.7.6}$$

$$= \rho_{11}|1\rangle\langle 1| + \rho_{12}|1\rangle\langle 2| + \rho_{21}|2\rangle\langle 1| + \rho_{22}|2\rangle\langle 2|, \tag{1.7.7}$$

  where $I = \sum_i |i\rangle\langle i|$.

- Density matrix

$$\rho = \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix}, \tag{1.7.8}$$

  where ρ_{11} and ρ_{22} are the populations, ρ_{12} and ρ_{21} are the coherence. Since ρ is hermitian, we have

$$\rho_{12} = \rho_{21}^*. \tag{1.7.9}$$

Example 1.7.1 (Example: Density Matrix of Pure State)**.** *We have*

$$|\Psi\rangle = |c_1||1\rangle + e^{i\phi}|c_2||2\rangle. \tag{1.7.10}$$

The corresponding density operator of the ***pure state*** *is $\hat{\rho} = |\Psi\rangle\langle\Psi|$. Then the corresponding density matrix is*

$$\rho = \begin{bmatrix} |c_1|^2 & |c_1||c_2|e^{-i\phi} \\ |c_1||c_2|e^{i\phi} & |c_2|^2 \end{bmatrix}, \tag{1.7.11}$$

where $|c_1||c_2|e^{-i\phi}$ and $|c_1||c_2|e^{i\phi}$ are relative phase between states $|1\rangle$ and $|2\rangle$.

specific example:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle + |2\rangle\right), \tag{1.7.12}$$

so

$$\rho = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}. \tag{1.7.13}$$

Example 1.7.2 (Example: Fully Incoherent Mixture)**.**

$$\hat{\rho} = \frac{1}{2}|1\rangle\langle 1| + \frac{1}{2}|2\rangle\langle 2| \tag{1.7.14}$$

with

$$\rho = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}, \tag{1.7.15}$$

where vanishingly coherence and the phase varies from 0 *to* 2π*. It means that we did not control phase.*

Useful Facts

- Expectation values: $\langle \hat{A} \rangle = \mathrm{Tr}(\hat{\rho}\hat{A}) = \mathrm{Tr}(\rho A)$
- Time evolution (von Neumann equation)

$$i\hbar \frac{\partial \hat{\rho}}{\partial t} = [\hat{H}, \hat{\rho}] \tag{1.7.16}$$

- Pure state: $\mathrm{Tr}(\rho^2) = 1$
- Mixed states: $\mathrm{Tr}(\rho^2) < 1$

1.8 Optical Bloch Equations

Time Evolution of Density Matrix How to calculate time evolution of density matrix?

$$i\hat{\hbar} \frac{\partial \hat{\rho}}{\partial t} = [\hat{H}, \hat{\rho}]. \tag{1.8.1}$$

Assume pure state

$$\begin{aligned}
\frac{d}{dt}\rho_{11} &= \frac{d}{dt}(c_1 c_1^*) && (1.8.2)\\
&= \dot{c}_1 c_1^* + c_1 \dot{c}_1^* && (1.8.3)\\
&= i\frac{\Omega_0}{2}\left(e^{i\delta t}\rho_{21} - e^{-i\delta t}\rho_{12}\right) && (1.8.4)
\end{aligned}$$

Transformation to rotality frame of light

$$= i\frac{\Omega_0}{2}\left(\tilde{\rho}_{21} - \tilde{\rho}_{12}\right), \tag{1.8.5}$$

where

$$\dot{c}_1(t) = i\frac{\Omega_0}{2}e^{+i\delta t}c_2(t) \tag{1.8.6}$$
$$\dot{c}_2(t) = i\frac{\Omega_0}{2}e^{-i\delta t}c_1(t) \tag{1.8.7}$$
$$\tilde{\rho}_{12} = e^{-i\delta t}\rho_{12} \tag{1.8.8}$$
$$\tilde{\rho}_{21} = e^{+i\delta t}\rho_{21}. \tag{1.8.9}$$

Other elements obtained in analogy!

$$\frac{d}{dt}\rho_{11} = i\frac{\Omega_0}{2}\left(\tilde{\rho}_{21} - \tilde{\rho}_{12}\right) \tag{1.8.10}$$
$$\frac{d}{dt}\rho_{22} = i\frac{\Omega_0}{2}\left(\tilde{\rho}_{12} - \tilde{\rho}_{21}\right) \tag{1.8.11}$$
$$\frac{d}{dt}\tilde{\rho}_{12} = -i\delta\tilde{\rho}_{12} + i\frac{\Omega_0}{2}\left(\rho_{22} - \rho_{11}\right) \tag{1.8.12}$$
$$\frac{d}{dt}\tilde{\rho}_{21} = +i\delta\tilde{\rho}_{21} + i\frac{\Omega_0}{2}\left(\rho_{11} - \rho_{22}\right). \tag{1.8.13}$$

Noting that $\tilde{\rho}_{12} = \tilde{\rho}_{21}$ due to hermitian matrix, the third and the forth equations are the same. So we have

$$\frac{d}{dt}\rho_{11} = i\frac{\Omega_0}{2}\left(\tilde{\rho}_{21} - \tilde{\rho}_{12}\right) \tag{1.8.14}$$
$$\frac{d}{dt}\rho_{22} = i\frac{\Omega_0}{2}\left(\tilde{\rho}_{12} - \tilde{\rho}_{21}\right) \tag{1.8.15}$$
$$\frac{d}{dt}\tilde{\rho}_{12} = -i\delta\tilde{\rho}_{12} + i\frac{\Omega_0}{2}\left(\rho_{22} - \rho_{11}\right). \tag{1.8.16}$$

Optical Bloch Equations with Damping Phenomenological damping and spontaneous emission in the figure. Combine the decay, we have

$$\frac{d}{dt}\rho_{11} = i\frac{\Omega_0}{2}\left(\tilde{\rho}_{21} - \tilde{\rho}_{12}\right) + \gamma\rho_{22} \tag{1.8.17}$$
$$\frac{d}{dt}\rho_{22} = i\frac{\Omega_0}{2}\left(\tilde{\rho}_{12} - \tilde{\rho}_{21}\right) - \gamma\rho_{22} \tag{1.8.18}$$
$$\frac{d}{dt}\tilde{\rho}_{12} = -i\delta\tilde{\rho}_{12} + i\frac{\Omega_0}{2}\left(\rho_{22} - \rho_{11}\right) - (\gamma/2)\tilde{\rho}_{12}. \tag{1.8.19}$$

We now define the inversion $w = \rho_{22} - \rho_{11}$. We have Optical Bloch Equations with Damping

$$\frac{d}{dt}\tilde{\rho}_{21} = -\left(\gamma/2 - i\delta\right)\tilde{\rho}_{21} - \frac{i\omega\Omega_0}{2} \tag{1.8.20}$$
$$\frac{d}{dt}\omega = -\gamma\left(\omega + 1\right) - i\Omega_0\left(\tilde{\rho}_{21} - \tilde{\rho}_{12}\right) \tag{1.8.21}$$

in the Density Matrix Form.

1.9 Optical Bloch Equations - Dynamics and Steady State

Dynamical Evolution of System Shown in the figure in the picture.

Steady State Solution Conditions: $\frac{d}{dt}\tilde{\rho}_{21} = 0$ and $\frac{d}{dt}\omega = 0$. Then we have the solutions

$$\omega = -\frac{1}{1+S} \tag{1.9.1}$$

$$\tilde{\rho}_{21} = \frac{2\Omega_0}{2\left(\gamma/2 - -\delta\right)\left(1+S\right)} \tag{1.9.2}$$

$$S = \frac{\Omega_0^2/2}{\delta^2 + \gamma^2/4} = \frac{S_0}{1+4\delta^2/\gamma^2} \tag{1.9.3}$$

$$S_0 = \frac{2\Omega_0^2}{\gamma^2} = \frac{I}{O_{sat}}, \tag{1.9.4}$$

where S is called the saturation parameter, S_0 is called resonant saturation parameter.

Limiting Cases:

- $S \leq\leq 1$: $w \to -1$ where $w = \rho_{22} - \rho_{11}$. Atom is mainly in ground state.
- $S \geq\geq 1$: $S \to \infty$, $w \to 0$.
- Excited State Population:

$$\rho_{22} \tag{1.9.5}$$

$$\overset{\text{Combine with } \rho_{22}+\rho_{11}=1}{=} \frac{1}{2}(1+w) \tag{1.9.6}$$

$$= \frac{S}{2(1+S)} \tag{1.9.7}$$

$$= \frac{S_0/2}{1+S_0+4\delta^2/\gamma^2} \tag{1.9.8}$$

$$\xrightarrow{S_0\to\infty,\delta=0} \frac{1}{2}. \tag{1.9.9}$$

- Photon Scattering Rate: $\Gamma_{ph} = \gamma\rho_{22} = \frac{\gamma}{2}\frac{S_0}{1+S_0+4\delta^2/\gamma^2}$. $\Gamma_{ph} \to \gamma/2$ for $S_0 \to \infty$ and $\delta = 0$. We rewrite it as

$$\Gamma_{ph} = \left(\frac{S_0}{1+S_0}\right)\left(\frac{\gamma/2}{1+4\delta^2/\gamma'^2}\right) \tag{1.9.10}$$

$$\gamma' = \gamma\sqrt{1+S_0}. \tag{1.9.11}$$

It has a figure in the video. The saturation brodening is shown in the figure.

1.10 Lambert-Beer Law

Attenuation of Light It is shown in the figure.

Scattered Light from Slab of Atoms scattered light power by slab of length dz

$$dP_{sc} = \Gamma_{ph} \times nAdz \times \hbar\omega, \tag{1.10.1}$$

where Γ_{ph} is the single atom photon scattering rate, $\hbar\omega$ is the energy of single atom, $nAdz$ is the number of atoms. Then we have

$$\frac{dP_{sc}}{dz} = \Gamma_{ph} \times nA \times \hbar\omega. \tag{1.10.2}$$

Scattered Light from Slab of Atoms Energy conservation requires

$$\frac{dP}{dz} = -\frac{dP_{sc}}{dz} \tag{1.10.3}$$

$$\frac{dP}{dz} = \frac{dI}{dI}A. \tag{1.10.4}$$

Put every thing together:

$$\frac{dI}{dz} = -\Gamma n\hbar\omega. \tag{1.10.5}$$

We have

$$\frac{dI(z)}{dz} = -n\sigma I(z), \tag{1.10.6}$$

where σ is the atomic scattering cross section.

Lambert-Beer Law (no saturation) We compute the solutions

$$I(z) = I(0)e^{-n\sigma z}, \tag{1.10.7}$$

which is the Lambert-Beer Law of Absorption.

Laser induced Fluorescence Shown in a video.

1.11 Bloch Vector

Density Matrix Revisited Density Matrix of TLA

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \tag{1.11.1}$$

Density Matrix hermitian

$$\rho = \rho^\dagger = (\rho^T)^*, \tag{1.11.2}$$

so we have

$$\rho = \begin{pmatrix} \rho_{11} & \mathrm{Re}\rho_{12} + i\mathrm{Im}\rho_{12} \\ \mathrm{Re}\rho_{12} - i\mathrm{Im}\rho_{12} & \rho_{22} \end{pmatrix}. \tag{1.11.3}$$

Pauli matrices are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \sigma_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.11.4}$$

The decomposition of Density matrix into Pauli matrices

$$\rho = \frac{1}{2}\left(I + b_x\sigma_x + b_y\sigma_y + b_z\sigma_z\right), \tag{1.11.5}$$

where $b_x, b_y, b_y \in \mathbb{R}$.

Bloch Vector We have the density matrix in rotating frame of light

$$\tilde{\rho} = \begin{pmatrix} \rho_{11} & \tilde{\rho}_{12} \\ \tilde{\rho}_{21} & \rho_{22} \end{pmatrix}, \tag{1.11.6}$$

where $\tilde{\rho}_{12} = \rho_{12}e^{-i\omega t}$. We use following sign convention and have

$$\tilde{\rho} = \frac{1}{2}\left(I + u\sigma_x - v\sigma_y - w\sigma_z\right), \tag{1.11.7}$$

and the bloch vector is defined as

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}. \tag{1.11.8}$$

It can be easily shown that

$$u = 2\mathrm{Re}(\tilde{\rho}_{12}) = \tilde{\rho}_{12} + \tilde{\rho}_{12}^* \tag{1.11.9}$$

$$v = 2\mathrm{Im}(\tilde{\rho}_{12}) = i(\tilde{\rho}_{12}^* + \tilde{\rho}_{12}) \tag{1.11.10}$$

$$w = \rho_{22} - \rho_{11}, \tag{1.11.11}$$

$$\tag{1.11.12}$$

where u is the dispersive component, v is the absorption component and w is the inversion.

Bloch vector can be used to describe any state of TLA density matrix!

Properties of Bloch Vector

- Mixed State: $u^2 + v^2 + w^2 < 1$
- Pure State: $u^2 + v^2 + w^2 = 1$

1.12 Understanding Bloch Vector

What physical behaviour do the components stand for?

- $w = -1$ atom in ground state. $w = +1$ atom in excited state.
- What about u, v?

$$\langle \hat{d}_i(t) = \mathrm{Tr}(\hat{\rho}\hat{d}) \tag{1.12.1}$$

$$= \mathrm{Tr}\left[\begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{12}^* & \rho_{22} \end{pmatrix}\begin{pmatrix} 0 & d_{12}^i \\ d_{12}^i & 0 \end{pmatrix}\right], \tag{1.12.2}$$

where $d_{12}^x = \langle 1| - q\hat{x}|2\rangle$.

Written in the vector form, we have

$$\langle \hat{d}\rangle(t) = d_{12}\left(\rho_{12} + \rho_{12}^*\right) \tag{1.12.3}$$

$$= d_{12}\left(\tilde{\rho}_{12}e^{i\omega t} + \tilde{\rho}_{12}^* e^{-i\omega t}\right) \tag{1.12.4}$$

$$= d_{12}\left[u\cos(\omega t) - v\sin(\omega t)\right], \tag{1.12.5}$$

where we use $\rho_{12} = \tilde{\rho}_{12}e^{i\omega t}$, u denotes in phase and v denotes $90°$ out of phase component.

Reminder: $E(t) = \epsilon E_0\cos(\omega t)$.

- Which component responsible for absorption/emission? We have a figure in the video to show the classical picture.

Average absorbed power per atom (classical ensemble average)

$$\langle \frac{dW}{dt} = \epsilon E_0\cos(\omega t)\langle -q\frac{dr}{dt}\rangle \tag{1.12.6}$$

$$= \epsilon E_0\cos(\omega t)\langle \dot{d}\rangle. \tag{1.12.7}$$

Quantum mechanical analogue (Ehrenfest)

$$\langle \frac{dW}{dt}\rangle = \epsilon E_0\cos(\omega t)\langle \dot{d}\rangle \tag{1.12.8}$$

$$\langle \hat{d}\rangle(t) = d_{12}[u\cos(\omega t) - v\sin(\omega t)]. \tag{1.12.9}$$

$$\langle \frac{dW}{dt}\rangle = -d_{12}\cdot\epsilon E_0\omega(u\cos(\omega t)\sin(\omega t) + v\sin(\omega t)^2) \tag{1.12.10}$$

$$\overline{\langle \frac{dW}{dt}\rangle} = \frac{1}{T}\int dt\langle \frac{dW}{dt}\rangle \tag{1.12.11}$$

$$= -\frac{d_{12}\cdot\epsilon E_0\omega v}{2} \tag{1.12.12}$$

$$= -\hbar\frac{d_{12}\epsilon E_0}{\hbar}\omega\frac{v}{2} \tag{1.12.13}$$

$$= -\hbar\Omega_0\omega\frac{v}{2}, \tag{1.12.14}$$

which is the absorption.

1.13 Optical Bloch Equations using Bloch Vector

1.14 Interlude: The Mach-Zehnder Interferometer

1.15 Ramsey Interferometer

1.16 Review: QM of the Harmonic Oscillator

[SZQ: 2023.04.20: I have understandard the content in this video.]

1.17 Wave equation and energy density of classical radiation field

This section is also known as the review of Maxwell equations vector potentials.

Fundamentaals Maxwell equations in free space

$$\nabla \cdot E = 0, \nabla \times E = -\frac{\partial B}{\partial t} \tag{1.17.1}$$

$$\nabla \cdot B = 0, \nabla \times B = \frac{1}{c^2}\frac{\partial E}{\partial t}. \tag{1.17.2}$$

Lemma 1.17.1 (Coulomb Gauge). *Considering Coulomb Gauge, we have*

$$\nabla \cdot A = 0. \tag{1.17.3}$$

Then we can express the eletric field and magnetic field in terms of the vector potential

$$B(r,t) = \nabla \times A(r,t) \tag{1.17.4}$$

$$E(r,t) = -\frac{\partial A(r,t)}{\partial t}. \tag{1.17.5}$$

Lemma 1.17.2 (Wave equation). *Considering Coulomb Gauge, the wave equation is*

$$\nabla^2 A - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} A = 0. \tag{1.17.6}$$

Proof. Using 1.17.1 and the forth equation in 1.17.1, we have

$$\nabla \times B = \nabla \times (\nabla \times A(r,t)), \tag{1.17.7}$$

and

$$\nabla \times B = \frac{1}{c^2}\frac{\partial E}{\partial t} \tag{1.17.8}$$

$$= -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}A(r,t). \tag{1.17.9}$$

So we have

$$\nabla \times (\nabla \times A) = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}. \tag{1.17.10}$$

Then use the rule in vector Calculus

$$\nabla \times (\nabla \times A) = \nabla(\nabla \cdot A) - \Delta A. \tag{1.17.11}$$

Use lemma 1.17.1, we then have

$$-\Delta A = -\frac{1}{c^2}\frac{\partial^2}{}\partial t^2 A \tag{1.17.12}$$

$$= -\nabla^2 A. \tag{1.17.13}$$

So we have

$$\nabla^2 A - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = 0. \tag{1.17.14}$$

□

Solutions of Wave Equation

Lemma 1.17.3 (Solutions of Wave Equation)**.** *Plane waves:*

$$\boldsymbol{A}_{\boldsymbol{k},\alpha} = \epsilon_{\boldsymbol{k},\alpha} A_{\boldsymbol{k},\alpha} \exp\left[i(\boldsymbol{kr} - \omega_k t)\right], \tag{1.17.15}$$

where $\epsilon_{\boldsymbol{k},\alpha}$ is polarization, $A_{\boldsymbol{k},\alpha}$ is complex amplitude, $|k| = \frac{2\pi}{\lambda}$ is wavenumber i.e., the magnitude of the wave vector, $\boldsymbol{k}$ is the wave vector, $\omega_k = ck$.

Which wave vectors are possible? (a). in finite space, $\mathbf{k}$ distributed continuous; (b). finite box of length L, $\mathbf{k}$ distributed discretely (periodic boundary conditions)

$$k_x = \frac{2\pi}{L}n_x, k_y = \frac{2\pi}{L}n_y, k_z = \frac{2\pi}{L}n_z \tag{1.17.16}$$

$$\mathbf{A}(\mathbf{r},t) = \sum_{\mathbf{k},\alpha} \epsilon_{\mathbf{k},\alpha}\left(A_{\mathbf{k},\alpha}\exp[i(\mathbf{kr} - \omega_k t)] + A^*_{\mathbf{k},\alpha}\exp[-i(\mathbf{kr} - \omega_k t)]\right) \tag{1.17.17}$$

$$\mathbf{E}(\mathbf{r},t) = -\frac{\partial}{\partial t}\mathbf{A}(r,t) = \sum_{\mathbf{k},\alpha} \epsilon_{\mathbf{k},\alpha} i\omega_k \left[A_{\mathbf{k},\alpha}\exp[i(\mathbf{kr} - \omega_k t)] - A^*_{\mathbf{k},\alpha}\exp[-i(\mathbf{kr} - \omega_k t)]\right] \tag{1.17.18}$$

$$\mathbf{B}(\mathbf{r},t) = \sum_{\mathbf{k},\alpha} i(\mathbf{k} \times \epsilon_{\mathbf{k},\alpha})\left[A_{\mathbf{k},\alpha}\exp[i(\mathbf{kr} - \omega_k t)] - A^*_{\mathbf{k},\alpha}\exp[-i(\mathbf{kr} - \omega_k t)]\right]. \tag{1.17.19}$$

[SZQ: 2023.04.20: The complex conjugate term is used to eliminate the imaginary part.]

Total Energy of Radiation Field Total energy of radiation field in volume $V = L^3$. [SZQ: 2023.04.20: The total energy is the integration of the electric density and magenetic density over the volume.]

The electric density is

$$\frac{1}{2}\varepsilon_0 E(r,t)^2. \tag{1.17.20}$$

The magenetic density is

$$\frac{1}{2\mu_0} B(r,t)^2. \tag{1.17.21}$$

Then the total energy of radiation field in volume $V = L^3$ is

$$\begin{aligned} H &= \frac{1}{2}\int_V dV \left[\varepsilon_0 E(r,t)^2 + \frac{1}{\mu_0} B(r,t)^2\right] && (1.17.22)\\ &= \sum_{\mathbf{k},\alpha} \varepsilon_0 V \omega_k^2 \left[A_{\mathbf{k},\alpha} A^*_{\mathbf{k},\alpha} + A^*_{\mathbf{k},\alpha} A_{\mathbf{k},\alpha}\right] && (1.17.23)\\ &= \sum_{\mathbf{k},\alpha} E_{\mathbf{k},\alpha}, && (1.17.24) \end{aligned}$$

where

$$E_{\mathbf{k},\alpha} = \varepsilon_0 V \omega_k^2 \left[A_{\mathbf{k},\alpha} A^*_{\mathbf{k},\alpha} + A^*_{\mathbf{k},\alpha} A_{\mathbf{k},\alpha}\right]. \tag{1.17.25}$$

[SZQ: 2023.04.20: This expression is similar to the quantum harmonic oscillators.] [SZQ: 2023.04.20: I ignore the bold. So you should understand where you should use the bold.]

1.18 Quantization of the e.m. field

Fundamental Idea RadiationMode (k, α)

- **To every radiation mode, we associate a harmonic oscillator!** Creation and annihilation operators can change the degree of excitation of mode (occupation with photons)
- **A photon is an excitation quantum of the harmonic oscillator associated with a mode!**

Creation and Annihilation Operators $\hat{a}_k|n_k\rangle = \sqrt{n_k}|n_k - 1\rangle$: decrease photon number by one photon.

$\hat{a}_k^\dagger|n_k\rangle = \sqrt{n_k+1}|n_k+1\rangle$: increase photon number by one photon.

Number operator: $\hat{n}_k|n_k\rangle = n_k|n_k\rangle$. $\hat{a}_k^\dagger\hat{a}_k = \hat{n}_k$.

Fock state: $|n_k\rangle$. Fock state is the eigenstate of quantum harmonic oscillator.

Hamitonian of Radiation Field The Hamitonian of Radiation Field is the sum of the hamitonian of harmonic oscillator of each mode as

$$\hat{H}_R = \sum_k \hat{H}_k, \tag{1.18.1}$$

where

$$\hat{H}_k = \frac{1}{2}\hbar\omega_k \left(\hat{a}_k\hat{a}_k^\dagger + \hat{a}_k^\dagger\hat{a}_k\right). \tag{1.18.2}$$

We can compare it with classicall expression

$$E_{k,\alpha} = \epsilon_0 V\omega_k^2 \left(A_{k,\alpha}A_{k,\alpha}^* + A_{k,\alpha}^* A_{k,\alpha}\right). \tag{1.18.3}$$

If we replace A_k with

$$A_k = \sqrt{\frac{\hbar}{2\epsilon_0 V\omega_k}}\hat{a}_k, \tag{1.18.4}$$

and replace A_k^* with

$$A_k^* = \sqrt{\frac{\hbar}{2\epsilon_0 V\omega_k}}\hat{a}_k^\dagger. \tag{1.18.5}$$

We will arrive at $\hat{H}_k$. Also we can obtain the quantum version of vector potential operator. The classical vector potential operator is

$$A_k(r,t) = \epsilon_k \left[A_k \exp[i(kr - \omega_k t)] + A_k^* \exp[-i(kr - \omega_k t)]\right]. \tag{1.18.6}$$

The quantum version will be

$$\hat{A}_k(r,t) = \epsilon_k\sqrt{\frac{\hbar}{2\epsilon_0 V\omega_k}}\left[\hat{a}_k \exp[i(kr - \omega_k t)] + \hat{a}_k^\dagger \exp[-i(kr - \omega_k t)]\right]. \tag{1.18.7}$$

Use the quantum vector potential, we can derive the quantum electric field operator as

$$\hat{E}_k(r,t) = -\frac{\partial}{\partial t}\hat{A}_k(r,t) \tag{1.18.8}$$

$$= -\epsilon_k\sqrt{\frac{\hbar}{2\epsilon_0\omega_k V}}(-\omega_k)\left[i\hat{a}_k \exp[i(kr - \omega_k t)] - i\hat{a}_k^\dagger \exp[-i(kr - \omega t)]\right]. \tag{1.18.9}$$

Recall that $i = \exp[i\pi/2]$ and define

$$\chi_k(r,t) = -kr + \omega_k t - \pi/2. \tag{1.18.10}$$

We then have the compact form

$$\hat{E}(r,t) = \sum_k \epsilon_k\sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}}\left[\hat{a}_k \exp[-i\chi_k(r,t)] + \hat{a}_k^\dagger \exp[i\chi_k(r,t)]\right] \tag{1.18.11}$$

$$= \sum_k \hat{E}_k(r,t) \tag{1.18.12}$$

$$:= \hat{E}^+(r,t) + \hat{E}^-(r,t). \tag{1.18.13}$$

Hamiltonian of Radiation Field The Hamiltonian of Radiation Field is

$$\hat{H}_R = \frac{1}{2}\int_V dV \left[\epsilon_0 \hat{E}\cdot\hat{E} + \frac{1}{\mu_0}\hat{B}\cdot\hat{B}\right] \tag{1.18.14}$$

$\hat{B}$, $\hat{E}$ are the quantum operator of B, E

$$= \sum_k \frac{\hbar\omega_k}{2}\left[\hat{a}_k^\dagger \hat{a}_k + \hat{a}_k \hat{a}_k^\dagger\right] \tag{1.18.15}$$

Use the commutation relation

$$= \sum_k \left(\hat{a_k}^\dagger \hat{a}_k + 1/2\right). \tag{1.18.16}$$

Use this hamiltonian, we derive the energy of multi-mode Fock states as

$$\hat{H}_R|n_{k_1}, n_{k_2}, \ldots\rangle = \sum_k \hbar\omega_k \left(n_k + \frac{1}{2}\right)|n_{k_1}, n_{k_2}, \ldots\rangle \tag{1.18.17}$$

using the fact that $\hat{a}_k^\dagger \hat{a}_k$ is the number operator $\hat{n}_k$.

Also the vacuum state energy will be

$$E_0 = \sum_k \frac{1}{2}\hbar\omega_k \tag{1.18.18}$$

corresponds to

$$|0\rangle = |0\rangle \otimes \cdots \otimes |0\rangle. \tag{1.18.19}$$

This is divergent, but do not worry. When we calculate the difference, this term will be canceled.

1.19 Field state of single radiation field mode: Fock States

We focus discussion on a **single mode of the radiation field (wave vector k)** We define the phase factor

$$\chi = \chi_k(r, t) = \omega_k t - \mathbf{kr} - \pi/2. \tag{1.19.1}$$

Then we have

$$\hat{E}(\chi) = \hat{E}^+(\chi) + \hat{E}^-(-\chi) \tag{1.19.2}$$

$$= \left(\frac{\hbar\omega}{2\varepsilon_0 V}\right)^{1/2} (\hat{a}\exp[-i\chi] + \hat{a}^\dagger \exp[i\chi]). \tag{1.19.3}$$

We write the field operator in natural units $2\left(\frac{\hbar\omega}{2\varepsilon_0 V}\right)^{1/2}$, which is also called vacuum field strength. We then have

$$\hat{E}(\chi) = \frac{1}{2}\left(\hat{a}\exp[-i\chi] - \hat{a}^\dagger \exp[i\chi]\right). \tag{1.19.4}$$

Fock states: $|n\rangle$ means n photons in radiation mode, also means eigenstate of number operator $\hat{n}$.

$$\hat{H}|n\rangle = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})|n\rangle \tag{1.19.5}$$
$$= \hbar\omega(\hat{n} + \frac{1}{2})|n\rangle \tag{1.19.6}$$
$$= \hbar\omega(n + \frac{1}{2})|n\rangle. \tag{1.19.7}$$

Lemma 1.19.1. *Fluctuations in n is*

$$(\Delta n)^2 = 0. \tag{1.19.8}$$

Proof.

$$(\Delta n)^2 = \langle n|(\hat{n} - \langle\hat{n}\rangle)^2|n\rangle \tag{1.19.9}$$
$$= \langle n|(\hat{n}^2 - 2\langle\hat{n}\rangle\hat{n} + \langle\hat{n}\rangle^2)|n\rangle \tag{1.19.10}$$
$$= \langle n|\hat{n}^2|n\rangle - 2\langle\hat{n}\rangle^2 + \langle\hat{n}\rangle^2 \tag{1.19.11}$$
$$= \langle n|\hat{n}^2|n\rangle - \langle n|\hat{n}|n\rangle^2 \tag{1.19.12}$$
$$= \langle n|n^2|n\rangle - \langle n|n|n\rangle^2 \tag{1.19.13}$$
$$= n^2 - n^2 \tag{1.19.14}$$
$$= 0, \tag{1.19.15}$$

where we use

$$\langle n|\hat{n}|n\rangle = \langle\hat{n}\rangle. \tag{1.19.16}$$

□

Lemma 1.19.2. *The expectation value of the field is*

$$E = \langle n|\hat{E}(\chi)|n\rangle = 0. \tag{1.19.17}$$

Proof.

$$E = \langle n|\hat{E}(\chi)|n\rangle \tag{1.19.18}$$
$$= \frac{1}{2}\langle n|\hat{a}\exp[-i\chi] + \hat{a}^\dagger\exp[i\chi]|n\rangle \tag{1.19.19}$$
$$= \frac{1}{2}\langle n|\hat{a}\exp[-i\chi]|n\rangle + \frac{1}{2}\langle n|\hat{a}^\dagger\exp[i\chi]|n\rangle \tag{1.19.20}$$
$$= 0 + 0 \tag{1.19.21}$$
$$= 0. \tag{1.19.22}$$

□

Lemma 1.19.3. *Field fluctuations is*

$$(\Delta E(\chi))^2 = \frac{1}{2}(n+\frac{1}{2}). \tag{1.19.23}$$

Proof.

$$\begin{aligned}
(\Delta E(\chi))^2 &= \langle n|\hat{E}(\chi)^2|n\rangle - \langle n|\hat{E}(\chi)|n\rangle^2 &(1.19.24)\\
&= \langle n|\hat{E}(\chi)^2|n\rangle - 0 &(1.19.25)\\
&= \frac{1}{4}\langle n|(\hat{a}\exp[-i\chi] + \hat{a}^\dagger \exp[i\chi])^2|n\rangle &(1.19.26)\\
&= \frac{1}{4}\langle n|\hat{a}^2\exp[-2i\chi] + \hat{a}^{\dagger 2}\exp[2i\chi] + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}|n\rangle &(1.19.27)\\
&= \frac{1}{4}\langle n|2\hat{n}+2|n\rangle &(1.19.28)\\
&= \frac{1}{4}\langle n|2n+1|n\rangle &(1.19.29)\\
&= \frac{1}{2}(n+\frac{1}{2}). &(1.19.30)
\end{aligned}$$

□

When $n=0$, which is vacuum state, we have the standard deviation as $1/2$, which is half of the unit, i.e., vacuum field strength.

1.20 Field state of single radiation field mode: Coherent States

How to reproduce classical motion Superposition of Fock states reproduces oscillating wavepacket motion!

$$\begin{aligned}
|\alpha\rangle &\propto \exp[-i\frac{1}{2}\omega t]|0\rangle + \exp[-i\frac{3}{2}\omega t]\alpha|1\rangle + \exp[-i\frac{5}{2}\omega t]\frac{\alpha^2}{\sqrt{2}}|2\rangle + \dots &(1.20.1)\\
&= \sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\exp\left[-iE_n t/\hbar\right]. &(1.20.2)
\end{aligned}$$

What we want: states of light, whose expectation value corresponds to classical e.m. waves!

Solotion: Coherent States

Definition 1.20.1. *Coherent States:*

$$|\alpha\rangle = \exp\left[-\frac{1}{2}|\alpha|^2\right]\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle, \tag{1.20.3}$$

where α is complex number, amplitude of coherent state. $\alpha = |\alpha|\exp[i\theta]$, $|n\rangle$ is the fock state.

Lemma 1.20.1. *Coherent state is normalized:* $\langle\alpha|\alpha\rangle = 1$.

Proof.

$$\langle\alpha|\alpha\rangle = \exp\left[-|\alpha|^2\right] \sum_{n,n'} \frac{(\alpha^*)^n \alpha^{n'}}{\sqrt{n!}\sqrt{n'!}} \langle n|n'\rangle \tag{1.20.4}$$

$$= \exp\left[-|\alpha|^2\right] \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} \tag{1.20.5}$$

$$= \exp\left[-|\alpha|^2\right] \exp\left[+|\alpha|^2\right] \tag{1.20.6}$$

$$= 1. \tag{1.20.7}$$

□

Lemma 1.20.2. *Coherent state is quasi orthogonal:* $|\alpha - \beta| >> 1 \longrightarrow |\langle\alpha|\alpha\rangle| \to 0$.

Proof.

$$\langle\alpha|\beta\rangle = \exp\left[-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2 + \alpha^*\beta\right]. \tag{1.20.8}$$

$$|\langle\alpha|\beta\rangle|^2 = \langle\alpha|\beta\rangle^* \langle\alpha|\beta\rangle \tag{1.20.9}$$

$$= \exp\left[-|\alpha|^2 - |\beta|^2 + \alpha^*\beta + \beta^*\alpha\right] \tag{1.20.10}$$

$$= \exp\left[-|\alpha - \beta|^2\right]. \tag{1.20.11}$$

We have $|\alpha - \beta| >> 1 \longrightarrow |\langle\alpha|\alpha\rangle| \to 0$, which is called quasi-orthogonal. □

Lemma 1.20.3. *Coherent states are eigenstates of destruction operator* $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$.

Proof.

$$\hat{a}|\alpha\rangle = \exp\left[-\frac{1}{2}|\alpha|^2\right] \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \tag{1.20.12}$$

$$= \exp\left[-\frac{1}{2}|\alpha|^2\right] \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \hat{a}|n\rangle \tag{1.20.13}$$

$$= \exp\left[-\frac{1}{2}|\alpha|^2\right] \sum_{n=1}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \sqrt{n}|n-1\rangle \tag{1.20.14}$$

$$= \alpha \exp\left[-\frac{1}{2}|\alpha|^2\right] \sum_{n=1}^{\infty} \frac{\alpha^{n-1}}{\sqrt{(n-1)!}} |n-1\rangle \tag{1.20.15}$$

k:= n-1

$$= \alpha \exp\left[-\frac{1}{2}|\alpha|^2\right] \sum_{k=0}^{\infty} \frac{\alpha^k}{\sqrt{k!}} |k\rangle \tag{1.20.16}$$

$$= \alpha|\alpha\rangle. \tag{1.20.17}$$

□

Lemma 1.20.4. *The average photon number of coherent states:* $\overline{n} = |\alpha|^2$.

Proof.

$$\overline{n} = \langle\alpha|\hat{n}|\alpha\rangle \tag{1.20.18}$$

$$= \langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle \tag{1.20.19}$$

Use 1.20.3

$$= \langle\alpha^*\alpha|\alpha\rangle \tag{1.20.20}$$

$$= |\alpha|^2\langle\alpha|\alpha\rangle \tag{1.20.21}$$

$$= |\alpha|^2. \tag{1.20.22}$$

□

[SZQ: 2023.04.20: coherent state is robust!]

Lemma 1.20.5. *Photon number variance of coherent states is*

$$(\Delta n)^2 = \overline{n}. \tag{1.20.23}$$

Proof.

$$(\Delta n)^2 = \langle\alpha|\hat{n}^2|\alpha\rangle - \langle\alpha|\hat{n}|\alpha\rangle^2 \tag{1.20.24}$$

$$= \langle\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}|\alpha\rangle - |\alpha|^4 \tag{1.20.25}$$

$\hat{a}\hat{a}^\dagger = \hat{a}^\dagger\hat{a} + 1$

$$= \langle\alpha|\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}|\alpha\rangle + \langle\hat{a}^\dagger\hat{a}|\alpha\rangle - |\alpha|^4 \tag{1.20.26}$$

$$= |\alpha|^2 \tag{1.20.27}$$

$$= \overline{n}. \tag{1.20.28}$$

□

Lemma 1.20.6. *The photon number distribution is*

$$P(n) = |\langle n|\alpha\rangle|^2 \tag{1.20.29}$$
$$= |\langle n| \exp\left[-\frac{|\alpha|^2}{2}\right] \sum_{n'=0}^{\infty} \frac{\alpha^{n'}}{\sqrt{n!}} |n'\rangle|^2 \tag{1.20.30}$$
$$= |\exp\left[-\frac{|\alpha|^2}{2}\right] \sum_{n'=0}^{\infty} \frac{\alpha^{n'}}{\sqrt{n!}} \langle n|n'\rangle|^2 \tag{1.20.31}$$
$$= |\exp\left[-\frac{|\alpha|^2}{2}\right] \frac{\alpha^{n}}{\sqrt{n!}}|^2 \tag{1.20.32}$$
$$= \exp\left[-|\alpha|^2\right] \frac{|\alpha|^{2n}}{n!} \tag{1.20.33}$$

Use lemma 1.20.4

$$= \exp\left[-\overline{n}\right] \frac{\overline{n}^{n}}{n!}. \tag{1.20.34}$$

This is known as **Poisson distribution**. The standard deviation is

$$\Delta n = \sqrt{\overline{n}}. \tag{1.20.35}$$

The standard deviation relative to the mean is

$$\frac{\Delta n}{\overline{n}} = \frac{1}{\sqrt{\overline{n}}}, \tag{1.20.36}$$

which shows that the fluctuations is smaller and smaller when the mean becomes larger and larger.

For large $\overline{n}$, we have

$$P(n) \simeq \frac{1}{\sqrt{2\pi\overline{n}}} \exp\left[-\frac{1}{2}\frac{(n-\overline{n})^2}{\overline{n}}\right]. \tag{1.20.37}$$

[SZQ: $\simeq$ means asymptoticallly equal to.]

Lemma 1.20.7. *The expectation value of the field operator*

$$\langle\alpha\hat{E}(\chi)|\alpha\rangle = \frac{1}{2}(\langle\alpha|\hat{a}\exp\left[-i\chi\right] + \hat{a}^{\dagger}\exp\left[i\chi\right]|\alpha\rangle) \tag{1.20.38}$$
$$= \frac{1}{2}\left(\langle\alpha|\hat{a}\exp\left[-i\chi\right]|\alpha\rangle + \langle\alpha|\hat{a}^{\dagger}\exp\left[i\chi\right]|\alpha\rangle\right) \tag{1.20.39}$$
$$= \frac{1}{2}\left(\alpha\exp\left[-i\chi\right] + \alpha^{*}\exp\left[i\chi\right]\right) \tag{1.20.40}$$
$$= |\alpha|\cos\left(\chi-\theta\right), \tag{1.20.41}$$

where α is the complex amplitude and $\alpha = |\alpha|\exp\left[i\theta\right]$.

We can plot $\langle\alpha|\hat{E}(\chi)|\alpha\rangle$ in Phasor Diagram in terms of $|\alpha|$ and θ.

Lemma 1.20.8. *The fluctuations (variance) of the E-field*

$$(\Delta\hat{E}(\chi))^2 = \langle\alpha|\hat{E}(\chi)^2|\alpha\rangle - \langle\alpha|\hat{E}(\chi)|\alpha\rangle^2 = \frac{1}{4}. \tag{1.20.42}$$

Proof.

$$\begin{aligned}
(\Delta\hat{E}(\chi))^2 &= \langle\alpha|\hat{E}(\chi)^2|\alpha\rangle - \langle\alpha|\hat{E}(\chi)|\alpha\rangle^2 && (1.20.43)\\
&= \frac{1}{4}\langle\alpha|\left[\hat{a}\exp\left[-i\chi\right] + \hat{a}^\dagger\exp\left[i\chi\right]\right]^2|\alpha\rangle - |\alpha|^2\cos^2(\chi-\theta) && (1.20.44)\\
&= \frac{1}{4}\langle\alpha|\hat{a}^2\exp\left[-2i\chi\right] + \hat{a}^{\dagger 2}\exp\left[2i\chi\right] + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}|\alpha\rangle - |\alpha|^2\cos^2(\chi-\theta) && (1.20.45)\\
&= \frac{1}{4}\left(\alpha^2\exp\left[-2i\chi\right] + \alpha^{*2}\exp\left[2i\chi\right] + 2|\alpha|^2 + 1\right) - |\alpha|^2\cos^2(\chi-\theta) && (1.20.46)\\
&= \frac{1}{4}\left(|\alpha\exp\left[-i\chi\right] + \alpha^*\exp\left[i\chi\right]|^2 + 1\right) - |\alpha|^2\cos^2(\chi-\theta) && (1.20.47)\\
&= \frac{1}{4}\left(4|\alpha|^2\cos^2(\chi-\theta) + 1\right) - |\alpha|^2\cos^2(\chi-\theta) && (1.20.48)\\
&= |\alpha|^2\cos^2(\chi-\theta) + \frac{1}{4} - |\alpha|^2\cos^2(\chi-\theta) && (1.20.49)\\
&= \frac{1}{4}. && (1.20.50)
\end{aligned}$$

□

Lemma 1.20.9. *The expectation value of the Energy of Coherent States*

$$\langle\alpha|\hat{H}|\alpha\rangle = \hbar\omega(\overline{n} + 1/2). \tag{1.20.51}$$

Proof.

$$\begin{aligned}
\langle\alpha|\hat{H}|\alpha\rangle &= \hbar\omega\langle\alpha|\hat{n} + \frac{1}{2}|\alpha\rangle && (1.20.52)\\
&= \hbar\omega\left(|\alpha|^2 + 1/2\right) && (1.20.53)\\
&= \hbar\omega\left(\overline{n} + \frac{1}{2}\right). && (1.20.54)
\end{aligned}$$

□

Lemma 1.20.10. *The fluctuations of the energy of coherent States*

$$\Delta H = \hbar\omega|\alpha|. \tag{1.20.55}$$

Proof.

$$(\Delta H)^2 = \langle\alpha|\hat{H}^2|\alpha\rangle - \langle\alpha|\hat{H}|\alpha\rangle^2, \tag{1.20.56}$$

where

$$\langle\alpha|\hat{H}^2|\alpha\rangle = \langle\alpha|(\hbar\omega(\hat{n}+\frac{1}{2}))^2|\alpha\rangle \tag{1.20.57}$$
$$= \hbar^2\omega^2\langle\alpha|\hat{n}^2+\hat{n}+\frac{1}{4}|\alpha\rangle \tag{1.20.58}$$
$$= \hbar^2\omega^2\left(|\alpha|^4+|\alpha|^2+\frac{1}{4}\right). \tag{1.20.59}$$

Then

$$(\Delta H)^2 = \hbar^2\omega^2\left(|\alpha|^4+|\alpha|^2+\frac{1}{4}\right) - \hbar^2\omega^2\left(|\alpha|^2+\frac{1}{2}\right)^2 \tag{1.20.60}$$
$$= 0 \tag{1.20.61}$$

□

[SZQ: 2023.04.21: I derive wrong answer. Where is the mistake?]

1.21 Quadrature Operators and Phase Space of Field States

Definition 1.21.1. *Classical eletromagnetic field is*

$$E(t) = E_0\cos(\omega t+\theta) \tag{1.21.1}$$
$$= E_0\cos\theta\cos\omega t - E_0\sin\theta\sin\omega t \tag{1.21.2}$$
$$= X_1\cos\omega t + X_2\sin\omega t, \tag{1.21.3}$$

where X_1, X_2 are quadrature variables defined as

$$X_1 = E_0\cos\theta \tag{1.21.4}$$
$$X_2 = -E_0\sin\theta. \tag{1.21.5}$$

Definition 1.21.2. *Phasor representation of field*

$$a(t) = E_0\exp[-i\theta]\exp[-i\omega t] = a\exp[-i\omega t], \tag{1.21.6}$$

where a is defined as $E_0\exp[-i\theta]$. $a(t)$ ***is called the phaor****.*

Lemma 1.21.1 (Relations between the Phasor representation and quadrature variables)**.**

$$a = X_1 + iX_2. \tag{1.21.7}$$

Proof. By definition 1.21.2,

$$
\begin{aligned}
a &= E_0 \exp\left[-i\theta\right] && (1.21.8)\\
&= E_0 \left(\cos\theta - i\sin\theta\right) && (1.21.9)\\
&= E_0 \cos\theta - iE_0 \sin\theta && (1.21.10)\\
&\text{By definition 1.21.1}\\
&= X_1 + iX_2. && (1.21.11)
\end{aligned}
$$

□

[SZQ: 2023.04.21: a is determined by E_0 and θ.]

Corollary 1.21.1.

$$X_1 = Re(a) = \frac{1}{2}\left(a + a^*\right) \tag{1.21.12}$$

$$X_2 = Im(a) = \frac{1}{2i}\left(a - a^*\right). \tag{1.21.13}$$

Proof. By lemma 1.21.1, we can prove this corollary. □

Quantum-Quadrature Operators

Definition 1.21.3. *Quadrature operators:*

$$\hat{x} \hat{=} \hat{X}_1 = \frac{1}{2}\left(\hat{a} + \hat{a}^\dagger\right) \tag{1.21.14}$$

$$\hat{p} \hat{=} \hat{X}_2 = \frac{1}{2i}\left(\hat{a} - \hat{a}^\dagger\right). \tag{1.21.15}$$

[SZQ: $\hat{=}$ means "define"] [SZQ: Quadrature operators are **Hermitian operators**, i.e., observables.]

Lemma 1.21.2. *The commutation and uncertainty relations:*

$$\left[\hat{X}_1, \hat{X}_2\right] = \frac{i}{2} \tag{1.21.16}$$

$$\Delta X_1 \Delta X_2 \geq \frac{1}{4}. \tag{1.21.17}$$

Proof. □

Definition 1.21.4. *Generalized quadrature operators have the same commutation relations with quadrature operators:*

$$\hat{X}_\phi = \frac{1}{2}\left(\hat{a}\exp\left[-i\phi\right] + \hat{a}^\dagger \exp\left[i\phi\right]\right) \tag{1.21.18}$$

$$\hat{X}_{\phi+\frac{\pi}{2}} = \frac{1}{2i}\left(\hat{a}\exp\left[-i\phi\right] - \hat{a}^\dagger \exp\left[i\phi\right]\right). \tag{1.21.19}$$

Phase space distribution of field states

Lemma 1.21.3. *In vacuum state $|0\rangle$, we have*

$$P^{(0)}(X_1) = |\langle X_1|0\rangle|^2 = \sqrt{\frac{2}{\pi}} \exp\left[-2X_1^2\right] \tag{1.21.20}$$

$$P^{(0)}(X_2) = |\langle X_2|0\rangle|^2 = \sqrt{\frac{2}{\pi}} \exp\left[-2X_2^2\right]. \tag{1.21.21}$$

$$\textit{Fluctuations:} \Delta X_1 = \sqrt{\langle 0|\hat{X}_1^2|0\rangle - \langle 0|\hat{X}_1|0\rangle^2} = \frac{1}{2}. \tag{1.21.22}$$

Lemma 1.21.4. *In fock state $|n\rangle$, we have*

$$P^{(n)}(X_1) = |\langle X_1|n\rangle|^2 \tag{1.21.23}$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{2^n n!} \exp\left[-2X_1^2\right] \left(H_n(\sqrt{2}X_1)\right)^2 \tag{1.21.24}$$

$$\Delta X_1 = \frac{1}{2}\sqrt{2n+1}. \tag{1.21.25}$$

Coherent state - A displaced vacuum

Definition 1.21.5. *Displacement operator (shifts any coherent state by α)*

$$\hat{D}(\alpha) = \exp\left[\alpha\hat{a}^\dagger - \alpha^*\hat{a}\right]. \tag{1.21.26}$$

Corollary 1.21.2. *Coherent state from vacuum state*

$$|\alpha\rangle = \hat{D}(\alpha)|0\rangle. \tag{1.21.27}$$

Squeezed states of light We have nice pictures in the video.

We have phase squeezed state and amplitude squeezed state.

1.22 The Classical Beamsplitter

Assume all beams have same polarization and frequency. We input E_1, E_2 and output E_3, E_4. There is a nice picture to illustrate it. Then we have

$$E_3 = RE_1 + TE_2 \tag{1.22.1}$$

$$E_4 = T'E_1 + R'E_2, \tag{1.22.2}$$

which is

$$\begin{pmatrix} E_3 \\ E_4 \end{pmatrix} = \begin{pmatrix} R & T \\ T' & R' \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}, \tag{1.22.3}$$

where R, R' denote reflection coefficients, which are complex coefficients, T, T' denote transmission coefficients, which are complex coefficients.

The simplified case: symmetric Beamsplitter

$$\begin{pmatrix} E_3 \\ E_4 \end{pmatrix} = \begin{pmatrix} R & T \\ T & R \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}. \tag{1.22.4}$$

Lemma 1.22.1. *The energy conservation*

$$|E_1|^2 + |E_2|^2 = |E_3|^2 + |E_4|^2 \tag{1.22.5}$$

leads to

$$|R|^2 + |T|^2 = 1 \tag{1.22.6}$$

$$RT^* + TR^* = 0. \tag{1.22.7}$$

Proof.

$$|E_3|^2 + |E_4|^2 = (RE_1 + TE_2)(RE_1 + TE_2)^* + (TE_1 + RE_2)(TE_1 + RE_2)^* \tag{1.22.8}$$

$$= |R|^2|E_1|^2 + |T|^2|E_2|^2 + RT^*E_1E_2^* + TR^*E_2E_1^* \tag{1.22.9}$$

$$+ |T|^2|E_1|^2 + |R|^2|E_2|^2 + TR^*E_1E_2^* + RT^*E_2E_1^* \tag{1.22.10}$$

$$= (|R|^2 + |T|^2)|E_1|^2 + (|R|^2 + |T|^2)|E_2|^2 \tag{1.22.11}$$

$$+ (RT^* + TR^*)E_1E_2^* + (RT^* + TR^*)E_2E_1^* \tag{1.22.12}$$

$$= |E_1|^2 + |E_2|^2. \tag{1.22.13}$$

Therefore,

$$|R|^2 + |T|^2 = 1 \tag{1.22.14}$$

$$RT^* + TR^* = 0. \tag{1.22.15}$$

□

Lemma 1.22.2. *Define* $R = |R|e^{i\phi_R}$ *and* $T = |T|e^{i\phi_T}$*. We have* $\phi_R - \phi_T = \frac{\pi}{2}$*.*

Proof. By the second requirement in 1.22.1, we have

$$|R||T|e^{i(\phi_R - \phi_T)} + |R||T|e^{-i(\phi_R - \phi_T)} = 0. \tag{1.22.16}$$

So

$$2\cos(\phi_R - \phi_T) = 0, \tag{1.22.17}$$

leads to

$$\phi_R - \phi_T = \frac{\pi}{2}. \tag{1.22.18}$$

□

Set $\phi_T = 0$, then $\phi_R = \frac{\pi}{2}$. We then have

$$E_3 = i|R|E_1 + |T|E_2 \tag{1.22.19}$$

$$E_4 = |T|E_1 + i|R|E_2. \tag{1.22.20}$$

50/50 beamsplitter input-output Symmetrized beamsplitter input-output relations

$$\begin{pmatrix} E_3 \\ E_4 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}. \tag{1.22.21}$$

In this case, light is always equally splited in port 3 and port 4.

1.23 The Quantum Beamsplitter

In the quantum beamsplitter, E_1, E_2, E_3, E_4 are replaced by $\hat{a}_1, \hat{a}_2, \hat{a}_3, \hat{a}_4$. There is a nice picture. We have

$$\hat{a}_3 = R\hat{a}_1 + T\hat{a}_2 \tag{1.23.1}$$

$$\hat{a}_4 = T\hat{a}_1 + R\hat{a}_2, \tag{1.23.2}$$

which is

$$\begin{pmatrix} \hat{a}_3 \\ \hat{a}_4 \end{pmatrix} = \begin{pmatrix} R & T \\ T & R \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix}, \tag{1.23.3}$$

where R denotes reflection coefficient, which is complex coefficient, T denotes transmission coefficient, which is complex coefficient.

What about $\hat{a}_1$ and $\hat{a}_2$?

Lemma 1.23.1.

$$\hat{a}_1 = R^*\hat{a}_3 + T^*\hat{a}_4 \tag{1.23.4}$$

$$\hat{a}2 = T^*\hat{a}_3 + R^*\hat{a}_4. \tag{1.23.5}$$

Proof. Recall that

$$\hat{a}_3 = R\hat{a}_1 + T\hat{a}_2 \tag{1.23.6}$$

$$\hat{a}_4 = T\hat{a}_1 + R\hat{a}_2. \tag{1.23.7}$$

We have

$$R^*\hat{a}_3 = |R|^2\hat{a}_1 + R^*T\hat{a}_2 \tag{1.23.8}$$

$$T^*\hat{a}_4 = |T|^2\hat{a}_1 + T^*R\hat{a}_2. \tag{1.23.9}$$

Sum two equations, we have

$$(|R|^2 + |T|^2)\hat{a}_1 + R^*T\hat{a}_2 + T^*R\hat{a}_2 = R^*\hat{a}_3 + T^*\hat{a}_4. \tag{1.23.10}$$

Use lemma 1.22.1, we have

$$\hat{a}_1 = R^*\hat{a}_3 + T^*\hat{a}_4. \tag{1.23.11}$$

Similarly, we can get

$$\hat{a}_2 = T^*\hat{a}_3 + R^*\hat{a}_4. \tag{1.23.12}$$

□

Lemma 1.23.2. *We have the creation operators*

$$\hat{a}_1^\dagger = R\hat{a}_3^\dagger + T\hat{a}_4^\dagger \tag{1.23.13}$$
$$\hat{a}_2^\dagger = T\hat{a}_3^\dagger + R\hat{a}_4^\dagger \tag{1.23.14}$$
$$\hat{a}_3^\dagger = R^*\hat{a}_1^\dagger + T^*\hat{a}_2^\dagger \tag{1.23.15}$$
$$\hat{a}_4^\dagger = T^*\hat{a}_1^\dagger + R^*\hat{a}_2^\dagger. \tag{1.23.16}$$

Single photon on BS **Input state** We have a nice picture in the video.

$$|1\rangle_1|0\rangle_2 \tag{1.23.17}$$
$$|1\rangle_1|0\rangle_2 = \hat{a}_1^\dagger|0\rangle_1|0\rangle_2 \tag{1.23.18}$$

Output state

$$\hat{a}_1^\dagger = R\hat{a}_3^\dagger + T\hat{a}_4^\dagger. \tag{1.23.19}$$

This shows how $\hat{a}_1^\dagger$ is splitted in port 3 and port 4. We have the output state as

$$\left(R\hat{a}_3^\dagger + T\hat{a}_4^\dagger\right)|0\rangle_3|0\rangle_4 = R|1\rangle_3|0\rangle_4 + T|0\rangle_3|1\rangle_4, \tag{1.23.20}$$

which is an entangled state of photon between field modes.

Single photon on 50/50 BS We have

$$\frac{1}{\sqrt{2}}(i|1\rangle_3|0\rangle_4 + |0\rangle_3|1\rangle_4). \tag{1.23.21}$$

Lemma 1.23.3. *Average output photon number at port 3*

$$\langle\hat{n}_3\rangle = \frac{1}{2}, \tag{1.23.22}$$

and at port 4

$$\langle\hat{n}_4\rangle = \frac{1}{2}, \tag{1.23.23}$$

Proof.

$$\langle \hat{n}_3 \rangle = \langle \hat{a}_3^\dagger \hat{a}_3 \rangle \tag{1.23.24}$$
$$=_2 \langle 0|_1 \langle 1|\hat{a}_3^\dagger \hat{a}_3 |1\rangle_1 |0\rangle_2 \tag{1.23.25}$$
$$=_2 \langle 0|_1 \langle 1|(R^* \hat{a}_1^\dagger + T^* \hat{a}_2^\dagger)(R\hat{a}_1 + T\hat{a}_2)|1\rangle_1 |0\rangle_2 \tag{1.23.26}$$

number operator $\hat{n} = \hat{a}^\dagger \hat{a}$

$$= |R|_2^2 \langle 0|_1 \langle 1|1\rangle_1 |0\rangle_2 \tag{1.23.27}$$
$$= \frac{1}{2}. \tag{1.23.28}$$

Similiarly, we have

$$\langle \hat{n}_4 \rangle = \frac{1}{2}. \tag{1.23.29}$$

□

Lemma 1.23.4. *Correlations*

$$\langle \hat{n}_3 \hat{n}_4 \rangle =_2 \langle 0|_1 \langle 1|\hat{n}_3^\dagger \hat{n}_4 |1\rangle_1 |0\rangle_2 = 0. \tag{1.23.30}$$

Remark 1.23.1.
- *non-classical Correlations.*
- *due to entangled single photon state between field modes.*

What about coherent states?

1.24 Balanced Homodyne Detection

Input states: $|in\rangle = |\Psi\rangle_1 |\beta\rangle_2$, where $|\beta\rangle$ is the coherent state.

Lemma 1.24.1. *Difference photoncurrent*

$$i_{34} \propto -2|\beta| \langle \hat{X} \left(\phi + \pi/2\right) |\Psi\rangle. \tag{1.24.1}$$

Proof.

$$\hat{n}_4 = \hat{a}_4^\dagger \hat{a}_4 \tag{1.24.2}$$
$$= \tag{1.24.3}$$
$$\hat{n}_3 = \hat{a}_3^\dagger \hat{a}_3 \tag{1.24.4}$$
$$= \tag{1.24.5}$$
$$\tag{1.24.6}$$

$$i_{34} = i_3 - i_4 \propto \langle in|\hat{n}_3 - \hat{n}_4|in\rangle = -2{}_1\langle\Psi|{}_2\langle\beta|\frac{1}{2i}\left(\hat{a}_2^\dagger\hat{a}_1 - \hat{a}_1^\dagger\hat{a}_2\right)|\beta\rangle_2|\Psi\rangle_1 \tag{1.24.7}$$

$$=, \tag{1.24.8}$$

where $|\beta|$ is the amplification of quadrature signal. □