# Chapter 1

# Prerequisites

## 1.1 Hilbert Spaces and Linear Operators

Throughout this course, $\mathcal{H}$ denotes a finite-dimensional Hilbert space (complex vector space with an associated inner product). Using Dirac's "bra-ket" notation we denote elements of the Hilbert space (called kets) as

$$|\psi\rangle \in \mathcal{H}. \tag{1.1}$$

The elements of the dual Hilbert space are called bras and are denoted

$$\langle\psi| \in \mathcal{H}^*, \tag{1.2}$$

where $\langle\psi| = (|\psi\rangle)^\dagger$. Here, $X^\dagger := \bar{X}^T$ denotes the Hermitian adjoint (also called the conjugate transpose). We denote

$$B(\mathcal{H}_1, \mathcal{H}_2) := \{\text{linear maps from } \mathcal{H}_1 \text{ to } \mathcal{H}_2\} \tag{1.3}$$

and the set of all linear maps to and from the same space will be denoted $B(\mathcal{H}) = B(\mathcal{H}, \mathcal{H})$. An operator $X \in B(\mathcal{H})$ is *normal* if $XX^T = X^T X$. Every normal operator has a *spectral decomposition.* That is, there exists a unitary $U$ and a diagonal matrix $D$ whose entries are the eigenvalues $\lambda_1, \ldots, \lambda_d \in \mathbb{C}$ of $X$ such that

$$X = UDU^\dagger. \tag{1.4}$$

In other words,

$$X = \sum_{i=1}^{d} \lambda_i |\psi_i\rangle \langle\psi_i| \tag{1.5}$$

where $X|\psi_i\rangle = \lambda_i |\psi_i\rangle$ and $U = (|\psi_i\rangle, \ldots, |\psi_d\rangle)$. If $X$ is Hermitian, $X = X^\dagger$, then $\lambda_i \in \mathbb{R}$. An operator $X$ is positive semi-definite (PSD) if

$$\langle\varphi| X |\varphi\rangle \geq 0 \qquad \forall |\varphi\rangle \in \mathcal{H}. \tag{1.6}$$

As a consequence, $X \geq 0$ and $\lambda_i \geq 0$. It holds that PSD $\implies$ Hermitian $\implies$ normal. Unless otherwise stated, we will always assume we are working in an orthonormal basis.

## 1.2 Quantum States

A quantum state $\rho$ in a Hilbert space $\mathcal{H}$ is a PSD linear operator with

$$\rho \in B(\mathcal{H}), \quad \rho \geq 0, \quad \mathrm{tr}\rho = 1. \tag{1.7}$$

This means that the state has eigenvalues $\{\lambda_i\}_{i=1}^d$ satisfying $\lambda_i \geq 0$ and $\sum_{i=1}^d \lambda_i = 1$. Thus, $\{\lambda_i\}_{i=1}^d$ forms a probability distribution.

A *pure quantum state* $\psi$ is a quantum state with rank 1. We can find $|\psi\rangle \in \mathcal{H}$ such that $\psi = |\psi\rangle\langle\psi|$. In this case, $\psi$ is called a *projector*. A *mixed state* is a quantum state with rank $> 1$. Mixed states are convex combinations of pure states. That is, for every quantum state $\rho$ with $r = \mathrm{rank}(\rho)$ there are pure states $|\psi_i\rangle_{i=1}^k \quad (k \geq r)$ and a probability distribution $\{p_i\}_{i=1}^k$ such that

$$\rho = \sum_{i=1}^k p_i |\psi_i\rangle\langle\psi_i|. \tag{1.8}$$

The spectral decomposition of $\rho$ is a special case of this property.

## 1.3 Composite systems, partial trace, entanglement

Let $A$ and $B$ be two quantum systems with Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$. The *joint system* $AB$ is described by the Hilbert space $\mathcal{H}_{AB} := \mathcal{H}_A \otimes \mathcal{H}_B$. We denote quantum states of the joint system as $\rho_{AB} \in \mathcal{H}_{AB}$. The marginal of the bipartite state, denoted $\rho_A$, is uniquely defined as the operator satisfying

$$\rho_A := \mathrm{tr}_B \rho_{AB}, \tag{1.9}$$

which is defined via $\mathrm{tr}(\rho_{AB}(X_A \otimes \mathbb{I}_B)) = \mathrm{tr}\rho_A X_A \quad \forall X_A \in B(\mathcal{H}_A)$. For a Hilbert space with $|B| := \dim \mathcal{H}_B$, the explicit form of the partial trace is

$$\mathrm{tr}_B \rho_{AB} = \sum_{i=1}^{|B|} (\mathbb{I}_A \otimes \langle i|_B) \rho_{AB} (\mathbb{I}_A \otimes |i\rangle_B), \tag{1.10}$$

for some basis $\{|i\rangle_B\}_{i=1}^{|B|}$ of $\mathcal{H}_B$.

A *product state* on $AB$ is a state of the form $\rho_A \otimes \sigma_B$. The state is called *separable* if it lies in the convex hull of product states:

$$\rho_{AB} = \sum_i p_i \rho_A^i \otimes \sigma_B^i \tag{1.11}$$

for some states $\{\rho_A^i\}_i$ and $\{\sigma_B^i\}_i$ and probability distribution $\{p_i\}_i$. A state is called *entangled*, if it is not separable. An entangled state of particular interest is the *maximally entangled state*. Let $d = \dim \mathcal{H}$, $\{|i\rangle\}_{i=1}^d$ be a basis for $\mathcal{H}$. A maximally entangled state is expressed as

$$|\phi^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^d |i\rangle \otimes |i\rangle \quad \in \mathcal{H} \otimes \mathcal{H} \tag{1.12}$$

## 1.4 Measurements

The most general measurement is given by a *positive operator-valued measure* (POVM) $E = \{E_i\}_i$ where $E_i \geq 0 \quad \forall i$ and $\sum_i E_i = \mathbb{I}$. Then, for a quantum system $\mathcal{H}$ in state $\rho$, the probability of obtaining measurement outcome $i$ is given by $p_i = \mathrm{tr}[\rho E_i]$. So, we have

$$\sum_i p_i = \sum_i \mathrm{tr}[\rho E_i] = \mathrm{tr}\left[\rho \sum_i E_i\right] = \mathrm{tr}[\rho \mathbb{I}] = \mathrm{tr}\rho = 1, \tag{1.13}$$

for all normalized quantum states. A *projective measurement* $\Pi = \{\Pi_i\}$ is a POVM with the added property of orthogonality, which for projectors means

$$\Pi_i \Pi_j = \delta_{ij} \Pi_i. \tag{1.14}$$

Any basis $\{|e_i\rangle\}_{i=1}^{\dim \mathcal{H}}$ gives rise to a projective measurement $\Pi = \{|e_i\rangle \langle e_i|\}_{i=1}^{\dim \mathcal{H}}$.

## 1.5 Entropies

The *Shannon entropy* $H(p)$ of a probability distribution $p = \{p_i, \ldots, p_d\}$ is defined as $H(p) = -\sum_{i=1}^d p_i \log p_i$, where the logarithm is base 2 unless otherwise specified. Note that when the logarithm is base 2, the entropy has units of *bits*. The *von Neumann entropy* $S(\rho)$ of a quantum state $\rho$ is defined as

$$S(\rho) = -\mathrm{tr}\left[\rho \log \rho\right] = H(\{\lambda_i, \ldots, \lambda_d\}), \tag{1.15}$$

where $\rho = \sum_i \lambda_i |\psi_i\rangle \langle \psi_i|$ is a spectral decomposition of $\rho$ and where the logarithm of an operator is obtained by first diagonalizing the matrix representing the operator and then taking the logarithm of the diagonal elements. That is,

$$\log \rho = \sum_{i:\lambda_i > 0} \log\left(\lambda_i\right) |\psi_i\rangle \langle \psi_i| \,. \tag{1.16}$$