Chapter 1

Functional Analysis

1.1 Metric Space

Definition 1.1.1 (Metrix Spaces)**.** *Define set X. Define a metric: $d : X \times X \to [0, \infty)$*

- $d(x, y) = 0 \iff x = y$
- $d(x, y) = d(y, x)$
- $d(x, y) \leq d(x, z) + d(z, y)$ *(triangle inequality)*

Summary: X set + $d : X \times X \to [0, \infty)$ metric = metric space (X, d).

Example 1.1.1.

- $X = \mathbb{C}, d(x, y) = |x - y|$
- $X = \mathbb{R}^n, d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$ *(Euclidean metric)*
- $X = \mathbb{R}^n, d(x, y) = \max\{|x_1 - y_1|, ..., |x_n - y_n|\}$. *It is possible that* $d(x, y) = d(x, z)$.
- X *any set* ($\neq \emptyset$)*, we define*

$$d(x, y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases} \tag{1.1}$$

Now, let's check whether this is a metrix space. Conditions (1) and (2) are verified easily. We focus on the verification of (3): choose $x, y, z \in X$.

For the first case: $x = y$: $d(x, y) = 0 \leq d(x, z) + d(z, y)$.

For the second case: $x \neq y$: $d(x, y) = 1 \leq \{d(x, z)$ *or* $d(z, y)\} = d(x, z) + d(z, y)$.

This is called ***discrete matric****.*

1.2 Open and Closed Sets

Definition 1.2.1 (Open Ball)**.** *(X, d) a metrix space. Define*

$$B_\epsilon(x) := \{y \in X | d(x, y) < \epsilon\}, \tag{1.2}$$

whihc is called the open ball of radiue $\epsilon > 0$ centered at x.

Definition 1.2.2 (Open Sets)**.** *$A \subseteq X$ is called open is for each $x \in A$ there is an open ball with $B_\epsilon(x) \subseteq A$.*

We have nice pic for this.

Definition 1.2.3 (Boundary Points)**.** *$A \subseteq X$, $x \in X$ is called a boundary point for A if for all $\epsilon > 0$: $B_\epsilon(x) \cap A = \emptyset$ and $B_\epsilon(x) \cap A^c = \emptyset$. [$A^c := X \backslash A$]*

Notation: $\partial A : \{x \in X | x \textit{ is boundary point for } A\}$

We have nice pic for this.

Remember: A open $\iff A \cap \partial A = \emptyset$.

Definition 1.2.4 (Closed Sets)**.** *$A \subseteq X$ is called closed if $A^c := X \backslash A$ is open.*

Definition 1.2.5 (Closure)**.** $\overline{A} := A \cup \partial A$ *(always closed!)*

Example 1.2.1. *$X := (1,3] \cup (4, \infty), d(x,y) := |x-y|, (X,d)$ is a metrix space.*

- *$A := (1,3] \subseteq X$ open?*
  For $x \in A, x \neq 3$, define $\epsilon := \frac{1}{2} \min(|1-x|, |3-x|)$. Then $B_\epsilon(x) \subseteq A$.
  For $x = 3$: $B_1(x) = \{y \in X | d(x,y) < 1\} = (2,3] \subseteq A$.
- *A is also closed!*
- *$C := (1,2]$, $\partial C = \{2\}$, $\overline{C} = C$.*

1.3 Sequence, Limits and Closed Sets

Definition 1.3.1 (Sequence)**.** *Sequence in X: $x_1, x_2, \dots$ or $(x_n)_{n \in \mathbb{N}}$ or map $x : \mathbb{N} \to X$ / $n \mapsto x_n$.*

Definition 1.3.2 (Convergence)**.** *A sequence $(x_n)_{n \in \mathbb{N}}$ in a metrix space (X,d) is called convergence if there is $\tilde{x} \in X$ with $\forall \varepsilon \geq 0, \exists N \in \mathbb{N}, \forall n \geq N : d(x_n, \tilde{x}) < \varepsilon$. We write: $X_n \overset{n\to\infty}{\longrightarrow} \tilde{x}$ or $\lim_{n \to} x_n = \tilde{x}$.*

Proposition 1.3.1 (Closed Sets)**.** *$A \subseteq X$ is closed $\iff$ For every convergent sequence $(a_n)_{n \in \mathbb{N}} \subseteq A$, one has $\lim_{n\to\infty} a_n \in A$.*

Proof. ($\Longleftarrow$): Show it by contraposition! Assume A is not closed.

$$\implies A^c := X \backslash A \text{ is not open.} \tag{1.3}$$

$$\implies \text{There is an } \tilde{x} \in A^c \text{ with } B_\varepsilon(\tilde{x}) \cap A \neq \emptyset, \forall \varepsilon > 0. \tag{1.4}$$

$$\implies \text{There is a sequence } (a_n)_{n \in \mathbb{N}} \textit{ with } a_n \in B_{1/n}(\tilde{x}) \cap A \tag{1.5}$$

$$\implies \lim_{n\to\infty} a_n = \tilde{x} \neq A. \tag{1.6}$$

($\Longrightarrow$): Show it by contraposition! Assume there is $(a_n)_{n \in \mathbb{N}} \subseteq A$ with $\tilde{x} := \lim_{n\to\infty} \notin A$.

$$\implies B_\varepsilon(\tilde{x}) \neq \emptyset, \forall \varepsilon > 0 \tag{1.7}$$

$$\implies A^c \text{ is not open} \tag{1.8}$$

$$\implies A \text{ is not closed.} \tag{1.9}$$

$\square$

[SZQ: This proof use many contrapositions. It is nice to get familiar with how to use contrapositions to prove. It also needs to know the inverse of $\forall$.]

1.4 Cauchy Sequence and complete metric spaces

Example 1.4.1. *$X = (0,3)$ with $d(x,y) = |x-y|$.*

$(0,3)$ is closed:

- *complement $\emptyset$ is open*
- *each convergent sequence $(x_n)_{n\in\mathbb{N}} \subseteq (0,3)$ (with limit $\tilde{x} \in X$) satisfies $\tilde{x} \in (0,3)$*

What is about the sequence $(\frac{1}{n})_{n\in\mathbb{N}}$?

- *sequence in X*
- $d(x_n, x_m) \overset{n,m\to\infty}{\longrightarrow} 0$
- *it does not converge $\Longrightarrow$ (X,d) is not complete*

Definition 1.4.1 (Cauchy Sequence and Complete)**.** *Let (X,d) be a metrix space. A sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ is called Cauchy sequence if $\forall \varepsilon 0, \exists N \in \mathbb{N}, \forall n, m \geq N : d(x_n, x_m) < \varepsilon$. (X,d) is called complete if all Cauchy sequences converge.*

Example 1.4.2.

- *$X = [0,3]$ with $d(x,y) = |x-y|$ is complete.*
- *$X = (0,3)$ with*

$$\begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases} \tag{1.10}$$

is complete.

Proof. Let $(x_n)_{n\in\mathbb{N}} \subseteq X$ be a Cauchy sequence. Take $\varepsilon = 1/2$. Then there is an $N \in \mathbb{N}$ such that for all $n, m \geq N$, we have $d(x_n, x_m) < \frac{1}{2}$. By definition, $d(x_n, x_m) = 0$. Hence, $x_n = x_m$. $\square$

1.5 Norms and Banach Spaces

Definition 1.5.1 (Norm)**.** $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$*. Let X be a $\mathbb{F}$-vector space. A map $||\cdot|| : X \to [0,\infty)$ is called norm if*

- *$||x|| = 0 \Longleftrightarrow x = 0$ (postitive definite)*
- *$||\lambda \cdot x|| = |\lambda| ||x||$ for all $\lambda \in \mathbb{F}$, $x \in X$ (absolutely homogeneous)*
- *$||x|| \leq ||x|| + ||y||$ for all $x, y \in X$ (triangle inequality)*

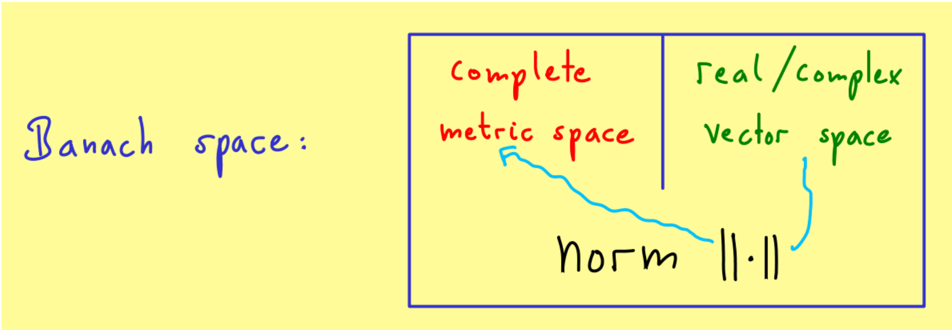
Definition 1.5.2 (Normed Space)**.** *$(X, ||\cdot||)$ is then called a normed space.*

Normed space is a special case of metric space!

Lemma 1.5.1 (Relationship between normed space and metric space)**.** *If* $||\cdot||$ *is a norm for the* $\mathbb{F}$*-vector space* X*, then* $d_{||\cdot||}(x,y) := ||x-y||$ *defines a metric for the set* X*.*

Proof. This can be proved by the defnition of norm. □

Definition 1.5.3 (Banach space)**.** *If* $(X, d_{||\cdot||})$ *is a complete metric space, then the normed space* $(X, ||\cdot||)$ *is called a Banach space.*



The figure is very impressive and informative!

1.6 Examples of Banach Spaces

1.7 Part 12: Continuity

Definition 1.7.1 (Continuity for metric spaces)**.** (X, d_Y)*,* (Y, d_Y) *are two metric spaces. A map* $f : X \to Y$ *is called:*

- *continuous if* $f^{-1}[B]$ *is open in* X *for all open sets* $B \subseteq Y$*.*
- *sequentially continuous if for all* $\tilde{x} \in X$ *and* $(x_n)_{n\in\mathbb{N}} \subseteq X$ *with* $x_n \overset{n\to\infty}{\longrightarrow} \tilde{x}$ *holds* $f(x_n) \overset{n\to\infty}{\longrightarrow} f(\tilde{x})$*.*

These are shown by figs.

Lemma 1.7.1. *For metrc spaces, continuous and sequentially continuous are equivalent. But for topological spaces, they are different.*

Example 1.7.1.

- (X, d_X) *discrete metric space,* (Y, d_Y) *any metrix space* $\Longrightarrow$ *all* $f : X \to Y$ *are continuous.*
- (X, d_X)*,* (Y, d_Y) *metric spaces,* $Y_0 \in Y$ *fixed.* $\Longrightarrow$ $f : X \to Y$*,* $x \mapsto y_0$ *is always continuous.*
- $(X, \|\cdot\|)$ *normed space,* $Y = \mathbb{R}$ *with standard metric.* $\Longrightarrow$ $f : X \to \mathbb{R}$*,* $x \mapsto \|x\|$ *is continuous.*

Proof. Let $(x_n)_{n\in\mathbb{N}} \subseteq X$ sequence with limit $\tilde{x} \in X$. Then:

$$\begin{aligned}
f(x_n) &= \|x_n\| && (1.11)\\
&= \|x_n - \tilde{x} + \tilde{x}\| && (1.12)\\
&\overset{\text{By triangle inequality 1.5.1}}{\leq} \|x_n - \tilde{x}\| + \|\tilde{x}\| && (1.13)\\
&= d(x_n, \tilde{x}) + f(\tilde{x}) && (1.14)\\
&\implies \lim n \to \infty f(x_n) \leq f(\tilde{x}). && (1.15)
\end{aligned}$$

We also hold:

$$\begin{aligned}
f(\tilde{x}) &= \|\tilde{x}\| && (1.16)\\
&= \|\tilde{x} - x_n + x_n\| && (1.17)\\
&\overset{\text{By triangle inequality 1.5.1}}{} \|\tilde{x} - x_n\| + \|x_n\| && (1.18)\\
&= d(\tilde{x}, x_n) + f(x_n) && (1.19)\\
&\implies f(\tilde{x}) \leq \lim_{n\to\infty} f(x_n). && (1.20)
\end{aligned}$$

□

- $(X, \langle\cdot,\cdot\rangle)$ *inner product space,* $Y \in \mathbb{C}$ *with the standard metric,* $x_0 \in X$ *fixed.* $\implies$ $f : X \to \mathbb{C}$, $x \mapsto \langle x_0, x\rangle$ *is continuous.*

Proof. Let $(x_n)_{n\in\mathbb{N}} \subseteq X$ sequence with limit $\tilde{x} \in X$. Then:

$$\begin{aligned}
|f(x_n) - f(\tilde{x})| &= |\langle x_0, x_n\rangle - \langle x_0, \tilde{x}\rangle| && (1.21)\\
&= |\langle x_0, x_n - \tilde{x}\rangle && (1.22)\\
&\overset{\text{By Cauthy Schiwz inequality}}{\leq} \|x_0\| \cdot \|x_n - \tilde{x}\| \overset{n\to\infty}{\longrightarrow} 0. && (1.23)
\end{aligned}$$

Analogously, $g : X \to \mathbb{C}$, $x \mapsto \langle x, x_0\rangle$ is continuous. □

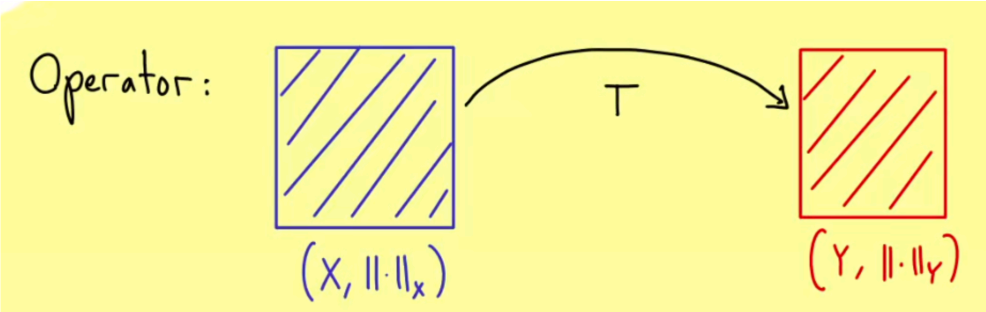
Lemma 1.7.2 (Orthogonal complement is closed)**.** $(X, \langle\cdot,\cdot\rangle)$ *inner product space,* $U \subseteq X$*. Then* $U^\perp$ *is closed.*

Proof. Let $(x_n)_{n\in\mathbb{N}} \subseteq U^\perp$ with limit $\tilde{x} \in X$.

$$\begin{aligned}
&\implies \langle x_n, u\rangle = 0, \forall u \in U && (1.24)\\
&\implies \lim_{n\to\infty} \langle x_n, u\rangle = 0, \forall u \in U && (1.25)\\
&\implies \langle \tilde{x}, u\rangle = 0, \forall u \in U. && (1.26)\\
&\implies \tilde{x} \in U^\perp. && (1.27)
\end{aligned}$$

□

1.8 Part 13: Bounded Operators



$T : X \to Y$, which satisfies

- linear (conserves the algebraic structure)
- continuous (bounded) (conserves the topological structure)

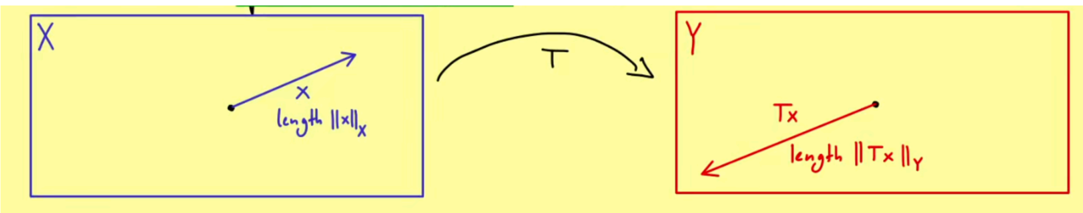
Definition 1.8.1 (Operator norm and bounded). *$(X, \|\cdot\|_X)$, $(Y, \|\cdot\|)$ two normed spaces, $T : X \to Y$ linear, which means*

$$T(x + \tilde{x}) = Tx + T\tilde{x} \tag{1.28}$$
$$T(\lambda x) = \lambda Tx \tag{1.29}$$

for all $x, \tilde{x} \in X$, $\lambda \in \mathbb{F}$. Then

$$\|T\| = \|T\|_{X \to Y} \tag{1.30}$$
$$:= \sup \left\{ \frac{\|Tx\|_Y}{\|x\|_X} | x \in X, x \neq 0 \right\} \tag{1.31}$$

is called the operator norm of T. If $\|T\| < \infty$, T is called bounded.



Proposition 1.8.1 (Continuous equivalent to bounded). *Let $(X, \|\cdot\|_X)$,$(Y, \|\cdot\|_Y)$ two normed spaces, $T : X \to Y$ linear. Then the following claims are equivalent:*

- *T is continuous.*
- *T is continuous at $x = 0$.*
- *T is bounded.*

Proof. (a) $\Longrightarrow$ (b) is easily seen.

(b) $\Longrightarrow$ (c): proposition (*): For all sequence $(x_n)_{n \in \mathbb{N}} \subseteq X$ with $x_n \overset{n \to \infty}{\longrightarrow} 0$, we have $Tx_n \overset{n \to \infty}{\longrightarrow} 0$ due the properties of linear map.

Claim: proposition (*) $\Longrightarrow$ proposition (+): there is a $\delta > 0$ such that $\|Tx\|_Y < 1$ for all $x \in X$ with $\|x\|_X < \delta$.

proof of this claim: we prove its contraposition such that $\neg(*) \Longrightarrow$ For all $n \in \mathbb{N}$, we find $x_n \in X$ with $\|x_n\|_X < \frac{1}{n}$ and $\|Tx_n\|_Y \geq 1 \Longrightarrow \neg(*)$.

$$\frac{\|Tx\|_Y}{\|x\|_X} = \frac{\|Tx\|_Y \cdot \frac{\delta}{2} \cdot \frac{1}{\|x\|_X}}{\|x\|_X \cdot \frac{\delta}{2} \cdot \frac{1}{\|x\|_X}} \tag{1.32}$$

$$= \frac{\|T(\frac{\delta}{2}\frac{x}{\|x\|_X})\|_Y}{\|\frac{\delta}{2}\frac{x}{\|x\|_X}\|_X} \tag{1.33}$$

$$\leq \frac{2}{\delta} \tag{1.34}$$

$$\Longrightarrow \|T\| = \sup\left\{\frac{\|Tx\|_Y}{\|x\|_X} | x \in X, x \neq 0\right\} \leq \frac{2}{\delta} < \infty. \tag{1.35}$$

(c) $\Longrightarrow$ (a): Let $(x_n)_{n\in\mathbb{N}} \subseteq X$ be convergent with limit $\tilde{x} \in X$. Then $\|Tx_n - T\tilde{x}\|_Y = \|T(x_n - \tilde{x})\|_Y \leq \|T\| \cdot \|x_n - \tilde{x}\|_X \overset{n\to\infty}{\longrightarrow} 0$. □

[SZQ: 2023.04.02: This proof needs to be understood later.]

[SZQ: 2023.04.02: Is should be proved that $\|T\|$ defined above is indeed a norm.]

1.9 Part 14: Example Operator Norm

Example 1.9.1. *$X = (C[0,1), \mathbb{F}, \|\cdot\|_\infty)$, $Y = (\mathbb{F}, |\cdot|)$. For $g \in X$ with $g(t) \neq 0$ for all $t \in [0,1]$, define $T_g : X \to Y$ by $T_g(f) := \int_0^1 g(t) \cdot f(t)\mathrm{d}t$. So what is $\|T_g\|$?*

Proof. Recall that

$$\|F_g\| = \sup\left\{\frac{|T_g(f)|}{\|f\|_\infty} | f \in X, f \neq 0\right\} \tag{1.36}$$

This trick has been used before.

$$= \sup\left\{\frac{|T_g(f)|}{\|f\|_\infty} | f \in X, f \neq 0\right\} \tag{1.37}$$

$$= \sup\left\{|T_g(f)| | f \in X, \|f\|_\infty = 1\right\} \tag{1.38}$$

$$= \sup\left\{|\int_0^1 g(t) \cdot f(t)\mathrm{d}t| | f \in X, \|f\|_\infty = 1\right\} \tag{1.39}$$

Since $|\int_0^1 g(t) \cdot f(t)\mathrm{d}t| \leq \int_0^1 |g(t)| \cdot |f(t)|\mathrm{d}t$ and $|f(t)| \leq \|f\|_\infty = 1$

$$\leq \int_0^1 |g(t)|\mathrm{d}t \tag{1.40}$$

$$< \infty. \tag{1.41}$$

Check the other inequality: $h(t) := \frac{\overline{g(t)}}{|g(t)|}$ with $\|h\|_{\infty=1}$. We then have

$$\|T_g\| \geq |T_g(h)| \tag{1.42}$$
$$= |\int_0^1 g(t)\frac{\overline{g(t)}}{|g(t)|}\mathrm{d}t \tag{1.43}$$
$$= \int_0^1 \frac{|g(t)|^2}{|g(t)|}\mathrm{d}t \tag{1.44}$$
$$= \int_0^1 |g(t)|\mathrm{d}t. \tag{1.45}$$

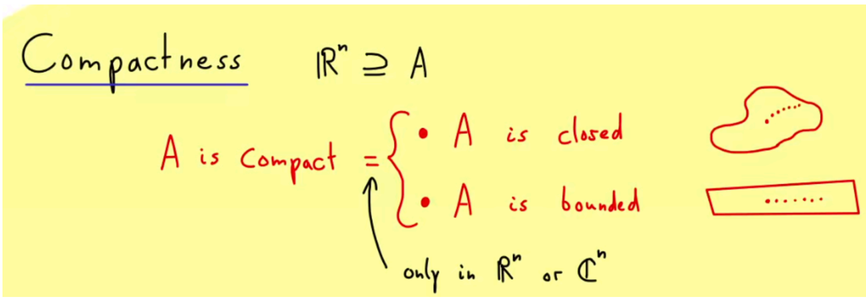
□

1.10 Part 16: Compact Sets

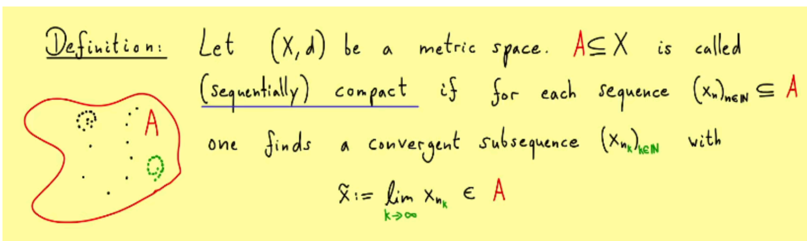
Example 1.10.1. *Compactness $A \subseteq \mathbb{R}^n$. A is compact such that*

- *A is closed.*
- *A is bounded.*

This is only true in $\mathbb{R}^n$ or $\mathbb{C}^n$.



Definition 1.10.1 (Sequentially compact)**.** *Let (X, d) be a metric space. $A \subseteq X$ is called (sequentially) compact if for each sequence $(x_n)_{n\in\mathbb{N}} \subseteq A$, one finds a convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$ with $\tilde{x} := \lim_{k\to\infty} x_{n_k} \in A$.*



Example 1.10.2.
- $(\mathbb{R}, d_{eucl.})$, *$A = [0, 1]$ compact by Bolzano-Weierstrass theorem.*
- $(\mathbb{R}, d_{discr})$, *$A = [0, 1]$ not compact because: The sequence $(x_n)_{n\in\mathbb{N}} \subseteq A$ with $x_n = \frac{1}{n}$ satisfies $d_{discr.}(x_n, x_m) = 1$ for all $n, m \in \mathbb{N}$ with $n \neq m$. $\implies$ no convergent subsequence.*

Proposition: Let (X,d) be a metric space and $A \subseteq X$ compact.
Then A is closed and bounded. There is an $x \in X$ and an $\varepsilon > 0$ such that $B_\varepsilon(x) \supseteq A$

Definition 1.10.2 (Bounded). *Let (X, d) be a metrix space and $A \subseteq X$ compact. A is bounded means that there is an $x \in X$ and an $\varepsilon > 0$ such that $A \subseteq B_\varepsilon(x)$.*

Proposition 1.10.1 (Compact implies closed and bounded). *Let (X, d) be a metrix space and $A \subseteq X$ compact. Then A is closed and bounded.*

Proof. Let $A \subseteq X$ be compact.

(1) Let $(x_n)_{n\in\mathbb{N}} \subseteq A$ be convergent with limit $\tilde{x} \in X$.

$$\Longrightarrow \text{There is aa convergent subsequence } (x_{n_k})_{k\in\mathbb{N}} \text{ with limit } \tilde{\tilde{x}} \in A \tag{1.46}$$
$$\Longrightarrow \tilde{x} = \tilde{\tilde{x}} \in A \tag{1.47}$$
$$\Longrightarrow A \textit{ is closed}! \tag{1.48}$$

(2) contraposition: A is not bounded.

$\Longrightarrow$ For given $a \in A$, there are $x_n \in A$ with $d(a, x_n) > n$.

$\Longrightarrow$ For any subsequence $(x_{n_k})_{k\in\mathbb{N}}$ and any point $b \in A$:

$$n_K < d(a, x_{n_k}) \tag{1.49}$$
$$\leq d(a, b) + d(b, x_{n_k}) \tag{1.50}$$

$\Longrightarrow n_k - d(a, b) \leq d(b, x_{n_k})$.

$\Longrightarrow d(b, x_{n_k}) \xrightarrow{k\to\infty} 0$ for all $b \in A$

$\Longrightarrow A$ not compact! □

1.11 Part 18: Compact operators

Definition 1.11.1 (Compact operators). *Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be two normed spaces. A bounded linear operator $T : X \to Y$ is called compact if $\overline{T[B_1(0)]} \subseteq Y$ is a compact set.*

[SZQ: 2023.04.02: the defintion of $\overline{T[B_1(0)]} \subseteq Y$ should be learned first.]

1.12 Part 19: Holder's inequality

Lemma 1.12.1 (Young's inequality). *For all $a, b > 0$, we have $a, b \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}$.*

Proof. Note that function $f : x \mapsto e^x$ is convex, we have

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y). \tag{1.51}$$

Let $\lambda = \frac{1}{p}$, $x = \ln a^p$, $1 - \lambda = \frac{1}{p'}$, and $y = \ln b^{p'}$. We have

$$a \cdot b = f(\frac{1}{p}\ln a^p + \frac{1}{p'}\ln b^{p'}) = f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \tag{1.52}$$
$$= \frac{1}{p}f(\ln a^p) + \frac{1}{p'}f(\ln b^{p'}) \tag{1.53}$$
$$= \frac{a^p}{p} + \frac{b'}{p'}. \tag{1.54}$$

□

Theorem 1 (Holder's inequality). *For all $x, y \in \mathbb{F}^n$, we have*

$$\|xy\|_1 \leq \|x\|_p \cdot \|y\|_{p'}, \tag{1.55}$$

where $x \in \mathbb{F}^n$ and the p-norm of x is

$$\|x\|_q := \left(\sum_{j=1}^{n} |x_j|^q\right)^{\frac{1}{q}}, \tag{1.56}$$

$q \in [1, \infty)$, $\frac{1}{p} + \frac{1}{p'} = 1$, *and for $x, y \in \mathbb{F}^n$, we write*

$$xy := \begin{pmatrix} x_1y_1 \\ x_2y_2 \\ \cdots \\ x_ny_n \end{pmatrix}. \tag{1.57}$$

Proof. Case 1: $x = 0$ or $y = 0$.

Case 2: We have

$$\frac{1}{\|x\|_p\|y\|_{p'}}\|xy\|_1 = \frac{1}{\|x\|_p\|y\|_{p'}}\sum_{j=1}^{n}|x_jy_j| \tag{1.58}$$
$$= \sum_{j=1}^{n}\frac{|x_j|}{\|x\|_p}\frac{|y_j|}{\|y\|_{p'}} \tag{1.59}$$

By Young's lemma (1.12.1)

$$\leq \sum_{j=1}^{n}\frac{1}{p}\cdot\frac{|x_j|^p}{\|x\|_p^p} + \sum_{j=1}^{n}\frac{1}{p'}\cdot\frac{|y_j|^{p'}}{\|x\|_{p'}^{p'}} \tag{1.60}$$
$$= \frac{1}{p} + \frac{1}{p'} \tag{1.61}$$
$$= 1. \tag{1.62}$$

□

1.13 Part 21: Isomorphism?

Definition 1.13.1 (Homomorphism). ***What is Homomorphism: map that preserves structures.***

[SZQ: When we talk about homophism, we must know the underlying structures!]

Example 1.13.1.

- *Let X, Y be vector spaces and $f : X \to Y$ be a map. We want*

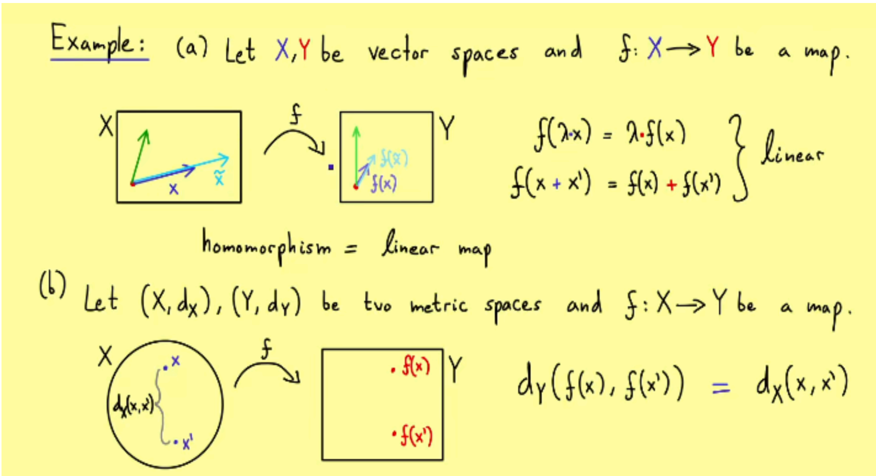
$$f(\lambda \cdot x) = \lambda \cdot f(x) \tag{1.63}$$
$$f(x + x') = f(x) + f(x'), \tag{1.64}$$

which is called linear. Thus homomorphism = linear map!

- *Let $(X, d_X), (Y, d_Y)$ be two metric spaces and $f : X \to Y$ be a map. We want*

$$d_Y(f(x), f(x')) \leq d_X(x, x'). \tag{1.65}$$

homomorphism = map that satisfies (1.65)



Definition 1.13.2 (Isomorphism)**.** *Isomorphism = homomorphism + bijective + inverse map is also homomorphism*

Definition 1.13.3 (Isomorphism for Banach spaces)**.** *Isomorphism for banach spaces X, Y: $f : X \to Y$ with: linear + bijective + $\|f(x)\|_Y = \|x\|_X$ (oftern called isometric isomorphism).*

Example 1.13.2.

- $S_R : l^p(\mathbb{N}) \to l^p(\mathbb{N})$, $(x_1, x_2, x_3, \dots) \mapsto (0, x_1, x_2, \dots)$

$$\implies \textit{linear, } \|S_R x\|_p = \|x\|_p \textit{ not surjective} \tag{1.66}$$
$$\implies \textit{not an isomorphism} \tag{1.67}$$

- $S : l^p(\mathbb{Z}) \to l^p(\mathbb{Z})$, $(\dots, x_{-1}, x_0, x_1, \dots) \mapsto (\dots, x_{-2}, x_{-1}, x_0, \dots)$

$$\implies \textit{linear, } \|Sx\|_p = \|x\|_p \textit{ and bijective} \tag{1.68}$$
$$\implies \textit{isomorphism} \tag{1.69}$$

1.14 Part 22: Dual spaces

Proposition 1.14.1. *Let X be a normed space. Then $(X', \|\cdot\|_{X \to \mathbb{F}})$ is a Banach space.*

Proof. □

[SZQ: 2023.04.02: I need to learn Riesz representation theorem!]

1.15 Part 28: Spectrum for bounded linear operators

Recall: $A \in \mathbb{C}^{n\times n}$ matrix with n rows and n columns. $\lambda \in \mathbb{C}$ is called an eigenvalue of A if:

$$\exists x \in \mathbb{C}^n \setminus \{0\} : Ax = \lambda x \tag{1.70}$$
$$\iff \exists x \in \mathbb{C}^n \setminus \{0\} : (A - \lambda I)x = 0 \tag{1.71}$$
$$\iff \mathrm{Ker}(A - \lambda I) \neq \{0\} \tag{1.72}$$
$$\iff \textit{map } x \mapsto (A - \lambda I)x \textit{ not injective.} \tag{1.73}$$

Theorem 2 (Rank-nullity theorem)**.** *For any matrix $M \in \mathbb{C}^{m\times n}$:*

$$dim(Ran(M)) + dim(Ker(M)) = n. \tag{1.74}$$

Definition 1.15.1 (Spectrum and resolvent)**.** *Let X be a complex Banach space and $T : X \to X$ be a bounded linear operator. Then the spectrum of T is defined by:*

$$\sigma(T) := \{\lambda \in \mathbb{C} | (T - \lambda I) not\ bijective\}. \tag{1.75}$$

The resolvent of T is defined by:

$$\rho(T) := \left\{\lambda \in \mathbb{C} | (T - \lambda I)\ bijective\ and\ (T - \lambda I)^{-1}\ bounded\right\}. \tag{1.76}$$

Corollary 1.15.1. *By bounded inverse theorem, we have*

$$\sigma(T) = \mathbb{C} \setminus \rho(T). \tag{1.77}$$

Definition 1.15.2 (Point/continuous/residual spectrum)**.** *We have the disjoint union: $\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T)$. We have*

$$Point\ spectrum : \sigma_p := \{\lambda \in \mathbb{C} | (T - \lambda I)\ not\ injective\}, \tag{1.78}$$
$$Continuous\ spectrum : \sigma_c := \{\lambda \in \mathbb{C} | \tag{1.79}$$
$$(T - \lambda I)\ injective\ but\ not\ surjective\ with\ \overline{Ran(T - \lambda I)} = X\}, \tag{1.80}$$
$$Residual\ spectrum : \sigma_r := \{\lambda \in \mathbb{C} | \tag{1.81}$$
$$(T - \lambda I)\ injective\ but\ not\ surjective\ with\ \overline{Ran(T - \lambda I)} \neq X\}. \tag{1.82}$$
$$\tag{1.83}$$

[SZQ: 2023.04.02: I need to learn injective, surjective, bijective first!]

1.16 Part 31: Spectral Radius

Definition 1.16.1 (Spectral radius)**.** *X complex Banach space. $T : X \to X$ bounded lineara operator. We define the spectral radius as*

$$r(T) := \sup\{|\lambda|\}, \tag{1.84}$$

where $\lambda \in \sigma(T)$.

Here, we have a fig to show this def.

Theorem 3. *X complex Banach space, $T : X \to X$ bounded linear operator. Then*

- *$\sigma(T) \subseteq \mathbb{C}$ is compact*
- *$X \neq \{0\} \implies \sigma(T) \neq \emptyset$*
- *$r(T) := \sup |\lambda| = \lim_{k\to\infty} \|T^k\|^{\frac{1}{k}} = \inf_{k\to\mathbb{N}} \|T^k\|^{\frac{1}{k}} \leq \|T\| < \infty$,*

where $\lambda \in \sigma(T)$

Proof. □

[SZQ: 2023.04.02: I need to learn the properties of sepctrum, dual spaces, Vom-Neuman series, Liouville's theorem, Hahn-Banach Theorem!]

1.17 Part 32: Normal and Self-Adjoint Operators

Definition 1.17.1 (Adjoint operator)**.** *Let X be a Hilbert space and $T : X \to X$ a bounded lineaar operator. The bounded linear operator $T^* : X \to X$ defined by*

$$\langle y, Tx\rangle = \langle T^*y, x\rangle, \forall x, y \in X \tag{1.85}$$

is called the adjoint operator of T.

Definition 1.17.2 (Self Adjoint operator)**.** *Let X be a Hilbert space and $T : X \to X$ a bounded lineaar operator. T is called*

- *self-adjoint if $T^* = T$*
- *skew-adjoint if $T^* = -T$*
- *normal if $T^*T = TT^*$*

Proposition 1.17.1. *T is normal $\implies r(T) = \|T\|$.*