Chapter 1

Measure Theory

1.1 Sigma algebra

Example 1.1.1. *We define* $\mathcal{P}(X)$ *as the power set of set* X. *Assume that set* $X = \{a, b\}$, *the power set* $P(X)$ *would be* $\{\emptyset, X, \{a\}, \{b\}\}$

Definition 1.1.1 (Sigma algebra)**.** $\mathcal{A} \subseteq \mathcal{P}(X)$ *is called a* $\sigma -$ algebra*:*

$$(a)\ \emptyset, X \in \mathcal{A} \tag{1.1}$$
$$(b)\ A \in \mathcal{A} \Longrightarrow A^c := X \setminus A \in \mathcal{A} \tag{1.2}$$
$$(c)\ A_i \in \mathcal{A},\ i \in \mathcal{N} \Longrightarrow \cup_{i=1}^{\infty} A_i \in \mathcal{A}. \tag{1.3}$$

Definition 1.1.2 (Measurable sets)**.** $A \in \mathcal{A}$ *is called a* $\mathcal{A}$*-measurable set.*

Example 1.1.2.

$$(1)\ \mathcal{A} = \{\emptyset, X\} \tag{1.4}$$
$$(2)\ \mathcal{A} = \{\mathcal{P}(X)\}. \tag{1.5}$$

Lemma 1.1.1. *Assume* $\mathcal{A}_i$ *is* σ*-algebra on* X, $i \in I$ *(index set). Then, we have* $\cap_{i \in I} \mathcal{A}_i$ *is also a* σ*-algebra on* X.

Definition 1.1.3 (Sigma algebra generated by $\mathcal{M}$)**.** *For* $\mathcal{M} \subseteq \mathcal{P}(X)$, *there is a smallest* σ*-algebra that contains* $\mathcal{M}$*:*

$$\sigma(\mathcal{M}) := \cap_{\mathcal{A} \supseteq \mathcal{M},\ a\ \sigma-algebra} \mathcal{A}. \tag{1.6}$$

Example 1.1.3. *We define* $X = \{a, b, c, d\}$ *and* $\mathcal{M} = \{\{a\}, \{b\}\}$. *Then we have*

$$\sigma(\mathcal{M}) = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c, d\}, \{a, c, d\}, \{c, d\}\}. \tag{1.7}$$

Definition 1.1.4 (Borel sigma algebra)**.** *Let* $(X, \mathcal{T})$ *be a topological space (Let* X *be a metric space/Let* X *be a subset of* $\mathbb{R}^n$*; We need "open sets".). We then define* $\mathcal{B}(X)$ *is the borel* σ*-algebra on* X *as*

$$\mathcal{B}(X) := \sigma(\mathcal{T}), \tag{1.8}$$

which is the σ*-algebra generated by the open sets* $\mathcal{T}$.

1.2 What is a measure?

Definition 1.2.1 (Measure)**.** *$(X, \mathcal{A})$ is called a measurable space, where X is a set and $\mathcal{A}$ is a σ-algebra on X. A map $\mu : \mathcal{A} \to [0, \infty] := [0, \infty) + \{\infty\}$ is called a measure if it satisfies:*

$$(a)\ \mu(\emptyset) = 0 \tag{1.9}$$

$$(b)\ \mu(\cup_{i=1}^{\infty} \mathcal{A}_i) = \sum_{i=1}^{\infty} \mu(\mathcal{A}_i)\ with\ \mathcal{A}_i \cap \mathcal{A}_j = \emptyset,\ i \neq j\ for\ all\ \mathcal{A}_i \in \mathcal{A}.(\sigma - additive) \tag{1.10}$$

Definition 1.2.2. *$(X, \mathcal{A}, \mu)$ is called a measure space.*

Example 1.2.1. *Given X and $\mathcal{A} = \mathcal{P}(X)$.*

- *Counting measure ($A \in \mathcal{A}$) is defined as*

$$\mu(A) := \begin{cases} \#A, & A\ has\ finitely\ many\ elements \\ \infty & else \end{cases} \tag{1.11}$$

  where $\#A$ means the number of elements in A.

  Calculation rules in $[0, \infty]$:

$$x + \infty := \infty\ for\ all\ x \in [0, \infty] \tag{1.12}$$
$$x \cdot \infty := \infty\ for\ all\ x \in (0, \infty] \tag{1.13}$$
$$0 \cdot \infty := 0\ (only\ true\ in\ most\ cases\ in\ measure\ theory!) \tag{1.14}$$

- *Dirac measure for $p \in X$ is defined as*

$$\delta_p(A) := \begin{cases} 1, & p \in A \\ 0, & else \end{cases} \tag{1.15}$$

- *We search a measure on $X \in \mathcal{R}^n$ satisfying:*

$$(1)\ \mu([0, 1]^n) = 1 \tag{1.16}$$
$$(2)\ \mu(x + A) = \mu(A)\ for\ all\ x \in \mathcal{R}^n, \tag{1.17}$$

  which is known as Lebesgue measure where the σ-algebra is not equal to power set.

1.3 Not everything is lebesgue measurable

Measure problem: search measure μ on $\mathcal{P}(\mathbb{R})$ with:

- (1) $\mu([a, b]) = b - a,\ b > a,$
- (2) $\mu(x + A) = \mu(A),\ A \in \mathcal{P}(\mathbb{R}),\ x \in \mathbb{R}.$

$\implies \mu$ does not exist.

Claim: Let μ be a measure on $\mathcal{P}(\mathbb{R})$ with $\mu((0, 1]) < \infty$ and (2). $\implies \mu = 0$.

Proof. (a) Definitions: $I \in (0,1]$ with equivalence relation on I: $x \sim y \iff x - y \in \mathbb{Q}$ i.e., $[x] := \{x + r | r \in \mathbb{Q},\ x + r \in I\}$. Following this definition, we have a disjoint decomposition of I into boxes, possibly uncontable many of them! We then pick one element a_n from each box $[x_n]$ and form a set $A \in I$, i.e., $\{a_1, a_2, \cdots\} = A$. We have $A \in I$ with prperty:

- (1) For each $[x]$, there is an $a \in A$ with $a \in [x]$.
- (2) For all $a, b \in A:\ a, b \in [x] \Longrightarrow a = b$.

In uncountable case, the existence of $A \in I$ with the above property is guaranted by the axiom of choice of set theory.

We define $A_n := r_n + A$, where $(r_n)_{n\in\mathbb{N}}$ enumeration of $\mathbb{Q}_n(-1,1]$.

(b) We then claim that $A_n \cap A_m = \emptyset \Longleftarrow n \neq m$. The proof is as follows: $x \in A_n \cap A_m \Longrightarrow x = r_n + a_n,\ a_n \in A$ and $x = r_m + a_m,\ a_m \in A. \Longrightarrow r_n + a_n = r_m + a_m \Longrightarrow a_n - a_m = r_n - r_m \in \mathbb{Q} \Longrightarrow a_n \sim a_m \Longrightarrow a_m, a_n \in [a_m] \Longrightarrow a_n = a_m \Longrightarrow r_n = r_m \Longrightarrow n = m.$

(c) We claim that $(0,1] \subseteq \cup_{n\in\mathbb{N}} A_n \subseteq (-1,2]$. The proof is as follows:

Assume now: μ measure on $\mathcal{P}(\mathbb{R})$ with $\mu((0,1]) < \infty$ and (2).

By (2): $\mu(1 + A) = \mu(A)$ for all $n \in \mathbb{N}$.

By (c): we have

$$\mu((0,1]) \leq \mu(\cup_{n\in\mathbb{N}} A_n) \leq \mu((-1,2]) \tag{1.18}$$

We know: $\mu((0,1]) =: C < \infty$. By using (2) and σ-additivity, we get $\mu((-1,2]) = \mu\left((-1,0] \cup (0,1] \cup (1,2] = 3C\right)$. $\Longrightarrow_{1.18,(b)} C \leq \sum_{n=1}^{\infty} \mu(A_n) \leq 3C \Longrightarrow C \leq \sum_{n=1}^{\infty} \mu(A) \leq 3C \Longrightarrow \mu(A) = 0 \Longrightarrow C = 0(\text{henceL } \mu\left((0,1]\right) = 0) \Longrightarrow \mu(\mathbb{R}) = \mu\left(\cup_{n\in\mathbb{Z}}(m, m+1]\right) = 0 \Longrightarrow \mu = 0.$ □

1.4 Measurable maps

Definition 1.4.1 (Measurable maps). *$(\Omega_1, \mathcal{A}_1)$ and $(\Omega_2, \mathcal{A}_2)$ are measurable spaces. $f : \Omega_1 \to \Omega_2$ is a measurable map w.r.t. $\mathcal{A}_1$ and $\mathcal{A}_2$ if $f^{-1}(A_2) \in \mathcal{A}_1$ for all $A_2 \in \mathcal{A}_2$.*

Example 1.4.1.

- *$(\Omega, \mathcal{A})$ and $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ are two measurable spaces. We define characteristic fucntion (aksi indicator function) as $\chi_A : \Omega \to \mathbb{R}$, where*

$$\chi_A(w) := \begin{cases} 1, & w \in A \\ 0, & w \notin A \end{cases} \tag{1.19}$$

For all measurable $A \in \mathcal{A}$, χ_A is a measurable map. We have

$$\chi_A^{-1}(\emptyset) = \emptyset \in \mathcal{A},\ \chi_A^{-1}(\mathbb{R}) = \Omega \in \mathcal{A} \tag{1.20}$$

$$\chi_A^{-1}(\{A\}) = A,\ \chi_A^{-1}(\{0\}) = A^c \in \mathcal{A}. \tag{1.21}$$

- *Composition of measurable maps.*

Lemma 1.4.1. $(\Omega_1, \mathcal{A}_1)$, $(\Omega_2, \mathcal{A}_2)$, $(\Omega_3, \mathcal{A}_3)$ *are measurable space. We define* $\Omega_1 \xrightarrow{f} \Omega_2 \xrightarrow{g} \Omega_3$. *Then* f, g *are measurable implies* $g \circ f$ *is measurable.*

Proof.

$$(g \circ f)^{-1}(A_3) = f^{-1}(g^{-1}(A_3)) \tag{1.22}$$
$$\in \mathcal{A}_1 \tag{1.23}$$

□

Important measurable maps

Lemma 1.4.2. $(\Omega, \mathcal{A})$ *and* $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ *are measurable spaces.* $f, g : \Omega \to \mathbb{R}$ *are measurable maps indicates that* $f + g$, $f - g$, $f \cdot g$, $|f|$ *are measurable maps.*

1.5 Lebesgue integral

Example 1.5.1. *Define Characteristic function* $\chi_A : X \to \mathbb{R}$, $A \in \mathcal{A}$. *We define* $I(A) := \mu(A)$. *Surprisingly,* $I(A)$ *is nothing but the integral of* χ_A *over* A.

Definition 1.5.1 (Simple/Step/Staircsae functions,...). *For* $A_1, A_2, \ldots, A_n \in \mathcal{A}$, *and* $c_1, c_2, \cdots, c_n \in \mathbb{R}$. *We define*

$$f(x) := \sum_{i=1}^{n} c_i \cdot \chi_{A_i}(x). \tag{1.24}$$

We then have $f(x)$ *is measurable and the integraal of* f *is defined as* $I(f) := \sum_{i=1}^{n} c_i \mu(A_i)$.

Remark 1.5.1. *The problem of the integral* $I(f)$ *is that it is undefined when* $\mu(A_i) = \infty$. *The problem can be solved by exclude* ∞ *by defintion or the following way.*

Definition 1.5.2 (Lebesgue integral). *Define* $S^+ := \{f : X \to \mathbb{R} | f \ simple\ function,\ f \geq 0\}$. $f \in S^+$ *and choose representation* $f(x) = \sum_{i=1}^{n} c_i \chi_{A_i}(x)$, $c_i \geq 0$. *The lebesgue integral of* f *w.r.t.* μ *is defined as*

$$\int_X f(x)\ \mathrm{d}\mu(x) = \int_X f\ \mathrm{d}\mu \tag{1.25}$$
$$= I(f) \tag{1.26}$$
$$= \sum_{i=1}^{n} c_i \cdot \mu(A_i) \tag{1.27}$$
$$= [0, \infty]. \tag{1.28}$$

Property 1.5.1.
- $I(\alpha f + \beta g) = \alpha I(f) + \beta I(g)$, $\alpha, \beta \geq 0$.
- $f \leq g \Longrightarrow I(f) \leq I(g)$ *(monotomicity)*

Definition 1.5.3. *Define a measurable map* $f : X \to [0, \infty)$. $h = \sum_{i=1}^{n} c_i \cdot \chi_{A_i}$. *The lebesgue integral of* f *w.r.t.* μ *is defined as*

$$\int_X f\ \mathrm{d}\mu := \sup \left\{ I(h) | h \in S^+,\ h \leq f \right\} \tag{1.29}$$
$$\in [0, \infty]. \tag{1.30}$$

f *is called* μ*-integrable if* $\int_X f\ \mathrm{d}\mu < \infty$.

Property 1.5.2. *Define measurable maps $f, g : X \to [0, \infty)$, we have*

- *1. $f = g$ for μ-almost everywhere(a.e.), which satisfies $\mu\left(\{x \in X | f(x) \neq g(x)\}\right) =$ $\implies \int_X f \; \mathrm{d}\mu = \int_X g \; \mathrm{d}\mu$.*
- *2. $f \leq g$ for μ a.e. $\implies \int_X f \; \mathrm{d}\mu \leq \int_X g \; \mathrm{d}\mu$*
- *3. $f = 0$ for μ-a.e. $\iff \int_X f \; \mathrm{d}\mu = 0$.*

Proof of 2.: monotonicity. Let $h := X \to [0, \infty)$ be a simple function, i.e.,

$$h(x) = \sum_{i=1}^{n} c_i \chi_{A_i}(x) \tag{1.31}$$

$$= \sum_{t \in h(X)} t \cdot \chi_{\{x \in X | h(x) = t\}}. \tag{1.32}$$

Let $X = \tilde{X}^c \cup \tilde{X}$ with $\mu(\tilde{X}^c) = 0$,

$$\tilde{h}(x) := \begin{cases} h(x), & x \in \tilde{X} \\ a, & x \in \tilde{X}^c \end{cases} \tag{1.33}$$

$$\tilde{h}(x) = \sum_{t \in h(X)} t \cdot \chi_{\left\{x \in \tilde{X} | h(x) = t\right\}} + a \cdot \chi_{\tilde{X}^c} \tag{1.34}$$

$$I(\tilde{h}) = \sum_{t \in h(X)} t \cdot \mu(\left\{x \in \tilde{X} | h(x) = t\right\}) + a \cdot \mu(\tilde{X}^c) \tag{1.35}$$

$$= \sum_{t \in h(X)} t \left[\mu\left(\left\{x \in \tilde{X} | h(x) = t\right\}\right) + \mu\left(\left\{x \in \tilde{X}^c | h(x) = t\right\}\right)\right] \tag{1.36}$$

$$= \sum_{t \in h(X)} t \left[\mu\left(\left\{x \in \tilde{X} | h(x) = t\right\} \cup \left\{x \in \tilde{X}^c | h(x) = t\right\}\right)\right] \tag{1.37}$$

$$I(h) = \sum_{t \in h(X) \setminus \{0\}} t \cdot \mu\left(\{x \in X | h(x) = t\}\right). \tag{1.38}$$

We define

$$\tilde{X} := \{x \in X | f(x) \leq g(x)\}, \tag{1.39}$$

$$\mu(\tilde{X}^c) = 0 \tag{1.40}$$

$$\int_X f \; \mathrm{d}\mu = \sup\left\{I(h) | h \in S^+, h \leq f\right\} \tag{1.41}$$

$$= \sup\{I(\tilde{h}) | \tilde{h} \in S^+, \tilde{h} \leq f \text{ on } \tilde{X}\} \tag{1.42}$$

$$\leq \sup\{I(\tilde{h}) | \tilde{h} \in S^+, h \leq g \text{ on } \tilde{X}\} \tag{1.43}$$

$$= \sup\{I(h) | h \in S^+, h \leq g\} \tag{1.44}$$

$$= \int_X g \; \mathrm{d}\mu. \tag{1.45}$$

□

Theorem 1 (Monotone convergence theorem). *$(X, \mathcal{A}, \mu)$ measurable spaces, $f_n : X \to [0, \infty]$, $(f : X \to [0, \infty])$ measurable for all $n \in \mathbb{N}$ with*

$$f_1 \leq f_2 \leq f_3 \leq \cdots \quad \mu - a.e. \tag{1.46}$$

$$(\lim_{n\to\infty} \int_X f_n \, \mathrm{d}\mu = \int_X f \, \mathrm{d}\mu \quad \mu - a.e.(x \in X)) \tag{1.47}$$

This implies that

$$\lim_{n\to\infty} \int_X f_n \, \mathrm{d}\mu = \int_X \lim_{n\to} f_n \, \mathrm{d}\mu. \tag{1.48}$$

Proof. $\int_X f_1 \, \mathrm{d}\mu \leq \int_X f_2 \, \mathrm{d}\mu \leq \cdots$ and $\int_X f_n \, \mathrm{d}\mu \leq \int_X f \, \mathrm{d}\mu$ for $n \in \mathbb{N}$. Then we have

$$\lim_{n\to\infty} \int_X f_n \, \mathrm{d}\mu \leq \int_X f \, \mathrm{d}\mu, \tag{1.49}$$

which is the first part of 1.48.

Let h be a simple function $0 \leq h \leq f$ and $\varepsilon > 0$. We define

$$X_n := \{x \in X | f_n(x) \geq (1 - \varepsilon)h(x)\} \tag{1.50}$$

with $\cup_{n=1}^{\infty} X_n = \tilde{X}$, and $\mu(\tilde{X}^c) = 0$. We have

$$\int_X f_n \, \mathrm{d}\mu \geq \int_{X_n} f_n \, \mathrm{d}\mu \geq \int_{X_n} (1 - \varepsilon)h \, \mathrm{d}\mu \tag{1.51}$$

$$\lim_{n\to\infty} \int_X f_n \, \mathrm{d}\mu \geq \lim_{n\to\infty} \int_{X_n} (1 - \varepsilon)h \, \mathrm{d}\mu \tag{1.52}$$

$$= \int_{\tilde{X}} (1 - \varepsilon)h \, \mathrm{d}\mu \tag{1.53}$$

$$= \int_X (1 - \varepsilon)h \, \mathrm{d}\mu. \tag{1.54}$$

This implies

$$\lim_{n\to\infty} \int_X f_n \, \mathrm{d}\mu \geq \int_X h \, \mathrm{d}\mu, \tag{1.55}$$

since $\varepsilon > 0$ arbitrarily. Then we have

$$\lim_{n\to\infty} \int_X f_n \, \mathrm{d}\mu \geq \int_X f \mathrm{d}\mu, \tag{1.56}$$

since h is arbitrary and $h \leq f$, which is second part of 1.48. □

Applictions Given a series $(g_n)_{n\in\mathbb{N}}$, $g_n : X \to [0, \infty]$ measurable for all n. Then we have $\sum_{n=1}^{\infty} g_n : X \to [0, \infty]$ measurable and

$$\int_X \sum_{n=1}^{\infty} g_n \, \mathrm{d}\mu = \sum_{n=1}^{\infty} \int_X g_n \, \mathrm{d}\mu, \tag{1.57}$$

which means the integral and sum can exchange.

1.6 Fatou' lemma

Lemma 1.6.1 (Fatou' lemma)**.** *Given* $(X, \mathcal{A}, \mu)$ *measurable space,* $f_n : X \to [0, \infty]$ *measurable for all* $n \in \mathbb{N}$*. Then we have*

$$\int_X \liminf_{n\to\infty} f_n \, \mathrm{d}\mu \leq \liminf_{n\to\infty} \int_X f_n \, \mathrm{d}\mu. \tag{1.58}$$

Remark 1.6.1. $\liminf_{n\to\infty} f_n : X \to [0, \infty]$ *is a function. This is*

$$\begin{aligned} g(x) &:= \left(\liminf_{n\to\infty} f_n\right)(x) &(1.59)\\ &:= \lim_{n\to\infty}\left(\inf_{k\geq n} f_k(x)\right) &(1.60)\\ &\in [0,\infty] &(1.61)\\ g_n(x) &:= \inf_{k\geq n} f_k(x). &(1.62)\end{aligned}$$

We have

$$g_1 \leq g_2 \leq g_3 \leq \cdots, \tag{1.63}$$

which is monotonically increasing. All these functions are measurable.

Proof.

Since (1),

$$\begin{aligned} \int_X \lim_{n\to\infty} g_n \, \mathrm{d}\mu &= \lim_{n\to\infty} \int_X g_n \, \mathrm{d}\mu &(1.64)\\ &= \liminf_{n\to\infty} \int_X g_n \, \mathrm{d}\mu. &(1.65)\end{aligned}$$

We know that $g_n \leq f_n$ for all $n \in \mathbb{N}$. By (1.5.2), we have

$$\int_X g_n \, \mathrm{d}\mu \leq \int_X f_n \, \mathrm{d}\mu, \tag{1.66}$$

for all $n \in \mathbb{N}$. Then we have

$$\begin{aligned} \int_X \liminf_{n\to\infty} f_n \, \mathrm{d}\mu &= \liminf_{n\to\infty} \int_X g_n \, \mathrm{d}\mu &(1.67)\\ &\leq \liminf_{n\to\infty} \int_X f_n \, \mathrm{d}\mu. &(1.68)\end{aligned}$$

□

1.7 Lebesgue's dominated convergence theorem

$(X, \mathcal{A}, \mu)$, $\mathcal{L}^1 := \left\{f : X \to \mathbb{R} \text{ } \textit{measurable} \middle| \int_X |f|^1 \, \mathrm{d}\mu < \infty\right\}$. For $f \in \mathcal{L}^1(\mu)$, write $f = f^+ - f^-$, where $f^+, f^- \geq 0$. Define $\int_X f \, \mathrm{d}\mu := \int_X f^+ \, \mathrm{d}\mu - \int_X f^- \, \mathrm{d}\mu$.

Theorem 2 (Lebesgue's dominated convergence theorem)**.** *$f_n : X \to \mathbb{R}$ measurable for all $n \in \mathbb{N}$. $f : X \to \mathbb{R}$ with $\overset{n\to\infty}{f(x)}$ for $x \in X$ (μ-a.e.) and $|f_n| \leq g$ with $g \in \mathcal{L}^1(\mu)$ for all $n \in \mathbb{N}$, where g is called integral majorant. Then: we have $f_1, f_2, \dots \in \mathcal{L}^1(\mu)$, $f \in \mathcal{L}^1(\mu)$ and*

$$\lim_{n\to\infty} \int_X f_n \, \mathrm{d}\mu = \int_X f \, \mathrm{d}\mu. \tag{1.69}$$

Proof.

$$|f_n| \leq g \overset{monotonicity}{\Longrightarrow} \int_X g \, \mathrm{d}\mu < \infty \tag{1.70}$$

$$\Longrightarrow f_1, f_2, \dots \in \mathcal{L}^1(\mu) \tag{1.71}$$

$$|f| \leq g \; for \; \mu - \text{a.e.} \Longrightarrow f \in \mathcal{L}^1(\mu) \tag{1.72}$$

We will show $\int_X |f_n - f| \, \mathrm{d}\mu \overset{n\to\infty}{\Longrightarrow} 0$.

$$|f_n - f| \leq |f_n| + |f| \leq 2g \tag{1.73}$$

$$\Longrightarrow h_n := 2g - |f_n - f| \geq 0 \tag{1.74}$$

Hence: $h_n : X \to [0, \infty]$ measurable for all $n \in \mathbb{N}$. Then by (1.6.1),

$$\Longrightarrow \int_X \liminf_{n\to\infty} h_n \, \mathrm{d}\mu \leq \liminf_{n\to\infty} \int_X h_n \, \mathrm{d}\mu \tag{1.75}$$

$$\Longrightarrow \int_X 2g \, \mathrm{d}\mu \leq \int_X 2g \, \mathrm{d}\mu - \limsup_{n\to\infty} \int_X |f_n - f| \, \mathrm{d}\mu \tag{1.76}$$

$$\Longrightarrow 0 \leq \liminf_{n\to\infty} \int_X |f_n - f| \, \mathrm{d}\mu \leq \limsup_{n\to\infty} \int_X |f_n - f| \, \mathrm{d}\mu \leq 0 \tag{1.77}$$

$$\Longrightarrow \tag{1.78}$$

Limits exists and $\lim_{n\to\infty} |f_n - f| \, \mathrm{d}\mu = 0$. We conclude that

$$\tag{1.79}$$

$$0 \leq |\int_X f_n \, \mathrm{d}\mu - \int_X f \, \mathrm{d}\mu| = |\int_X (f_n - f) \, \mathrm{d}\mu| \leq \int_X |f_n - f| \, \mathrm{d}\mu \overset{n\to\infty}{\longrightarrow} 0, \tag{1.80}$$

where the third inequality is due to the integral's triangle inequality.

$$\tag{1.81}$$

$$\Longrightarrow \lim_{n\to\infty} \int_X f_n \, \mathrm{d}\mu = \int_X f \, \mathrm{d}\mu. \tag{1.82}$$

□

1.8 Caratheodory's extension theorem

Theorem 3 (Caratheodory's extension theorem)**.** *X set, $\mathcal{A} \in \mathcal{P}(X)$ semiring of sets. A map $\mu : \mathcal{A} \to [0, \infty]$. Note that μ is not a measure, it is called A pre-measure.*

- *Then μ has an extension $\tilde{\mu} : \sigma(\mathcal{A}) \to [0, \infty]$, where $\tilde{\mu}$ is a measure and $\sigma(\mathcal{A})$ is a σ-algebra generated by $\mathcal{A}$, i.e., $\mu(A) = \tilde{\mu}(A)$.*

- *If there is sequence (S_j) with $S_j \in \mathcal{A}$, $\cup_{j=1}^{\infty} S_j = X$, then the extension $\tilde{\mu}$ from (a) is unique. ($\tilde{\mu}$ is also σ-finite)*

Definition 1.8.1 (Semiring set)**.** *Semiring of sets $\mathcal{A} \subseteq \mathcal{P}(X)$:*

- *$\emptyset \in \mathcal{A}$ (as for σ-algebra)*
- *$A, B \in \mathcal{A} \Longrightarrow A \cap B \in \mathcal{A}$*
- *For $A, B \in \mathcal{A}$, there are pairwise disjoint sets $S_1, S_2, \dots, S_n \in \mathcal{A} : \cup_{j=1}^{n} S_j = A \backslash B$*

Example 1.8.1. *$A := \{[a, b) | a, b \in \mathbb{R}, a \leq b\}$ not a σ-algebra because $\mathbb{R} \notin \mathcal{A}$. But $\sigma(\mathcal{A}) = \mathcal{B}(\mathbb{R})$ (Borel σ-algebra). Check that $\mathcal{A}$ is semiring set:*

- $\emptyset \in \mathcal{A}$
- $$[a, b) \cap [c, d) = \begin{cases} \emptyset, & b \leq c, d \leq a \\ [c, b), & c \in [a, b), d \notin [a, b) \\ \cdots \end{cases} \tag{1.83}$$
- $$[a, b) \backslash [c, d) = \begin{cases} [a, b), & d \leq a, b \leq c \\ [a, c), & c \in [a, b), d \notin [a, b) \\ [a, c) \cup [d, b), & c > a, d < b \\ \cdots \end{cases} \tag{1.84}$$

Definition 1.8.2 (Pre-measure)**.** *$\mu : \mathcal{A} \to [0, \infty]$ with $\mathcal{A}$ semiring os sets:*

- $\mu(\emptyset) = 0$
- $\mu(\cup_{j=1}^{\infty}) = \sum_{j=1}^{\infty} \mu(A_j)$, *for $A_j \in \mathcal{A}$, $A_i \cap A_j = \emptyset$ for $i \neq j$ and $\cup_{j=1}^{\infty} A_j \in \mathcal{A}$.*

Application: $A := \{[a, b) | a, b \in \mathbb{R}, a \leq b\}$, $\mu : \mathcal{A} \to [0, \infty]$, $\mu([a, b)) = b - a$ is a pre-measure (We can check by the definition of pre-measure). Then by (3), there is a unique extension to $\mathcal{B}(\mathbb{R}) \Longrightarrow$ lebesgue measure.

1.9 Lebesgue-Stieltjes measures

$F : \mathbb{R} \to \mathbb{R}$ monotonically increasing (non-decreasing).$[a, b)$ is the length of the interval. Now we consider new kinds of intervals:

$$F(b^-) - F(a^-) =: \mu_F([a, b)), \tag{1.85}$$

where $F(a^-) := \lim_{\varepsilon \to 0^+} F(a - \varepsilon)$. Alternatively, we also have

$$F(b^+) - F(a^+) =: \mu_F((a, b]), \tag{1.86}$$

where $F(a^+) := \lim_{\varepsilon \to 0^+} F(a + \varepsilon)$. We consider the previous one hereafter.

Definition 1.9.1. *$\mathcal{A} := \{[a,b) : a, b \in \mathbb{R}, a \leq b\}$ semiring of sets. Then by Caratheodory' theorem, we have that there exists exactly one measure*

$$\mu_F : \mathcal{B}(\mathbb{R}) \to [0, \infty] \tag{1.87}$$

with $\mu_F([a,b))$

$$. \tag{1.88}$$

Example 1.9.1.

- *$F(x) = x$, $\mu_F([a,b)) = b - a \to$ Lebesgue measure.*
- *$F(x) = 1$, $\mu_F([a,b)) = 0 \to$ zero measure.*
-
$$F(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases} \tag{1.89}$$
  $\mu_F([-\varepsilon, \varepsilon)) = 1 \to$ Dirac measure δ_0.
- *$F : \mathbb{R} \to \mathbb{R}$ monotonically increasing + continuously differentiable. Then we have*
$$F' : \mathbb{R} \to [0, \infty) \tag{1.90}$$
  and
$$\mu_F([a,b)) = F(b) - F(a) \tag{1.91}$$
$$= \int_a^b F'(x) \, \mathrm{d}x, \tag{1.92}$$
  which implies
$$\mu_F : A \longmapsto \int_A F'(x) \, \mathrm{d}x, \tag{1.93}$$
  where $F'(x)$ is called the density function.

1.10 Radon-Nikodym theorem and Lebesgue's decomposition theorem

$(X, \mathcal{A}, \lambda)$ measure space. Special case: $X = \mathcal{R}$, $\mathcal{A} = \mathcal{B}(\mathbb{R})$, and λ is lebesgue measure. Recall that $\lambda([a,b)) = b - a$. Another measure $\mu : \mathcal{B}(\mathbb{R}) \to [0, \infty]$. We will look how μ acts w.r.t. the given reference measure: lebesgue measure.

Definition 1.10.1.

- *μ is called absolutely continuous (w.r.t. λ) if $\lambda(A) = 0 \Longrightarrow \mu(A) = 0$ for all $A \in \mathcal{B}(\mathbb{R})$. One writes: $\mu << \lambda$.*
- *μ is called singular (w.r.t. λ) if there is $N \in \mathcal{B}(\mathbb{R})$ with $\lambda(N) = 0$ and $\mu(N^c) = 0$. One writes: $\mu \perp \lambda$.*

  Example 1.10.1. *δ_0 Dirac measure ($\delta_0(\{0\}) = 1$) $\Longrightarrow \delta_0 \perp \lambda$ (Choose $N = \{0\}$).*

Theorem 4 (Lebesgue's decomposition theorem). *$\mu : \mathcal{B}(\mathbb{R}) \to [0, \infty]$ (σ-finite)*

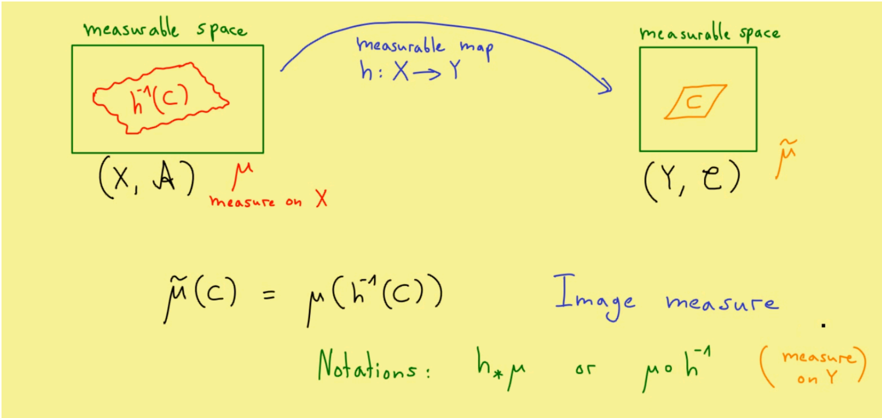
- *There are measures (uniquely determined) $\mu_{ac}, \mu_s : \mathcal{B}(\mathbb{R}) \to [0, \infty]$ with $\mu = \mu_{ac} + \mu_s$, $\mu_{ac} << \lambda, \mu_s \perp \lambda$.*

Theorem 5 (Radon-Nikodym theorem). *$\mu : \mathcal{B}(\mathbb{R}) \to [0, \infty]$ (σ-finite)*

- *There is a measurable map $h : \mathbb{R} \to [0, \infty)$ with $\mu_{ac} = \int_A d \, \mathrm{d}\lambda$ for all $A \in \mathcal{B}(\mathbb{R})$, where h is called the density function.*

1.11 Image measure and substitution formula

Image measure is also called pushforward measure. Substitution formula is also called change of variable.



Definition 1.11.1 (Image measure)**.** *Measure space* $(X, \mathcal{A})$*,* μ *is a measure on* X*. Measure space* $(Y, \mathcal{E})$*,* $\tilde{\mu}$ *is a measure on* Y*. Define a measure map* $h : X \to Y$*. See the above figure. We then define the image measure as*

$$\tilde{\mu}(c) = \mu(h^{-1}(c)). \tag{1.94}$$

The notations: $h * \mu$ *or* $\mu \circ h^{-1}$*.* $h * \mu$ *means pushforward and* $\mu \circ h^{-1}$ *is readble. Remember that* $\tilde{\mu}$ *is aa measure on* Y*.*

Lemma 1.11.1 (Substitution formula)**.** *A integrable function* $g : Y \to \mathbb{R}$*. We have*

$$\int_Y g \ \mathrm{d}(h * \mu) = \int_X g \circ h \ \mathrm{d}\mu, \tag{1.95}$$

which can also be written as

$$\int_Y g(y) \ \mathrm{d}(\mu \circ h^{-1})(y) = \int_X g(h(x)) \ \mathrm{d}\mu(x), \tag{1.96}$$

which is called the change of variables: $y = h(x)$*.*

Example 1.11.1. *F is a strictly monotonically increasing and continuously differentiable and surjective function from* $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ *with* μ_F *as* $\mu_F(A) = \int_A F'(x) \ \mathrm{d}x$ *to* $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$*. We have*

$$
\begin{aligned}
(F * \mu_F)([a, b)) &= \mu_F(F^{-1}([a, b))) && (1.97)\\
&= \mu_F([F^{-1}(a), F^{-1}(b)]) && (1.98)\\
&= \int_{F^{-1}(a)}^{F^{-1}(b)} F'(x) \ \mathrm{d}x && (1.99)\\
&= \int_a^b \mathrm{d}y && (1.100)\\
&= \lambda([a, b)) && (1.101)\\
&\Longrightarrow F_x \mu_F = \lambda, && (1.102)
\end{aligned}
$$

Substitution formula:

$$\int_Y g \,\mathrm{d}(F * \mu_F) = \int_X g \circ F \,\mathrm{d}\mu_F \tag{1.103}$$

$$\Longrightarrow \int_{\mathbb{R}} g(y) \,\mathrm{d}y = \int_{\mathbb{R}} g(F(x))F'(x) \,\mathrm{d}x. \tag{1.104}$$

Proof. (1) Let $g = \chi_c$ with $C \subseteq Y$ measurable. For the left hand side, we have

$$\int_Y \chi_c \,\mathrm{d}(h * \mu) = (h * \mu)(c) \tag{1.105}$$

$$= \mu(h^{-1}(c)). \tag{1.106}$$

For the right hand side, we have

$$\int_X \chi_c \circ h \,\mathrm{d}\mu = \int_X \chi_c \circ h \,\mathrm{d}\mu \tag{1.107}$$

$$= \int_X \chi_c(h(x)) \,\mathrm{d}\mu(x) \tag{1.108}$$

$$= \int_X \chi_{h^{-1}(c)} \,\mathrm{d}\mu \tag{1.109}$$

$$= \mu(h^{-1}(c)), \tag{1.110}$$

where

$$\chi_c(h(x)) = \begin{cases} 1, & x \in h^{-1}(c) \\ 0, & x \notin h^{-1}(c) \end{cases} \tag{1.111}$$

(2) Let g be a simple function, i.e.,$g = \sum_{i=1}^n \chi_{c_i}$. We then obtatin

$$\int_Y \sum_{i=1}^n \lambda_i \chi_{c_i} \,\mathrm{d}(h * \mu) = \sum_{i=1}^n \lambda_i \int_Y \chi_{c_i} \,\mathrm{d}(h * \mu) \tag{1.112}$$

By (1)

$$= \sum_{i=1}^n \lambda_i \int_X \chi_{c_i}(h(x)) \,\mathrm{d}\mu(x) \tag{1.113}$$

$$= \int_X (\sum_{i=1}^n \lambda_i \chi_{c_i})(h(x)) \,\mathrm{d}\mu(x) \tag{1.114}$$

$$= \int_X g \circ h \,\mathrm{d}\mu. \tag{1.115}$$

(3) Let $g : Y \to [0, \infty)$ measurable. We have

$$\int_Y g \,\mathrm{d}(h * \mu) = \sup \left\{ \int_Y \tilde{s} \,\mathrm{d}(h * \mu) | \tilde{s} : Y \to [0, \infty) \ \textit{simple}, \ \tilde{s} \leq g \right\}. \tag{1.116}$$

We have the following equivalence relation:

$$\forall y \in h(x) : \tilde{s}(y) \leq g(y) \tag{1.117}$$

$$\Longleftrightarrow \forall x \in X : \tilde{s}(h(x)) \leq g(h(x)) \tag{1.118}$$

$$[i.e., \tilde{s} \circ h \leq (g \circ h)(x)]. \tag{1.119}$$

Then we have

$$\int_Y g \; \mathrm{d}(h * \mu) = \sup \left\{ \int_X \tilde{s} \circ \; \mathrm{d}\mu | \tilde{s} : Y \to [0, \infty) \; \textit{simple}, \tilde{s} \circ h \leq g \circ h \right\} \tag{1.120}$$

$$\overset{\text{Left as exercise}}{=} \sup \left\{ \int_X s \circ \; \mathrm{d}\mu | s : X \to [0, \infty) \; \textit{simple}, s \circ h \leq g \circ h \right\} \tag{1.121}$$

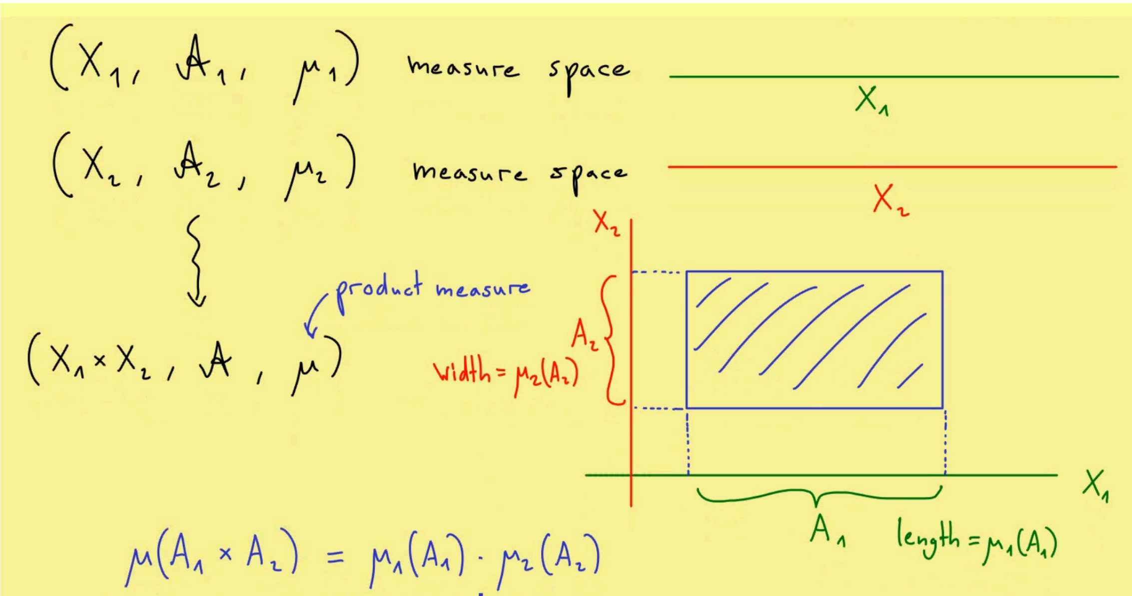
$$= \int_X g \circ h \; \mathrm{d}\mu. \tag{1.122}$$

□

1.12 Product measure and Cavalieri's principle

$(X_1, \mathcal{A}_1, \mu_1)$ measure space and $(X_2, \mathcal{A}_2, \mu_2)$ measure space,

$$\implies (X_1 \times X_2, \mathcal{A}, \mu), \; \textit{where } \mu \textit{ is the product measure.} \tag{1.123}$$



We have

$$\mu(A_1 \times A_2) = \mu_1(A_1) \cdot \mu_2(A_2). \tag{1.124}$$

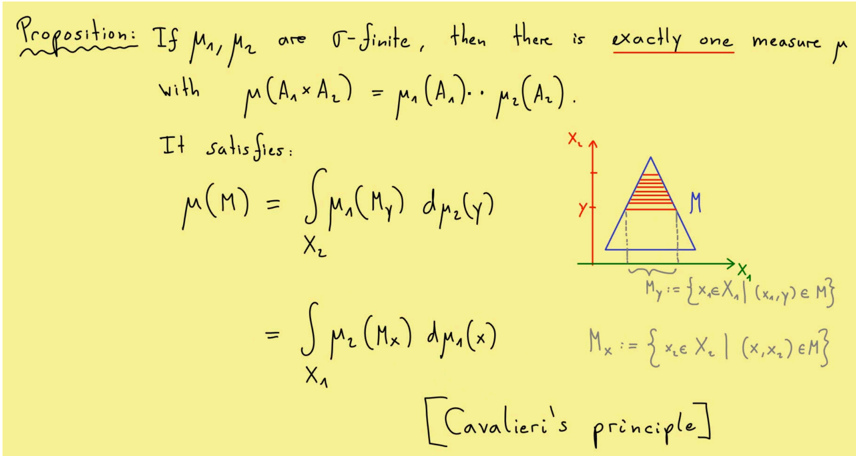
Definition 1.12.1 (Product σ-algebra)**.**

$$\mathcal{A} = \sigma(\mathcal{A}_1 \times \mathcal{A}_2). \tag{1.125}$$

Remark 1.12.1. *Set of rectangles (=$\mathcal{A}_1 \times \mathcal{A}_2$) are not a σ-algebra (but a semiring of sets)*

Definition 1.12.2. *Define product measure μ as $\mu(A_1 \times A_2) = \mu_1(A_1) \times \mu_2(A_2)$ for all $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$, and use (3).*

Remark 1.12.2. *Product measure in general not unique.*

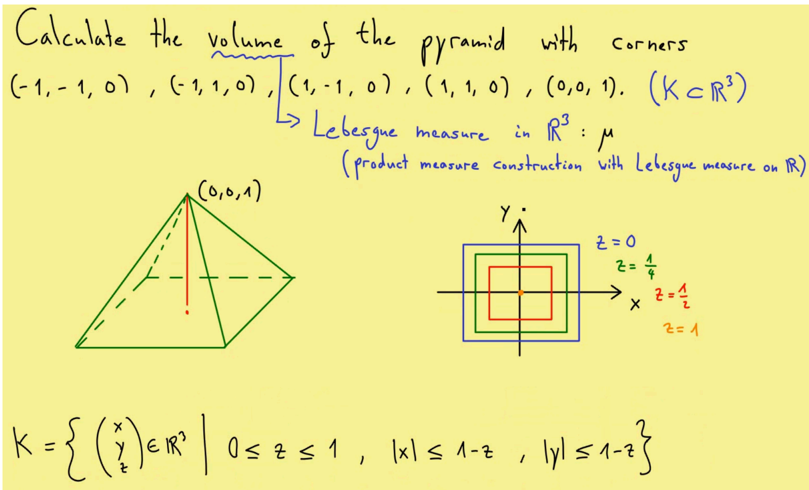


Proposition 1.12.1 (Cavalieri's principle)**.** *If μ_1, μ_2 are σ-finite, then there is exactly one measure μ with $\mu(A_1 \times A_2) = \mu_1(A_1) \cdot \mu_2(A_2)$. Is satisfies:*

$$\mu(M) = \int_{X_2} \mu_1(M_y)\ \mathrm{d}\mu_2(y) \tag{1.126}$$

$$= \int_{X_1} \mu_2(M_x)\ \mathrm{d}\mu_1(x). \tag{1.127}$$

Example 1.12.1 (An example for Cavalieri's principle)**.** *Calculate the volume of the pyramid with corners (-1,-1,0), (-1,1,0),(1,-1,0),(1,1,0),(0,0,1), $K \subset \mathbb{R}^3$, where the volume if the lebesgue measure in $\mathbb{R}^3$: μ(Recall product measure construction with lebesgus measure on $\mathbb{R}$).*



Proof. Set

$$K = \left\{(x,y,z)^T \in \mathbb{R}^3 | 0 \le z \le 1, |x| \le 1-z, |y| \le 1-z\right\}. \tag{1.128}$$

Define μ as a product measure of μ_1 and μ_2, where μ_1 is the lebesgue measure in $\mathbb{R}$(z-coordinate) and μ_2 is the lebesgue measure on $\mathbb{R}^2$(x- and y-coordinate). Following the definition of product measure, we have the volume of K as

$$\mu(k) = \int_{\mathbb{R}} \mu_2(M_{z_0}) \, \mathrm{d}\mu_1(z_0) \tag{1.129}$$

$$= \int_{[0,1]} 4 \cdot (1 - z_0)^2 \mathrm{d}\mu_1(z_0) \tag{1.130}$$

$$= \frac{4}{3}, \tag{1.131}$$

where

$$M_{z_0} := \left\{ (x,y)^T \in \mathbb{R}^2 | |x| \leq 1 - z_0, |y| \leq 1 - z_0 \right\}, \tag{1.132}$$

and $\mu_2(M_{z_0})$ is the area of the square only for $z_0 \in [0,1]$. □

1.13 Fubini's theorem

Theorem 6 (Fubini's theorem)**.** *Let μ_1 and μ_2 be σ-finite, μ be the product measure and*

$$f : X_1 \times X_2 \to [0, \infty] \textit{ measurable } [\textit{or } f \in \mathcal{L}^1(\mu)], \tag{1.133}$$

then:

$$\int_{X_1 \times X_2} f \, \mathrm{d}\mu = \int_{X_2} \left(\int_{X_1} f(x,y) \, \mathrm{d}\mu_1(x) \right) \, \mathrm{d}\mu_2(x) \tag{1.134}$$

$$= \int_{X_1} \left(\int_{X_2} f(x,y) \, \mathrm{d}\mu_2(x) \right) \, \mathrm{d}\mu_1(x). \tag{1.135}$$

Example 1.13.1. *μ lebesgue measure for $\mathbb{R}^2$. Calculate $\int_A f \, \mathrm{d}\mu =?$, where*

$$A = \left\{ (x,y) \in [0,1] \times [0,1] | x \geq y \geq x^2 \right\}, \tag{1.136}$$

$$f(x,y) = 2xy. \tag{1.137}$$

We have

$$\int_A f \, \mathrm{d}\mu = \int_{\mathbb{R}^2} f \cdot \chi_A \, \mathrm{d}\mu \tag{1.138}$$

$$= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x,y) \chi_A(x,y) \, \mathrm{d}y \right) \, \mathrm{d}x \tag{1.139}$$

$$= \int_0^1 \left(\int_x^{x^2} 2xy \, \mathrm{d}y \right) \, \mathrm{d}x \tag{1.140}$$

$$= \frac{1}{12}. \tag{1.141}$$

1.14 Outer measure

- tools for the proof of (3)

- "outer measure" is a new notion. "Outer measure" is not an attribute for "measure"! "Outer mesure" do not have to be measures!

Definition 1.14.1 (Outer measure)**.** *A map $\phi : \mathcal{P}(X) \to [0, \infty]$ is called an outer measure if:*

- *(a)* $\phi(\emptyset) = 0$
- *(b)* $A \subseteq B \implies \phi(A) \leq \phi(B)$. *(monotonicity)*
- *(c)* $A_1, A_2, \ldots, \in \mathcal{P}(X) \implies \phi\left(\cup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \phi(A_n)$. *($\sigma$-subadditivity)*

Question: $\phi : \mathcal{P}(X) \to [0, \infty]$ outer measure $\overset{?}{\longrightarrow}$ μ measrue?

Definition 1.14.2 (ϕ-measurable)**.** *Let ϕ be an outer measure. $A \in \mathcal{P}(X)$ is called ϕ-measurable if for all $Q \in \mathcal{P}(X)$ we have:*

$$\phi(Q) \geq \phi(Q \cap A) + \phi(Q \cap A^c). \tag{1.142}$$

Proposition 1.14.1. *If $\phi : \mathcal{P}(X) \to [0, \infty]$ is an outer measure, then:*

- $\mathcal{A}_\phi := \{A \subseteq X | A \ \phi \text{ measurable}\}$ *is a σ-algebra.*
- $\mu : \mathcal{A}_\phi \to [0, \infty]$, $\mu(A) := \phi(A)$, *is a measrue.*

- union with two sets: $A_1, A_2 \in \mathcal{A}_\varphi$

$$\varphi(Q) = \varphi(Q \cap A_1) + \varphi(\overbrace{Q \cap A_1^c}^{\tilde{Q}}) = \varphi(Q \cap A_1) + \varphi(\tilde{Q} \cap A_2) + \varphi(\tilde{Q} \cap A_2^c)$$

$$\geq \varphi\left((Q \cap A_1) \cup (\tilde{Q} \cap A_2)\right) + \varphi(\underbrace{\tilde{Q} \cap A_2^c}_{Q \cap (A_1^c \cap A_2^c)})$$

Proof.

- $\phi \in \mathcal{A}_\phi$? Is $\emptyset$ ϕ -measurable?

$$\phi(Q) = \phi(Q \cap \emptyset) + \phi(Q \cup \emptyset^c) \tag{1.143}$$
$$= 0 + \phi(Q) \tag{1.144}$$

- $X \in \mathcal{A}_\phi$? Is X ϕ-measurable?

$$\phi(Q) = \phi(Q \cap X) + \phi(Q \cap X^c) \tag{1.145}$$
$$= \phi(Q) + \phi(\emptyset). \tag{1.146}$$

- $A \in \mathcal{A}_\phi \implies$

$$\phi(Q) = \phi(Q \cap A) + \phi(Q \cap A^c) \tag{1.147}$$
$$= \phi(Q \cap A^c) + \phi(Q \cap (A^c)^c) \tag{1.148}$$
$$\implies A^c \in \mathcal{A}_\phi. \tag{1.149}$$

- union with two sets: $A_1, A_2 \in \mathcal{A}$

$$
\begin{aligned}
\phi(Q) &= \phi(Q \cap A_1) + \phi(Q \cap A_1^c) && (1.150)\\
&\overset{\text{Define } \tilde{Q} := Q \cap A_1^c}{=} \phi(Q \cap A_1) + \phi(\tilde{Q} \cap A_2) + \phi(\tilde{Q} \cap A_2^c) && (1.151)\\
&\geq \phi\Big((Q \cap A_1) \cup (\tilde{Q} \cap A_2)\Big) + \phi(\tilde{Q} \cap A_2^c) && (1.152)\\
&= \phi(Q \cap (A_1 \cup A_2)) + \phi(Q \cap (A_1 \cup A_2)^c), && (1.153)\\
&\implies \phi(Q) \geq \phi(Q \cap (A_1 \cup A_2)) + \phi(Q \cap (A_1 \cup A_2)^c) && (1.154)\\
&\implies A_1 \cup A_2 \in \mathcal{A}_\phi, && (1.155)
\end{aligned}
$$

where the fouth equation is obtain by the above figure.

- countable union: $A_1, A_2, \cdots \in \mathcal{A}_\phi$, $A := \cup_{j=1}^{\infty} A_j \in \mathcal{A}_\phi$?

$$
\begin{aligned}
\phi(Q) &= \phi(Q \cap A_1) + \phi(Q \cap A_1^c) && (1.156)\\
&\overset{\text{Set } Q = \hat{Q} \cap (A_1 \cup A_2)}{=} \phi(\hat{Q} \cap A_1) + \phi(\hat{Q} \cap A_2). && (1.157)
\end{aligned}
$$

Induction: $\phi(\hat{Q} \cap \cup_{j=1}^{n} A_j) = \sum_{j=1}^{n} \phi(\hat{Q} \cap A_j)$. We have:

$$
\begin{aligned}
\phi(\hat{Q}) &= \phi(\hat{Q} \cap \cup_{j=1}^{n}) + \phi(\hat{Q} \cap (\cup_{j=1}^{n} A_j)^c) && (1.158)\\
&\geq \sum_{j=1}^{n} \phi(\hat{Q} \cap A_j) + \phi(\hat{Q} \cap A^c) && (1.159)\\
\implies \phi(\hat{Q}) &\geq \sum_{j=1}^{n} \phi(\hat{Q} \cap A_j) + \phi(\hat{Q} \cap A^c) && (1.160)\\
&\geq \phi(\hat{Q} \cap A) + \phi(\hat{Q} \cap A^c) && (1.161)\\
&\geq \phi(\hat{Q}) && (1.162)\\
&\implies A \in \mathcal{A}_\phi. && (1.163)
\end{aligned}
$$

$\square$

Example 1.14.1. *(1)* $\phi : \mathcal{P}(\mathbb{R}) \to [0, \infty]$,

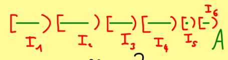
$$
\phi(A) = \begin{cases} 0, & A = \emptyset \\ 1, & A \neq \emptyset. \end{cases} \qquad \implies \textit{outer measure but not a measure!} \qquad (1.164)
$$

Example 1.14.2. $\phi : \mathcal{P}(\mathbb{N}) \to [0, \infty]$,

$$
\begin{aligned}
\phi(A) &= \left\{\begin{matrix} |A|, & A \textit{ finite} \\ \infty, & A \textit{ not finite.} \end{matrix}\right\} && (1.165)\\
&\implies \textit{outer measure but a measure!(counting measure)} && (1.166)
\end{aligned}
$$

(3) $\mathcal{I} = \{ [a,b) \mid a,b \in \mathbb{R} , \ a \leq b \}$, $\mu([a,b)) = b - a$ ("length")

Define $\varphi : \mathcal{P}(\mathbb{R}) \longrightarrow [0,\infty)$ by :



$$\varphi(A) := \inf \left\{ \sum_{j=1}^{\infty} \mu(I_j) \;\middle|\; I_j \in \mathcal{I} , \ A \subseteq \bigcup_{j=1}^{\infty} I_j \right\}$$

$\rightsquigarrow \varphi$ is an outer measure!

Example 1.14.3. *$\mathcal{I} = \{[a,b)|a,b \in \mathbb{R}, \ a \leq b\}$, $\mu([a,b)) = b - a$("length").*

Define $\phi : \mathcal{P}(\mathbb{R}) \to [0,\infty)$ by:

$$\phi(A) := \inf \left\{ \sum_{j=1}^{\infty} \mu(I_j) | I_j \in \mathcal{I}, \ A \subseteq \cup_{j=1}^{\infty} I_j \right. \tag{1.167}$$

$$\Longrightarrow \phi \textit{ is an outer measure!} \tag{1.168}$$

Proof. check (a) of (1.14.1): $\phi(\emptyset) = 0$.

check (b) of (1.14.1): monotonicity,

$$A \subseteq B \Longrightarrow \phi(B) \tag{1.169}$$
$$= \inf \left\{ \sum_{j=1}^{\infty} \mu(I_j) | I_j \in \mathcal{I}, \ B \subseteq U_{j=1}^{\infty} I_j \right\} \tag{1.170}$$
$$\geq \inf \left\{ \sum_{j=1}^{\infty} \mu(I_j) | I_j \in \mathcal{I}, \ A \subseteq U_{j=1}^{\infty} I_j \right\}, \tag{1.171}$$

since $A \subseteq B$.

check (c) of (1.14.1): show that $\phi(\cup_{n \in \mathbb{N}} A_n) \leq \sum_{n \in \mathbb{N}} \phi(A_n)$. Let $\varepsilon > 0$. Choose $\varepsilon_n > 0$ with $\sum_{n \in \mathbb{N}} \varepsilon_n = \varepsilon$. Then there are intervals $I_{j,n}$ with:

$$\phi(A_n) \geq \sum_{j=1}^{\infty} \mu(I_{j,n}) - \varepsilon_n, \tag{1.172}$$

and

$$A_n \subseteq \cup_{j=1}^{\infty} I_{j,n}. \tag{1.173}$$

Then: $\cup_{n\in\mathbb{N}} \subseteq \cup_{n\in\mathbb{N}} \cup_{j\in\mathbb{N}} I_{j,n} = \cup_{j,n} I_{j,n}$.

$$\implies \phi(\cup_{n\in\mathbb{N}}) \overset{(b)}{\leq} \phi(\cup_{j,n} I_{j,n}) \tag{1.174}$$

$$\leq \sum_{j,n} \mu(I_{j,n}) \tag{1.175}$$

$$= \sum_{n\in\mathbb{N}} \left\{ \sum_{j\in\mathbb{N}} \mu(I_{j,n}) \right\} \tag{1.176}$$

$$\leq \sum_{n\in\mathbb{N}} (\phi(A_n) + \varepsilon_n) \tag{1.177}$$

$$= \sum_{n\in\mathbb{N}} \phi(A_n) + \varepsilon. \tag{1.178}$$

□