

Chapter 1

Quantum Optics by Carlos

1.1 Prerequisites

This section is used to collect the supplementary material to understand the content when I learn this this course.

1.1.1 Position and momentum operators in quantum mechanics

In quantum mechanics **position** and **momentum** are **operators**.

Definition 1.1.1 (Position operator). *Position operator*

$$\hat{X}\psi(x) = x\psi(x), \quad (1.1.1)$$

where x is the eigenvalue of \hat{X} acting on $\psi(x)$.

\hat{X} and x are different things, i.e., \hat{X} is an operator and x is a variable.

Definition 1.1.2 (Momentum operator). *The momentum operator is defined as*

$$\hat{P}\psi(x) = -i\hbar \frac{\partial}{\partial x}\psi(x). \quad (1.1.2)$$

When we consider the three dimensional and time dependent case, we have the **orthogonal components** of a three-dimensional system as

$$\hat{P}_x\psi(x, y, z, t) = -i\hbar \frac{\partial}{\partial x}\psi(x, y, z, t) \quad (1.1.3)$$

$$\hat{P}_y\psi(x, y, z, t) = -i\hbar \frac{\partial}{\partial y}\psi(x, y, z, t) \quad (1.1.4)$$

$$\hat{P}_z\psi(x, y, z, t) = -i\hbar \frac{\partial}{\partial z}\psi(x, y, z, t). \quad (1.1.5)$$

Theorem 1 (De Broglie principle). ψ can be expressed as

$$\psi = \exp[i(kx - \omega t)]. \quad (1.1.6)$$

Then we have

$$\hat{P}_x \psi = \hbar k \psi, \quad (1.1.7)$$

where wave-vector $k = 2\pi/\lambda$ and λ is the wavelength. Thus, the eigenvalue of momentum operator is $\hbar k$.

Proof.

$$\hat{P}_x \psi = -i\hbar \frac{\partial}{\partial x} \psi \quad (1.1.8)$$

$$= -i\hbar(ik\psi) \quad (1.1.9)$$

$$= \hbar k \psi. \quad (1.1.10)$$

□

Proposition 1.1.1 (De Broglie relation). We have

$$\lambda = \frac{h}{p}, \quad (1.1.11)$$

where λ is the wavelength, h is Plank constant and p is momentum.

Corollary 1.1.1. We have

$$p = \hbar k. \quad (1.1.12)$$

Proof. Start from 1.1.1, we have

$$p = \frac{h}{\lambda} \quad (1.1.13)$$

$$= \frac{h}{2\pi/k} \quad (1.1.14)$$

$$= \frac{hk}{2\pi} \quad (1.1.15)$$

$$= \hbar k. \quad (1.1.16)$$

□

Property 1.1.1. • $\hat{S} = \hat{A} + \hat{B} \implies \hat{S}\psi = \hat{A}\psi + \hat{B}\psi$

• $\hat{C} = \beta \hat{B} \implies \hat{C}\psi = \beta(\hat{B}\psi)$

• $\hat{A} = \hat{A}^\dagger$, all operators that represent observable are Hermitian (self-adjoint)

- $\hat{A}\hat{B} \neq \hat{B}\hat{A} \implies \text{If } \hat{M} = \hat{A}\hat{B}, \hat{M}\psi = \hat{A}(\hat{B}\psi)$
- *The power of an operator:* $\hat{X}^2\psi = \hat{X}(\hat{X}\psi) = \hat{X}(x\psi) = x^2\psi$
- *The commutator:* $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} \neq 0$, operators generally do not commute. Commutator is also a operator, then we have $[\hat{A}, \hat{B}]\psi = \hat{A}\hat{B}\psi - \hat{B}\hat{A}\psi$.

Theorem 2 (Fundamental commutator of position and momentum). *We have the fundamental commutator of position and momentum as*

$$[\hat{X}, \hat{P}] = i\hbar. \quad (1.1.17)$$

Proof.

$$[\hat{X}, \hat{P}]\psi = \hat{X}\hat{P}\psi - \hat{P}\hat{X}\psi \quad (1.1.18)$$

$$= \hat{X}(-i\hbar)(ik\psi) - \hat{P}(x\psi) \quad (1.1.19)$$

$$= \hat{X}(\hbar k\psi) - (-i\hbar(\psi + ikx\psi)) \quad (1.1.20)$$

$$= x\hbar k\psi - (-i\hbar\psi + \hbar kx\psi) \quad (1.1.21)$$

$$= i\hbar\psi. \quad (1.1.22)$$

This leads to $[\hat{X}, \hat{P}] = i\hbar$. □

Corollary 1.1.2. *We have Heisenberg Uncertainty Principle*

$$\Delta x \Delta p \geq \frac{\hbar}{2}. \quad (1.1.23)$$

Proof. This can be proved by 2. □

1.2 Ch 12: What are generators in classical mechanics? | Maths of Quantum Mechanics

Lagrangian $\mathcal{L}(t, x(t), \dot{x}(t))$ represents the classical state.

Lagrangian:

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - V(x) \quad (1.2.1)$$

Principle of stationary action.

$$S[\mathcal{L}] = \int_{t_i}^{t_f} dt \mathcal{L}(t, x(t), \dot{x}(t)). \quad (1.2.2)$$

Calculus of variations.

Euler-Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}}. \quad (1.2.3)$$

Theorem 3. *Euler-Lagrange equation is more abstract version of Newton's second law.*

Proof. □

Theorem 4. • *momentum is the generator of spatial change*

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt} p. \quad (1.2.4)$$

• *position is the generator of momentum change*

$$\frac{\partial \mathcal{L}}{\partial p} = \frac{d}{dt} x. \quad (1.2.5)$$

• *energy is the generator of time change*

$$\frac{\partial \mathcal{L}}{\partial t} = -\frac{d}{dt} E. \quad (1.2.6)$$

• *angular momentum is the generator of rotational change*

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt} L. \quad (1.2.7)$$

Proof. □

How is this related to quantum mechanics: recall we think about $\mathcal{L} \Longleftrightarrow |\psi\rangle$. They are both mathematical objects that contains all the information about the particle!

1.2.1 Ch 13: Where does the Schrödinger equation come from? | Maths of Quantum Mechanics

Time evolution in quantum mechanics?

Theorem 5. *The Schrödinger equation is derived as*

$$i\hbar \frac{d}{dt} |\psi\rangle = \hat{H} |\psi\rangle. \quad (1.2.8)$$

Proof. □

1.2.2 SPDC

This content is from 'Q-Leap Edu Quantum Communications' from YouTube.

Spontaneous parametric down-conversion (SPDC) is a process useful in generating entangled pairs of photons.

Use linear polarization to encode a qubit.

How do polarized photons interact with a BBO? That depends on the optical axis of the crystal.

Here we have two figures to show this.

Polarization of the photon pairs is opposite to the pump!

How do we create entangled photons?

- BBO crystals must be very thin to ensure indistinguishability.
- SPDC is a rare process. Expect 1 photon pair per 10^6 pump photons in state-of-the-art experiments.

1.2.3 Schrodinger vs. Heisenberg pictures of quantum mechanics

This content is from 'Professor M does Science' from YouTube.

Schrodinger picture: state $t_0 : |\psi_s(t_0)\rangle \implies t : |\psi_s(t)\rangle$. From the postulates of quantum mechanics, the evolution of state is described by Schrodinger picture:

$$i\hbar \frac{d}{dt} |\psi_s(t)\rangle = \hat{H}_s(t) |\psi_s(t)\rangle. \quad (1.2.9)$$

Schrodinger picture: observable

- Time independent observable e.g.

$$\text{momentum operator } \hat{P}_s \longrightarrow -i\hbar \frac{d}{dx} \quad (1.2.10)$$

$$\text{kinetic energy operator } \frac{\hat{P}_s^2}{2m} \longrightarrow -\frac{\hbar^2}{2m} \frac{d^2}{dx^2}. \quad (1.2.11)$$

- Time dependent observable e.g.

$$\hat{H}_s(t) = \frac{\hat{P}_s^2}{2m} + V(\hat{x}_s) \sin(\omega t). \quad (1.2.12)$$

Time evolution operator We have

$$|\psi_s(t)\rangle = \hat{U}(t, t_0)|\psi_s(t_0)\rangle, \quad (1.2.13)$$

where subindex s means we work in Schrödinger picture, $\hat{U}(t, t_0)$ is the time evolution operator which depends on the initial time t_0 and time t . This is the same of Schrodinger equation. They are both in Schrodinger picture. Combine the time evolution operator and Schrodinger picture, we have

$$i\hbar \frac{d}{dt} \hat{U}(t, t_0) = \hat{H}_s(t) \hat{U}(t, t_0), \quad (1.2.14)$$

satisfying $\hat{U}(t_0, t_0) = I$. For the time independent Hamiltonian $\hat{H}_s(t) = \hat{H}_s$, we have

$$\hat{U}(t, t_0) = \exp \left[-i \hat{H}_s \frac{t - t_0}{\hbar} \right]. \quad (1.2.15)$$

For time dependent case, we need Dyson series.

Time evolution in quantum mechanics: alternative picture From Schrodinger picture,

$$|\psi_s(t)\rangle = \hat{U}(t, t_0)|\psi_s(t_0)\rangle. \quad (1.2.16)$$

We then have the expectation value of observable \hat{A}_s as

$$\langle \psi_s(t) | \hat{A}_s | \psi_s(t) \rangle = \langle \psi_s(t_0) | \hat{U}^\dagger(t, t_0) \hat{A}_s \hat{U}(t, t_0) | \psi_s(t_0) \rangle \quad (1.2.17)$$

$$= \langle \psi_s(t_0) | \hat{A}_H(t) | \psi_s(t_0) \rangle \quad (1.2.18)$$

$$= \langle \psi_H(t_0) | \hat{A}_H(t) | \psi_H(t_0) \rangle, \quad (1.2.19)$$

where \hat{A}_s is the observable in Schrödinger picture, $|\psi_H\rangle$ is the state in Heisenberg picture, and \hat{A}_H is the observable in Heisenberg picture.

Unitary operator/transformation Unitary operator: $\hat{U}^{-1} = \hat{U}^\dagger$ satisfying $\hat{U}^\dagger \hat{U} = \hat{U} \hat{U}^\dagger = I$.

Unitary transformation:

$$|\psi\rangle = |\psi'\rangle = \hat{U}|\psi\rangle \quad (1.2.20)$$

$$|\phi\rangle = |\phi'\rangle = \hat{U}|\phi\rangle \quad (1.2.21)$$

$$\implies \langle \psi' | \phi' \rangle = \langle \psi | \phi \rangle, \quad (1.2.22)$$

which shows unitary transformation preserves inner product.

For observable \hat{A} , we have

$$\hat{A} \implies \hat{A}' = \hat{U} \hat{A} \hat{U}^\dagger \quad (1.2.23)$$

$$\implies \langle \phi' | \hat{A}' | \psi' \rangle = \langle \phi | \hat{A} | \psi \rangle. \quad (1.2.24)$$

Why we can have different pictures of time evolution in quantum mechanics

Prediction of quantum mechanics: inner products e.g.

$$|\psi_s\rangle, \hat{A}_s|(u_n)_s\rangle = \lambda_n|(u_n)_s\rangle \quad (1.2.25)$$

$$P(\lambda_n) = |\langle(u_n)_s|\psi_s\rangle|^2 \quad (1.2.26)$$

$$\langle\hat{A}\rangle = \langle\psi_s|\hat{A}_s|\psi_s\rangle, \quad (1.2.27)$$

where n denote n^{th} eigenstate, this leads to the fact that unitary transformations do not change the above results. Therefore,

$$\text{unitary transformation to Schrodinger picture} \quad (1.2.28)$$

$$\implies \text{alternative picture of quantum mechanics} \quad (1.2.29)$$

Heisenberg picture

Definition 1.2.1. *In the Heisenberg picture, quantum states are time independent.*

Lemma 1.2.1. *The relation between the state in Schrodinger picture and the state in Heisenberg picture is*

$$|\psi_H\rangle = \hat{U}^\dagger(t, t_0)|\psi_s(t)\rangle \quad (1.2.30)$$

$$= |\psi_s(t_0)\rangle, \quad (1.2.31)$$

where $|\psi_H\rangle$ is the state in Heisenberg picture, $|\psi_s(t)\rangle$ is the state in the Schrodinger picture and $|\psi_s(t_0)\rangle$ is the initial state in the Schrödinger picture.

Proof. Recall the time evolution:

$$|\psi_s(t)\rangle = \hat{U}(t, t_0)|\psi_s(t_0)\rangle. \quad (1.2.32)$$

By definition 1.2.1, we perform $\hat{U}^{-1}(t, t_0)$ on $|\psi_s(t)\rangle$ to get the time independent state

$$\hat{U}^{-1}(t, t_0)|\psi_s(t)\rangle = \hat{U}^{-1}(t, t_0)\hat{U}(t, t_0)|\psi_s(t_0)\rangle \quad (1.2.33)$$

$$= |\psi_s(t_0)\rangle. \quad (1.2.34)$$

Since $\hat{U}^{-1}(t, t_0) = \hat{U}^\dagger(t, t_0)$, we have

$$|\psi_H\rangle := \hat{U}^\dagger(t, t_0)|\psi_s(t)\rangle \quad (1.2.35)$$

$$= |\psi_s(t_0)\rangle. \quad (1.2.36)$$

□

Lemma 1.2.2. *The relation of the observable in the Schrodinger and Heisenberg picture is*

$$\hat{A}_H(t) = \hat{U}^\dagger(t, t_0) \hat{A}_s(t) \hat{U}(t, t_0), \quad (1.2.37)$$

where \hat{A}_H is the observable in the Heisenberg picture and \hat{A}_s is the observable in the Schrödinger picture.

Proof. Recall the unitary transformation

$$\hat{A}' = \hat{U} \hat{A} \hat{U}^\dagger, \quad (1.2.38)$$

and $\hat{U} = \hat{U}^\dagger(t, t_0)$. Then we define $\hat{A}_H = \hat{A}'$, and we get

$$\hat{A}_H(t) = \hat{U}^\dagger(t, t_0) \hat{A}_s(t) \hat{U}(t, t_0). \quad (1.2.39)$$

□

Lemma 1.2.3. *The expectation value of the observable in Schrodinger picture and Heisenberg picture are the same*

$$\langle \psi_s(t) | \hat{A}_s(t) | \psi_s(t) \rangle = \langle \psi_H | \hat{A}_H(t) | \psi_H \rangle. \quad (1.2.40)$$

Proof.

Insert the identity $\hat{U}(t, t_0) \hat{U}^\dagger(t, t_0) = I$

$$\langle \psi_s(t) | \hat{A}_s(t) | \psi_s(t) \rangle = \langle \psi_s(t) | \hat{U}(t, t_0) \hat{U}^\dagger(t, t_0) \hat{A}_s(t) \hat{U}(t, t_0) \hat{U}^\dagger(t, t_0) | \psi_s(t) \rangle \quad (1.2.41)$$

$$= \langle \psi_H | \hat{A}_H(t) | \psi_H \rangle, \quad (1.2.42)$$

where in the second equation we use the definitions of 1.2.1 and 1.2.2. □

Dynamics

Lemma 1.2.4. *We have the dynamics in Heisenberg picture as*

$$i\hbar \frac{d}{dt} \hat{A}_H(t) = [\hat{A}_H(t), \hat{H}_H(t)] + i\hbar \left[\frac{d}{dt} \hat{A}_s(t) \right]_H. \quad (1.2.43)$$

Proof.

$$\frac{d}{dt}\hat{A}_H(t) = \frac{d}{dt} \left[\hat{U}^\dagger(t, t_0) \hat{A}_s(t) \hat{U}(t, t_0) \right] \quad (1.2.44)$$

Use chain rule

$$= \left[\frac{d}{dt} \hat{U}^\dagger(t, t_0) \right] \hat{A}_s(t) \hat{U}(t, t_0) + \hat{U}^\dagger \left[\frac{d}{dt} \hat{A}_s(t) \right] \hat{U}(t, t_0) \quad (1.2.45)$$

$$+ \hat{U}^\dagger(t, t_0) \hat{A}_s(t) \left[\frac{d}{dt} \hat{U}(t, t_0) \right] \quad (1.2.46)$$

Use 1.2.14

$$= \left[-\frac{1}{i\hbar} \hat{U}^\dagger(t, t_0) \hat{H}_s(t) \right] \hat{A}_s(t) \hat{U}(t, t_0) + \hat{U}^\dagger(t, t_0) \left[\frac{d}{dt} \hat{A}_s(t) \right] \hat{U}(t, t_0) \quad (1.2.47)$$

$$+ \hat{U}^\dagger(t, t_0) \hat{A}_s(t) \left[\frac{1}{i\hbar} \hat{H}_s(t) \hat{U}(t, t_0) \right] \quad (1.2.48)$$

$$= \frac{1}{i\hbar} \left[\hat{U}^\dagger(t, t_0) \hat{A}_s(t) \hat{H}_s(t) \hat{U}(t, t_0) - \hat{U}^\dagger(t, t_0) \hat{H}_s(t) \hat{A}_s(t) \hat{U}(t, t_0) \right] \quad (1.2.49)$$

$$+ \hat{U}^\dagger(t, t_0) \left[\frac{d}{dt} \hat{A}_s(t) \right] \hat{U}(t, t_0) \quad (1.2.50)$$

Insert the identity $\hat{U}(t, t_0) \hat{U}^\dagger(t, t_0) = I$

$$= \frac{1}{i\hbar} [\hat{U}^\dagger(t, t_0) \hat{A}_s(t) \hat{U}(t, t_0) \hat{U}^\dagger(t, t_0) \hat{H}_s(t) \hat{U}(t, t_0) \quad (1.2.51)$$

$$- \hat{U}^\dagger(t, t_0) \hat{H}_s(t) \hat{U}(t, t_0) \hat{U}^\dagger(t, t_0) \hat{A}_s(t) \hat{U}(t, t_0)] + \left[\frac{d}{dt} \hat{A}_s(t) \right]_H \quad (1.2.52)$$

$$= \frac{1}{i\hbar} \left(\hat{A}_H(t) \hat{H}_H(t) - \hat{H}_H(t) \hat{A}_H(t) \right) + \left[\frac{d}{dt} \hat{A}_s(t) \right]_H \quad (1.2.53)$$

$$= \frac{1}{i\hbar} [\hat{A}_H(t), \hat{H}_H(t)] + \left[\frac{d}{dt} \hat{A}_s(t) \right]_H, \quad (1.2.54)$$

which lead to

$$i\hbar \frac{d}{dt} \hat{A}_H(t) = [\hat{A}_H(t), \hat{H}_H(t)] + i\hbar \left[\frac{d}{dt} \hat{A}_s(t) \right]_H. \quad (1.2.55)$$

□

Corollary 1.2.1. When $\hat{A}_s(t) = \hat{A}_s$, we have

$$i\hbar \frac{d}{dt} \hat{A}_H(t) = [\hat{A}_H(t), \hat{H}_H(t)]. \quad (1.2.56)$$

Conservative system

Definition 1.2.2. The conservative system have the following

$$\hat{H}_s(t) = \hat{H}_s. \quad (1.2.57)$$

Lemma 1.2.5. If we have $[\hat{B}, \hat{C}]$, then we have

$$[\hat{B}, F(\hat{C})] = 0, \quad (1.2.58)$$

where F denote any function.

Lemma 1.2.6. *Assume we have a conservative system and assume a time independent observable $\hat{A}_s(t) = \hat{A}_s$ satisfying $[\hat{A}_s, \hat{H}_s] = 0$, then we have*

$$\hat{A}_H(t) = \hat{A}_s. \quad (1.2.59)$$

Proof. Using 1.2.5 and

$$\hat{U}(t, t_0) = \exp \left[-i\hat{H}_s(t - t_0)/\hbar \right], \quad (1.2.60)$$

we have

$$[\hat{A}_s, \hat{U}(t, t_0)] = 0. \quad (1.2.61)$$

Then we have

$$\hat{A}_H(t) = \hat{U}^\dagger(t, t_0) \hat{A}_s \hat{U}(t, t_0) \quad (1.2.62)$$

Use $[\hat{A}_s, \hat{U}(t, t_0)] = 0$

$$= \hat{U}^\dagger(t, t_0) \hat{U}(t, t_0) \hat{A}_s \quad (1.2.63)$$

$$= \hat{A}_s. \quad (1.2.64)$$

□

Corollary 1.2.2. *The conservative system have the following*

$$\hat{H}_H = \hat{H}_s = \hat{H}. \quad (1.2.65)$$

Why we use the Heisenberg picture In the Heisenberg picture, it is easier to reach at Ehrenfest theorem.

Theorem 6. *The Ehrenfest theorem states that*

$$\frac{d}{dt} \langle \hat{A} \rangle = \frac{1}{i\hbar} \langle [\hat{A}_s(t), \hat{H}_s(t)] \rangle + \left\langle \frac{d}{dt} \hat{A}_s(t) \right\rangle. \quad (1.2.66)$$

Proof.

$$\frac{d}{dt} \langle \hat{A} \rangle = \frac{d}{dt} \langle \psi_H | \hat{A}_H(t) | \psi_H \rangle \quad (1.2.67)$$

$$= \langle \psi_H | \frac{d}{dt} \hat{A}_H(t) | \psi_H \rangle \quad (1.2.68)$$

Use lemma 1.2.4

$$= \frac{1}{i\hbar} \langle [\hat{A}_H(t), \hat{H}_H(t)] \rangle + \left\langle \left[\frac{d}{dt} \hat{A}_s(t) \right]_H \right\rangle. \quad (1.2.69)$$

We then use the fact that the expectation value is invariant under unitary transformation, which leads to the final result. □

Summary Here we have a nice table to summarize this subsection!

1.3 Part 3. Quantum oscillator in phase space, Wigner function, and Gaussian states

1.3.1 Quantum oscillator in phase space

1.3.2 Wigner function

Visualizing quantum states in phase space: Wigner function!

- \hat{X} and \hat{P} 's eigenstates are not physical
- \hat{X} and \hat{P} do not possess common eigenstates

\implies impossible to prepare state where the quadrature take definite values! Most we can offer:

- Position probability density function (PDF) $\langle x|\hat{\rho}|x\rangle = \int_{\mathbb{R}} dp w_{\rho}(\vec{r})$,
- Momentum PDF $\langle p|\hat{\rho}|p\rangle = \int_{\mathbb{R}} dx w_{\rho}(\vec{r})$,

where

$$\vec{r} = \begin{pmatrix} x \\ p \end{pmatrix}, \quad (1.3.1)$$

and x, p are eigenvalues of position and momentum. Is it possible to describe these through a combined PDF, i.e., **wigner function** $w_{\rho}(\vec{r})$ in phase space.

- $\vec{\hat{R}} = \begin{pmatrix} \hat{X} \\ \hat{P} \end{pmatrix}$, so $[\hat{R}_m, \hat{R}_n] = 2i\Omega_{mn}$, where $\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the symplectic form satisfying $\Omega = -\Omega^T = -\Omega^{-1}$.
- Displacement operator $\hat{D}(\vec{r}) = \exp\left[\frac{i}{2}\vec{\hat{R}}^T \Omega \vec{r}\right] = \exp\left[\frac{i}{2}(p\hat{X} - x\hat{P})\right]$
- Define quantum characteristic function $\chi_{\rho}(\vec{s}) = \langle \hat{D}(\vec{s}) \rangle = \text{Tr}[\hat{\rho}\hat{D}(\vec{s})]$

Then, we have

Definition 1.3.1 (Wigner function). *The wigner function is defined as*

$$w_{\rho}(\vec{r}) = \int_{\mathbb{R}^2} \frac{d^2\vec{s}}{(4\pi)^2} \exp\left[-\frac{i}{2}\vec{r}^T \Omega \vec{s}\right] \chi_{\rho}(\vec{s}). \quad (1.3.2)$$

Remark 1.3.1. $\vec{x}^T \Omega \vec{y} = x_1 y_2 - x_2 y_1$

$$\int_{\mathbb{R}} \frac{dx}{2\pi} \exp[ipx] = \delta(p) \quad (1.3.3)$$

$$\implies \int_{\mathbb{R}} \frac{dx}{4\pi} \exp\left[\frac{i}{2}xp\right] = \delta(p) \quad (1.3.4)$$

$$\int_{\mathbb{R}} \frac{d^2 \vec{r}}{(2\pi)^2} \exp[i\vec{r}^T \vec{s}] = \delta^{(2)}(\vec{s}) \quad (1.3.5)$$

$$\implies \int_{\mathbb{R}} \frac{d^2 \vec{r}}{(4\pi)^2} \exp\left[\frac{i}{2}\vec{r}^T \Omega \vec{s}\right] = \delta^{(2)}(\vec{s}) \quad (1.3.6)$$

Lemma 1.3.1. *We have the right marginals:*

$$\int_{\mathbb{R}} dp w_{\rho}(\vec{r}) = \langle x | \hat{\rho} | x \rangle. \quad (1.3.7)$$

Proof.

$$\int_{\mathbb{R}} dp w_{\rho}(\vec{r}) = \int_{\mathbb{R}} \frac{d^2 \vec{r}'}{4\pi} \exp\left[-\frac{i}{2}xp'\right] \int_{\mathbb{R}} \frac{dp}{4\pi} \exp\left[\frac{i}{2}px'\right] \chi_{\rho}(\vec{r}') \quad (1.3.8)$$

$$\begin{aligned} & \int_{\mathbb{R}} \frac{dp}{4\pi} \exp\left[\frac{i}{2}px'\right] = \delta(x') \\ & = \int_{\mathbb{R}} \frac{dp'}{4\pi} \exp\left[-\frac{i}{2}xp'\right] \text{Tr}[\hat{\rho} \hat{D}(0, p')] \end{aligned} \quad (1.3.9)$$

$$\begin{aligned} & \hat{D}(0, p') = \exp\left[\frac{i}{2}p' \hat{X}\right] \\ & = \int_{\mathbb{R}} \frac{dp'}{4\pi} \exp\left[-\frac{i}{2}xp'\right] \int_{\mathbb{R}} dx' \langle x' | \hat{\rho} \exp\left[\frac{i}{2}p' \hat{X}\right] | x' \rangle \end{aligned} \quad (1.3.10)$$

$$= \int_{\mathbb{R}} dx' \int_{\mathbb{R}} \frac{dp'}{4\pi} \exp\left[\frac{i}{2}(x' - x)p'\right] \langle x' | \hat{\rho} | x' \rangle \quad (1.3.11)$$

$$= \langle x | \hat{\rho} | x \rangle. \quad (1.3.12)$$

□

Property 1.3.1. •

$$\int_{\mathbb{R}} dp w_{\rho}(\vec{r}) = \langle x | \hat{\rho} | x \rangle. \quad (1.3.13)$$

- $w_{\rho}(\vec{r}) \in \mathbb{R}, \forall(\vec{r}, \hat{\rho})$
- $\int_{\mathbb{R}} d^2 \vec{r} w_{\rho}(\vec{r}) = 1$
- $\langle (\hat{X}^m \hat{P}^n)^{(s)} \rangle = \int_{\mathbb{R}^2} d^2 \vec{r} w_{\rho}(\vec{r}) x^m p^n$, where for example

$$(\hat{X}^2 \hat{P})^{(s)} = \frac{1}{3} \left(\hat{X}^2 \hat{P} + \hat{X} \hat{P} \hat{X} + \hat{P} \hat{X}^2 \right) \quad (1.3.14)$$

- $\int_{\mathbb{R}^2} d^2 \vec{r} w_\rho^2(\vec{r}) \leq \frac{1}{4\pi} \implies \text{No divergence!}$
- *Unique-state correspondence:* $\rho = \int_{\mathbb{R}^2} \frac{d^2 \vec{s}}{4\pi} \hat{D}^\dagger(\vec{s}) \chi_\rho(\vec{s})$
- *Wigner's original:* $w_\rho(\vec{r}) = \int_{\mathbb{R}} \frac{dy}{4\pi} \exp\left[-\frac{i}{2}py\right] \langle x + y/2 | \hat{\rho} | x - y/2 \rangle$
-

$$\text{Tr} [\hat{\rho}_1 \hat{\rho}_2] = 4\pi \int_{\mathbb{R}^2} w_{\rho_1}(\vec{r}) w_{\rho_2}(\vec{r}) \quad (1.3.15)$$

When $\hat{\rho}_j = |\psi_j\rangle\langle\psi_j|$

$$= |\langle\psi_1|\psi_2\rangle|^2 \quad (1.3.16)$$

Choose $\langle\psi_1|\psi_2\rangle = 0$

$$= 0, \quad (1.3.17)$$

which implies that $w_\rho(\vec{r})$ **is negative for some** (ρ, \vec{r}) . \implies It is not a true PDF! \implies quantum mechanics \neq noise on classical mechanics.

Example 1.3.1. The wigner function of number states can be derived as

$$w_{|n\rangle}(\vec{r}) = \frac{(-1)^n}{2\pi} L_n(\vec{r}^2) \exp\left[-\frac{\vec{r}^2}{2}\right]. \quad (1.3.18)$$

Negativity is a smoking gun for quantum phenomena in experiments! There is a nice figure in the video!

Proof. It can be found in the lecture notes. □

1.3.3 Gaussian states

Definition 1.3.2. Gaussian states are equivalent to statistics defined from 1st and 2nd order moments $\langle\hat{R}_j\rangle$, $\langle\hat{R}_j\hat{R}_l\rangle$. Its wigner function is

$$w_\rho(\vec{r}) = \frac{1}{2\pi\sqrt{\det V}} \exp\left[-\frac{1}{2}(\vec{r} - \vec{d})^T V^{-1}(\vec{r} - \vec{d})\right]. \quad (1.3.19)$$

Theorem 7 (Gaussian integral theorem). We have

$$\int_{\mathbb{R}} d^N \vec{r} \exp\left[-\frac{1}{2}\vec{r}^T A \vec{r} + \vec{x}^T \vec{r}\right] = \sqrt{\frac{(2\pi)^N}{\det A}} \exp\left[\frac{1}{2}\vec{x}^T A^{-1} \vec{x}\right]. \quad (1.3.20)$$

Lemma 1.3.2. The mean vector of Gaussian states is

$$d_j = \int_{\mathbb{R}^2} d^2 \vec{r} w_\rho(\vec{r}) r_j \quad (1.3.21)$$

$$= \langle\hat{R}_j\rangle \quad (1.3.22)$$

$$\longrightarrow \vec{d} = \langle\hat{\vec{R}}\rangle. \quad (1.3.23)$$

Lemma 1.3.3. *The covariance matrix of Gaussian states is*

$$V_{jl} = \int_{\mathbb{R}} d^2 \vec{r} w_{\rho}(\vec{r}) w_{\rho}(\vec{r}) \delta r_j \delta r_l \quad (1.3.24)$$

$$= \frac{1}{2} \langle \delta \hat{R}_j \delta \hat{R}_l + \delta \hat{R}_l \delta \hat{R}_j \rangle \quad (1.3.25)$$

with $\delta \vec{r} = \vec{r} - \vec{d}$ and $\delta \hat{A} = \hat{A} - \langle \hat{A} \rangle$ ($V(A) = \langle \delta \hat{A}^2 \rangle$). These leads to covariance matrix

$$V = \langle \delta \hat{R} \delta \hat{R}^T \rangle - i\Omega. \quad (1.3.26)$$

Theorem 8 (Gaussian moment theorem).

$$\int_{\mathbb{R}^2} d^2 \vec{r} w_{\rho}(\vec{r}) \delta r_{j1} \delta r_{j2} \dots \delta r_{jN} = \begin{cases} 0, & \text{for odd } N \\ \sum_{\{i_1, \dots, i_N\} \in \text{all } (N-1)!! \text{ pairings}} V_{i1i2} V_{i3i4} \dots V_{iN-1iN} & \text{for each } N \end{cases} \quad (1.3.27)$$

Example 1.3.2. $N = 4$: $\int_{\mathbb{R}^2} d^2 \vec{r} w_{\rho}(\vec{r}) \delta x \delta p \delta x \delta p = V_{12} V_{21} + V_{11} V_{22} + V_{12} V_{21}$.

Example 1.3.3.

$$\langle \hat{X}^3 \rangle = \langle (\delta \hat{X} + \langle \hat{X} \rangle)^3 \rangle \quad (1.3.28)$$

$$= \langle \delta \hat{X}^3 \rangle + 2 \langle \hat{X} \rangle \langle \delta \hat{X}^2 \rangle + 2 \langle \hat{X} \rangle^2 \langle \delta \hat{X} \rangle + \langle \hat{X} \rangle^3. \quad (1.3.29)$$