Chapter 1

Representation Theory of Finite Groups

1.1 Irreducible Representations

Definition 1.1.1. *Representation $\phi : G \to GL(V)$ is irreducible if $V \neq \{0\}$ and the only invariant subspaces are $\{0\}$ and V.*

Example 1.1.1. *Every 1-dimensional rep. is irreducible.*

Example 1.1.2. *$\rho : S_3 \to GL_3(\mathbb{C})$ standard rep. not irreducible.*

$$W = \mathbb{C}v,\ v = e_1 + e_2 + e_3 \tag{1.1}$$

$$U = \mathbb{C}x + \mathbb{C}y,\ x = e_1 + e_2,\ y = e_2 - e_3 \tag{1.2}$$

$$\Longrightarrow (\phi|_w) \text{ and } (\phi|_U) \text{ are irreducible } S_3 - reps. \tag{1.3}$$

Proof. $(\rho|_U) \sim \phi : S_3 \to GL_2(\mathbb{C})$.

$$\phi_{(12)} \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \tag{1.4}$$

$$\phi_{(123)} \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \tag{1.5}$$

Claim: ϕ irreducible. dim $\phi = 2$. Consider $H = \mathbb{C}_z,\ z = \mathbb{C}^2 \setminus \{0\}$. H invariant $\iff$ $\phi_{(12)}(z), \phi_{(123)}(z) \in \mathbb{C}_z$ i.e., z is an eigenvector of both matrixces. We can compute that $\phi_{(12)}$ and $\phi_{(123)}$ have no common eigenvectors. □

1.2 Morphism (Homomorphism) of Representations

Definition 1.2.1. *$\phi : G \to GL(V)$, $\psi : G \to GL(W)$, a morphism of representations from ϕ to ψ is a linear map $T : V \to W$ such that*

$$\psi_g \circ T = T \circ \phi_g,\ \forall g \in G. \tag{1.6}$$

(also called intertwining operator)

Note: if T isomorphism of vector spaces, $T^{-1}: W \to V \iff \psi_g = T \circ \phi_g \circ T^{-1}$.

Notations: V, W vector spaces. $Hom(V, W) := \{T : V \to W \textit{ linear maps}\}$, where $Hom(V, W)$ is also a vector space.

Definition 1.2.2. *Define $Hom_G(\phi, \psi) := \{T \in Hom(V, W) | \psi_g T = T\phi_g, \forall g \in G\}$, which is a vector space of $Hom(V, W)$ meaning that if $T_1, T_2 \in Hom_G(\phi, \psi)$, then $c_1T_1 + c_2T_2 \in Hom_G(\phi, \psi)$, $c_1, c_2 \in \mathbb{C}$.*

Proposition 1.2.1. *$T : V \to W$ is a morphism of representations from ϕ to ψ with $\phi : G \to GL(V)$, $\psi : G \to GL(W)$, then $ker(T) \leq V$ and $T(V) \leq W$ are invariant subspaces.*

Proof. If $V \in ker(T)$, $g \in G$. $T(\phi_g(v)) = \psi_g(T(v)) = \psi_g(0) = 0$ so $\phi_g(ker(T)) \subseteq ker(T)$, also applies to ϕ_g^{-1}. □

1.3 Schur's Lemma

Lemma 1.3.1 (Schur's Lemma)**.** *Let $\phi : G \to GL(V)$, $\psi : G \to GL(W)$ irreducible and $T \in Hom_G(\phi, \psi)$ a morphism. Then either $T = 0$ or T is an isomorphism.*

- *If $\phi \nsim \psi$, then $Hom_G(\phi, \psi) = \{0\}$.*
- *If $\phi \sim \psi$, then dim $Hom_G(\phi, \psi) = 1$. In particular, $Hom_G(\phi, \psi) = \{\lambda I, \lambda \in \mathbb{C}\}$.*

Proof. $T \in Hom_G(\phi, \psi)$, ϕ, ψ irreducible. $ker(T) \leq V$, $T(V) \leq W$ invariant subspaces. If $T \neq 0$, then $ker(T) \neq V$, $T(V) \neq \{0\}$,

$$\Longrightarrow ker(T) \neq V,\ T(V) \neq \{0\} \tag{1.7}$$
$$\Longrightarrow T \textit{ is isomorphism}! \tag{1.8}$$

- $\phi \nsim \psi$, irreducible. $T \in Hom_G(\phi, \psi) \Longrightarrow T = 0$.
- $\phi \nsim \phi$ irreducible. Special case: $\phi = \psi$, $T \in Hom_G(\phi, \psi)$. So $T : V \to V$, has an eigenvalue $\lambda \in \mathbb{C}$. So $T - \lambda I$ not an isomorphism of V,

$$\Longrightarrow T - \lambda I = 0 \Longrightarrow T = \lambda I, \tag{1.9}$$
$$so\ Hom_G(\phi, \psi) = \{\lambda I | \lambda \in \mathbb{C}\}. \tag{1.10}$$

General case: If $\phi \sim \psi$, irreducible. Pick $S \in Hom_G(\phi, \psi)$, equivalence, S^{-1} exists. Is $T \in Hom_G(\phi, \psi)$, the $TS \in Hom_G(\psi, \psi)$,

$$\Longrightarrow T = \lambda S^{-1}, Hom_G(\phi, \psi) = \left\{\lambda S^{-1}\right\}. \tag{1.11}$$

□

1.4 Trace of a Linear Operator

Definition 1.4.1 (Trace of a Matrix). *The trace of s square matrix* $A = (a_{ij}) \in \mathrm{Mat}_{n\times m}(\mathbb{C})$ *is*

$$\mathrm{Tr}[A] := \sum_{i=1}^{n} a_{ii} \tag{1.12}$$

$$= a_{11} + \ldots, a_{nn} \tag{1.13}$$

$$= \sum_{i,j=1}^{n} a_{ij}\delta_{ij}. \tag{1.14}$$

Lemma 1.4.1.

$$\mathrm{Tr}[AB] = \mathrm{Tr}[BA] \tag{1.15}$$

Corollary 1.4.1. $P \in GL_n(\mathbb{C})$, *we have* $\mathrm{Tr}[PAP^{-1}] = \mathrm{Tr}[A]$.

Lemma 1.4.2 (Trace of a Linear Operator). *Let* $T \in Hom(V, V)$, *dim* $V < \infty$. *The trace of* T *is defined: Choose basis* $B = \{v_1, ..., v_n\}$ *of* V *with* $A = [T]_B \in Mat_{n\times n}(\mathbb{C})$. *Define:* $\mathrm{Tr}[T] := \mathrm{Tr}[A]$. *Show that* $\mathrm{Tr}[T]$ *does not depend on choice of* B.

Proof. If $B' = \{v_1', ..., v_n'\}$ basis, then $[T]_{B'} = P[T]_B P^{-1}$ for some $P \in GL_n(\mathbb{C})$

$$\Longrightarrow Trace\ are\ same. \tag{1.16}$$

□

Proposition 1.4.1. $T \in Hom(V, V)$, $S \in Hom(W, W)$, *dim* V, *dim* W $<\infty$, *is* $U : V \to W$ *linear such that* $UTU^{-1} = S$, *then* $\mathrm{Tr}[T] = \mathrm{Tr}[S]$.

Proof. Basis $B = \{v_1, ..., v_n\}$ of V. Define $B' = \{v_1', ..., v_n'\}$, $v_k' := Uv_k$ basis of W.

$$\Longrightarrow [S]_{B'} = [T]_B \tag{1.17}$$

$$\Longrightarrow Trace\ is\ equal. \tag{1.18}$$

□

Proposition 1.4.2. *Let* $T_k : V_k \to V_k$, $k = 1, 2, ..., r$, *dim* $V_k < \infty$. *Let* $V := V_1 \bigoplus \cdots \bigoplus V_r$, *define* $T : V \to V$ *by* $T := T_1 \bigoplus \cdots \bigoplus T_r$. *Then* $\mathrm{Tr}[T_1 \bigoplus \cdots \bigoplus T_r] = \mathrm{Tr}[T_1] + \cdots + \mathrm{Tr}[T_r]$. *(*$T(v_1, \ldots, v_r) = (T_1(v_1), \ldots, T_r(v_r))$*)*

Proof. Choose basis B of $V = V_1 \bigoplus \cdots \bigoplus V_r$, so that $A = [T]_B$ is a block matrix such that

$$\begin{bmatrix} A_1 & \ldots & 0 \\ 0 & A_2 \ldots & 0 \\ & & \\ 0 & 0 \ldots & A_r \end{bmatrix} \tag{1.19}$$

where $A_k = [T_k]_{B_k}$

$$\Longrightarrow \mathrm{Tr}[A] = \mathrm{Tr}[A_1] + \ldots, \mathrm{Tr}[A_r]. \tag{1.20}$$

□

1.5 Unitary Groups

First, we define the group of unitary $n \times n$ matrices where the binary operation is matrix multiplication. A matrix is unitary if

$$U^{\dagger} = U^{-1}. \tag{1.21}$$

1.5.1 U(1), i.e., 1×1 Matrices

$U = [u]$ and $U^{\dagger} = [u^*]$. $UU^{\dagger} = [u][u^*] = [uu^*] = [1]$. $u(\theta) = e^{i\theta}$.

- Closure: $u(\alpha)u(\beta) = e^{i\alpha}e^{i\beta}$
- Association: $[u(\alpha)u(\beta)]u(\gamma) = u(\alpha)[u(\beta)u(\gamma)]$
- Identity: $I = u(0) = 1$
- Inverset: $u^{-1} = e^{-i\theta}$, consistent with $U^{-1} = U^{\dagger}$.

1.5.2 SU(1)

The ”S” stands for ”spectial”. We must have the determinant euqal to 1.

$$\det U = \det[u] = u = 1. \tag{1.22}$$

So the group SU(1) contains the single matrix: $A = [1]$.

1.5.3 SU(2)

This group consists of the spectial unitary 2×2 matrices. Start with

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{1.23}$$

and

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \tag{1.24}$$

As the determinant must be 1, we have

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{1.25}$$

and

$$A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \tag{1.26}$$

For the unitary matrices we must have

$$A^{\dagger} = A^{-1} \tag{1.27}$$

which means

$$A^{-1} = \begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix}. \tag{1.28}$$

The conditions are $a^* = d$ and $c = -b^*$ along with $|A| = ad - bc = 1$. Let's write the real and imaginary components out, applying these conditions. Then

$$A = \begin{bmatrix} a_r + ia_i & b_r + ib_i \\ -b_r + ib_i & a_r - ia_i \end{bmatrix} \tag{1.29}$$

for the general special unitary 2×2 matrix, where $a_r^2 + a_i^2 + b_r^2 + b_i^2 = 1$.

Our general form of a matrix in the group SU(2):

$$A = \begin{bmatrix} a_r + ia_i & b_r + ib_i \\ -b_r + ib_i & a_r - ia_i \end{bmatrix}. \tag{1.30}$$

Recall how you can write a vector in terms of basis unit vectors:

$$\overrightarrow{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}. \tag{1.31}$$

Check out the same trick with matrices:

$$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{1.32}$$

But there is a natural expansion for SU(2) matrices.

$$A = a_r\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + ia_i\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b_r\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + ib_i\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{1.33}$$

$$A = a_r\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + ia_i\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + ib_r\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + ib_i\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{1.34}$$

The Pauli matrices are

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \sigma_y = b_r\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \sigma_z ib_i\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{1.35}$$

Lemma 1.5.1. *We have*

$$\{\sigma_j, \sigma_k\} = 2\delta_{jk}I, \tag{1.36}$$

where δ_{jk} is the Kronecker Delta symbol and $\{A, B\} := AB + BA$ is the anticommutator.

Lemma 1.5.2. *We have*

$$[\sigma_j, \sigma_k] = 2i\varepsilon_{jkl}\sigma_l, \tag{1.37}$$

where ε_{jkl} is the Levi-Civita symbol and $[A, B] := AB - BA$ is the commutator. Thus SU(2) is non-abelian.

1.6 SO(2)

This group consists of the spectial unitary orthogonal matrices. These are matrices where the columns and rows, thought of as vectors, are orthogonal. Remember or rotation matrix in 2D:

$$R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \tag{1.38}$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}. \tag{1.39}$$

These matrices satisfy our orthogonal conditions since

$$a_{11}a_{12} + a_{21}a_{22} = \cos\theta\sin\theta - \cos\theta\sin\theta = 0 \tag{1.40}$$

$$a_{11}a_{21} + a_{12}a_{22} = -\cos\theta\sin\theta + \sin\theta\cos\theta = 0. \tag{1.41}$$

Note that now the transpose is the inverse:

$$A^{-1} = A^T, \tag{1.42}$$

$$AA^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \times \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = I. \tag{1.43}$$

Lemma 1.6.1. *SO(2) is an abelian group.*