Chapter 1

Introduction to Quantum Optics by Immanuel

1.1 Introduction

- classicaal: classical atom and light
- semiclassical: quantized atom and classical light
- quantum mechanical: quantized atom and light

Light-Atom Interaction Hamiltonian

- classical dipole in eletric field: dipole moment $\overrightarrow{d} = q\overrightarrow{r}$, $U_I = -\overrightarrow{d} \cdot \overrightarrow{E}$. We have

$$\hat{H}_I = -\hat{d} \cdot \overrightarrow{E}(\overrightarrow{v_0}, t), \tag{1.1}$$

  where $\hat{d} = q\hat{v}$ is the dipole operator.
- induced atomic dipole

1.2 Light Atom Quantum Evolution

Time Evolution We have the Schrodinger equation (both sides) as

$$i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = (\hat{H}_0 + \hat{H}_I(t))|\Psi(t)\rangle, \tag{1.2}$$

where the general ansatz (assumption) is

$$|\Psi(t)\rangle = \sum_n c_n(t)e^{-iE_nt/\hbar}|n\rangle, \tag{1.3}$$

and

$$\hat{H}_0|n\rangle = E_n|n\rangle \tag{1.4}$$

is the atomic eigenstates. Inserting $|\Psi(t)\rangle$ and $\hat{H}_0|n\rangle$ into Schrodinger equation, we get

$$i\hbar\sum_n\left\{\dot{c}_n e^{-iE_nt/\hbar}|n\rangle-\frac{iE_n}{\hbar}c_n e^{-iE_nt/\hbar}|n\rangle\right\}=\sum_n\left\{c_n e^{-iE_nt/\hbar}|n\rangle+c_n e^{-iE_nt/\hbar}\hat{H}_I|n\rangle\right\} \tag{1.5}$$

$$\Longrightarrow i\hbar\sum_n\dot{c_n}e^{-iE_nt/\hbar}|n\rangle=\sum_n c_n e^{-iE_nt/\hbar}\hat{H}_I|n\rangle \tag{1.6}$$

$$\Longrightarrow i\hbar\dot{c_n}e^{-iE_kt/\hbar}|=\sum_n c_n(t)e^{-iE_nt/\hbar}\langle k\hat{H}_I(t)|n\rangle \tag{1.7}$$

$$\Longrightarrow i\hbar\dot{c}_k=\sum_n c_n(t)e^{-iE_{n,k}t/\hbar}\langle k|\hat{H}_I(t)|n\rangle, \tag{1.8}$$

where we use

$$\langle k|n\rangle=\delta_{kn}, \tag{1.9}$$

$$E_{n,k}=E_n-E_k, \tag{1.10}$$

$$\omega_{nk}=(E_n-E_k)/\hbar. \tag{1.11}$$

and $\langle k|\hat{H}_I(t)|n\rangle$ is the matrix element.

1.3 Time Dependent Perturbation Theory

Recall the time evolution:

$$i\hbar\dot{c}_k=\sum_n c_n(t)e^{-i\omega_{nk}t}\langle k|\hat{H}_I(t)|n\rangle, \tag{1.12}$$

and

$$\omega_{nk}=(E_n-E_k)/\hbar. \tag{1.13}$$

Consider the Simplification (Perturbation Theory)

- System only in state $|1\rangle$ at $t=0\Longrightarrow c_1|0\rangle=1$ (only the ground state $|1\rangle$),
- Perturbative treatment of interaction term: weak perturbation $\forall|c_k(t)|^2<<1$.

We then have

$$i\hbar\dot{c}_k=e^{i\omega_{1k}t}\langle k|\hat{H}_I(t)|1\rangle, \tag{1.14}$$

with $c_k(0)=0$, we obtain:

$$c_k(t)=\frac{1}{i\hbar}\int_0^t e^{-i\omega_{1k}t}\langle k|\hat{H}_I(t')|1\rangle\mathrm{d}t'. \tag{1.15}$$

Example 1.3.1 (Sinusoidal perturbation)**.** *Define*

$$\hat{H}(t)=\hat{H}_I e^{-i\omega t}. \tag{1.16}$$

Given the figure in the video, we have

$$c_k(T) = \frac{1}{i\hbar}\int_0^T e^{i\Delta\omega t}\langle k|\hat{H}_I|1\rangle \mathrm{d}t \tag{1.17}$$

$$\implies \textit{Transition probability } P_{k1}(T) = |c_k(T)|^2 = \frac{1}{\hbar^2}|\langle k|\hat{H}_I|1\rangle|^2 Y(\Delta\omega, T), \tag{1.18}$$

with

$$Y(\Delta\omega, T) = \frac{\sin^2(\Delta\omega T/2)}{(\Delta\omega/2)^2} \tag{1.19}$$

$$\sim \operatorname{sinc}^2 x, \tag{1.20}$$

where $\Delta\omega = \omega - \omega_{1k}$ *is the detwining.*

Let's take a look at the sinc function $Y(\Delta\omega, T) = \operatorname{sinc}^2 x$. Transition for $\Delta\omega \leq \frac{2\pi}{T}$, we have $\Delta\omega \cdot T \leq 2\pi$, which implies

$$\Delta E \cdot T \leq h, \tag{1.21}$$

which is the time-frequency uncertainty. (The expression in the video seems wrong, so I make corrections abrove.) We have the following case

$$\frac{1}{2\pi T} Y(\Delta\omega, T) \overset{T\to\infty}{\to} \delta(\Delta\omega), \tag{1.22}$$

then we have

$$P_{k1}(T \to \infty) = \frac{2\pi}{\hbar^2}|\langle k|\hat{H}_I|i\rangle|^2 \delta(\Delta\omega) T. \tag{1.23}$$

Fermi's Golden Rule $|k\rangle$ Quasi continuum of final states. We have the transition probability

$$P_{k1} = \Gamma_{k1} T, \tag{1.24}$$

where

$$\Gamma_{k1} = \frac{2\pi}{\hbar}|\langle k|\hat{H}_I|1\rangle|^2 \rho(E_k = E_1 + \hbar\omega) \tag{1.25}$$

is called the Femi's Golden Rule,

$$|\langle k|\hat{H}_I|1\rangle|^2 \tag{1.26}$$

is the coupling strength $\propto E_0^2$ and $\propto I$,

$$\rho(E_k = E_1 + \hbar\omega) \tag{1.27}$$

is the density states which is number of availble final states to the system,

$$\Gamma_{k1} \hat{=} \textit{Transition Rate} = \frac{\mathrm{d}P_{k1}}{\mathrm{d}T}, \tag{1.28}$$

and density states

$$\rho(E) = \frac{\mathrm{d}N}{\mathrm{d}E}, \tag{1.29}$$

where ΔN is the number of states in an energy interval ΔE around energy E_k and we let ΔE approaches 0.

1.4 Two Level Atom (TLA)

Given by the figure, in state $|1\rangle$, we have $E_1 = \hbar\omega_1$ and in state $|2\rangle$, we have $E_2 = \hbar\omega_2$ and $E_2 - E_1 = \hbar(\omega_2 - \omega_1) = \omega_{21}$. We have the Hamiltonian

$$\hat{H} = \hat{H}_0 - \hat{d} \cdot E(t), \tag{1.30}$$

where

$$E(t) = \varepsilon E_0 \cos(\omega t), \tag{1.31}$$

where ε is the polarization vector, E_0 is the field amplitude, and ω is the frequency of the light field.

Ansatz for Solving TLA We have

$$|\Psi(t)\rangle = c_1(t)e^{-i\omega_1 t}|1\rangle + c_2(t)e^{-i\omega_2 t}|2\rangle. \tag{1.32}$$

Time Evolution Amplitude We have

$$\dot{c_1}(t) = i\frac{d_{12}^{\varepsilon}E_0}{\hbar}e^{-\omega_{21}}\cos(\omega t)c_2(t) \tag{1.33}$$

$$\dot{c_2}(t) = i\frac{d_{12}^{\varepsilon}E_0}{\hbar}e^{+\omega_{21}}\cos(\omega t)c_1(t), \tag{1.34}$$

where

$$d_{12}^{\varepsilon} = \langle 1|\hat{d} \cdot \varepsilon|2\rangle \tag{1.35}$$

$$= \langle 1|\hat{d}|2\rangle \cdot \varepsilon \tag{1.36}$$

$$= \langle 1|\hat{d}_x|2\rangle \cdot \varepsilon_x + \langle 1|\hat{d}_y|2\rangle \cdot \varepsilon_y + \langle 1|\hat{d}_z|2\rangle \cdot \varepsilon_z. \tag{1.37}$$

is the Dipole Matrix Element, which is the atomic property and we assume it's real. We also define

$$\Omega_0 = \frac{d_{12}^{\varepsilon}E_0}{\hbar} \tag{1.38}$$

as the Rubi frequency.

Time Evolution Using Euler' form, we have

$$\dot{c_1}(t) = i\frac{\Omega_0}{2}e^{-\omega_{21}}(e^{i\omega t} + e^{-i\omega t})c_2(t) \tag{1.39}$$

$$\dot{c_2}(t) = i\frac{\Omega_0}{2}e^{+\omega_{21}}(e^{i\omega t} + e^{-i\omega t})c_1(t) \tag{1.40}$$

by

$$\cos\alpha = \frac{1}{2}(e^{i\alpha} + e^{-i\alpha}) \tag{1.41}$$

and

$$e^{i\alpha} = \cos\alpha + i\sin\alpha. \tag{1.42}$$

Rotating Wave Approximation We have

$$\dot{c_1}(t) = i\frac{\Omega_0}{2}(e^{+i(\omega-\omega_{21})t} + e^{-i(\omega+\omega_{21})t})c_2(t) \tag{1.43}$$

$$\dot{c_2}(t) = i\frac{\Omega_0}{2}(e^{-i(\omega-\omega_{21})t} + e^{+i(\omega+\omega_{21})t})c_1(t), \tag{1.44}$$

and we ignore the sum frequency term and get

$$\dot{c_1}(t) = i\frac{\Omega_0}{2}e^{+i(\omega-\omega_{21})t}c_2(t) \tag{1.45}$$

$$\dot{c_2}(t) = i\frac{\Omega_0}{2}e^{-i(\omega-\omega_{21})t}c_1(t), \tag{1.46}$$

which is a good approcimation for detwining $\delta = \omega - \omega_{21} \approx 0$. We introduce

$$\tilde{c_1}(t) = c_1(t)e^{-i\frac{\delta}{2}t} \tag{1.47}$$

$$\tilde{c_2}(t) = c_2(t)e^{+i\frac{\delta}{2}t}. \tag{1.48}$$

$$\tag{1.49}$$

Ansatz Wavefunctions for TLA Whole time evolution in state amplitudes

$$|\Psi(t)\rangle = c_1'(t)|1\rangle + c_2'(t)|2\rangle. \tag{1.50}$$

Time evolution when field is off

$$|\Psi(t)\rangle = c_1'(0)e^{-i\omega_1 t}|1\rangle + c_2'(0)e^{-i\omega_2}|2\rangle. \tag{1.51}$$

However, this is boring. We chose different ansatz as

$$|\Psi(t)\rangle = c_1(t)e^{-i\omega_1 t}|1\rangle + c_2(t)e^{-i\omega_2 t}|2\rangle \tag{1.52}$$

$$\iff |\Psi(t)\rangle = c_1(t)|1\rangle + c_2(t)e^{-i\omega_{21}t}|2\rangle, \tag{1.53}$$

where $c_1(t)$ and $c_2(t)$ capture time evolution on top of eigenstate evolution! We now have

$$|\Psi(t)\rangle = c_1(t)|1\rangle + c_2(t)e^{-i\omega_{21}t}|2\rangle, \tag{1.54}$$

which is called the rotating frame of atom. We also have Rotating frame of light field as

$$|\Psi(t)\rangle = \tilde{c_1}(t)|1\rangle + \tilde{c_2}(t)e^{-i\omega t}|2\rangle, \tag{1.55}$$

where ω is the light frequency, $\tilde{c_1}$ and $\tilde{c_2}$ describe time evolution on top of fast light field oscilation.

Solving the TLA Dynamics We have the following equations:

$$\frac{d}{dt}\begin{pmatrix}\tilde{c_1}(t)\\ \tilde{c_2}(t)\end{pmatrix} = \frac{i}{2}\begin{pmatrix}-\delta & \Omega_0\\ \Omega_0 & +\delta\end{pmatrix}\begin{pmatrix}\tilde{c_1}(t)\\ \tilde{c_2}(t)\end{pmatrix}. \tag{1.56}$$

Considering the simplest case $\delta = 0$

$$\frac{d}{dt}\tilde{c_1}(t) = \frac{i}{2}\Omega_0\tilde{c_2}(t) \tag{1.57}$$

$$\frac{d}{dt}\tilde{c_2}(t) = \frac{i}{2}\Omega_0\tilde{c_1}(t). \tag{1.58}$$

Take time dirivative of the first equation, then we have

$$\ddot{\tilde{c_1}}(t) = -\frac{\Omega_0^2}{4}\tilde{c_1}(t), \tag{1.59}$$

the solutions of which are

$$\tilde{c_1}(t) = \cos(\Omega_0 t/2) \tag{1.60}$$

$$\tilde{c_2}(t) = i\sin(\Omega_0 t/2) \tag{1.61}$$

for $\tilde{c_1}(0) = 1$ and $\tilde{c_2}(0) = 0$. Also we can obtain the excited state probability as

$$P_2(t) = |c_2(t)|^2 \tag{1.62}$$

$$= |\tilde{c_2}(t)|^2. \tag{1.63}$$

Rabi Oscillations (Resonant Case) Nonlinear Response can be seen from the figure.

General Rabi Oscillations (with detuning) Given the figurem.

$$|\tilde{c_2}(t)|^2 = \frac{\Omega_0^2}{\Omega}\sin^2\left(\frac{1}{2}\Omega t\right) \tag{1.64}$$

$$= \frac{\Omega_0^2}{2\Omega^2}\left\{1 - \cos(\Omega t)\right\}, \tag{1.65}$$

where $\Omega = \sqrt{\Omega_0^2 + \delta^2}$ is the effective Rabi frequency.

Interesting Special Cases a) Pi-Puls $\Omega_0\tau = \pi$: swap population

$$|1\rangle \to i|2\rangle \tag{1.66}$$

$$|2\rangle \to i|1\rangle. \tag{1.67}$$

b) 2Pi-Puls $\Omega_0\tau = 2\pi$: flip the sign

c) Pi/2-Puls $\Omega_0\tau = \pi/2$: superposition state

1.5 Oscillating Dipoles

Atomic Eigenstates

$$|\Psi_{nlm}(t)\rangle = e^{-iE_{nlm}t/\hbar}|\Psi_{nlm}(0)\rangle, \tag{1.68}$$

$$\hat{H}_0|\Psi_{nlm}(0)\rangle = E_{nlm}|\Psi_{nlm}\rangle, \tag{1.69}$$

and the electron density is

$$\rho(r,\theta,\phi) = |\Psi(r,\theta,\phi,t=0)^2|. \tag{1.70}$$

Atomic Dipole Calculate (Oscillating) Dipole Moment for Atomic Eigenstate. We denote $|1\rangle = |\Psi_{nlm}\rangle$. We have

$$d(t) = \langle 1(t)|\hat{d}|1(t)\rangle \tag{1.71}$$
$$= \langle \hat{d}|1\rangle \tag{1.72}$$
$$= -e\langle 1|\hat{r}|1\rangle. \tag{1.73}$$

Then,

$$-e\langle 1|\hat{r}|1\rangle = -e\langle 1|\hat{P}\hat{P}^{-1}\hat{r}\hat{P}\hat{P}^{-1}|1\rangle \tag{1.74}$$
$$= +e\langle 1|\hat{r}|1\rangle, \tag{1.75}$$

which implies

$$\langle 1|\hat{r}|1\rangle = 0. \tag{1.76}$$

1.6 The Bloch Sphere