Note

Zanqiu Shen

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Contents

Part I

Mathematical Fundamentals

Chapter 1

Advent of Mathematical Symbols

- Kronecker delta:

$$\delta_{ij} := \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \tag{1.0.1}$$

- Levi-Civita symbol:

$$\varepsilon_{ijk} := \begin{cases} 1, & (i,j,k) = (1,2,3) \text{ or } (2,3,1) \text{ or } (3,1,2) \\ -1, & (i,j,k) = (3,2,1) \text{ or } (2,1,3) \text{ or } (1,3,2) \\ 0, & \text{else.} \end{cases} \tag{1.0.2}$$

  Example 1.0.1.

$$(a \times b)_i = \sum_{j,k=1}^{3} \varepsilon_{ijk} a_j b_k, \tag{1.0.3}$$

  where a, b are three dimensional vectors and "$\times$" denotes cross product.

- Nabla symbol:

$$\nabla := \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{pmatrix}. \tag{1.0.4}$$

- Factorial: $n! := n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$.

  Recursive definition: $0! := 1$, $n! := n \cdot (n-1)!,\ n \in \mathbb{N}$.

- Gamma function:

$$\Gamma(z) := \int_0^\infty x^{z-1} \cdot \mathrm{e}^{-x} \mathrm{d}x,\ \mathrm{Re}(z) \geq 0. \tag{1.0.5}$$

Property 1.0.1.

$$\Gamma(n) = (n-1)!,\ n \in \mathbb{N};\ \Gamma(z+1) = z \cdot \Gamma(z). \tag{1.0.6}$$

- Composition: $(g \circ f)(x) := g(f(x))$.

- Sum symbol: $\sum_{k=1}^{n} a_k := a_1 + a_2 + \cdots + a_n$.

  Recursive defintion: $\sum_{k=1}^{0} a_k := 0$, $\sum_{k=1}^{n} a_k := \left(\sum_{k=1}^{n-1} a_k\right) + a_n$.

- Product: $\Pi_{k=1} n a_k := a_1 \cdot a_2 \cdot \cdots \cdot a_n$.

  Recursive defintion: $\Pi_{k=1}^{0} a_k := 1$, $\Pi_{k=1}^{n} a_k := \left(\Pi_{k=1}^{n-1} a_k\right) \cdot a_n$

- Restriction: $f|_A : A \to Y$. For $f : X \to Y$ and $A \subseteq X$, we define $f|_A(x) = f(x)$ for all $x \in A$.

- Pauli matrices:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} . \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} . \sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{1.0.7}$$

Property 1.0.2. *We have* $\sigma_k^2 = I$ *and* $\sigma_j \sigma_k - \sigma_k \sigma_j = 2i\varepsilon_{jkl}\sigma_l$.

- Set brackets: $\{f(x) | x \in A\}$.

Example 1.0.2.

$$\{2x + 1 | x \in \{0, 1, 2, 3\}\} = \{1, 3, 5, 7\}. \tag{1.0.8}$$

- Big O: $f(x) = O(g(x))$, $(x \to a)$, which means that $|f(x)| \leq M \cdot |g(x)|$, i.e., $\limsup_{x \to a} \frac{f(x)}{g(x)} < \infty$.

$$x^2 + x + 2 = O(x^2),\ (x \to \infty) \tag{1.0.9}$$

$$x^2 + x + 2 = O(x^3),\ (x \to \infty). \tag{1.0.10}$$

- Binomial coefficient:

$$\binom{n}{k} = \frac{n \cdot (n-1) \cdots (n-k-1)}{k!} \tag{1.0.11}$$

$$= \frac{n!}{k!(n-k)!}. \tag{1.0.12}$$

- Modulo: $x \bmod n := r \in [0, n)$ with $x = n \cdot q + r$ where q is the integer.

Example 1.0.3.

$$5 \bmod 3 = 2 \tag{1.0.13}$$

$$6 \bmod 3 = 0 \tag{1.0.14}$$

$$7.1 \bmod 3 = 1.1 \tag{1.0.15}$$

$$9.7 \bmod 2.1 = 1.3. \tag{1.0.16}$$

- Beta function:

$$\beta(x, y) := \int_0^1 t^{x-1}(1-t)\ \mathrm{d}t, \tag{1.0.17}$$

where $x, y \in \mathbb{C}$, $\mathrm{Re}(x) > 0$ and $\mathrm{Re}(y) > 0$.

Lemma 1.0.1 (Identity between β func. and Γ func.)**.**

$$\beta(x, y) = \frac{\Gamma(x) \cdot \Gamma(y)}{\Gamma(x+y)}, \tag{1.0.18}$$

where $\Gamma(\cdot)$ is related to factorial and $\beta(x, y)$ is related to binomial coefficient.

- Map arrows: $f: \ X \to Y$ where X is the domain and Y is the codomain. This map can also be denoted as elementwise-mapping as $x \longmapsto f(x)$.

Example 1.0.4.

$$f := \mathbb{R} \to \mathbb{R} \tag{1.0.19}$$

$$x \longmapsto x^2. \tag{1.0.20}$$

- Little o: $f(x) = o(g(x))$, $(x \to a)$, which means $\lim_{x \to a} |\frac{f(x)}{g(x)}| = 0$.

Example 1.0.5.

$$8 \cdot x^2 \neq o(x^2),\ (x \to \infty) \tag{1.0.21}$$

$$8 \cdot x^2 \neq o(x^3),\ (x \to \infty). \tag{1.0.22}$$

- Outer product (Kronecker product for vectors):

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \otimes \begin{pmatrix} w_1 & w_2 & w_3 \end{pmatrix} = \begin{pmatrix} v_1 w_1 & v_1 w_2 & v_1 w_3 \\ v_2 w_1 & v_2 w_2 & v_2 w_3 \end{pmatrix}, \tag{1.0.23}$$

i.e. matrix entries $(V \otimes W)_{ij} = v_i \cdot w_j$.

- Euler's phi function: $\phi : \mathbb{N} \to \mathbb{N}$ defined as

$$\phi(n) = \text{count numbers } a \in \mathbb{N} \text{ with} \tag{1.0.24}$$

$$(1)\ a \leq n \tag{1.0.25}$$

$$(2)\ \gcd(a, n) = 1(\text{mutually prime}). \tag{1.0.26}$$

Example 1.0.6.

$$\phi(4) = 2 \tag{1.0.27}$$
$$\phi(5) = 4 \tag{1.0.28}$$
$$\phi(p) = p - 1 \ for \ p \ prime. \tag{1.0.29}$$

- Laplace operator (Laplacian):

$$\Delta f(x) = \frac{\partial^2 f}{\partial x_1^2}(x) + \frac{\partial^2 f}{\partial x_2^2}(x) + \frac{\partial^2 f}{\partial x_3^2}(x), \tag{1.0.30}$$

  where $f : \mathbb{R}^3 \to \mathbb{R}$.

- Convolution: $(f * g)(x) := \int_{-\infty}^{\infty} f(\tau) \cdot g(x - \tau)\mathrm{d}\tau$, where $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ and $f * g : \mathbb{R} \to \mathbb{R}$.

- Heaviside function:

$$H(x) := \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases} \tag{1.0.31}$$

  Property 1.0.3.

$$H' = \delta. \tag{1.0.32}$$

- Quaternions:

$$\mathbb{H} \supseteq \mathbb{C}, \tag{1.0.33}$$

  where $a, b, c, d \in \mathbb{R}$, the element in $\mathbb{H}$ is $a + i \cdot b + j \cdot c + k \cdot d$ with $i^2 = -1, j^2 = -1, k^2 = -1, ijk = -1$. $\mathbb{H}$ is not commutative in multiplication, i.e., $i \cdot j = -j \cdot i$.

- Infinity: ∞.

  Example 1.0.7. *In measure theory:* $[0, \infty]$. *We have*

$$a + \infty = \infty + a = \infty \ for \ a \in [a, \infty] \tag{1.0.34}$$
$$\tag{1.0.35}$$

- means equivalence reltation. For example, x y means x is equivalent to y for some conditions.

Chapter 2

Measure Theory

2.1 Sigma algebra

Definition 2.1.1 (Power set). *We define $\mathcal{P}(X)$ as the power set of set X. Assume that set $X = \{a, b\}$, the power set $P(X)$ would be $\{\emptyset, X, \{a\}, \{b\}\}$*

Definition 2.1.2 (Sigma algebra). *$\mathcal{A} \subseteq \mathcal{P}(X)$ is called a $\sigma - \text{algebra}$:*

$$(a)\ \emptyset, X \in \mathcal{A} \tag{2.1.1}$$

$$(b)\ A \in \mathcal{A} \Longrightarrow A^c := X \setminus A \in \mathcal{A} \tag{2.1.2}$$

$$(c)\ A_i \in \mathcal{A},\ i \in \mathcal{N} \Longrightarrow \cup_{i=1}^{\infty} A_i \in \mathcal{A}. \tag{2.1.3}$$

Definition 2.1.3 (Measurable sets). *$A \in \mathcal{A}$ is called a $\mathcal{A}$-measurable set.*

Example 2.1.1.

$$(1)\ \mathcal{A} = \{\emptyset, X\} \tag{2.1.4}$$

$$(2)\ \mathcal{A} = \{\mathcal{P}(X)\}. \tag{2.1.5}$$

Lemma 2.1.1. *Assume $\mathcal{A}_i$ is σ-algebra on X, $i \in I$ (index set). Then, we have $\cap_{i \in I} \mathcal{A}_i$ is also a σ-algebra on X.*

Definition 2.1.4 (Sigma algebra generated by $\mathcal{M}$). *For $\mathcal{M} \subseteq \mathcal{P}(X)$, there is a smallest σ-algebra that contains $\mathcal{M}$:*

$$\sigma(\mathcal{M}) := \cap_{\mathcal{A} \supseteq \mathcal{M},\ a\ \sigma-algebra} \mathcal{A}. \tag{2.1.6}$$

Example 2.1.2. *We define $X = \{a, b, c, d\}$ and $\mathcal{M} = \{\{a\}, \{b\}\}$. Then we have*

$$\sigma(\mathcal{M}) = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c, d\}, \{a, c, d\}, \{c, d\}\}. \tag{2.1.7}$$

Here is a very nice picture which shows the connections between the topological space, metric space and vector space. It is from this video:

Definition 2.1.5 (Topological spaxe). *Topological space (X, τ), where X is a set and τ is a collection of subsets of X. This τ is called the togology on X. The pair (X, τ) is called the topological space. In order to be a topological space, the collection of subsets must satisfy three properties:*

- *Empth set $\emptyset, X \in \tau$.*
- *Unions must be in τ, i.e., $\cup_{i=1}^{\infty} \tau_i \in \tau$.*
- *Intersections must be in τ, i.e., $\cap_{i=1}^{n} \tau_i \in \tau$.*

Definition 2.1.6 (Indiscrete topology). *Indiscrete topology is defined as $\tau = \{\emptyset, X\}$.*

Proof. This can proved by the definition of **??**. □

Definition 2.1.7 (Discrete topology). *Discrete topology is the power set* **??** *of X.*

Proposition 2.1.1. *Any topology τ on X satisfy the following relation: $\{\emptyset, X\} \subset \tau \subset \mathcal{P}(X)$, where $\{\emptyset, X\}$ is the indiscrete topology* **??** *and $\mathcal{P}(X)$ is the discrete topology* **??**.

Definition 2.1.8 (Borel sigma algebra). *Let $(X, \mathcal{T})$ be a topological space* **??** *(Let X be a metric space/Let X be a subset of $\mathbb{R}^n$; We need "open sets".). We then define $\mathcal{B}(X)$ is the borel σ-algebra on X as*

$$\mathcal{B}(X) := \sigma(\mathcal{T}), \tag{2.1.8}$$

which is the σ-algebra generated by the open sets $\mathcal{T}$.

Definition 2.1.9 (Borel sets). *Any set contained in Borel σ-algebra is called Borel set. If $A \in \mathcal{B}(X)$, then A is borel set.*

Proposition 2.1.2. *Let $\Omega = [0, 1)$ and $b \in \Omega$, then the singleton $\{b\}$ is a Borel set.*

Proof.

$$\{b\} = \cap_{n=1}^{\infty} \left[(b - \frac{1}{n}, b + \frac{1}{n}) \cap \Omega \right]. \tag{2.1.9}$$

□

[SZQ: 2023.04.08: This is a standard trick to prove.]

Proposition 2.1.3. *Let $\Omega = [0, 1)$ and $b \in \Omega$, then $(a, b]$, $[a, b]$ and $[a, b)$ are Borel sets.*

Proof. We write

$$(a,b] = \cap_{n=1}^{\infty}(a, b+\frac{1}{n}) \cap \Omega. \tag{2.1.10}$$

Then we can prove $(a,b]$ is a borel set.

We can also write $(a,b]$ as the union of singletons and there use **??** and the fact that the union of borel sets is also a borel set. □

Definition 2.1.10 (Borel measure)**.** *A borel measure on* $\mathbb{R}$ *is a measure on* $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$*.*

Definition 2.1.11 (Cumulative distribution function)**.** *A CDF(cumulative distribution function) is a function* $F : \mathbb{R} \to \mathbb{R}$ *such that*

- *F is nondecreasing* ($x \leq y \implies F(x) \leq F(y)$)
- *F is right continuous* ($\lim_{a\to a^+} = F(a)$)
- $\lim_{x\to\infty} F(x) = 1$
- $\lim_{x\to-\infty} F(x) = 0$

Theorem 1.

- *If F is a CDF then there si a unique Borel probability measure on* $\mathbb{R}$ *such that* $P((-\infty, x]) = F(x), \forall x \in \mathbb{R}$*.*
- *If P is a Borel probability measure on* $\mathbb{R}$ *then there is a unique CDF F such that* $F(x) = P((-\infty, x]), \forall x \in \mathbb{R}$*.*

That is, there is an equivalence between CDFs and Borel probability measure.

2.2 What is a measure?

Definition 2.2.1 (Measure)**.** $(X, \mathcal{A})$ *is called a measurable space, where X is a set and* $\mathcal{A}$ *is a* σ*-algebra on X. A map* $\mu : \mathcal{A} \to [0,\infty] := [0,\infty) + \{\infty\}$ *is called a measure if it satisfies:*

$$(a)\ \mu(\emptyset) = 0 \tag{2.2.1}$$

$$(b)\ \mu(\cup_{i=1}^{\infty}\mathcal{A}_i) = \sum_{i=1}^{\infty}\mu(\mathcal{A}_i)\ with\ \mathcal{A}_i \cap \mathcal{A}_j = \emptyset,\ i \neq j\ for\ all\ \mathcal{A}_i \in \mathcal{A}.(\sigma - additive) \tag{2.2.2}$$

Definition 2.2.2. $(X, \mathcal{A}, \mu)$ *is called a measure space.*

Example 2.2.1. *Given X and* $\mathcal{A} = \mathcal{P}(X)$*.*

- *Counting measure ($A \in \mathcal{A}$) is defined as*

$$\mu(A) := \begin{cases} \#A, & A \text{ has finitely many elements} \\ \infty & \text{else} \end{cases} \tag{2.2.3}$$

where $\#A$ means the number of elements in A.

Calculation rules in $[0, \infty]$:

$$x + \infty := \infty \ for\ all\ x \in [0, \infty] \tag{2.2.4}$$

$$x \cdot \infty := \infty \ for\ all\ x \in (0, \infty] \tag{2.2.5}$$

$$0 \cdot \infty := 0 \ (\text{only true in most cases in measure theory!}) \tag{2.2.6}$$

- *Dirac measure for $p \in X$ is defined as*

$$\delta_p(A) := \begin{cases} 1, & p \in A \\ 0, & \text{else} \end{cases} \tag{2.2.7}$$

- *We search a measure on $X \in \mathcal{R}^n$ satisfying:*

$$(1)\ \mu([0,1]^n) = 1 \tag{2.2.8}$$

$$(2)\ \mu(x + A) = \mu(A)\ for\ all\ x \in \mathcal{R}^n, \tag{2.2.9}$$

which is known as Lebesgue measure where the σ-algebra is not equal to power set.

2.3 Not everything is lebesgue measurable

Measure problem: search measure μ on $\mathcal{P}(\mathbb{R})$ with:

- (1) $\mu([a, b]) = b - a,\ b > a$,
- (2) $\mu(x + A) = \mu(A),\ A \in \mathcal{P}(\mathbb{R}),\ x \in \mathbb{R}$.

$\Longrightarrow \mu$ does not exist.

Claim: Let μ be a measure on $\mathcal{P}(\mathbb{R})$ with $\mu((0, 1]) < \infty$ and (2). $\Longrightarrow \mu = 0$.

Proof. (a) Definitions: $I \in (0, 1]$ with equivalence relation on I: $x \sim y \iff x - y \in \mathbb{Q}$ i.e., $[x] := \{x + r | r \in \mathbb{Q},\ x + r \in I\}$. Following this definition, we have a disjoint decomposition of I into boxes, possibly uncontable many of them! We then pick one element a_n from each box $[x_n]$ and form a set $A \in I$, i.e., $\{a_1, a_2, \cdots\} = A$. We have $A \in I$ with prperty:

- (1) For each $[x]$, there is an $a \in A$ with $a \in [x]$.
- (2) For all $a, b \in A: \ a, b \in [x] \Longrightarrow a = b$.

In uncountable case, the existence of $A \in I$ with the above property is guaranted by the axiom of choice of set theory.

We define $A_n := r_n + A$, where $(r_n)_{n\in\mathbb{N}}$ enumeration of $\mathbb{Q}_n(-1, 1]$.

(b) We then claim that $A_n \cap A_m = \emptyset \Longleftarrow n \neq m$. The proof is as follows: $x \in A_n \cap A_m \Longrightarrow x = r_n + a_n, \ a_n \in A$ and $x = r_m + a_m, \ a_m \in A. \Longrightarrow r_n + a_n = r_m + a_m \Longrightarrow a_n - a_m = r_n - r_m \in \mathbb{Q} \Longrightarrow a_n \sim a_m \Longrightarrow a_m, a_n \in [a_m] \Longrightarrow a_n = a_m \Longrightarrow r_n = r_m \Longrightarrow n = m$.

(c) We claim that $(0, 1] \subseteq \cup_{n\in\mathbb{N}} A_n \subseteq (-1, 2]$. The proof is as follows:

Assume now: μ measure on $\mathcal{P}(\mathbb{R})$ with $\mu((0, 1]) < \infty$ and (2).

By (2): $\mu(1 + A) = \mu(A)$ for all $n \in \mathbb{N}$.

By (c): we have

$$\mu((0, 1]) \leq \mu(\cup_{n\in\mathbb{N}} A_n) \leq \mu((-1, 2]) \tag{2.3.1}$$

We know: $\mu((0, 1]) =: C < \infty$. By using (2) and σ-additivity, we get $\mu((-1, 2]) = \mu\left((-1, 0] \cup (0, 1] \cup (1, 2] = 3C\right). \Longrightarrow_{??,(b)} C \leq \sum_{n=1}^{\infty} \mu(A_n) \leq 3C \Longrightarrow C \leq \sum_{n=1}^{\infty} \mu(A) \leq 3C \Longrightarrow \mu(A) = 0 \Longrightarrow C = 0(\text{henceL } \mu\left((0, 1]\right) = 0) \Longrightarrow \mu(\mathbb{R}) = \mu\left(\cup_{n\in\mathbb{Z}} (m, m + 1]\right) = 0 \Longrightarrow \mu = 0$. □

2.4 Measurable maps

Definition 2.4.1 (Measurable maps). *$(\Omega_1, \mathcal{A}_1)$ and $(\Omega_2, \mathcal{A}_2)$ are measurable spaces. $f : \Omega_1 \to \Omega_2$ is a measurable map w.r.t. $\mathcal{A}_1$ and $\mathcal{A}_2$ if $f^{-1}(A_2) \in \mathcal{A}_1$ for all $A_2 \in \mathcal{A}_2$.*

Example 2.4.1.

- *$(\Omega, \mathcal{A})$ and $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ are two measurable spaces. We define characteristic fucntion (aksi indicator function) as $\chi_A : \Omega \to \mathbb{R}$, where*

$$\chi_A(w) := \begin{cases} 1, & w \in A \\ 0, & w \notin A \end{cases} \tag{2.4.1}$$

For all measurable $A \in \mathcal{A}$, χ_A is a measurable map. We have

$$\chi_A^{-1}(\emptyset) = \emptyset \in \mathcal{A}, \ \chi_A^{-1}(\mathbb{R}) = \Omega \in \mathcal{A} \tag{2.4.2}$$

$$\chi_A^{-1}(\{A\}) = A, \ \chi_A^{-1}(\{0\}) = A^c \in \mathcal{A}. \tag{2.4.3}$$

- *Composition of measurable maps.*

Lemma 2.4.1. $(\Omega_1, \mathcal{A}_1)$, $(\Omega_2, \mathcal{A}_2)$, $(\Omega_3, \mathcal{A}_3)$ *are measurable space. We define* $\Omega_1 \xrightarrow{f} \Omega_2 \xrightarrow{g} \Omega_3$. *Then* f, g *are measurable implies* $g \circ f$ *is measurable.*

Proof.

$$(g \circ f)^{-1}(A_3) = f^{-1}(g^{-1}(A_3)) \tag{2.4.4}$$

$$\in \mathcal{A}_1 \tag{2.4.5}$$

□

Important measurable maps

Lemma 2.4.2. $(\Omega, \mathcal{A})$ *and* $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ *are measurable spaces.* $f, g : \Omega \to \mathbb{R}$ *are measurable maps indicates that* $f+g,\ f-g,\ f \cdot g,\ |f|$ *are measurable maps.*

2.5 Lebesgue integral

Example 2.5.1. *Define Characteristic function* $\chi_A : X \to \mathbb{R},\ A \in \mathcal{A}$. *We define* $I(A) := \mu(A)$. *Surprisingly,* $I(A)$ *is nothing but the integral of* χ_A *over* A.

Definition 2.5.1 (Simple/Step/Staircsae functions,...)**.** *For* $A_1, A_2, \ldots, A_n \in \mathcal{A}$, *and* $c_1, c_2, \cdots, c_n \in \mathbb{R}$. *We define*

$$f(x) := \sum_{i=1}^{n} c_i \cdot \chi_{A_i}(x). \tag{2.5.1}$$

We then have $f(x)$ *is measurable and the integraal of* f *is defined as* $I(f) := \sum_{i=1}^{n} c_i \mu(A_i)$.

Remark 2.5.1. *The problem of the integral* $I(f)$ *is that it is undefined when* $\mu(A_i) = \infty$. *The problem can be solved by exclude* ∞ *by defintion or the following way.*

Definition 2.5.2 (Lebesgue integral)**.** *Define* $S^+ := \{f : X \to \mathbb{R} | f \ simple\ function,\ f \geq 0\}$. $f \in S^+$ *and choose representation* $f(x) = \sum_{i=1}^{n} c_i \chi_{A_i}(x),\ c_i \geq 0$. *The lebesgue integral of* f *w.r.t.* μ *is defined as*

$$\int_X f(x)\ \mathrm{d}\mu(x) = \int_X f\ \mathrm{d}\mu \tag{2.5.2}$$

$$= I(f) \tag{2.5.3}$$

$$= \sum_{i=1}^{n} c_i \cdot \mu(A_i) \tag{2.5.4}$$

$$= [0, \infty]. \tag{2.5.5}$$

Property 2.5.1.
- $I(\alpha f + \beta g) = \alpha I(f) + \beta I(g),\ \alpha, \beta \geq 0.$

- $f \leq g \Longrightarrow I(f) \leq I(g)$ *(monotomicity)*

Definition 2.5.3. *Define a measurable map $f : X \to [0, \infty)$. $h = \sum_{i=1}^{n} c_i \cdot \chi_{A_i}$. The lebesgue integral of f w.r.t. μ is defined as*

$$\int_X f \, \mathrm{d}\mu := \sup \left\{ I(h) | h \in S^+, \ h \leq f \right\} \tag{2.5.6}$$

$$\in [0, \infty]. \tag{2.5.7}$$

f is called μ-integrable if $\int_X f \, \mathrm{d}\mu < \infty$.

Property 2.5.2. *Define measurable maps $f, g : X \to [0, \infty)$, we have*

- *1. $f = g$ for μ-almost everywhere(a.e.), which satisfies $\mu\left(\{x \in X | f(x) \neq g(x)\}\right) = \Longrightarrow \int_X f \, \mathrm{d}\mu = \int_X g \, \mathrm{d}\mu$.*

- *2. $f \leq g$ for μ a.e. $\Longrightarrow \int_X f \, \mathrm{d}\mu \leq \int_X g \, \mathrm{d}\mu$*

- *3. $f = 0$ for μ-a.e. $\Longleftrightarrow \int_X f \, \mathrm{d}\mu = 0$.*

Proof of 2.: monotonicity. Let $h := X \to [0, \infty)$ be a simple function, i.e.,

$$h(x) = \sum_{i=1}^{n} c_i \chi_{A_i}(x) \tag{2.5.8}$$

$$= \sum_{t \in h(X)} t \cdot \chi_{\{x \in X | h(x) = t\}}. \tag{2.5.9}$$

Let $X = \tilde{X}^c \cup \tilde{X}$ with $\mu(\tilde{X}^c) = 0$,

$$\tilde{h}(x) := \begin{cases} h(x), & x \in \tilde{X} \\ a, & x \in \tilde{X}^c \end{cases} \tag{2.5.10}$$

$$\tilde{h}(x) = \sum_{t \in h(X)} t \cdot \chi_{\left\{x \in \tilde{X} | h(x) = t\right\}} + a \cdot \chi_{\tilde{X}^c} \tag{2.5.11}$$

$$I(\tilde{h}) = \sum_{t \in h(X)} t \cdot \mu(\left\{x \in \tilde{X} | h(x) = t\right\}) + a \cdot \mu(\tilde{X}^c) \tag{2.5.12}$$

$$= \sum_{t \in h(X)} t \left[\mu\left(\left\{x \in \tilde{X} | h(x) = t\right\}\right) + \mu\left(\left\{x \in \tilde{X}^c | h(x) = t\right\}\right)\right] \tag{2.5.13}$$

$$= \sum_{t \in h(X)} t \left[\mu\left(\left\{x \in \tilde{X} | h(x) = t\right\} \cup \left\{x \in \tilde{X}^c | h(x) = t\right\}\right)\right] \tag{2.5.14}$$

$$I(h) = \sum_{t \in h(X) \setminus \{0\}} t \cdot \mu\left(\{x \in X | h(x) = t\}\right). \tag{2.5.15}$$

We define

$$\tilde{X} := \{x \in X | f(x) \leq g(x)\}, \tag{2.5.16}$$

$$\mu(\tilde{X}^c) = 0 \tag{2.5.17}$$

$$\int_X f \; \mathrm{d}\mu = \sup\left\{I(h)|h \in S^+, h \leq f\right\} \tag{2.5.18}$$

$$= \sup\{I(\tilde{h})|\tilde{h} \in S^+, \tilde{h} \leq f \textit{ on } \tilde{X}\} \tag{2.5.19}$$

$$\leq \sup\{I(\tilde{h})|\tilde{h} \in S^+, h \leq g \textit{ on } \tilde{X}\} \tag{2.5.20}$$

$$= \sup\{I(h)|h \in S^+, h \leq g\} \tag{2.5.21}$$

$$= \int_X g \; \mathrm{d}\mu. \tag{2.5.22}$$

□

Theorem 2 (Monotone convergence theorem)**.** *$(X, \mathcal{A}, \mu)$ measurable spaces, $f_n : X \to [0, \infty]$, $(f : X \to [0, \infty])$ measurable for all $n \in \mathbb{N}$ with*

$$f_1 \leq f_2 \leq f_3 \leq \cdots \quad \mu - a.e. \tag{2.5.23}$$

$$(\lim_{n\to\infty} \int_X f_n \; \mathrm{d}\mu = \int_X f \; \mathrm{d}\mu \quad \mu - a.e.(x \in X)) \tag{2.5.24}$$

This implies that

$$\lim_{n\to\infty} \int_X f_n \; \mathrm{d}\mu = \int_X \lim_{n\to} f_n \; \mathrm{d}\mu. \tag{2.5.25}$$

Proof. $\int_X f_1 \; \mathrm{d}\mu \leq \int_X f_2 \; \mathrm{d}\mu \leq \cdots$ and $\int_X f_n \; \mathrm{d}\mu \leq \int_X f \; \mathrm{d}\mu$ for $n \in \mathbb{N}$. Then we have

$$\lim_{n\to\infty} \int_X f_n \; \mathrm{d}\mu \leq \int_X f \; \mathrm{d}\mu, \tag{2.5.26}$$

which is the first part of **??**.

Let h be a simple function $0 \leq h \leq f$ and $\varepsilon > 0$. We define

$$X_n := \{x \in X | f_n(x) \geq (1 - \varepsilon)h(x)\} \tag{2.5.27}$$

with $\cup_{n=1}^{\infty} X_n = \tilde{X}$, and $\mu(\tilde{X}^c) = 0$. We have

$$\int_X f_n \; \mathrm{d}\mu \geq \int_{X_n} f_n \; \mathrm{d}\mu \geq \int_{X_n} (1 - \varepsilon)h \; \mathrm{d}\mu \tag{2.5.28}$$

$$\lim_{n\to\infty} \int_X f_n \; \mathrm{d}\mu \geq \lim_{n\to\infty} \int_{X_n} (1 - \varepsilon)h \; \mathrm{d}\mu \tag{2.5.29}$$

$$= \int_{\tilde{X}} (1 - \varepsilon)h \; \mathrm{d}\mu \tag{2.5.30}$$

$$= \int_X (1 - \varepsilon)h \; \mathrm{d}\mu. \tag{2.5.31}$$

This implies

$$\lim_{n\to\infty} \int_X f_n \, \mathrm{d}\mu \geq \int_X h \, \mathrm{d}\mu, \tag{2.5.32}$$

since $\varepsilon > 0$ arbitrarily. Then we have

$$\lim_{n\to\infty} \int_X f_n \, \mathrm{d}\mu \geq \int_X f \mathrm{d}\mu, \tag{2.5.33}$$

since h is arbitrary and $h \leq f$, which is second part of **??**. □

Applictions Given a series $(g_n)_{n\in\mathbb{N}}$, $g_n : X \to [0,\infty]$ measurable for all n. Then we have $\sum_{n=1}^{\infty} g_n : X \to [0,\infty]$ measurable and

$$\int_X \sum_{n=1}^{\infty} g_n \, \mathrm{d}\mu = \sum_{n=1}^{\infty} \int_X g_n \, \mathrm{d}\mu, \tag{2.5.34}$$

which means the integral and sum can exchange.

2.6 Fatou' lemma

Lemma 2.6.1 (Fatou' lemma)**.** *Given $(X, \mathcal{A}, \mu)$ measurable space, $f_n : X \to [0,\infty]$ measurable for all $n \in \mathbb{N}$. Then we have*

$$\int_X \liminf_{n\to\infty} f_n \, \mathrm{d}\mu \leq \liminf_{n\to\infty} \int_X f_n \, \mathrm{d}\mu. \tag{2.6.1}$$

Remark 2.6.1. $\liminf_{n\to\infty} f_n : X \to [0,\infty]$ *is a function. This is*

$$g(x) := \left(\liminf_{n\to\infty} f_n\right)(x) \tag{2.6.2}$$

$$:= \lim_{n\to\infty} \left(\inf_{k\geq n} f_k(x)\right) \tag{2.6.3}$$

$$\in [0,\infty] \tag{2.6.4}$$

$$g_n(x) := \inf_{k\geq n} f_k(x). \tag{2.6.5}$$

We have

$$g_1 \leq g_2 \leq g_3 \leq \cdots, \tag{2.6.6}$$

which is monotonically increasing. All these functions are measurable.

Proof.

Since (**??**),

$$\int_X \lim_{n\to\infty} g_n \, \mathrm{d}\mu = \lim_{n\to\infty} \int_X g_n \, \mathrm{d}\mu \tag{2.6.7}$$

$$= \liminf_{n\to\infty} \int_X g_n \, \mathrm{d}\mu. \tag{2.6.8}$$

We know that $g_n \leq f_n$ for all $n \in \mathbb{N}$. By (**??**), we have

$$\int_X g_n \, \mathrm{d}\mu \leq \int_X f_n \, \mathrm{d}\mu, \tag{2.6.9}$$

for all $n \in \mathbb{N}$. Then we have

$$\int_X \liminf_{n\to\infty} f_n \, \mathrm{d}\mu = \liminf_{n\to\infty} \int_X g_n \, \mathrm{d}\mu \tag{2.6.10}$$

$$\leq \liminf_{n\to\infty} \int_X f_n \, \mathrm{d}\mu. \tag{2.6.11}$$

$\square$

2.7 Lebesgue's dominated convergence theorem

$(X, \mathcal{A}, \mu)$, $\mathcal{L}^1 := \left\{ f : X \to \mathbb{R} \textit{ measurable} | \int_X |f|^1 \, \mathrm{d}\mu < \infty \right\}$. For $f \in \mathcal{L}^1(\mu)$, write $f = f^+ - f^-$, where $f^+, f^- \geq 0$. Define $\int_X f \, \mathrm{d}\mu := \int_X f^+ \, \mathrm{d}\mu - \int_X f^- \, \mathrm{d}\mu$.

Theorem 3 (Lebesgue's dominated convergence theorem)**.** *$f_n : X \to \mathbb{R}$ measurable for all $n \in \mathbb{N}$. $f : X \to \mathbb{R}$ with $\overset{n\to\infty}{f(x)}$ for $x \in X$ (μ-a.e.) and $|f_n| \leq g$ with $g \in \mathcal{L}^1(\mu)$ for all $n \in \mathbb{N}$, where g is called integral majorant. Then: we have $f_1, f_2, \cdots \in \mathcal{L}^1(\mu)$, $f \in \mathcal{L}^1(\mu)$ and*

$$\lim_{n\to\infty} \int_X f_n \, \mathrm{d}\mu = \int_X f \, \mathrm{d}\mu. \tag{2.7.1}$$

Proof.

$$|f_n| \leq g \overset{monotonicity}{\Longrightarrow} \int_X g \, \mathrm{d}\mu < \infty \tag{2.7.2}$$

$$\Longrightarrow f_1, f_2, \cdots \in \mathcal{L}^1(\mu) \tag{2.7.3}$$

$$|f| \leq g \ for \ \mu - \text{a.e.} \Longrightarrow f \in \mathcal{L}^1(\mu) \tag{2.7.4}$$

We will show $\int_X |f_n - f| \, \mathrm{d}\mu \overset{n\to\infty}{\Longrightarrow} 0$.

$$|f_n - f| \leq |f_n| + |f| \leq 2g \tag{2.7.5}$$

$$\Longrightarrow h_n := 2g - |f_n - f| \geq 0 \tag{2.7.6}$$

Hence: $h_n : X \to [0,\infty]$ measurable for all $n \in \mathbb{N}$. Then by (**??**),

$$\implies \int_X \liminf_{n\to\infty} h_n \; \mathrm{d}\mu \leq \liminf_{n\to\infty} \int_X h_n \; \mathrm{d}\mu \tag{2.7.7}$$

$$\implies \int_X 2g \; \mathrm{d}\mu \leq \int_X 2g \; \mathrm{d}\mu - \limsup_{n\to\infty} \int_X |f_n - f| \; \mathrm{d}\mu \tag{2.7.8}$$

$$\implies 0 \leq \liminf_{n\to\infty} \int_X |f_n - f| \; \mathrm{d}\mu \leq \limsup_{n\to\infty} \int_X |f_n - f| \; \mathrm{d}\mu \leq 0 \tag{2.7.9}$$

$$\implies \tag{2.7.10}$$

Limits exists and $\lim_{n\to\infty} |f_n - f| \; \mathrm{d}\mu = 0$. We conclude that

$$\tag{2.7.11}$$

$$0 \leq | \int_X f_n \; \mathrm{d}\mu - \int_X f \; \mathrm{d}\mu| = | \int_X (f_n - f) \; \mathrm{d}\mu| \leq \int_X |f_n - f| \; \mathrm{d}\mu \overset{n\to\infty}{\longrightarrow} 0, \tag{2.7.12}$$

where the third inequality is due to the integral's triangle inequality.

$$\tag{2.7.13}$$

$$\implies \lim_{n\to\infty} \int_X f_n \; \mathrm{d}\mu = \int_X f \; \mathrm{d}\mu. \tag{2.7.14}$$

□

2.8 Caratheodory's extension theorem

Theorem 4 (Caratheodory's extension theorem)**.** *X set, $\mathcal{A} \in \mathcal{P}(X)$ semiring of sets. A map $\mu : \mathcal{A} \to [0,\infty]$. Note that μ is not a measure, it is called A pre-measure.*

- *Then μ has an extension $\tilde{\mu} : \sigma(\mathcal{A}) \to [0,\infty]$, where $\tilde{\mu}$ is a measure and $\sigma(\mathcal{A})$ is a σ-algebra generated by $\mathcal{A}$, i.e., $\mu(A) = \tilde{\mu}(A)$.*
- *If there is sequence (S_j) with $S_j \in \mathcal{A}$, $\cup_{j=1}^{\infty} S_j = X$, then the extension $\tilde{\mu}$ from (a) is unique. ($\tilde{\mu}$ is also σ-finite)*

Definition 2.8.1 (Semiring set)**.** *Semiring of sets $\mathcal{A} \subseteq \mathcal{P}(X)$:*

- *$\emptyset \in \mathcal{A}$ (as for σ-algebra)*
- *$A, B \in \mathcal{A} \implies A \cap B \in \mathcal{A}$*
- *For $A, B \in \mathcal{A}$, there are pairwise disjoint sets $S_1, S_2, \ldots, S_n \in \mathcal{A} : \cup_{j=1}^{n} S_j = A \backslash B$*

Example 2.8.1. *$A := \{[a,b) | a, b \in \mathbb{R}, a \leq b\}$ not a σ-algebra because $\mathbb{R} \notin \mathcal{A}$. But $\sigma(\mathcal{A}) = \mathcal{B}(\mathbb{R})$ (Borel σ-algebra). Check that $\mathcal{A}$ is semiring set:*

- *$\emptyset \in \mathcal{A}$*

•

$$[a,b)\cap[c,d)=\begin{cases}\emptyset, & b\le c, d\le a\\ [c,b), & c\in[a,b), d\notin[a,b)\\ \dots\end{cases} \tag{2.8.1}$$

•

$$[a,b)\backslash[c,d)=\begin{cases}[a,b), & d\le a, b\le c\\ [a,c), & c\in[a,b), d\notin[a,b)\\ [a,c)\cup[d,b), & c>a, d<b\\ \dots\end{cases} \tag{2.8.2}$$

Definition 2.8.2 (Pre-measure)**.** $\mu:\mathcal{A}\to[0,\infty]$ *with* $\mathcal{A}$ *semiring os sets:*

- $\mu(\emptyset)=0$
- $\mu(\cup_{j=1}^{\infty})=\sum_{j=1}^{\infty}\mu(A_j)$*, for* $A_j\in\mathcal{A}$*,* $A_i\cap A_j=\emptyset$ *for* $i\neq j$ *and* $\cup_{j=1}^{\infty}A_j\in\mathcal{A}$*.*

Application: $A:=\{[a,b)|a,b\in\mathbb{R}, a\le b\}$, $\mu:\mathcal{A}\to[0,\infty]$, $\mu([a,b))=b-a$ is a pre-measure (We can check by the definition of pre-measure). Then by (**??**), there is a unique extension to $\mathcal{B}(\mathbb{R})\Longrightarrow$ lebesgue measure.

2.9 Lebesgue-Stieltjes measures

$F:\mathbb{R}\to\mathbb{R}$ monotonically increasing (non-decreasing).$[a,b)$ is the length of the interval. Now we consider new kinds of intervals:

$$F(b^-)-F(a^-)=:\mu_F([a,b)), \tag{2.9.1}$$

where $F(a^-):=\lim_{\varepsilon\to0^+}F(a-\varepsilon)$. Alternatively, we also have

$$F(b^+)-F(a^+)=:\mu_F((a,b]), \tag{2.9.2}$$

where $F(a^+):=\lim_{\varepsilon\to0^+}F(a+\varepsilon)$. We consider the previous one hereafter.

Definition 2.9.1. $\mathcal{A}:=\{[a,b):a,b\in\mathbb{R},a\le b\}$ *semiring of sets. Then by Caratheodory' theorem, we have that there exists exactly one measure*

$$\mu_F:\mathcal{B}(\mathbb{R})\to[0,\infty] \tag{2.9.3}$$

with $\mu_F([a,b))$

$$. \tag{2.9.4}$$

Example 2.9.1.

- *$F(x) = x$, $\mu_F([a,b)) = b - a \to$ Lebesgue measure.*
- *$F(x) = 1$, $\mu_F([a,b)) = 0 \to$ zero measure.*
- $$F(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases} \tag{2.9.5}$$
  $\mu_F([-\varepsilon, \varepsilon)) = 1 \to$ Dirac measure δ_0.
- *$F : \mathbb{R} \to \mathbb{R}$ monotonically increasing + continuously differentiable. Then we have*
  $$F' : \mathbb{R} \to [0, \infty) \tag{2.9.6}$$
  and
  $$\mu_F([a,b)) = F(b) - F(a) \tag{2.9.7}$$
  $$= \int_a^b F'(x) \, \mathrm{d}x, \tag{2.9.8}$$
  which implies
  $$\mu_F : A \longmapsto \int_A F'(x) \, \mathrm{d}x, \tag{2.9.9}$$
  where $F'(x)$ is called the density function.

2.10 Radon-Nikodym theorem and Lebesgue's decomposition theorem

$(X, \mathcal{A}, \lambda)$ measure space. Special case: $X = \mathcal{R}$, $\mathcal{A} = \mathcal{B}(\mathbb{R})$, and λ is lebesgue measure. Recall that $\lambda([a,b)) = b - a$. Another measure $\mu : \mathcal{B}(\mathbb{R}) \to [0, \infty]$. We will look how μ acts w.r.t. the given reference measure: lebesgue measure.

Definition 2.10.1.

- *μ is called absolutely continuous (w.r.t. λ) if $\lambda(A) = 0 \Longrightarrow \mu(A) = 0$ for all $A \in \mathcal{B}(\mathbb{R})$. One writes: $\mu << \lambda$.*
- *μ is called singular (w.r.t. λ) if there is $N \in \mathcal{B}(\mathbb{R})$ with $\lambda(N) = 0$ and $\mu(N^c) = 0$. One writes: $\mu \perp \lambda$.*

  Example 2.10.1. *δ_0 Dirac measure ($\delta_0(\{0\}) = 1$) $\Longrightarrow \delta_0 \perp \lambda$ (Choose $N = \{0\}$).*

Theorem 5 (Lebesgue's decomposition theorem)**.** *$\mu : \mathcal{B}(\mathbb{R}) \to [0, \infty]$ (σ-finite)*

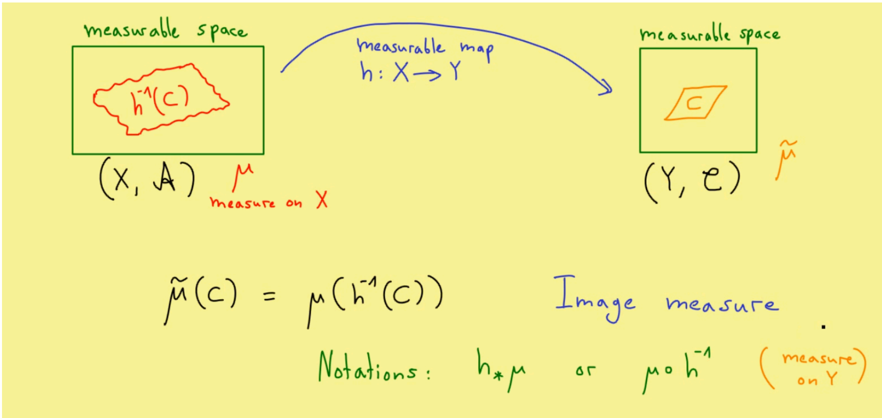
- *There are measures (uniquely determined) $\mu_{ac}, \mu_s : \mathcal{B}(\mathbb{R}) \to [0, \infty]$ with $\mu = \mu_{ac} + \mu_s$, $\mu_{ac} << \lambda, \mu_s \perp \lambda$.*

Theorem 6 (Radon-Nikodym theorem)**.** $\mu : \mathcal{B}(\mathbb{R}) \to [0, \infty]$ *(σ-finite)*

- *There is a measurable map* $h : \mathbb{R} \to [0, \infty)$ *with* $\mu_{ac} = \int_A d \,\mathrm{d}\lambda$ *for all* $A \in \mathcal{B}(\mathbb{R})$*, where* h *is called the density function.*

2.11 Image measure and substitution formula

Image measure is also called pushforward measure. Substitution formula is also called change of variable.



Definition 2.11.1 (Image measure)**.** *Measure space* $(X, \mathcal{A})$*,* μ *is a measure on* X*. Measure space* $(Y, \mathcal{E})$*,* $\tilde{\mu}$ *is a measure on* Y*. Define a measure map* $h : X \to Y$*. See the above figure. We then define the image measure as*

$$\tilde{\mu}(c) = \mu(h^{-1}(c)). \tag{2.11.1}$$

The notations: $h * \mu$ *or* $\mu \circ h^{-1}$*.* $h * \mu$ *means pushforward and* $\mu \circ h^{-1}$ *is readble. Remember that* $\tilde{\mu}$ *is aa measure on* Y*.*

Lemma 2.11.1 (Substitution formula)**.** *A integrable function* $g : Y \to \mathbb{R}$*. We have*

$$\int_Y g \,\mathrm{d}(h * \mu) = \int_X g \circ h \,\mathrm{d}\mu, \tag{2.11.2}$$

which can also be written as

$$\int_Y g(y) \,\mathrm{d}(\mu \circ h^{-1})(y) = \int_X g(h(x)) \,\mathrm{d}\mu(x), \tag{2.11.3}$$

which is called the change of variables: $y = h(x)$*.*

Example 2.11.1. F *is a strictly monotonically increasing and continuously differentiable and surjective function from* $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ *with* μ_F *as* $\mu_F(A) = \int_A F'(x) \,\mathrm{d}x$ *to*

$(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. *We have*

$$(F * \mu_F)([a,b)) = \mu_F(F^{-1}([a,b))) \tag{2.11.4}$$
$$= \mu_F([F^{-1}(a), F^{-1}(b)]) \tag{2.11.5}$$
$$= \int_{F^{-1}(a)}^{F^{-1}(b)} F'(x) \ \mathrm{d}x \tag{2.11.6}$$
$$= \int_a^b \mathrm{d}y \tag{2.11.7}$$
$$= \lambda([a,b)) \tag{2.11.8}$$
$$\Longrightarrow F_x \mu_F = \lambda, \tag{2.11.9}$$

Substitution formula:

$$\int_Y g \ \mathrm{d}(F * \mu_F) = \int_X g \circ F \ \mathrm{d}\mu_F \tag{2.11.10}$$
$$\Longrightarrow \int_{\mathbb{R}} g(y) \ \mathrm{d}y = \int_{\mathbb{R}} g(F(x))F'(x) \ \mathrm{d}x. \tag{2.11.11}$$

Proof. (1) Let $g = \chi_c$ with $C \subseteq Y$ measurable. For the left hand side, we have

$$\int_Y \chi_c \ \mathrm{d}(h * \mu) = (h * \mu)(c) \tag{2.11.12}$$
$$= \mu(h^{-1}(c)). \tag{2.11.13}$$

For the right hand side, we have

$$\int_X \chi_c \circ h \ \mathrm{d}\mu = \int_X \chi_c \circ h \ \mathrm{d}\mu \tag{2.11.14}$$
$$= \int_X \chi_c(h(x)) \ \mathrm{d}\mu(x) \tag{2.11.15}$$
$$= \int_X \chi_{h^{-1}(c)} \ \mathrm{d}\mu \tag{2.11.16}$$
$$= \mu(h^{-1}(c)), \tag{2.11.17}$$

where

$$\chi_c(h(x)) = \begin{cases} 1, \ x \in h^{-1}(c) \\ 0, \ x \notin h^{-1}(c) \end{cases} \tag{2.11.18}$$

(2) Let g be a simple function, i.e.,$g = \sum_{i=1}^{n} \chi_{c_i}$. We then obtatin

$$\int_Y \sum_{i=1}^{n} \lambda_i \chi_{c_i} \, \mathrm{d}(h * \mu) = \sum_{i=1}^{n} \lambda_i \int_Y \chi_{c_i} \, \mathrm{d}(h * \mu) \tag{2.11.19}$$

By (1)

$$= \sum_{i=1}^{n} \lambda_i \int_X \chi_{c_i}(h(x)) \, \mathrm{d}\mu(x) \tag{2.11.20}$$

$$= \int_X (\sum_{i=1}^{n} \lambda_i \chi_{c_i})(h(x)) \, \mathrm{d}\mu(x) \tag{2.11.21}$$

$$= \int_X g \circ h \, \mathrm{d}\mu. \tag{2.11.22}$$

(3) Let $g : Y \to [0, \infty)$ measurable. We have

$$\int_Y g \, \mathrm{d}(h * \mu) = \sup \left\{ \int_Y \tilde{s} \, \mathrm{d}(h * \mu) | \tilde{s} : Y \to [0, \infty) \ simple, \ \tilde{s} \leq g \right\}. \tag{2.11.23}$$

We have the following equivalence relation:

$$\forall y \in h(x) : \tilde{s}(y) \leq g(y) \tag{2.11.24}$$

$$\iff \forall x \in X : \tilde{s}(h(x)) \leq g(h(x)) \tag{2.11.25}$$

$$[i.e., \tilde{s} \circ h \leq (g \circ h)(x)]. \tag{2.11.26}$$

Then we have

$$\int_Y g \, \mathrm{d}(h * \mu) = \sup \left\{ \int_X \tilde{s} \circ \ \mathrm{d}\mu | \tilde{s} : Y \to [0, \infty) \ simple, \tilde{s} \circ h \leq g \circ h \right\} \tag{2.11.27}$$

Left as exercise

$$= \sup \left\{ \int_X s \circ \ \mathrm{d}\mu | s : X \to [0, \infty) \ simple, s \circ h \leq g \circ h \right\} \tag{2.11.28}$$

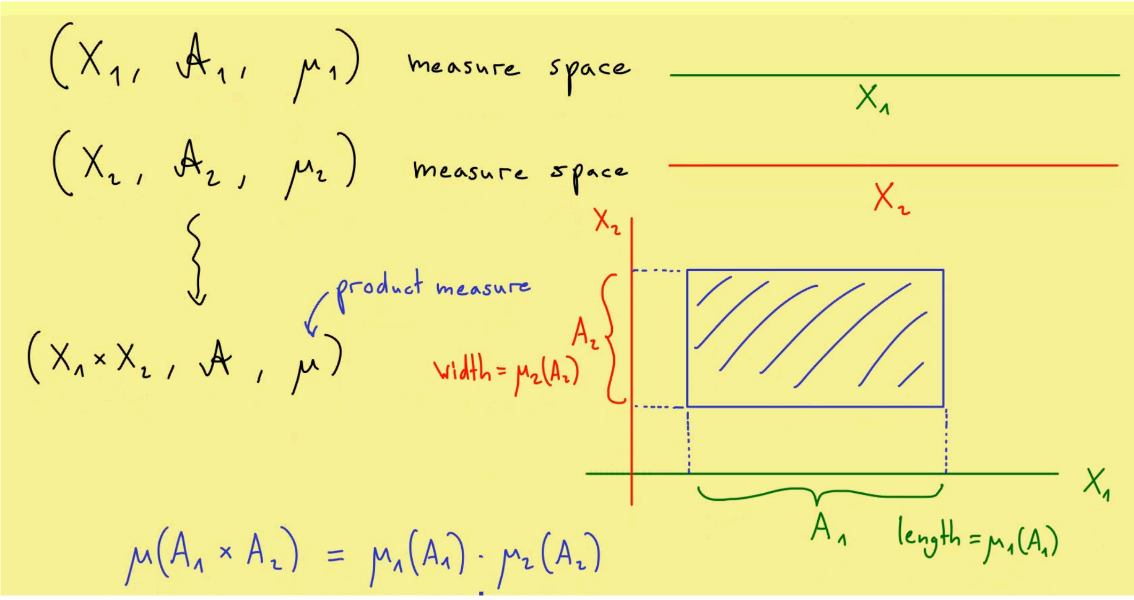
$$= \int_X g \circ h \, \mathrm{d}\mu. \tag{2.11.29}$$

□

2.12 Product measure and Cavalieri's principle

$(X_1, \mathcal{A}_1, \mu_1)$ measure space and $(X_2, \mathcal{A}_2, \mu_2)$ measure space,

$$\Longrightarrow (X_1 \times X_2, \mathcal{A}, \mu), \ where \ \mu \ is \ the \ product \ measure. \tag{2.12.1}$$



We have

$$\mu(A_1 \times A_2) = \mu_1(A_1) \cdot \mu_2(A_2). \tag{2.12.2}$$

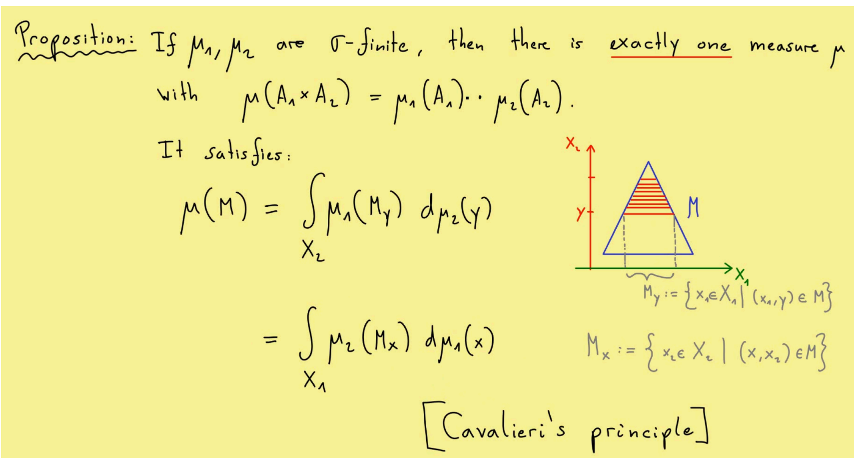
Definition 2.12.1 (Product σ-algebra)**.**

$$\mathcal{A} = \sigma(\mathcal{A}_1 \times \mathcal{A}_2). \tag{2.12.3}$$

Remark 2.12.1. *Set of rectangles (=$\mathcal{A}_1 \times \mathcal{A}_2$) are not a σ-algebra (but a semiring of sets)*

Definition 2.12.2. *Define product measure μ as $\mu(A_1 \times A_2) = \mu_1(A_1) \times \mu_2(A_2)$ for all $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$, and use (**??**).*

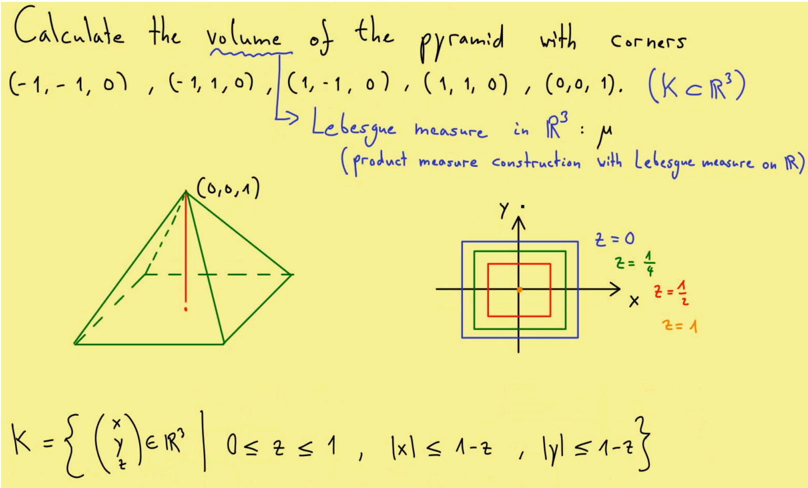
Remark 2.12.2. *Product measure in general not unique.*



Proposition 2.12.1 (Cavalieri's principle)**.** *If μ_1, μ_2 are σ-finite, then there is exactly one measure μ with $\mu(A_1 \times A_2) = \mu_1(A_1) \cdot \mu_2(A_2)$. Is satisfies:*

$$\mu(M) = \int_{X_2} \mu_1(M_y) \, \mathrm{d}\mu_2(y) \tag{2.12.4}$$
$$= \int_{X_1} \mu_2(M_x) \, \mathrm{d}\mu_1(x). \tag{2.12.5}$$

Example 2.12.1 (An example for Cavalieri's principle)**.** *Calculate the volume of the pyramid with corners (-1,-1,0), (-1,1,0),(1,-1,0),(1,1,0),(0,0,1), $K \subset \mathbb{R}^3$, where the volume if the lebesgue measure in $\mathbb{R}^3$: μ(Recall product measure construction with lebesgus measure on $\mathbb{R}$).*



Proof. Set

$$K = \left\{(x, y, z)^T \in \mathbb{R}^3 | 0 \leq z \leq 1, |x| \leq 1 - z, |y| \leq 1 - z\right\}. \tag{2.12.6}$$

Define μ as a product measure of μ_1 and μ_2, where μ_1 is the lebesgue measure in $\mathbb{R}$(z-coordinate) and μ_2 is the lebesgue measure on $\mathbb{R}^2$(x- and y-coordinate). Following the definition of product measure, we have the volume of K as

$$\mu(k) = \int_{\mathbb{R}} \mu_2(M_{z_0}) \, \mathrm{d}\mu_1(z_0) \tag{2.12.7}$$
$$= \int_{[0,1]} 4 \cdot (1 - z_0)^2 \mathrm{d}\mu_1(z_0) \tag{2.12.8}$$
$$= \frac{4}{3}, \tag{2.12.9}$$

where

$$M_{z_0} := \left\{(x, y)^T \in \mathbb{R}^2 ||x| \leq 1 - z_0, |y| \leq 1 - z_0\right\}, \tag{2.12.10}$$

and $\mu_2(M_{z_0})$ is the area of the square only for $z_0 \in [0, 1]$. □

2.13 Fubini's theorem

Theorem 7 (Fubini's theorem)**.** *Let μ_1 and μ_2 be σ-finite, μ be the product measure and*

$$f : X_1 \times X_2 \to [0,\infty] \text{ measurable } [\text{or } f \in \mathcal{L}^1(\mu)], \tag{2.13.1}$$

then:

$$\int_{X_1\times X_2} f \ \mathrm{d}\mu = \int_{X_2} \left(\int_{X_1} f(x,y) \ \mathrm{d}\mu_1(x) \right) \ \mathrm{d}\mu_2(x) \tag{2.13.2}$$

$$= \int_{X_1} \left(\int_{X_2} f(x,y) \ \mathrm{d}\mu_2(x) \right) \ \mathrm{d}\mu_1(x). \tag{2.13.3}$$

Example 2.13.1. *μ lebesgue measure for $\mathbb{R}^2$. Calculate $\int_A f \ \mathrm{d}\mu =?$, where*

$$A = \left\{ (x,y) \in [0,1] \times [0,1] | x \geq y \geq x^2 \right\}, \tag{2.13.4}$$

$$f(x,y) = 2xy. \tag{2.13.5}$$

We have

$$\int_A f \ \mathrm{d}\mu = \int_{\mathbb{R}^2} f \cdot \chi_A \ \mathrm{d}\mu \tag{2.13.6}$$

$$= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x,y)\chi_A(x,y) \ \mathrm{d}y \right) \ \mathrm{d}x \tag{2.13.7}$$

$$= \int_0^1 \left(\int_x^{x^2} 2xy \ \mathrm{d}y \right) \ \mathrm{d}x \tag{2.13.8}$$

$$= \frac{1}{12}. \tag{2.13.9}$$

2.14 Outer measure

- tools for the proof of (**??**)
- "outer measure" is a new notion. "Outer measure" is not an attribute for "measure"! "Outer mesure" do not have to be measures!

Definition 2.14.1 (Outer measure)**.** *A map $\phi : \mathcal{P}(X) \to [0,\infty]$ is called an outer measure if:*

- *(a)* $\phi(\emptyset) = 0$
- *(b)* $A \subseteq B \implies \phi(A) \leq \phi(B)$. *(monotonicity)*
- *(c)* $A_1, A_2, \ldots, \in \mathcal{P}(X) \implies \phi\left(\cup_{n=1}^\infty A_n\right) \leq \sum_{n=1}^\infty \phi(A_n)$. *($\sigma$-subadditivity)*

Question: $\phi : \mathcal{P}(X) \to [0,\infty]$ outer measure $\xrightarrow{?}$ μ measrue?

Definition 2.14.2 (ϕ-measurable)**.** *Let ϕ be an outer measure. $A \in \mathcal{P}(X)$ is called ϕ-measurable if for all $Q \in \mathcal{P}(X)$ we have:*

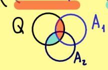
$$\phi(Q) \geq \phi(Q \cap A) + \phi(Q \cap A^c). \tag{2.14.1}$$

Proposition 2.14.1. *If $\phi : \mathcal{P}(X) \to [0,\infty]$ is an outer measure, then:*

- *$\mathcal{A}_\phi := \{A \subseteq X | A\ \phi \text{ measurable}\}$ is a σ-algebra.*
- *$\mu : \mathcal{A}_\phi \to [0,\infty]$, $\mu(A) := \phi(A)$, is a measrue.*

- union with two sets: $A_1, A_2 \in \mathcal{A}_\varphi$

$$\varphi(Q) = \varphi(Q \cap A_1) + \varphi(\underbrace{Q \cap A_1^c}_{\tilde{Q}}) = \varphi(Q \cap A_1) + \varphi(\tilde{Q} \cap A_2) + \varphi(\tilde{Q} \cap A_2^c)$$
$$\geq \varphi\big((Q \cap A_1) \cup (\tilde{Q} \cap A_2)\big) + \varphi(\underbrace{\tilde{Q} \cap A_2^c}_{Q \cap (A_1^c \cap A_2^c)})$$



Proof.

- $\phi \in \mathcal{A}_\phi$? Is $\emptyset$ ϕ -measurable?

$$\phi(Q) = \phi(Q \cap \emptyset) + \phi(Q \cup \emptyset^c) \tag{2.14.2}$$
$$= 0 + \phi(Q) \tag{2.14.3}$$

- $X \in \mathcal{A}_\phi$? Is X ϕ-measurable?

$$\phi(Q) = \phi(Q \cap X) + \phi(Q \cap X^c) \tag{2.14.4}$$
$$= \phi(Q) + \phi(\emptyset). \tag{2.14.5}$$

- $A \in \mathcal{A}_\phi \implies$

$$\phi(Q) = \phi(Q \cap A) + \phi(Q \cap A^c) \tag{2.14.6}$$
$$= \phi(Q \cap A^c) + \phi(Q \cap (A^c)^c) \tag{2.14.7}$$
$$\implies A^c \in \mathcal{A}_\phi. \tag{2.14.8}$$

- union with two sets: $A_1, A_2 \in \mathcal{A}$

$$\phi(Q) = \phi(Q \cap A_1) + \phi(Q \cap A_1^c) \tag{2.14.9}$$

Define $\tilde{Q} := Q \cap A_1^c$

$$= \phi(Q \cap A_1) + \phi(\tilde{Q} \cap A_2) + \phi(\tilde{Q} \cap A_2^c) \tag{2.14.10}$$
$$\geq \phi\Big((Q \cap A_1) \cup (\tilde{Q} \cap A_2)\Big) + \phi(\tilde{Q} \cap A_2^c) \tag{2.14.11}$$
$$= \phi(Q \cap (A_1 \cup A_2)) + \phi(Q \cap (A_1 \cup A_2)^c), \tag{2.14.12}$$
$$\implies \phi(Q) \geq \phi(Q \cap (A_1 \cup A_2)) + \phi(Q \cap (A_1 \cup A_2)^c) \tag{2.14.13}$$
$$\implies A_1 \cup A_2 \in \mathcal{A}_\phi, \tag{2.14.14}$$

where the fouth equation is obtain by the above figure.

- countable union: $A_1, A_2, \cdots \in \mathcal{A}_\phi$, $A := \cup_{j=1}^\infty A_j \in \mathcal{A}_\phi$?

$$\phi(Q) = \phi(Q \cap A_1) + \phi(Q \cap A_1^c) \tag{2.14.15}$$

Set $Q = \hat{Q} \cap (A_1 \cup A_2)$

$$= \phi(\hat{Q} \cap A_1) + \phi(\hat{Q} \cap A_2). \tag{2.14.16}$$

Induction: $\phi(\hat{Q} \cap \cup_{j=1}^n A_j) = \sum_{j=1}^n \phi(\hat{Q} \cap A_j)$. We have:

$$\phi(\hat{Q}) = \phi(\hat{Q} \cap \cup_{j=1}^n) + \phi(\hat{Q} \cap (\cup_{j=1}^n A_j)^c) \tag{2.14.17}$$

$$\geq \sum_{j=1}^n \phi(\hat{Q} \cap A_j) + \phi(\hat{Q} \cap A^c) \tag{2.14.18}$$

$$\Longrightarrow \phi(\hat{Q}) \geq \sum_{j=1}^n \phi(\hat{Q} \cap A_j) + \phi(\hat{Q} \cap A^c) \tag{2.14.19}$$

$$\geq \phi(\hat{Q} \cap A) + \phi(\hat{Q} \cap A^c) \tag{2.14.20}$$

$$\geq \phi(\hat{Q}) \tag{2.14.21}$$

$$\Longrightarrow A \in \mathcal{A}_\phi. \tag{2.14.22}$$

□

Example 2.14.1. *(1)* $\phi : \mathcal{P}(\mathbb{R}) \to [0, \infty]$,

$$\phi(A) = \begin{cases} 0, & A = \emptyset \\ 1, & A \neq \emptyset. \end{cases} \qquad \Longrightarrow \textit{outer measure but not a measure!} \tag{2.14.23}$$

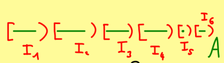
Example 2.14.2. $\phi : \mathcal{P}(\mathbb{N}) \to [0, \infty]$,

$$\phi(A) = \left\{\begin{matrix} |A|, & A \textit{ finite} \\ \infty, & A \textit{ not finite}. \end{matrix}\right\} \tag{2.14.24}$$

$$\Longrightarrow \textit{outer measure but a measure!(counting measure)} \tag{2.14.25}$$

(3) $\mathcal{I} = \{ [a,b) \mid a,b \in \mathbb{R},\ a \leq b \}$, $\mu([a,b)) = b - a$ ("length")

Define $\varphi : \mathcal{P}(\mathbb{R}) \longrightarrow [0,\infty)$ by :



$$\varphi(A) := \inf \left\{ \sum_{j=1}^\infty \mu(I_j) \;\middle|\; I_j \in \mathcal{I},\ A \subseteq \bigcup_{j=1}^\infty I_j \right\}$$

$\rightsquigarrow$ φ is an outer measure!

Example 2.14.3. *$\mathcal{I} = \{[a,b)|a,b \in \mathbb{R},\ a \leq b\}$, $\mu([a,b)) = b - a$("length").*

Define $\phi : \mathcal{P}(\mathbb{R}) \to [0,\infty)$ by:

$$\phi(A) := \inf\left\{\sum_{j=1}^{\infty} \mu(I_j)|I_j \in \mathcal{I},\ A \subseteq \cup_{j=1}^{\infty} I_j\right. \tag{2.14.26}$$

$$\Longrightarrow \phi \textit{ is an outer measure}! \tag{2.14.27}$$

Proof. check (a) of (**??**): $\phi(\emptyset) = 0$.

check (b) of (**??**): monotonicity,

$$A \subseteq B \Longrightarrow \phi(B) \tag{2.14.28}$$

$$= \inf\left\{\sum_{j=1}^{\infty} \mu(I_j)|I_j \in \mathcal{I},\ B \subseteq U_{j=1}^{\infty} I_j\right\} \tag{2.14.29}$$

$$\geq \inf\left\{\sum_{j=1}^{\infty} \mu(I_j)|I_j \in \mathcal{I},\ A \subseteq U_{j=1}^{\infty} I_j\right\}, \tag{2.14.30}$$

since $A \subseteq B$.

check (c) of (**??**): show that $\phi(\cup_{n\in\mathbb{N}} A_n) \leq \sum_{n\in\mathbb{N}} \phi(A_n)$. Let $\varepsilon > 0$. Choose $\varepsilon_n > 0$ with $\sum_{n\in\mathbb{N}} \varepsilon_n = \varepsilon$. Then there are intervals $I_{j,n}$ with:

$$\phi(A_n) \geq \sum_{j=1}^{\infty} \mu(I_{j,n}) - \varepsilon_n, \tag{2.14.31}$$

and

$$A_n \subseteq \cup_{j=1}^{\infty} I_{j,n}. \tag{2.14.32}$$

Then: $\cup_{n\in\mathbb{N}} \subseteq \cup_{n\in\mathbb{N}} \cup_{j\in\mathbb{N}} I_{j,n} = \cup_{j,n} I_{j,n}$.

$$\Longrightarrow \phi(\cup_{n\in\mathbb{N}}) \overset{(b)}{\leq} \phi(\cup_{j,n} I_{j,n}) \tag{2.14.33}$$

$$\leq \sum_{j,n} \mu(I_{j,n}) \tag{2.14.34}$$

$$= \sum_{n\in\mathbb{N}} \left\{\sum_{j\in\mathbb{N}} \mu(I_{j,n})\right\} \tag{2.14.35}$$

$$\leq \sum_{n\in\mathbb{N}} (\phi(A_n) + \varepsilon_n) \tag{2.14.36}$$

$$= \sum_{n\in\mathbb{N}} \phi(A_n) + \varepsilon. \tag{2.14.37}$$

□

Chapter 3

Linear algebra

3.1 Exercises

- 3.C.6: Are the construced v_i form a basis of vector space V?
-

Chapter 4

Functional Analysis

4.1 Metric Space

Definition 4.1.1 (Metrix Spaces)**.** *Define set X. Define a metric: $d : X \times X \to [0, \infty)$*

- $d(x, y) = 0 \Longleftrightarrow x = y$
- $d(x, y) = d(y, x)$
- $d(x, y) \leq d(x, z) + d(z, y)$ *(triangle inequality)*

Summary: X set + $d : X \times X \to [0, \infty)$ metric = metric space (X, d).

Example 4.1.1.

- $X = \mathbb{C}, d(x, y) = |x - y|$
- $X = \mathbb{R}^n, d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$ *(Euclidean metric)*
- $X = \mathbb{R}^n, d(x, y) = \max\{|x_1 - y_1|, ..., |x_n - y_n|\}$. *It is possible that* $d(x, y) = d(x, z)$.
- X *any set* ($\neq \emptyset$), *we define*

$$d(x, y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases} \tag{4.1.1}$$

Now, let's check whether this is a metrix space. Conditions (1) and (2) are verified easily. We focus on the verification of (3): choose $x, y, z \in X$.

For the first case: $x = y$: $d(x, y) = 0 \leq d(x, z) + d(z, y)$.

For the second case: $x \neq y$: $d(x, y) = 1 \leq \{d(x, z)$ or $d(z, y)\} = d(x, z) + d(z, y)$.

*This is called **discrete matric**.*

4.2 Open and Closed Sets

Definition 4.2.1 (Open Ball). *(X, d) a metrix space. Define*

$$B_\epsilon(x) := \{y \in X | d(x, y) < \epsilon\}, \tag{4.2.1}$$

whihc is called the open ball of radiue $\epsilon > 0$ centered at x.

Definition 4.2.2 (Open Sets). *$A \subseteq X$ is called open is for each $x \in A$ there is an open ball with $B_\epsilon(x) \subseteq A$.*

We have nice pic for this.

Definition 4.2.3 (Boundary Points). *$A \subseteq X$, $x \in X$ is called a boundary point for A if for all $\epsilon > 0$: $B_\epsilon(x) \cap A = \emptyset$ and $B_\epsilon(x) \cap A^c = \emptyset$. [$A^c := X \backslash A$]*

Notation: $\partial A : \{x \in X | x \ is\ boundary\ point\ for\ A\}$

We have nice pic for this.

Remember: A open $\iff A \cap \partial A = \emptyset$.

Definition 4.2.4 (Closed Sets). *$A \subseteq X$ is called closed if $A^c := X \backslash A$ is open.*

Definition 4.2.5 (Closure). *$\overline{A} := A \cup \partial A$ (always closed!)*

Example 4.2.1. *$X := (1, 3] \cup (4, \infty), d(x, y) := |x - y|, (X, d)$ is a metrix space.*

- *$A := (1, 3] \subseteq X$ open?*

  For $x \in A, x \neq 3$, define $\epsilon := \frac{1}{2} \min(|1 - x|, |3 - x|)$. Then $B_\epsilon(x) \subseteq A$.

  For $x = 3$: $B_1(x) = \{y \in X | d(x, y) < 1\} = (2, 3] \subseteq A$.

- *A is also closed!*

- *$C := (1, 2]$, $\partial C = \{2\}$, $\overline{C} = C$.*

4.3 Sequence, Limits and Closed Sets

Definition 4.3.1 (Sequence). *Sequence in X: $x_1, x_2, \ldots$ or $(x_n)_{n \in \mathbb{N}}$ or map $x : \mathbb{N} \to X$ / $n \mapsto x_n$.*

Definition 4.3.2 (Convergence). *A sequence $(x_n)_{n \in \mathbb{N}}$ in a metrix space (X, d) is called convergence if there is $\tilde{x} \in X$ with $\forall \varepsilon \geq 0, \exists N \in \mathbb{N}, \forall n \geq N : d(x_n, \tilde{x}) < \varepsilon$. We write: $X_n \overset{n \to \infty}{\longrightarrow} \tilde{x}$ or $\lim_{n \to} x_n = \tilde{x}$.*

Proposition 4.3.1 (Closed Sets)**.** *$A \subseteq X$ is closed $\Longleftrightarrow$ For every convergent sequence $(a_n)_{n\in\mathbb{N}} \subseteq A$, one has $\lim_{n\to\infty} a_n \in A$.*

Proof. ($\Longleftarrow$): Show it by contraposition! Assume A is not closed.

$$\Longrightarrow A^c := X\backslash A \text{ is not open.} \tag{4.3.1}$$

$$\Longrightarrow \text{There is an } \tilde{x} \in A^c \text{ with } B_\varepsilon(\tilde{x}) \cap A \neq \emptyset, \forall \varepsilon > 0. \tag{4.3.2}$$

$$\Longrightarrow \text{There is a sequence } (a_n)_{n\in\mathbb{N}} \; with \; a_n \in B_{1/n}(\tilde{x}) \cap A \tag{4.3.3}$$

$$\Longrightarrow \lim_{n\to\infty} a_n = \tilde{x} \neq A. \tag{4.3.4}$$

($\Longrightarrow$): Show it by contraposition! Assume there is $(a_n)_{n\in\mathbb{N}} \subseteq A$ with $\tilde{x} := \lim_{n\to\infty} \notin A$.

$$\Longrightarrow B_\varepsilon(\tilde{x}) \neq \emptyset, \forall \varepsilon > 0 \tag{4.3.5}$$

$$\Longrightarrow A^c \text{ is not open} \tag{4.3.6}$$

$$\Longrightarrow A \text{ is not closed.} \tag{4.3.7}$$

□

[SZQ: This proof use many contrapositions. It is nice to get familiar with how to use contrapositions to prove. It also needs to know the inverse of $\forall$.]

4.4 Cauchy Sequence and complete metric spaces

Example 4.4.1. *$X = (0, 3)$ with $d(x, y) = |x - y|$.*

$(0, 3)$ is closed:

- *complement $\emptyset$ is open*
- *each convergent sequence $(x_n)_{n\in\mathbb{N}} \subseteq (0, 3)$ (with limit $\tilde{x} \in X$) satisfies $\tilde{x} \in (0, 3)$*

What is about the sequence $(\frac{1}{n})_{n\in\mathbb{N}}$?

- *sequence in X*
- *$d(x_n, x_m) \overset{n,m\to\infty}{\longrightarrow} 0$*
- *it does not converge $\Longrightarrow$ (X, d) is not complete*

Definition 4.4.1 (Cauchy Sequence and Complete)**.** *Let (X, d) be a metrix space. A sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ is called Cauchy sequence if $\forall \varepsilon 0, \exists N \in \mathbb{N}, \forall n, m \geq N : d(x_n, x_m) < \varepsilon$. (X, d) is called complete if all Cauchy sequences converge.*

Example 4.4.2.
- *$X = [0, 3]$ with $d(x, y) = |x - y|$ is complete.*
- *$X = (0, 3)$ with*

$$\begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases} \tag{4.4.1}$$

is complete.

Proof. Let $(x_n)_{n\in\mathbb{N}} \subseteq X$ be a Cauchy sequence. Take $\varepsilon = 1/2$. Then there is an $N \in \mathbb{N}$ such that for all $n, m \geq N$, we have $d(x_n, x_m) < \frac{1}{2}$. By definition, $d(x_n, x_m) = 0$. Hence, $x_n = x_m$. □

4.5 Norms and Banach Spaces

Definition 4.5.1 (Norm)**.** $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. *Let X be a $\mathbb{F}$-vector space. A map $||\cdot|| : X \to [0, \infty)$ is called norm if*

- *$||x|| = 0 \iff x = 0$ (postitive definite)*
- *$||\lambda \cdot x|| = |\lambda| ||x||$ for all $\lambda \in \mathbb{F}$, $x \in X$ (absolutely homogeneous)*
- *$||x|| \leq ||x|| + ||y||$ for all $x, y \in X$ (triangle inequality)*

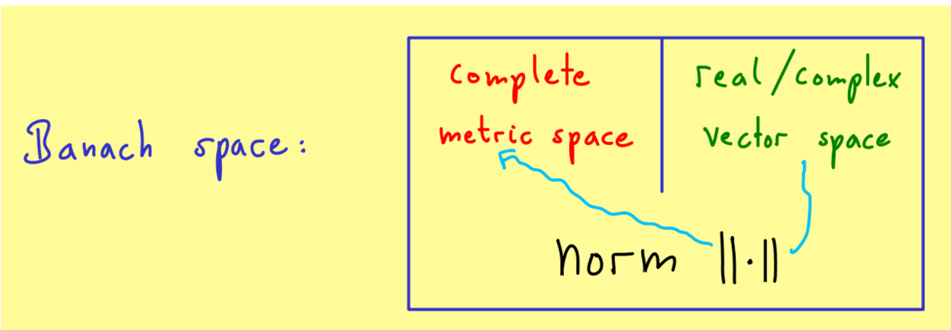
Definition 4.5.2 (Normed Space)**.** *$(X, ||\cdot||)$ is then called a normed space.*

Normed space is a special case of metric space!

Lemma 4.5.1 (Relationship between normed space and metric space)**.** *If $||\cdot||$ is a norm for the $\mathbb{F}$-vector space X, then $d_{||\cdot||}(x, y) := ||x - y||$ defines a metric for the set X.*

Proof. This can be proved by the defnition of norm. □

Definition 4.5.3 (Banach space)**.** *If $(X, d_{||\cdot||})$ is a complete metric space, then the normed space $(X, ||\cdot||)$ is called a Banach space.*



The figure is very impressive and informative!

4.6 Examples of Banach Spaces

4.7 Part 12: Continuity

Definition 4.7.1 (Continuity for metric spaces)**.** *(X, d_Y), (Y, d_Y) are two metric spaces. A map $f : X \to Y$ is called:*

- *continuous if $f^{-1}[B]$ is open in X for all open sets $B \subseteq Y$.*
- *sequentially continuous if for all $\tilde{x} \in X$ and $(x_n)_{n\in\mathbb{N}} \subseteq X$ with $x_n \overset{n\to\infty}{\longrightarrow} \tilde{x}$ holds $f(x_n) \overset{n\to\infty}{\longrightarrow} f(\tilde{x})$.*

These are shown by figs.

Lemma 4.7.1. *For metrc spaces, continuous and sequentially continuous are equivalent. But for topological spaces, they are different.*

Example 4.7.1.

- *(X, d_X) discrete metric space, (Y, d_Y) any metrix space $\implies$ all $f : X \to Y$ are continuous.*
- *(X, d_X), (Y, d_Y) metric spaces, $Y_0 \in Y$ fixed. $\implies$ $f : X \to Y$, $x \mapsto y_0$ is always continuous.*
- *$(X, \|\cdot\|)$ normed space, $Y = \mathbb{R}$ with standard metric. $\implies$ $f : X \to \mathbb{R}$, $x \mapsto \|x\|$ is continuous.*

Proof. Let $(x_n)_{n\in\mathbb{N}} \subseteq X$ sequence with limit $\tilde{x} \in X$. Then:

$$f(x_n) = \|x_n\| \tag{4.7.1}$$

$$= \|x_n - \tilde{x} + \tilde{x}\| \tag{4.7.2}$$

By triangle inequality ??

$$\leq \|x_n - \tilde{x}\| + \|\tilde{x}\| \tag{4.7.3}$$

$$= d(x_n, \tilde{x}) + f(\tilde{x}) \tag{4.7.4}$$

$$\implies \lim n \to \infty f(x_n) \leq f(\tilde{x}). \tag{4.7.5}$$

We also hold:

$$
\begin{aligned}
f(\tilde{x}) &= \|\tilde{x}\| && (4.7.6)\\
&= \|\tilde{x} - x_n + x_n\| && (4.7.7)\\
&\text{By triangle inequality ??}\\
&\|\tilde{x} - x_n\| + \|x_n\| && (4.7.8)\\
&= d(\tilde{x}, x_n) + f(x_n) && (4.7.9)\\
&\implies f(\tilde{x}) \leq \lim_{n\to\infty} f(x_n). && (4.7.10)
\end{aligned}
$$

□

- $(X, \langle\cdot,\cdot\rangle)$ *inner product space,* $Y \in \mathbb{C}$ *with the standard metric,* $x_0 \in X$ *fixed.* $\implies$ $f : X \to \mathbb{C}$, $x \mapsto \langle x_0, x\rangle$ *is continuous.*

Proof. Let $(x_n)_{n\in\mathbb{N}} \subseteq X$ sequence with limit $\tilde{x} \in X$. Then:

$$
\begin{aligned}
|f(x_n) - f(\tilde{x})| &= |\langle x_0, x_n\rangle - \langle x_0, \tilde{x}\rangle| && (4.7.11)\\
&= |\langle x_0, x_n - \tilde{x}\rangle && (4.7.12)\\
&\text{By Cauthy Schiwz inequality}\\
&\leq \|x_0\| \cdot \|x_n - \tilde{x}\| \overset{n\to\infty}{\longrightarrow} 0. && (4.7.13)
\end{aligned}
$$

Analogously, $g : X \to \mathbb{C}$, $x \mapsto \langle x, x_0\rangle$ is continuous. □

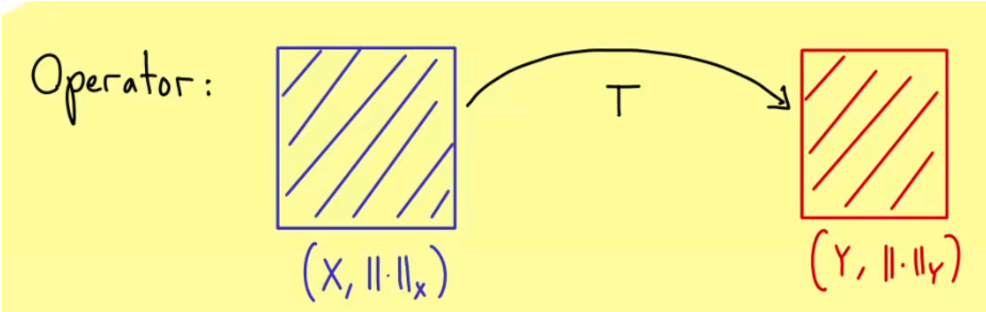
Lemma 4.7.2 (Orthogonal complement is closed)**.** $(X, \langle\cdot,\cdot\rangle)$ *inner product space,* $U \subseteq X$*. Then* $U^\perp$ *is closed.*

Proof. Let $(x_n)_{n\in\mathbb{N}} \subseteq U^\perp$ with limit $\tilde{x} \in X$.

$$
\begin{aligned}
&\implies \langle x_n, u\rangle = 0, \forall u \in U && (4.7.14)\\
&\implies \lim_{n\to\infty} \langle x_n, u\rangle = 0, \forall u \in U && (4.7.15)\\
&\implies \langle \tilde{x}, u\rangle = 0, \forall u \in U. && (4.7.16)\\
&\implies \tilde{x} \in U^\perp. && (4.7.17)
\end{aligned}
$$

□

4.8 Part 13: Bounded Operators



$T : X \to Y$, which satisfies

- linear (conserves the algebraic structure)
- continuous (bounded) (conserves the topological structure)

Definition 4.8.1 (Operator norm and bounded)**.** $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|)$ *two normed spaces,* $T : X \to Y$ *linear, which means*

$$T(x + \tilde{x}) = Tx + T\tilde{x} \tag{4.8.1}$$

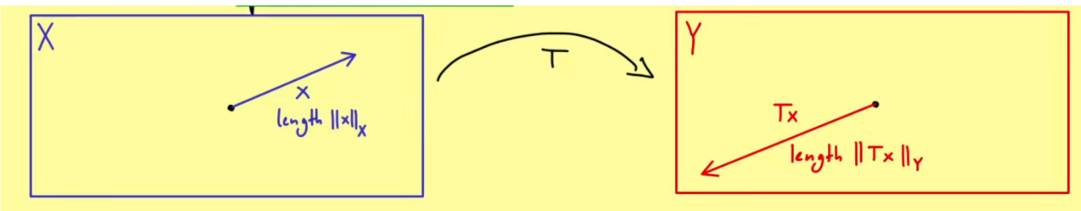
$$T(\lambda x) = \lambda Tx \tag{4.8.2}$$

for all $x, \tilde{x} \in X$, $\lambda \in \mathbb{F}$. *Then*

$$\|T\| = \|T\|_{X\to Y} \tag{4.8.3}$$

$$:= \sup\left\{\frac{\|Tx\|_Y}{\|x\|_X} | x \in X, x \neq 0\right\} \tag{4.8.4}$$

is called the operator norm of T. *If* $\|T\| < \infty$, T *is called bounded.*



Proposition 4.8.1 (Continuous equivalent to bounded)**.** *Let* $(X, \|\cdot\|_X)$,$(Y, \|\cdot\|_Y)$ *two normed spaces,* $T : X \to Y$ *linear. Then the following claims are equivalent:*

- T *is continuous.*
- T *is continuous at* $x = 0$.
- T *is bounded.*

Proof. (a) $\Longrightarrow$ (b) is easily seen.

(b) $\Longrightarrow$ (c): proposition (*): For all sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ with $x_n \overset{n\to\infty}{\longrightarrow} 0$, we have $Tx_n \overset{n\to\infty}{\longrightarrow} 0$ due the properties of linear map.

Claim: proposition (*) $\Longrightarrow$ proposition (+): there is a $\delta > 0$ such that $\|Tx\|_Y < 1$ for all $x \in X$ with $\|x\|_X < \delta$.

proof of this claim: we prove its contraposition such that $\neg(*) \Longrightarrow$ For all $n \in \mathbb{N}$, we find $x_n \in X$ with $\|x_n\|_X < \frac{1}{n}$ and $\|Tx_n\|_Y \geq 1 \Longrightarrow \neg(*)$.

$$\frac{\|Tx\|_Y}{\|x\|_X} = \frac{\|Tx\|_Y \cdot \frac{\delta}{2} \cdot \frac{1}{\|x\|_X}}{\|x\|_X \cdot \frac{\delta}{2} \cdot \frac{1}{\|x\|_X}} \tag{4.8.5}$$

$$= \frac{\|T(\frac{\delta}{2}\frac{x}{\|x\|_X})\|_Y}{\|\frac{\delta}{2}\frac{x}{\|x\|_X}\|_X} \tag{4.8.6}$$

$$\leq \frac{2}{\delta} \tag{4.8.7}$$

$$\Longrightarrow \|T\| = \sup\left\{\frac{\|Tx\|_Y}{\|x\|_X} | x \in X, x \neq 0\right\} \leq \frac{2}{\delta} < \infty. \tag{4.8.8}$$

(c) $\Longrightarrow$ (a): Let $(x_n)_{n\in\mathbb{N}} \subseteq X$ be convergent with limit $\tilde{x} \in X$. Then $\|Tx_n - T\tilde{x}\|_Y = \|T(x_n - \tilde{x})\|_Y \leq \|T\| \cdot \|x_n - \tilde{x}\|_X \overset{n\to\infty}{\longrightarrow} 0$. □

[SZQ: 2023.04.02: This proof needs to be understood later.]

[SZQ: 2023.04.02: Is should be proved that $\|T\|$ defined above is indeed a norm.]

4.9 Part 14: Example Operator Norm

Example 4.9.1. *$X = (C[0,1), \mathbb{F}, \|\cdot\|_\infty)$, $Y = (\mathbb{F}, |\cdot|)$. For $g \in X$ with $g(t) \neq 0$ for all $t \in [0,1]$, define $T_g : X \to Y$ by $T_g(f) := \int_0^1 g(t) \cdot f(t)\mathrm{d}t$. So what is $\|T_g\|$?*

Proof. Recall that

$$\|F_g\| = \sup\left\{\frac{|T_g(f)|}{\|f\|_\infty} | f \in X, f \neq 0\right\} \tag{4.9.1}$$

This trick has been used before.

$$= \sup\left\{\frac{|T_g(f)|}{\|f\|_\infty} | f \in X, f \neq 0\right\} \tag{4.9.2}$$

$$= \sup\left\{|T_g(f)| | f \in X, \|f\|_\infty = 1\right\} \tag{4.9.3}$$

$$= \sup\left\{|\int_0^1 g(t) \cdot f(t) \mathrm{d}t| | f \in X, \|f\|_\infty = 1\right\} \tag{4.9.4}$$

Since $|\int_0^1 g(t) \cdot f(t)\mathrm{d}t| \leq \int_0^1 |g(t)| \cdot |f(t)|\mathrm{d}t$ and $|f(t)| \leq \|f\|_\infty = 1$

$$\leq \int_0^1 |g(t)|\mathrm{d}t \tag{4.9.5}$$

$$< \infty. \tag{4.9.6}$$

Check the other inequality: $h(t) := \frac{\overline{g(t)}}{|g(t)|}$ with $\|h\|_{\infty=1}$. We then have

$$\|T_g\| \geq |T_g(h)| \tag{4.9.7}$$

$$= |\int_0^1 g(t)\frac{\overline{g(t)}}{|g(t)|}\mathrm{d}t \tag{4.9.8}$$

$$= \int_0^1 \frac{|g(t)|^2}{|g(t)|}\mathrm{d}t \tag{4.9.9}$$

$$= \int_0^1 |g(t)|\mathrm{d}t. \tag{4.9.10}$$

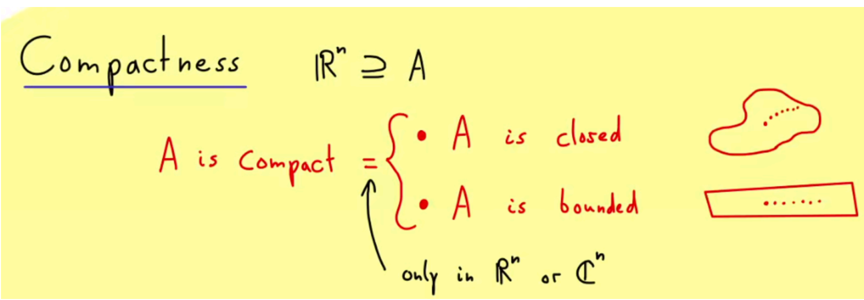
□

4.10 Part 16: Compact Sets

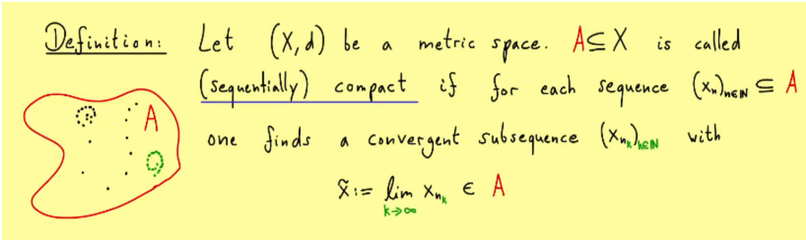
Example 4.10.1. *Compactness $A \subseteq \mathbb{R}^n$. A is compact such that*

- *A is closed.*
- *A is bounded.*

This is only true in $\mathbb{R}^n$ or $\mathbb{C}^n$.

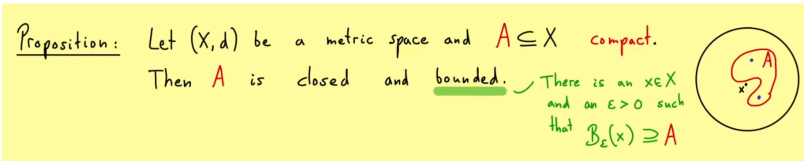


Definition 4.10.1 (Sequentially compact)**.** *Let (X,d) be a metric space. $A \subseteq X$ is called (sequentially) compact if for each sequence $(x_n)_{n\in\mathbb{N}} \subseteq A$, one finds a convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$ with $\tilde{x} := \lim_{k\to\infty} x_{n_k} \in A$.*



Example 4.10.2.

- $(\mathbb{R}, d_{eucl.})$, *$A = [0,1]$ compact by Bolzano-Weierstrass theorem.*
- $(\mathbb{R}, d_{discr})$, *$A = [0,1]$ not compact because: The sequence $(x_n)_{n\in\mathbb{N}} \subseteq A$ with $x_n = \frac{1}{n}$ satisfies $d_{discr.}(x_n, x_m) = 1$ for all $n, m \in \mathbb{N}$ with $n \neq m$. $\implies$ no convergent subsequence.*



Definition 4.10.2 (Bounded)**.** *Let (X,d) be a metrix space and $A \subseteq X$ compact. A is bounded means that there is an $x \in X$ and an $\varepsilon > 0$ such that $A \subseteq B_\varepsilon(x)$.*

Proposition 4.10.1 (Compact implies closed and bounded)**.** *Let (X,d) be a metrix space and $A \subseteq X$ compact. Then A is closed and bounded.*

Proof. Let $A \subseteq X$ be compact.

(1) Let $(x_n)_{n\in\mathbb{N}} \subseteq A$ be convergent with limit $\tilde{x} \in X$.

$$\implies \text{There is aa convergent subsequence } (x_{n_k})_{k\in\mathbb{N}} \text{ with limit } \tilde{\tilde{x}} \in A \tag{4.10.1}$$
$$\implies \tilde{x} = \tilde{\tilde{x}} \in A \tag{4.10.2}$$
$$\implies A \textit{ is closed}! \tag{4.10.3}$$

(2) contraposition: A is not bounded.

$\implies$ For given $a \in A$, there are $x_n \in A$ with $d(a, x_n) > n$.

$\implies$ For any subsequence $(x_{n_k})_{k\in\mathbb{N}}$ and any point $b \in A$:

$$n_K < d(a, x_{n_k}) \tag{4.10.4}$$
$$\leq d(a,b) + d(b, x_{n_k}) \tag{4.10.5}$$

$\implies n_k - d(a,b) \le d(b, x_{n_k})$.

$\implies d(b, x_{n_k}) \overset{k\to\infty}{\longrightarrow} 0$ for all $b \in A$

$\implies A$ not compact! $\square$

4.11 Part 18: Compact operators

Definition 4.11.1 (Compact operators). *Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be two normed spaces. A bounded linear operator $T : X \to Y$ is called compact if $\overline{T[B_1(0)]} \subseteq Y$ is a compact set.*

[SZQ: 2023.04.02: the defintion of $\overline{T[B_1(0)]} \subseteq Y$ should be learned first.]

4.12 Part 19: Holder's inequality

Lemma 4.12.1 (Young's inequality). *For all $a, b > 0$, we have $a, b \le \frac{a^p}{p} + \frac{b^{p'}}{p'}$.*

Proof. Note that function $f : x \mapsto e^x$ is convex, we have

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y). \tag{4.12.1}$$

Let $\lambda = \frac{1}{p}$, $x = \ln a^p$, $1 - \lambda = \frac{1}{p'}$, and $y = \ln b^{p'}$. We have

$$\begin{aligned} a \cdot b = f(\frac{1}{p}\ln a^p + \frac{1}{p'}\ln b^{p'}) = f(\lambda x + (1-\lambda)y) &\le \lambda f(x) + (1-\lambda) f(y) && (4.12.2)\\ &= \frac{1}{p} f(\ln a^p) + \frac{1}{p'} f(\ln b^{p'}) && (4.12.3)\\ &= \frac{a^p}{p} + \frac{b'}{p'}. && (4.12.4) \end{aligned}$$

$\square$

Theorem 8 (Holder's inequality). *For all $x, y \in \mathbb{F}^n$, we have*

$$\|xy\|_1 \le \|x\|_p \cdot \|y\|_{p'}, \tag{4.12.5}$$

where $x \in \mathbb{F}^n$ and the p-norm of x is

$$\|x\|_q := \left(\sum_{j=1}^{n} |x_j|^q \right)^{\frac{1}{q}}, \tag{4.12.6}$$

$q \in [1, \infty)$, $\frac{1}{p} + \frac{1}{p'} = 1$, and for $x, y \in \mathbb{F}^n$, we write

$$xy := \begin{pmatrix} x_1 y_1 \\ x_2 y_2 \\ \cdots \\ x_n y_n \end{pmatrix}. \tag{4.12.7}$$

Proof. Case 1: $x = 0$ or $y = 0$.

Case 2: We have

$$\begin{aligned} \frac{1}{\|x\|_p \|y\|_{p'}} \|xy\|_1 &= \frac{1}{\|x\|_p \|y\|_{p'}} \sum_{j=1}^{n} |x_j y_j| && (4.12.8) \\ &= \sum_{j=1}^{n} \frac{|x_j|}{\|x\|_p} \frac{|y_j|}{\|y\|_{p'}} && (4.12.9) \\ &\overset{\text{By Young's lemma (??)}}{\leq} \sum_{j=1}^{n} \frac{1}{p} \cdot \frac{|x_j|^p}{\|x\|_p^p} + \sum_{j=1}^{n} \frac{1}{p'} \cdot \frac{|y_j|^{p'}}{\|x\|_{p'}^{p'}} && (4.12.10) \\ &= \frac{1}{p} + \frac{1}{p'} && (4.12.11) \\ &= 1. && (4.12.12) \end{aligned}$$

□

4.13 Part 21: Isomorphism?

Definition 4.13.1 (Homomorphism)**.** ***What is Homomorphism: map that preserves structures.***

[SZQ: When we talk about homophism, we must know the underlying structures!]

Example 4.13.1.

- *Let X, Y be vector spaces and $f : X \to Y$ be a map. We want*

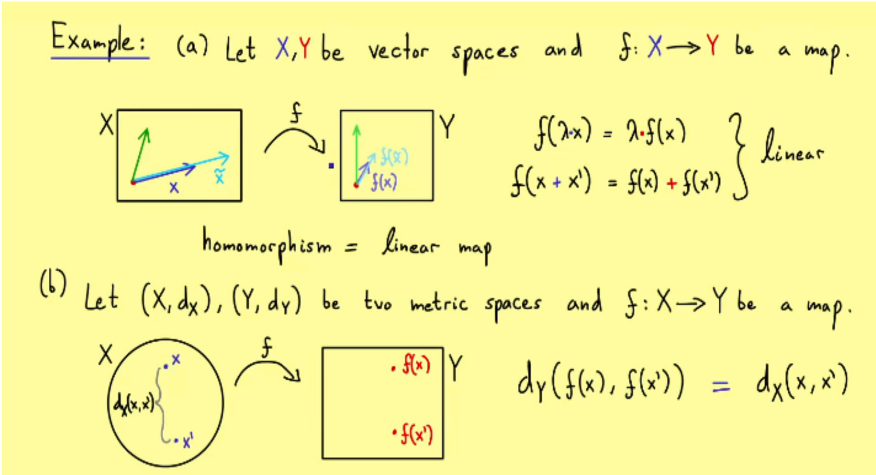
$$f(\lambda \cdot x) = \lambda \cdot f(x) \tag{4.13.1}$$
$$f(x + x') = f(x) + f(x'), \tag{4.13.2}$$

which is called linear. Thus homomorphism = linear map!

- *Let $(X, d_X), (Y, d_Y)$ be two metric spaces and $f : X \to Y$ be a map. We want*

$$d_Y(f(x), f(x')) \leq d_X(x, x'). \tag{4.13.3}$$

*homomorphism = map that satisfies (**??**)*



Definition 4.13.2 (Isomorphism)**.** *Isomorphism = homomorphism + bijective + inverse map is also homomorphism*

Definition 4.13.3 (Isomorphism for Banach spaces)**.** *Isomorphism for banach spaces X, Y: $f : X \to Y$ with: linear + bijective + $\|f(x)\|_Y = \|x\|_X$ (oftern called isometric isomorphism).*

Example 4.13.2.

- $S_R : l^p(\mathbb{N}) \to l^p(\mathbb{N})$, $(x_1, x_2, x_3, \dots) \mapsto (0, x_1, x_2, \dots)$

$$\implies \textit{linear, } \|S_R x\|_p = \|x\|_p \textit{ not surjective} \tag{4.13.4}$$
$$\implies \textit{not an isomorphism} \tag{4.13.5}$$

- $S : l^p(\mathbb{Z}) \to l^p(\mathbb{Z})$, $(\dots, x_{-1}, x_0, x_1, \dots) \mapsto (\dots, x_{-2}, x_{-1}, x_0, \dots)$

$$\implies \textit{linear, } \|Sx\|_p = \|x\|_p \textit{ and bijective} \tag{4.13.6}$$
$$\implies \textit{isomorphism} \tag{4.13.7}$$

4.14 Part 22: Dual spaces

Proposition 4.14.1. *Let X be a normed space. Then $(X', \|\cdot\|_{X\to\mathbb{F}})$ is a Banach space.*

Proof. □

[SZQ: 2023.04.02: I need to learn Riesz representation theorem!]

4.15 Part 28: Spectrum for bounded linear operators

Recall: $A \in \mathbb{C}^{n\times n}$ matrix with n rows and n columns. $\lambda \in \mathbb{C}$ is called an eigenvalue of A if:

$$\exists x \in \mathbb{C}^n \setminus \{0\} : Ax = \lambda x \tag{4.15.1}$$

$$\iff \exists x \in \mathbb{C}^n \setminus \{0\} : (A-\lambda I)x = 0 \tag{4.15.2}$$

$$\iff \mathrm{Ker}(A-\lambda I) \neq \{0\} \tag{4.15.3}$$

$$\iff \textit{map } x \mapsto (A-\lambda I)x \textit{ not injective.} \tag{4.15.4}$$

Theorem 9 (Rank-nullity theorem)**.** *For any matrix $M \in \mathbb{C}^{m\times n}$:*

$$dim(Ran(M)) + dim(Ker(M)) = n. \tag{4.15.5}$$

Definition 4.15.1 (Spectrum and resolvent)**.** *Let X be a complex Banach space and $T : X \to X$ be a bounded linear operator. Then the spectrum of T is defined by:*

$$\sigma(T) := \{\lambda \in \mathbb{C} | (T-\lambda I) not\ bijective\}. \tag{4.15.6}$$

The resolvent of T is defined by:

$$\rho(T) := \left\{\lambda \in \mathbb{C} | (T-\lambda I)\ bijective\ and\ (T-\lambda I)^{-1}\ bounded\right\}. \tag{4.15.7}$$

Corollary 4.15.1. *By bounded inverse theorem, we have*

$$\sigma(T) = \mathbb{C} \backslash \rho(T). \tag{4.15.8}$$

Definition 4.15.2 (Point/continuous/residual spectrum)**.** *We have the disjoint union: $\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T)$. We have*

$$Point\ spectrum : \sigma_p := \{\lambda \in \mathbb{C} | (T-\lambda I)\ not\ injective\}, \tag{4.15.9}$$

$$Continuous\ spectrum : \sigma_c := \{\lambda \in \mathbb{C} | \tag{4.15.10}$$

$$(T-\lambda I)\ injective\ but\ not\ surjective\ with\ \overline{Ran(T-\lambda I)} = X\}, \tag{4.15.11}$$

$$Residual\ spectrum : \sigma_r := \{\lambda \in \mathbb{C} | \tag{4.15.12}$$

$$(T-\lambda I)\ injective\ but\ not\ surjective\ with\ \overline{Ran(T-\lambda I)} \neq X\}. \tag{4.15.13}$$

$$\tag{4.15.14}$$

[SZQ: 2023.04.02: I need to learn injective, surjective, bijective first!]

4.16 Part 31: Spectral Radius

Definition 4.16.1 (Spectral radius). *X complex Banach space. $T : X \to X$ bounded lineara operator. We define the spectral radius as*

$$r(T) := \sup \{|\lambda|\}, \tag{4.16.1}$$

where $\lambda \in \sigma(T)$.

Here, we have a fig to show this def.

Theorem 10. *X complex Banach space, $T : X \to X$ bounded linear operator. Then*

- *$\sigma(T) \subseteq \mathbb{C}$ is compact*
- *$X \neq \{0\} \implies \sigma(T) \neq \emptyset$*
- *$r(T) := \sup |\lambda| = \lim_{k\to\infty} \|T^k\|^{\frac{1}{k}} = \inf_{k\to\mathbb{N}} \|T^k\|^{\frac{1}{k}} \leq \|T\| < \infty$,*

where $\lambda \in \sigma(T)$

Proof. □

[SZQ: 2023.04.02: I need to learn the properties of sepctrum, dual spaces, Vom-Neuman series, Liouville's theorem, Hahn-Banach Theorem!]

4.17 Part 32: Normal and Self-Adjoint Operators

Definition 4.17.1 (Adjoint operator). *Let X be a Hilbert space and $T : X \to X$ a bounded lineaar operator. The bounded linear operator $T^* : X \to X$ defined by*

$$\langle y, Tx\rangle = \langle T^* y, x\rangle, \forall x, y \in X \tag{4.17.1}$$

is called the adjoint operator of T.

Definition 4.17.2 (Self Adjoint operator). *Let X be a Hilbert space and $T : X \to X$ a bounded lineaar operator. T is called*

- *self-adjoint if $T^* = T$*
- *skew-adjoint if $T^* = -T$*
- *normal if $T^*T = TT^*$*

Proposition 4.17.1. *T is normal $\implies r(T) = \|T\|$.*

Part II

Physics Fundamentals

Chapter 5

Theory of Electrodynamics

5.1 零、数学与物理常量基础公式

1. 矢量叉乘公式

$$a \times (b \times c) = b(a \cdot c) - c(a \cdot b) \tag{5.1.1}$$

[* 请自行结合 ∇ 的微分性与线性叠加，得到 $\nabla \times (a \times b), \nabla \times (\nabla \times a)$ 等]

2. 关于 ∇ 的相关公式 (推导用):

$$\nabla \cdot \left(\frac{R}{R^3}\right) = 4\pi\delta(R) \quad \nabla f(r) = \frac{\partial f}{\partial r}\nabla r = \frac{\partial f}{\partial r}\hat{r} \tag{5.1.2}$$

3. 关于 ∇ 积分的相关公式:

$$\int_{\mathrm{V}} \nabla \cdot A \mathrm{d}\tau = \oint_{\mathrm{S}} A \cdot \mathrm{d}S \quad \int_{\mathrm{V}} \nabla\psi \mathrm{d}\tau = \oint_{\mathrm{S}} \psi \mathrm{d}S \tag{5.1.3}$$

$$\int_{\mathrm{V}} \nabla \times A d\tau = \oint_{\mathrm{S}} \mathrm{d}S \times A \tag{5.1.4}$$

$$\int_{\mathrm{S}} \nabla \times A \cdot \mathrm{d}S = \oint_{\mathrm{c}} A \cdot \mathrm{d}l \quad \int_{\mathrm{S}} \mathrm{d}S \times \nabla\psi = \oint_{\mathrm{c}} \psi \mathrm{d}l \tag{5.1.5}$$

4. 物理常量基本公式

$$\mu_0\epsilon_0 = \frac{1}{c^2} \tag{5.1.6}$$

5.2 麦克斯韦方程组

1. 电场及标量势，及 2 者关系

$$E(r) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho\left(r'\right)}{R^3} R d\tau' = -\nabla\varphi \quad \varphi(r) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho\left(r'\right)}{R} d\tau' \tag{5.2.1}$$

2. 电偶极子 ($p = ql$) 的电场及电势

$$\varphi = \frac{1}{4\pi\varepsilon_0}\frac{p\cdot r}{r^3} \quad E = -\frac{1}{4\pi\varepsilon_0}\left[\frac{p - 3(p\cdot\hat{r})\hat{r}}{r^3}\right] \tag{5.2.2}$$

3. 电荷守恒/连续性方程:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot j = 0 \quad j = \rho v \tag{5.2.3}$$

4. 磁场及矢势，及 2 者关系

$$B = \frac{\mu_0}{4\pi}\int\frac{j\left(r'\right)d\tau' \times R}{R^3} = \nabla\times A \quad A = \frac{\mu_0}{4\pi}\int\frac{j\left(r'\right)}{R}d\tau' \tag{5.2.4}$$

5. 点电荷在电磁场中的受力

$$F = q(E + v\times B) \tag{5.2.5}$$

6. 磁偶极子 ($m = IS$) 的矢势及磁场

$$A = \frac{\mu_0}{4\pi}\frac{m\times r}{r^3} \quad B = -\frac{\mu_0}{4\pi}\left[\frac{m - 3(m\cdot\hat{r})\hat{r}}{r^3}\right] \tag{5.2.6}$$

7. 真空及介质中的 Maxwell 方程组

$$\begin{cases} \nabla\cdot E = \rho/\epsilon_0 \\ \nabla\times E = -\frac{\partial}{\partial t}B \\ \nabla\cdot B = 0 \\ \nabla\times B = \mu_0 j + \mu_0\epsilon_0\frac{\partial}{\partial t}E \end{cases} \qquad \begin{cases} \nabla\cdot D = \rho_f \\ \nabla\times E = -\frac{\partial}{\partial t}B \\ \nabla\cdot B = 0 \\ \nabla\times H = j_f + \frac{\partial}{\partial t}D \end{cases} \tag{5.2.7}$$

8. 各项同性材料中的本构关系

$$D = \epsilon E, H = B/\mu \text{ (无色散)} \tag{5.2.8}$$

$$D_\omega(r) = \epsilon(\omega)E_\omega(r), B_\omega(r) = \mu(\omega)H_\omega(r) \text{ (色散介质)} \tag{5.2.9}$$

$$(j_\omega = \sigma(\omega)E_\omega) \tag{5.2.10}$$

9. 材料的线性响应

$$P = \epsilon_0\chi_e E, \quad \epsilon = \epsilon_r\epsilon_0 = (1+\chi_e)\,\epsilon_0 \tag{5.2.11}$$

$$M = \frac{1}{\mu_0}\frac{\chi_m}{1+\chi_m}B, \quad \mu = \mu_r\mu_0 = (1+\chi_m)\,\mu_0, \tag{5.2.12}$$

10. 介质中的电荷和电流 (自由、极化、磁化)

$$\rho = \rho_f + \rho_p, \rho_p = -\nabla\cdot P \tag{5.2.13}$$

$$j = j_f + j_m + j_\rho, j_m = \nabla\times M, j_p = \frac{\partial P}{\partial t} \tag{5.2.14}$$

11. 麦克斯韦方程组的边界条件 ($\nabla \to n$)

$$\begin{cases} n \cdot (D_1 - D_2) = \sigma_f \text{ 自由电荷面密度} \\ n \times (E_1 - E_2) = 0 \\ n \cdot (B_1 - B_2) = 0 \\ n \times (H_1 - H_2) = \alpha_f \text{ 面电流密度} \end{cases} \tag{5.2.15}$$

5.3 电磁场的守恒定律和对称性

1. 电 (磁) 场对电荷做功:

$$dR = E \cdot \mathrm{j} \mathrm{d}\tau \mathrm{d}t \tag{5.3.1}$$

2. 真空中电磁场的能量守恒定律及各项的含义

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[W_m + \int u \ \mathrm{d}\tau\right] = -\oint S_p \cdot \mathrm{d}S \tag{5.3.2}$$

$$u(r,t) = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right), \quad S_p(r,t) = \frac{1}{\mu_0}E \times B \tag{5.3.3}$$

其物理意义为: 在一个闭合空间内物理量 (总能) $W_m + \int u \ \mathrm{d}\tau$ 的增加等于从边界流入闭合空间的 S_p 的大小。其中 $u(r,t)$ 是 r 点处 t 时刻电磁场的能量密度, S_p 即为相应的能流密度，也称做坡印廷矢量。

3. 真空中电磁场的动量守恒定律及各项的含义

$$\frac{\mathrm{d}G_m}{\mathrm{d}t} = -\oint \mathrm{d}S \cdot \overleftrightarrow{T} - \frac{\mathrm{d}}{\mathrm{d}t}\int g \mathrm{d}\tau \tag{5.3.4}$$

$$\text{电磁场的动量密度} g = \epsilon_0(E \times B) = \frac{1}{c^2}S_p \tag{5.3.5}$$

$$\text{动量流密度} \overleftrightarrow{T} = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) - \epsilon_0 EE - \frac{1}{\mu_0}BB \tag{5.3.6}$$

$$\text{受力} F_m = \frac{\mathrm{d}G_m}{\mathrm{d}t} \tag{5.3.7}$$

4. 带电的运动粒子在外磁场中的总动量

$$p = mv + qA \tag{5.3.8}$$

5. 介质中的电磁能量守恒定律

$$u(r,t) = \frac{1}{2}\left(\epsilon E^2 + \mu H^2\right), \quad S_p(r,t) = E \times H \tag{5.3.9}$$

6. 介质中电磁场的动量守恒定律

$$g' = D \times B, \quad \overleftrightarrow{T} = \frac{1}{2}(E \cdot D + B \cdot H)\overleftrightarrow{I} - DE - BH \tag{5.3.10}$$

5.4 导体静电学

1. 导体静电问题中电势满足的方程及边界条件

$$\begin{cases} \nabla^2\varphi = -\rho/\epsilon \\ \varphi \text{ 在边界有限} \\ \left.\frac{\partial\varphi}{\partial n}\right|_{\text{边界}} = -\frac{\sigma}{\epsilon}, \quad Q = -\epsilon \oint \frac{\partial\varphi}{\partial n}\,\mathrm{d}S \end{cases} \tag{5.4.1}$$

2. 格林互易定理给定一个有 m 个导体组成的体系，假设当导体上的电荷为 $q_1, q_2, \cdots$ 时，它们的电势等于 ϕ_i ，而对应另外一种电荷分布 q_i', 导体的电势分布为 ϕ_i' ，那么有关系式 $\sum_i q_i\phi_i' = \sum_i q_i'\phi_i$.

3. 导体系中的总能和相互作用能

$$\text{总能}W = \frac{1}{2}\int \varphi\rho \mathrm{d}\tau = \frac{1}{2}\sum_i \phi_i Q_i \tag{5.4.2}$$

$$\text{相互作用能}W = \frac{1}{2}\sum_i q_i\phi_i', \quad \phi_i' = \sum_{j\neq i}\phi_j \text{ 为其余电荷在}q_i\text{ 处电势} \tag{5.4.3}$$

4. 电容的定义

$$q_i = \sum_j C_{ij}\phi_j, \phi_i = \sum_j C_{ij}^{-1}q_j \to W = \frac{1}{2}\sum_{i,j} C_{ij}\phi_i\phi_j \tag{5.4.4}$$

5. 静电体系的稳定性一汤姆逊定理和恩肖定理

   汤姆逊定理: 若导体系中每个导体的位置固定不变，电荷可再分布，则体系基态对应，电荷的分布使所有导体均为等势体。

   恩肖定理：静电体系的平衡条件是体系中任一导体所处的位置的电场均为 0 ，因此无约束下静电体系没有平衡态。

6. 导体表面所受静电力

$$F_s = \frac{\epsilon_0}{2}E^2\hat{e}_n = \frac{1}{2\epsilon_0}\sigma^2\hat{e}_n \tag{5.4.5}$$

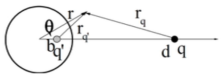
5.5 电介质静电学

1. 电介质的边界条件

$$\begin{cases} \varphi_1 = \varphi_2 \\ \epsilon_1 \cdot \frac{\partial\varphi_1}{\partial n} - \epsilon_2 \cdot \frac{\partial\varphi_2}{\partial n} = -\sigma_f(\vec{n} : 2 \to 1) \end{cases} \tag{5.5.1}$$

2. 唯一性定理对于确定的静电 (磁) 问题 (已知电荷分布和电介质分布 $\rho(r), \epsilon(r)$ 或电流和磁介质分布)，边条唯一决定解 (前提: 介质中 D 与 E(H 与 A) 之间的本构关系为单调单值)

3. 镜像法 (会做)



$$q' = \frac{R}{d}q, \quad b = \frac{R^2}{d} \tag{5.5.2}$$

4. 本征函数展开法

外场为均匀电场时，对于球/柱体系，解只具有 $l = 1$ 的项。

$$\text{轴对称球坐标系}\varphi = \left(Ar + \frac{B}{r^2}\right)\cos\theta \tag{5.5.3}$$

$$\text{与z 无关的柱对称}\varphi = B_0 \ln\rho + \left(A_1\rho + B_1\rho^{-1}\right)\cos\phi + \left(C_1\rho + D_1\rho^{-1}\right)\sin\phi \tag{5.5.4}$$

5. 退极场

$$E_d = E_0 - E_{\text{in}} = -L \cdot P/\epsilon_0 \tag{5.5.5}$$

L 为退极因子，取决于物体形状，对于球 $L = 1/3$.

6. 多极矩展开

$$\varphi = \frac{Q}{4\pi\epsilon_0 r} - \frac{P}{4\pi\epsilon_0}\nabla\frac{1}{r} + \frac{1}{4\pi\epsilon_0}\frac{1}{6}\overrightarrow{\overrightarrow{D}} : \nabla\nabla\frac{1}{r} \tag{5.5.6}$$

$$Q = \int \rho\left(r'\right) \mathrm{d}\tau', P = \int \rho\left(r'\right) r' \mathrm{d}\tau', \overrightarrow{\overrightarrow{D}} = 3\int \rho\left(r'\right) r'r' \mathrm{d}\tau' \tag{5.5.7}$$

7. 多极矩在外场中的作用

$$W = Q\varphi - p \cdot E + \frac{1}{6}\overrightarrow{\overrightarrow{D}} : \nabla E \tag{5.5.8}$$

8. 电偶极矩在外场中的受力与力矩

$$F_e = p \cdot \nabla E, \quad M = p \times E \tag{5.5.9}$$

5.6 静磁场

1. 磁场的矢势方程及边界条件

$$\nabla^2 A = -\mu_0 j \tag{5.6.1}$$

$$\hat{e}_n \times (A_1 - A_2) = 0 \tag{5.6.2}$$

$$\hat{e}_n \times \left[\frac{1}{\mu_1}(\nabla \times A_1) - \frac{1}{\mu_2}(\nabla \times A_2)\right] = \alpha_f \tag{5.6.3}$$

2. 静磁场总能

$$U_m = \frac{1}{2}\int B \cdot H \mathrm{d}\tau = \frac{1}{2}\int A \cdot j \mathrm{d}\tau \tag{5.6.4}$$

3. 磁场的标量势解法中磁标势的定义，引入磁标势的条件及其意义

$$H = -\nabla \varphi_m \tag{5.6.5}$$

1) 无传导电流；2) 引入磁壳使得空间为单连通;

从而保证了 H 是个保守场，且保证了磁标势单值性.

4. 磁介质中的边界条件

$$\begin{cases} \varphi_1 = \varphi_2 \\ \mu_1 \cdot \frac{\partial \varphi_1}{\partial n} = \mu_2 \cdot \frac{\partial \varphi_2}{\partial n} \end{cases} \tag{5.6.6}$$

5. 铁磁介质中的边界条件 (饱和磁化为 M_0^i)

$$\begin{cases} \varphi_1 = \varphi_2 \\ \frac{\partial \varphi_1}{\partial n} - \frac{\partial \varphi_2}{\partial n} = e_n \cdot \left(M_0^1 - M_0^2\right) \end{cases} \tag{5.6.7}$$

6. 磁多极展开

$$A = A^{(1)} = \frac{\mu_0}{4\pi}\frac{m \times r}{r^3}, m = \frac{1}{2}\int r' \times j \ \mathrm{d}\tau' \tag{5.6.8}$$

7. 磁偶极子的能量、受力和力矩

$$U = -m \cdot B; F = \nabla(m \cdot B); \tau = m \times B \tag{5.6.9}$$

5.7 似稳场 (准静)

1. 似稳条件

   (a) 电磁场变化频率远小于金属特征频率 $\omega << \omega_\sigma = \sigma_c/\epsilon$, 其中 $j = \sigma_c E$.

(b) $R << \lambda/2\pi$ 从而忽略位移电流与辐射

2. 似稳场方程

$$\frac{\partial}{\partial t}\begin{pmatrix} H \\ E \end{pmatrix} = \frac{1}{\mu\sigma_c}\nabla^2\begin{pmatrix} H \\ E \end{pmatrix} \tag{5.7.1}$$

3. 趋肤效应与趋肤深度

$$E = \hat{x}A\exp[pz - i\omega t] = \hat{x}E_0 e^{-\alpha z}\cos(\omega t - \alpha z) \tag{5.7.2}$$

$$\delta = \frac{1}{\alpha} = \sqrt{\frac{2}{\mu\omega\sigma_c}} \tag{5.7.3}$$

5.8 电磁波的传播

1. 电磁波的传播方程

$$\left(\nabla^2 - \epsilon\mu\frac{\partial}{\partial t^2}\right)\begin{pmatrix} E \\ B \end{pmatrix} = 0 \tag{5.8.1}$$

[* 请自行从 Maxwell 方程推导得到该式.]

2. 电磁波的解

$$E(r,t) = E_0\cos(k\cdot r - \omega t + \phi) \tag{5.8.2}$$

$$B(r,t) = B_0\cos(k\cdot r - \omega t + \phi) \tag{5.8.3}$$

3. 电磁波的色散关系、波速、波长和折射率

$$k^2 = \epsilon\mu\omega^2 \tag{5.8.4}$$

$$v^2 = \frac{1}{\epsilon\mu}, v_p = \omega/k \tag{5.8.5}$$

$$k = 2\pi/\lambda \tag{5.8.6}$$

$$n = \sqrt{\epsilon_r\mu_r} = k\omega/c \tag{5.8.7}$$

4. 波矢与电场、磁场的关系

$$k\cdot E_0 = 0, \quad k\cdot B_0 = 0 \tag{5.8.8}$$

$$k\times E_0 = \omega B_0, \quad k\times B_0 = -\epsilon\mu\omega E_0 \tag{5.8.9}$$

5. 阻抗的定义

$$Z = \sqrt{\mu/\epsilon}, |E_0| = Z\,|H_0| \tag{5.8.10}$$

6. 电磁波的能流

$$\langle S_p \rangle = \frac{1}{2}\mathrm{Re}\,(E \times H^*) = \frac{1}{2Z}E_0^2\hat{k} = \langle u \rangle \cdot v \tag{5.8.11}$$

$$\langle u \rangle = \frac{\epsilon}{2}E_0^2 \tag{5.8.12}$$

7. $E_0 = \hat{x}E_{x0}e^{i\phi_x} + \hat{y}E_{y0}e^{i\phi_y}$ 的线偏振和圆偏振条件

   线偏振: $\phi_x = \phi_y$;

   圆偏振 $\phi_x - \phi_y = \pm\pi/2, |E_{x0} = E_{y0}|$, 正负对应右旋、左旋偏振 $\hat{e}_{\mathrm{right}} = (\hat{x} - i\hat{y})/\sqrt{2}, \hat{e}_{\mathrm{eft}} = (\hat{x} + i\hat{y})/\sqrt{2}$

8. 金属的有效电导率—Drude 模型及其推导

   利用散射力模型 (平均 τ 时间受到一次异种粒子的散射而丢失所有的动量)

$$\sigma(\omega) = \frac{n_e e^2}{m(-i\omega + 1/\tau)} \tag{5.8.13}$$

9. 金属的有效介电函数及其不同频段的行为

$$\epsilon(\omega) = \epsilon_0\epsilon_v + i\frac{\sigma(\omega)}{\omega}, \epsilon_r = \epsilon_v - \frac{\omega_p^2}{\omega(\omega + i/\tau)}, \quad \epsilon_v \approx 1 \tag{5.8.14}$$

$$\epsilon_r(\omega) \approx \begin{cases} i\frac{\sigma_c}{\epsilon_0\omega}, & \leq \mathrm{GHz} \\ 1 - \left(\frac{\omega_p}{\omega}\right)^2, & \text{可见光波段} \end{cases} \tag{5.8.15}$$

   在可见光波段，电子高频下碰撞时间远大于周期，几乎不表现出损耗。在 GHz 波段，电子平均碰撞时间远小于电场周期，表现为欧姆损耗直流良导体特征。

10. 等离共振频率及其物理含义

$$\omega_p = \sqrt{\frac{n_e e^2}{\epsilon_0 m}} \tag{5.8.16}$$

    表示自由电子气在外场的驱动下集体震荡。

11. 导电介质的色散关系

$$k^2 = \left(\frac{\omega}{c}\right)^2 \epsilon_r(\omega) \tag{5.8.17}$$

12. 良导体在不同电磁波频率的行为

    (a) 在 GHz 以下,$1/\omega >> \tau$ 既震荡又衰减,电磁场有 $\pi/4$ 相位差,$\alpha = \sqrt{\sigma_c\mu_0\omega/2}$, 趋肤深度 $\delta = \alpha^{-1}$.

(b) 在光波段 $k^2=\left(\omega^2-\omega_p^2\right)/c^2$ ，电磁场 $\pi/2$ 相位差. 当
$\omega<\omega_p$ ，不传播，造成反射；
当 $\omega=\omega_p$ 隧穿至极值；
当 $\omega>\omega_p$ ，紫外透明，有一定反射.

13. 非良导体在不同电磁波频率的行为

非良导体 $\omega\ll 1/\tau$, 电磁场几乎无相位差，电磁波传播伴随很小的衰减。

14. 各向异性介质/旋光介质中的本构关系

$$D_\omega=\epsilon_0\vec{\epsilon}_r(\omega)E. \tag{5.8.18}$$

$$\overrightarrow{\epsilon}_r(\omega)=\begin{pmatrix}\epsilon_1 & i\epsilon_2 & 0\\ -i\epsilon_2 & \epsilon_1 & 0\\ 0 & 0 & \epsilon_3\end{pmatrix} \tag{5.8.19}$$

$$\epsilon_1=1-\frac{\omega_p^2}{\omega^2-\omega_B^2},\epsilon_2=\frac{\omega_p^2\omega_B}{\left(\omega^2-\omega_B^2\right)\omega},\epsilon_3=1-\frac{\omega_p^2}{\omega^2},\omega_B=-\frac{eB_0}{m} \tag{5.8.20}$$

15. 左 (k_-) 右 (k_+) 旋光的色散关系

$$k_\pm=\frac{\omega}{c}\sqrt{\epsilon_1\pm\epsilon_2} \tag{5.8.21}$$

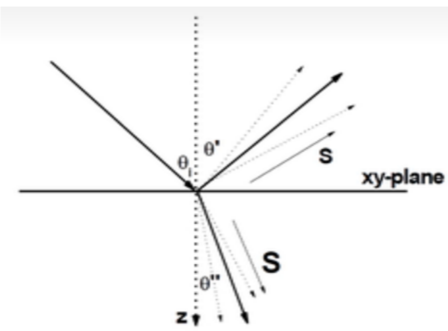
16. 法拉第效应

与入射光比较偏振方向旋转了如下的角度 $\Delta\phi=(k_+-k_-)\cdot d/2$.

17. 电磁波在介质界面反射、折射的基本规律及其物理本质

(a) 反射波、折射波与入射波 ω 相等——时间平移不变性
(b) 入射线、反射线和折射线在同一平面内, $k_{\|}$ 相同——空间平移不变性
(c) 在正常介质中, 反射波 k_z' 应取负根，折射波 k_z'' 应取正根——因果律
(d) 入射角等于反射角　一空间平移不变性
(e) Snell law　空间平移不变性

$$\frac{\sin\theta}{\sin\theta''}=\frac{k''}{k}=\frac{n_2}{n_1} \tag{5.8.22}$$



18. 电磁波在介质界面反射、折射的振幅关系

    (a) 横电波 S/TE ，电场垂直于入射面，有效阻抗 $Z_{eff} = Z/\cos\theta$

$$E_0' = \frac{Z_{eff,2} - Z_{eff,1}}{Z_{eff,2} + Z_{eff,1}} E_0 = r_s \cdot E_0, E_0'' = \frac{2Z_{eff,2}}{Z_{eff,2} + Z_{eff,1}} E_0 = t_s \cdot E_0 \tag{5.8.23}$$

    (b) 横磁波 P/TM，磁场垂直于入射面，$Z_{\text{eff}} = Z\cos\theta$

$$H_0' = \frac{Z_{eff,1} - Z_{eff,2}}{Z_{eff,2} + Z_{eff,1}} H_0 = r_p \cdot H_0, H_0'' = \frac{2Z_{eff,1}}{Z_{eff,2} + Z_{eff,1}} H_0 = t_p \cdot H_0 \tag{5.8.24}$$

19. 电磁波的反射率 R 与透射率 T

$$R = |r|^2, T = \begin{cases} |t_s|^2 \frac{Z_{eff,1}}{Z_{2ff,2}} \\ |t_p|^2 \frac{Z_{eff,2}}{Z_{ef,1}} \end{cases} \tag{5.8.25}$$

20. 布鲁斯特 Brewster 角及其意义

    当反射波与折射波相互垂直时，P 偏振的电磁波完全不被反射, 入射角

$$\theta_B = \arctan(n_2/n_1) \tag{5.8.26}$$

21. 全反射临界角

$$\theta_C = \arcsin(n_2/n_1) \tag{5.8.27}$$

5.9 波导和谐振腔

1. 波导管 (PEC) 的边界条件

$$n \cdot B = 0, n \times E = 0 \tag{5.9.1}$$

$$n \cdot D = \sigma, n \times H = j \tag{5.9.2}$$

2. 波导管中波的模式 (偏振)

    (a) 横电波 TE: $B_{0z} \neq 0, E_{0z} = 0$

    (b) 横电波 TM: $B_{0z} = 0, E_{0z} \neq 0$

3. TE 波的解及基模

$$B_{0z} = B_0 \cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right); B_z = B_{0z}\exp[ik_z z - \omega t] \tag{5.9.3}$$

$$k_c^2 = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right) = k_0^2 - k_z^2 \tag{5.9.4}$$

$$\text{截止频率, 电磁波能够传播的最低频率: } \omega_c = ck_c \tag{5.9.5}$$

$$\text{色散关系} k_z = \frac{\omega}{c} \sqrt{1 - \left(\frac{\omega_c}{\omega} \right)^2} \tag{5.9.6}$$

TE 模的最低阶模式 (基模) 为 TE_{01} 或 TE_{10}

4. TM 波的解及基模

$$E_{0z} = E_0 \sin\left(\frac{m\pi}{a} x \right) \sin\left(\frac{n\pi}{b} y \right); E_z = E_{0z} \exp\left[ik_z z - \omega t \right] \tag{5.9.7}$$

$$k_c^2 = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right) \tag{5.9.8}$$

TM 模的最低阶模式 (基模) 为 TM_{11}.

5. 谐振腔的频率及基模

$$\omega_{\{mnp\}} = c\pi \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2} + \frac{p^2}{d^2}} \tag{5.9.9}$$

谐振腔中 m, n, p 三个指标中只能有一个不是 0 ，故谐振腔的最低阶模式为 (110), (101) 或 (011).

5.10 电磁波的辐射

1. 用电势和矢势求得电场

$$E = -\frac{\partial}{\partial t} A - \nabla \varphi \tag{5.10.1}$$

[* 请自行从 Maxwell 方程推导得到]

2. 库仑规范条件与洛伦兹规范条件

$$\nabla \cdot A = 0 \tag{5.10.2}$$

$$\nabla \cdot A + \frac{1}{c^2} \frac{\partial \varphi}{\partial t} = 0 \tag{5.10.3}$$

3. 洛伦兹规范条件下势所满足的方程

$$\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\begin{pmatrix}\varphi\\ A\end{pmatrix}=\begin{pmatrix}-\rho/\epsilon_0\\ -\mu_0 j\end{pmatrix} \tag{5.10.4}$$

4. 推迟势

$$\varphi(r,t)=\frac{1}{4\pi\epsilon_0}\int\frac{[\rho]}{R}d\tau',\quad [\rho]=\rho\left(r',t'=t-R/c\right),R=r-r' \tag{5.10.5}$$

$$A=\frac{\mu_0}{4\pi}\int\frac{[j]}{R}d\tau' \tag{5.10.6}$$

5. 电偶极辐射 $\left([p]=p_0e^{-i\omega t}e^{ikr}\right)$

$$\varphi_1=-\nabla\cdot\frac{[p]}{4\pi\epsilon_0 r},[p]=\left[\int r'\rho d\tau'\right] \tag{5.10.7}$$

$$A_0=\frac{\mu_0}{4\pi r}[\dot{p}]=\frac{-i\omega\mu_0}{4\pi}\frac{[p]}{r} \tag{5.10.8}$$

远场下的电磁场为 $(\nabla\to ik\hat{r})$

$$B=\nabla\times A=\frac{\omega^2\mu_0}{4\pi cr}\hat{r}\times[p] \tag{5.10.9}$$

$$E=-\hat{r}\times(cB)=-\frac{\omega^2}{4\pi\epsilon_0c^2r}\hat{r}\times(\hat{r}\times[p]) \tag{5.10.10}$$

6. 辐射能流的角分布

$$\langle f(\theta,\phi)\rangle=\langle S_p\rangle r^2 \tag{5.10.11}$$

7. 远场下磁偶极辐射

$$A=\frac{i\mu_0\omega}{4\pi r^2c}r\times[m] \tag{5.10.12}$$

$$B=\nabla\times A=-\frac{\mu_0}{4\pi}\frac{k^2}{r}\hat{r}\times(\hat{r}\times[m]) \tag{5.10.13}$$

$$E=-\hat{r}\times(cB)=-\frac{\mu_0}{4\pi}\frac{k^2c}{r}\hat{r}\times[m] \tag{5.10.14}$$

8. 天线电流及矢势

$$I\left(z',t'\right)=I_0e^{-i\omega T}\sin\left(k\left(\frac{l}{2}-\left|z'\right|\right)\right) \tag{5.10.15}$$

$$A_z=\frac{\mu_0I_0}{2\pi kr}e^{-i\omega(t-r/c)}\left(\frac{\cos\left(\frac{kl}{2}\cos\theta\right)-\cos\frac{kl}{2}}{\sin^2\theta}\right) \tag{5.10.16}$$

9. 天线阵的辐射角分布

利用相邻路程差 $l\cos\theta,E_i=C(\theta)\exp\left[ikR_i\right]/R_i$ ，求和得到，总的辐射角分布

$$f_{\text{total}}(\theta,\phi)=f_{\text{single}}(\theta,\phi)\frac{\sin^2(m\alpha/2)}{\sin^2(\alpha/2)},\alpha=kl\cos\theta \tag{5.10.17}$$

5.11 相对论电动力学

1. 狭义相对论的两条基本假设

    (a) 相对性原理: 自然规律在不同惯性系中的表达式相同

    (b) 光速不变原理，选择 Maxwell 方程在一切惯性系中形式不变

2. 洛伦兹变换

$$x' = (x - vt)/\sqrt{1 - v^2/c^2}, t' = \Big(t - vx/c'/\sqrt{1 - v^2/c^2} \tag{5.11.1}$$

$$x_\mu = (x, y, z, ict), \beta = v/c, \gamma = /\sqrt{1 - \beta^2} \tag{5.11.2}$$

$$x_\mu = \alpha_{\nu\mu} x'_\nu, \alpha = \begin{pmatrix} \gamma & & & i\beta\gamma \\ & 1 & & \\ & & 1 & \\ -i\beta\gamma & & & \gamma \end{pmatrix} \tag{5.11.3}$$

$$\tag{5.11.4}$$

3. 电荷守恒定律的协变形式

$$J_\mu = (j, ic\rho), \quad \partial_\mu J_\mu = 0, \partial_\mu = \left(\nabla, -i\frac{\partial_t}{c}\right) \tag{5.11.5}$$

4. 协变形式的麦克斯韦方程组

$$\partial_\mu F_{\mu\nu} = -\mu_0 J_\nu, \partial_\mu F_{\nu\alpha} + \partial_\alpha F_{\mu\nu} + \partial_\nu F_{\alpha\mu} = 0(\mu \neq \nu \neq \alpha) \tag{5.11.6}$$

$$A_\mu = (A, i\varphi/c), F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{5.11.7}$$

Part III

Machine Learning Fundamentals

5.12 Computational learning theory

Part IV

Quantum Fundamentals

Chapter 6

Prerequisites

6.1 Hilbert Spaces and Linear Operators

Throughout this course, $\mathcal{H}$ denotes a finite-dimensional Hilbert space (complex vector space with an associated inner product). Using Dirac's "bra-ket" notation we denote elements of the Hilbert space (called kets) as

$$|\psi\rangle \in \mathcal{H}. \tag{6.1.1}$$

The elements of the dual Hilbert space are called bras and are denoted

$$\langle\psi| \in \mathcal{H}^*, \tag{6.1.2}$$

where $\langle\psi| = (|\psi\rangle)^\dagger$. Here, $X^\dagger := \bar{X}^T$ denotes the Hermitian adjoint (also called the conjugate transpose). We denote

$$B(\mathcal{H}_1, \mathcal{H}_2) := \{\text{linear maps from } \mathcal{H}_1 \text{ to } \mathcal{H}_2\} \tag{6.1.3}$$

and the set of all linear maps to and from the same space will be denoted $B(\mathcal{H}) = B(\mathcal{H}, \mathcal{H})$. An operator $X \in B(\mathcal{H})$ is *normal* if $XX^T = X^T X$. Every normal operator has a *spectral decomposition.* That is, there exists a unitary U and a diagonal matrix D whose entries are the eigenvalues $\lambda_1, \dots, \lambda_d \in \mathbb{C}$ of X such that

$$X = UDU^\dagger. \tag{6.1.4}$$

In other words,

$$X = \sum_{i=1}^{d} \lambda_i \, |\psi_i\rangle \langle\psi_i| \tag{6.1.5}$$

where $X|\psi_i\rangle = \lambda_i|\psi_i\rangle$ and $U = (|\psi_i\rangle, \ldots, |\psi_d\rangle)$. If X is Hermitian, $X = X^\dagger$, then $\lambda_i \in \mathbb{R}$. An operator X is positive semi-definite (PSD) if

$$\langle\varphi| X |\varphi\rangle \geq 0 \qquad \forall |\varphi\rangle \in \mathcal{H}. \tag{6.1.6}$$

As a consequence, $X \geq 0$ and $\lambda_i \geq 0$. It holds that PSD $\implies$ Hermitian $\implies$ normal. Unless otherwise stated, we will always assume we are working in an orthonormal basis.

6.2 Quantum States

A quantum state ρ in a Hilbert space $\mathcal{H}$ is a PSD linear operator with

$$\rho \in B(\mathcal{H}), \quad \rho \geq 0, \quad \mathrm{tr}\rho = 1. \tag{6.2.1}$$

This means that the state has eigenvalues $\{\lambda_i\}_{i=1}^d$ satisfying $\lambda_i \geq 0$ and $\sum_{i=1}^d \lambda_i = 1$. Thus, $\{\lambda_i\}_{i=1}^d$ forms a probability distribution.

A *pure quantum state* ψ is a quantum state with rank 1. We can find $|\psi\rangle \in \mathcal{H}$ such that $\psi = |\psi\rangle\langle\psi|$. In this case, ψ is called a *projector*. A *mixed state* is a quantum state with rank > 1. Mixed states are convex combinations of pure states. That is, for every quantum state ρ with $r = \mathrm{rank}(\rho)$ there are pure states $|\psi_i\rangle_{i=1}^k \quad (k \geq r)$ and a probability distribution $\{p_i\}_{i=1}^k$ such that

$$\rho = \sum_{i=1}^k p_i |\psi_i\rangle\langle\psi_i|. \tag{6.2.2}$$

The spectral decomposition of ρ is a special case of this property.

6.3 Composite systems, partial trace, entanglement

Let A and B be two quantum systems with Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$. The *joint system AB* is described by the Hilbert space $\mathcal{H}_{AB} := \mathcal{H}_A \otimes \mathcal{H}_B$. We denote quantum states of the joint system as $\rho_{AB} \in \mathcal{H}_{AB}$. The marginal of the bipartite state, denoted ρ_A, is uniquely defined as the operator satisfying

$$\rho_A := \mathrm{tr}_B \rho_{AB}, \tag{6.3.1}$$

which is defined via $\mathrm{tr}(\rho_{AB}(X_A \otimes \mathbb{I}_B)) = \mathrm{tr}\rho_A X_A \quad \forall X_A \in B(\mathcal{H}_A)$. For a Hilbert space with $|B| := \dim \mathcal{H}_B$, the explicit form of the partial trace is

$$\mathrm{tr}_B \rho_{AB} = \sum_{i=1}^{|B|} (\mathbb{I}_A \otimes \langle i|_B) \rho_{AB} (\mathbb{I}_A \otimes |i\rangle_B), \tag{6.3.2}$$

for some basis $\{|i\rangle_B\}_{i=1}^{|B|}$ of $\mathcal{H}_B$.

A *product state* on AB is a state of the form $\rho_A \otimes \sigma_B$. The state is called *separable* if it lies in the convex hull of product states:

$$\rho_{AB} = \sum_i p_i \rho_A^i \otimes \sigma_B^i \tag{6.3.3}$$

for some states $\{\rho_A^i\}_i$ and $\{\sigma_B^i\}_i$ and probability distribution $\{p_i\}_i$. A state is called *entangled*, if it is not separable. An entangled state of particular interest is the *maximally entangled state*. Let $d = \dim \mathcal{H}$, $\{|i\rangle\}_{i=1}^d$ be a basis for $\mathcal{H}$. A maximally entangled state is expressed as

$$|\phi^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle \otimes |i\rangle \quad \in \mathcal{H} \otimes \mathcal{H} \tag{6.3.4}$$

6.4 Measurements

The most general measurement is given by a *positive operator-valued measure* (POVM) $E = \{E_i\}_i$ where $E_i \geq 0 \quad \forall i$ and $\sum_i E_i = \mathbb{I}$. Then, for a quantum system $\mathcal{H}$ in state ρ, the probability of obtaining measurement outcome i is given by $p_i = \mathrm{tr}[\rho E_i]$. So, we have

$$\sum_i p_i = \sum_i \mathrm{tr}[\rho E_i] = \mathrm{tr}\left[\rho \sum_i E_i\right] = \mathrm{tr}[\rho \mathbb{I}] = \mathrm{tr}\rho = 1, \tag{6.4.1}$$

for all normalized quantum states. A *projective measurement* $\Pi = \{\Pi_i\}$ is a POVM with the added property of orthogonality, which for projectors means

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The *Shannon entropy* $H(p)$ of a probability distribution $p = \{p_i, \ldots, p_d\}$ is defined as $H(p) = -\sum_{i=1}^d p_i \log p_i$, where the logarithm is base 2 unless otherwise specified. Note that when the logarithm is base 2, the entropy has units of *bits*. The *von Neumann entropy* $S(\rho)$ of a quantum state ρ is defined as

$$S(\rho) = -\mathrm{tr}\left[\rho \log \rho\right] = H(\{\lambda_i, \ldots, \lambda_d\}), \tag{6.5.1}$$

where $\rho = \sum_i \lambda_i |\psi_i\rangle \langle\psi_i|$ is a spectral decomposition of ρ and where the logarithm of an operator is obtained by first diagonalizing the matrix representing the operator and then taking the logarithm of the diagonal elements. That is,

$$\log \rho = \sum_{i:\lambda_i>0} \log(\lambda_i) |\psi_i\rangle \langle\psi_i| . \tag{6.5.2}$$

Chapter 7

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$$\log \rho = \sum_{i:\lambda_i>0} \log(\lambda_i) |\psi_i\rangle \langle\psi_i| . \tag{7.5.2}$$

Chapter 8

Classes of Quantum Channels

8.1 Qubit channels

Note: please see Nielsen and Chuang for figures depicting these channels' actions on the Bloch sphere. I may make nice ones at some point but for now, I am too lazy.

1. **Bit flip channel.** First, we look at the classical bit flip channel, the properties of which were established by Claude Shannon in his seminal 1948 paper. The quantum version is simply given by the Pauli X operator,

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{8.1.1}$$

which acts as

$$X|0\rangle = |1\rangle, \tag{8.1.2}$$

$$X|1\rangle = |0\rangle. \tag{8.1.3}$$

It's eigenbasis is $|\pm\rangle := \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$. So, $X|+\rangle|+\rangle$ and $X|-\rangle = -|-\rangle$. The channel acts as

$$\mathcal{F}_p^X : \rho \mapsto (1-p)\rho + pX\rho X \tag{8.1.4}$$

So, the Kraus operators are $\sqrt{1-p}\mathbb{I}, \sqrt{p}X$

2. **Phase-flip/Z-dephasing channel**

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{8.1.5}$$

$$Z|0\rangle = |0\rangle \tag{8.1.6}$$

$$Z|1\rangle = -|1\rangle \tag{8.1.7}$$

$$Z|+\rangle = |-\rangle \tag{8.1.8}$$

$$Z|-\rangle = |+\rangle \tag{8.1.9}$$

The channel is given as

$$\mathcal{F}_p^Z : \rho \mapsto (1-p)\rho + pZ\rho Z \tag{8.1.10}$$

3. **Bit-phase flip/Y-dephasing channel**

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{8.1.11}$$

$$|\pm i\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm i|1\rangle) \tag{8.1.12}$$

$$Y|i\rangle = |i\rangle \tag{8.1.13}$$

$$Y|-i\rangle = -|-i\rangle \tag{8.1.14}$$

$$Y|0\rangle = i|1\rangle \tag{8.1.15}$$

$$Y|1\rangle = -i|0\rangle \tag{8.1.16}$$

The channel is then

$$\mathcal{F}_p^Y : \rho \mapsto (1-p)\rho + pY\rho Y \tag{8.1.17}$$

$$= (1-p)\rho + pXZ\rho ZX \tag{8.1.18}$$

4. **Depolarizing channel.** Corresponds to an error model in which each Pauli error occurs with equal probability. The channel is

$$\mathcal{D}_p : \rho \mapsto (1-p)\rho + \frac{p}{3}\left(X\rho X + Y\rho Y + Z\rho Z\right) \tag{8.1.19}$$

with equal probability $p/3$. The Kraus operators are then $\sqrt{1-p}\mathbb{I}$, $\sqrt{p/3}X$, $\sqrt{p/3}Y$, $\sqrt{p/3}Z$. Thus, the Kraus rank is 4 for this channel when $p \in (0,1)$. An alternative representation is given by

$$\rho \mapsto (1-q)\rho + q\mathrm{tr}(\rho)\frac{\mathbb{I}}{2} \tag{8.1.20}$$

This represents replacing the input state with the maximally mixed state with "probability" q (q can be greater that 1).

What about the relation $p \leftrightarrow q$? To deduce this, we can use the identity

$$\frac{1}{2}\mathbb{I}_2 = \frac{1}{4}\left(\rho + X\rho X + Y\rho Y + Z\rho Z\right) \forall \rho \in B(\mathbb{C}^2) \tag{8.1.21}$$

We can then derive the relationship between p and q. We have

$$\frac{1}{2}\mathbb{I}_2 = \frac{1}{4}\left(\rho + X\rho X + Y\rho Y + Z\rho Z\right) \quad \forall \rho \in B(\mathbb{C}^2) \tag{8.1.22}$$

$$(1-p)\rho + p\frac{\mathbb{I}}{2} = (1-q)\rho + \frac{q}{4}\left(\rho + X\rho X + Y\rho Y + Z\rho Z\right) \tag{8.1.23}$$

$$\implies \boxed{q = \frac{4}{3}p} \tag{8.1.24}$$

5. **Generalized Pauli channel.** Let $\vec{p} = (p_0, p_1, p_2, p_3)$ be a probability distribution. The generalized Pauli channel is

$$\mathcal{N}_{\vec{p}}(\rho) = p_0\rho + p_1 X\rho X + p_2 Y\rho Y + p_3 Z\rho Z \tag{8.1.25}$$

where we recover the depolarizing channel by setting $p_0 = 1 - p$ and $p_i = p/3$. Note that for any $\vec{p}$, this channel is unital.

Pauli channels are interesting from an information theoretic standpoint. Classically, they are very easily understood. Quantum mechanically, they very much are not understood (except in the case of flip or dephasing channels).

In what sense are these flip channels de-phasing? Let us look at the example of the phase flip channel

$$\mathcal{F}_p^Z : \rho \mapsto (1-p)\rho + pZ\rho\rho Z. \tag{8.1.26}$$

We can understand the justification of the term de-phasing by looking at the action on the density matrix

$$\begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \mapsto \begin{pmatrix} \rho_{11} & (1-2p)\rho_{12} \\ (1-2p)\rho_{21} & \rho_{22} \end{pmatrix} \tag{8.1.27}$$

where $\rho \geq 0, \mathrm{tr}(\rho) = 1 \implies \rho_{11} + \rho_{22} = 1$. We can make the following observations

1. If $\rho = x\left|0\right\rangle\left\langle 0\right| + (1-x)\left|1\right\rangle\left\langle 1\right|$, then

$$\mathcal{F}_p^Z(\rho) = \rho \quad \forall p \in [0,1] \tag{8.1.28}$$

2. $p = \frac{1}{2}$: the channel is diagonal in the z-basis for all input states

We can also think about sending classical information through this channel. Let us encode 0 as $|0\rangle\langle 0|$ and 1 as $|1\rangle\langle 1|$. That is, we are encoding one bit in one qubit. We can do this reliably with this channel because

$$\mathcal{F}_p^Z(|0\rangle\langle 0|) = |0\rangle\langle 0|, \tag{8.1.29}$$

$$\mathcal{F}_p^Z(|1\rangle\langle 1|) = |1\rangle\langle 1|. \tag{8.1.30}$$

This means that we can send one bit through the channel. The key idea is: this channel preserves classical information because it preserves the orthogonality of the basis states. The same would hold for the X channel but you would have to encode info in the $|\pm\rangle$ basis.

6. **Amplitude damping channel.** Physical model: 2-level system (e.g. an atom with a ground state $|0\rangle$ and excited state $|1\rangle$. If the system is in an excited state $|1\rangle$, it decays with a certain probability, γ, emitting a photon to the environment. How can we capture this decay process mathematically? Consider the isometry

$$|0\rangle_A \mapsto |0\rangle_B |0\rangle_E, \tag{8.1.31}$$

$$|1\rangle \mapsto \sqrt{1-\gamma}\,|1\rangle_B |0\rangle_E + \sqrt{\gamma}\,|0\rangle_B |1\rangle_E. \tag{8.1.32}$$

Compactly we can write

The Kraus operators for this channel are

$$K_0 = \langle 0|_E V = |0\rangle\langle 0|_B + \sqrt{1-\gamma}\,|1\rangle\langle 1|)_B = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}, \tag{8.1.33}$$

$$K_1 = \langle 1|_E V = \sqrt{\gamma}\,|0\rangle_B \langle 1|_A = \begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix}. \tag{8.1.34}$$

In the Kraus representation, then, we can write the amplitude damping channel

$$\mathcal{A}_\gamma : \rho \mapsto K_0 \rho K_0^\dagger + K_1 \rho K_1^\dagger. \tag{8.1.35}$$

We note that the amplitude damping channel is not unital. Therefore, it is not a Pauli (or even a mixed-unitary) channel. Interestingly, it is completely understood how to send *quantum information* through the amplitude damping channel; however, it is not known how much classical information can be sent through this channel.

7. **Erasure channel.** Classical model:

Quantum version: $\mathcal{E}_p : B(\mathcal{H}) \to B(\mathcal{H} \otimes \mathbb{C})$. The channel acts as

$$\rho \mapsto (1-p)\rho + p\mathrm{tr}(\rho)\,|e\rangle\langle e|. \tag{8.1.36}$$

This channel has Kraus operators given by $K_0\sqrt{1-p}\mathbb{I}, K_1\sqrt{p}\,|0\rangle\langle 0|\,, K_2\sqrt{p}\,|e\rangle\langle 1|\,.$ Note that $|e\rangle$ must be orthogonal to all $\rho \in B(\mathcal{H})$. That is, $\langle |e\rangle\langle e|\,,\rho\rangle = 0 \quad \forall \rho \in B(\mathcal{H})$. Bob can always tell whether erasure happened by performing a measurement! This is a very well-understood quantum channel.

8.2 Generalized dephasing channels

A generalized dephasing channel leaves a *fixed* orthonormal basis invariant and it dephases off-diagonal elements with respect to that fixed basis. That is, we lose quantum coherences under the action of a generalized dephasing channel.

Construction: $\mathcal{H} = \mathbb{C}^d$ with orthonormal basis $\{|i\rangle\}_{i=1}^d$. Choose an environment $\mathcal{H}_E$ with $\dim\mathcal{H}_E := |E| \geq 2$, and let $\{\varphi_i\}_{i=1}^{|E|}$ be *some* set of pure states on E. This set is normalized but not orthogonal. Then, define the isometry

$$V : |i\rangle_A \mapsto |i\rangle_B \otimes |\varphi_i\rangle_E\,. \tag{8.2.1}$$

The Stinespring extension is

$$\begin{aligned}\mathcal{N}(\rho_A) &= \mathrm{tr}_E V\rho_A V^\dagger, &(8.2.2)\\ &= \sum_{i,j} |i\rangle\,\rho\,\langle i|\,|i\rangle\,\langle j|_B\,\mathrm{tr}\left(|\varphi_i\rangle\,\langle\varphi_j|\right), &(8.2.3)\\ &= \sum_{i,j} \langle\varphi_i|\varphi_j\rangle\,\langle i|\rho|j\rangle\,|i\rangle\,\langle i|_B\,. &(8.2.4)\end{aligned}$$

Let us confirm this channel acts as expect.

$$\begin{aligned}[\mathcal{N}(\rho)]_{kk} &= \langle\varphi_k|\varphi_k\rangle\,\langle k|\,\rho\,|k\rangle = \rho_{kk}, &(8.2.5)\\ [\mathcal{N}(\rho)]_{jk} &= \langle\varphi_j|\varphi_k\rangle\,\langle j|\,\rho\,|k\rangle\,, &(8.2.6)\end{aligned}$$

but $0 < \langle\varphi_j|\varphi_k\rangle \leq 1$. Thus, we see that the diagonals are preserved but the off-diagonals are dephased.

Higher-dimensional example: Let $d \geq 2$, and define two unitaries

$$\begin{aligned}&X\,|i\rangle = |i+1 \bmod d\rangle \qquad \text{(shift operator)} &(8.2.7)\\ &Z\,|j\rangle = \omega^j\,|j\rangle\,, \text{ where} \quad \omega = \exp\left(\frac{2\pi i}{d}\right) \quad \text{(clock operator)} &(8.2.8)\end{aligned}$$

This generalizes the Pauli operators: $X^d = Z^d = \mathbb{I}$. These are the generators of the Heisenberg-Weyl group:

$$\{\omega^j Z^k X^l : j,k,l \in [d]\}. \tag{8.2.9}$$

So, for $\rho \in B(\mathbb{C}^d)$ we maps like

$$\rho \mapsto (1-p)\rho + pX\rho X^\dagger, \tag{8.2.10}$$

$$\rho \mapsto (1-p)\rho + \frac{p}{3}Z\rho Z^\dagger + \frac{2p}{3}Z^2\rho(Z^\dagger)^2. \tag{8.2.11}$$

The generalized de-phasing channels have full classical capacity

$$C(\mathcal{N}) = \log d \quad (\text{ in general, } C(\mathcal{N}) \leq \log d) \tag{8.2.12}$$

Sketch of a proof: we know that for a generalized dephasing channel, there exists an orthonormal basis $\{|i\rangle\}_i$ such that

$$\mathcal{N}(|i\rangle\langle i|) = |i\rangle\langle i| \tag{8.2.13}$$

of classical signals/messages $x_i, \ldots, x_d$. We can use the encoding

$$x_i \mapsto |i\rangle\langle i| \tag{8.2.14}$$

where $\langle i|j\rangle = \delta_{ij}$. Thus, d messages can be sent perfectly. So $\log d$ bits of classical info can be sent through N. Bob can measure the output to retrieve the classical message.

8.2.1 Detour: Holevo information ($\mathcal{X}$-quantity)

Recall: the von Neumann entropy is defined as

$$S(\rho) = -\mathrm{tr}\rho\log\rho. \tag{8.2.15}$$

When one has the spectral decomposition of $\rho = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$, the von Neumann entropy is just the Shannon entropy for the eigenvalue distribution

$$S(\rho) = -\sum_i \lambda_i \log \lambda_i. \tag{8.2.16}$$

If we then have an ensemble of quantum states $E = \{p_x, \rho_x\}$, we have

$$\mathcal{X}(E, \mathcal{N}) = S(\sum_x p_x \mathcal{N}(\rho_x)) - \sum_x p_x S(\mathcal{N}(\rho_x)) \tag{8.2.17}$$

The Holevo information is then defined as

$$\boxed{\text{Holevo Information, } \mathcal{X}(\mathcal{N}) = \max_E \mathcal{X}(E, \mathcal{N})} \tag{8.2.18}$$

A fundamental result due to Holevo, Schumacher, and Westmoreland is

$$\boxed{C(\mathcal{N}) \geq \mathcal{X}(\mathcal{N})} \tag{8.2.19}$$

If $\mathcal{N}$ is a generalized dephasing channel, there exists an orthonormal basis $\{|i\rangle\}$ such that $\mathcal{N}(|i\rangle\langle i|) = |i\rangle\langle i|$ for all i. Let our ensemble be

$$\rho_i = |i\rangle\langle i| \tag{8.2.20}$$

$$p_i = \frac{1}{d}, \tag{8.2.21}$$

which implies

$$\bar{\rho} = \sum_i p_i \rho_i = \frac{1}{d}\mathbb{I} \tag{8.2.22}$$

Then, $E_n = \{\frac{1}{d}, \rho_i\}$, so the Holevo quantity is

$$\mathcal{X}(E_n, \mathcal{N}) = S(\sum_i p_i \mathcal{N}(\rho_i)) - \sum_i p_i S(\mathcal{N}(\rho_i)) \tag{8.2.23}$$

$$= S(\sum_i p_i \rho_i) - \sum_i p_i S(\rho_i) \tag{8.2.24}$$

$$= \log d - 0 \tag{8.2.25}$$

$$= \log d \tag{8.2.26}$$

Thus, we have the following chain of inequalities

$$\log d \leq \mathcal{X}(\mathcal{N}) \leq C(\mathcal{N}) \leq \log d \implies \boxed{C(\mathcal{N}) = \log d} \tag{8.2.27}$$

as desired.

8.2.2 Some comments on last lecture

- The rank of the Kraus operators is not unitarily invariant. We can see this by examining the 50-50 dephasing channel.

$$\rho \mapsto \frac{1}{2}\rho + \frac{1}{2}Z\rho Z \implies K_0 = \frac{1}{\sqrt{2}}\mathbb{I}, \quad K_1 = \frac{1}{\sqrt{2}}Z \tag{8.2.28}$$

  We can apply a Hadamard transform to the operators to obtain new Kraus operators

$$L_i = \sum_j H_{ij} K_j, \tag{8.2.29}$$

$$\implies L_0 = |0\rangle\langle 0|, L_1 = |1\rangle\langle 1|, \tag{8.2.30}$$

  which are rank-1 operators now when the previous Kraus operators were full rank.

- Every unital qubit channel is unitarily equivalent to a Pauli channel. That is, if $\mathcal{N} : B(\mathbb{C}^2) \to B(\mathbb{C}^2)$ is a unital channel, then there are unitaries U, V such that

$$M(\rho) = U\mathcal{N}(V\rho V^\dagger)U^\dagger \tag{8.2.31}$$

  is Pauli.

- mixed unitary channels

$$\rho \mapsto \sum_i p_i U_i \rho U_i^\dagger, \quad U_i \text{ unitary} \tag{8.2.32}$$

$$\mathbb{I} \mapsto \sum_i p_i U_i U_i^\dagger = \mathbb{I} \quad \text{unital!} \tag{8.2.33}$$

Every unital qubit channel is mixed unitary. Watrous' book provides a nice example of a unital channel that is *not* mixed unitary:

$$X \in B(\mathbb{C}^3), \quad \rho \mapsto \frac{1}{2}\mathrm{tr}(X)\mathbb{I} - \frac{1}{2}X^T \tag{8.2.34}$$

8.3 Entanglement-breaking channels

Reminder: A bipartite ρ_{AB} is called separable if it lies in the convex hull of product states. That is,

$$\rho_{AB} \in \mathrm{conv}\{\omega_A \otimes \sigma_B : \omega_{A(B)} \text{ state on } \mathcal{H}_{A(B)}\} \tag{8.3.1}$$

Explicitly,

$$\rho_{AB} = \sum_i p_i \omega_A^i \otimes \sigma_B^i \tag{8.3.2}$$

Definition 8.3.1. *A channel $\mathcal{N} : A \to B$ is entanglement-breaking if*

$$(\mathbb{I}_R \otimes \mathcal{N})(\rho_{RA}) \tag{8.3.3}$$

is separable for any ρ_{RA}.

Proposition 8.3.1. *The following are all equivalent*

- *$\mathcal{N} : A \to B$ is entanglement breaking*
- *$\tau_{AB}^{\mathcal{N}} = (\mathbb{I}_A \otimes \mathcal{N})(\gamma_{AA'})$ is separable*
- *$\mathcal{N}$ has a Kraus representation with rank-1 Kraus operators*
- *$\mathcal{N}$ is a measure-and-prepare channel: there exists POVM $E = \{E_i\}_i$ and states $\{\sigma_i\}_i$ such that*

$$\mathcal{N}(\rho) = \sum_i tr(\rho E_i)\sigma_i \tag{8.3.4}$$

where we recall that a POVM must satisfy $E_i \geq 0, \sum_i E_i = \mathbb{I}$.

Proof. (a $\implies$ b): $(\mathbb{I}_R \otimes \mathcal{N})(\rho_{RA})$ is separable for all ρ_{RA}. In particular, for γ_{RA}, $(|\gamma\rangle_{RA}) = \sum_i |i\rangle_R |i\rangle_A)$.

(b $\implies$ c): $\tau^{\mathcal{N}}_{AB}$ is separable: there exist pure states $\psi_i = |\psi_i\rangle\langle\psi_i|_A$ and $\varphi_i = |\varphi_i\rangle\langle\varphi_i|_B$ such that $\frac{1}{d}\tau^{\mathcal{N}}_{AB} = \sum_i p_i \psi_i \otimes \varphi_i$, where $d = |A|$. Then, set $K_i = \sqrt{p_i d}\,|\varphi_i\rangle_B \langle\bar{\psi}_i|_A$. We can then check the action of these operators

$$
\begin{aligned}
\frac{1}{d}\sum_i (\mathbb{I}\otimes K_i)(\gamma_{AA'})(\mathbb{I}_A \otimes K_i)^\dagger &= \frac{1}{d}\sum_{i,j,k} |j\rangle\langle k|_A \otimes K_i |j\rangle\langle k|_{A'} K_i^\dagger && (8.3.5)\\
&= \sum_{i,j,k} dp_i |j\rangle\langle k|_A \otimes \langle\bar{\psi}_i|\,|j\rangle\langle k|\,|\bar{\psi}_i\rangle\,|\varphi_i\rangle\langle\varphi_i|_B && (8.3.6)\\
&= \sum_{i,j,k} p_i |j\rangle\langle k|_A \otimes \langle\bar{\psi}_i|\,|j\rangle\langle k|\,|\bar{\psi}_i\rangle\,|\varphi_i\rangle\langle\varphi_i|_B && (8.3.7)\\
&= \sum_i p_i \left(\sum_{j,k} \langle k|\bar{\psi}_i|j\rangle\,|j\rangle\langle k|\right) \otimes |\varphi_i\rangle\langle\varphi_i|_B && (8.3.8)\\
&= \sum_i p_i(\psi_i) \otimes \varphi_i && (8.3.9)\\
&= \frac{1}{d}\tau^{\mathcal{N}}_{AB} && (8.3.10)
\end{aligned}
$$

$$
\begin{aligned}
\sum_i K_i^\dagger K_i &= \mathbb{I} && (8.3.11)\\
&= d\sum_i p_i |\bar{\psi}_i\rangle\langle\varphi_i|\varphi_i\rangle\langle\bar{\psi}_i| && (8.3.12)\\
&= d\sum_i p_i\bar{\psi}_i && (8.3.13)\\
&= d(\sum_i p_i\bar{\psi}_i) && (8.3.14)\\
&= d\frac{1}{d}\bar{\mathbb{I}}_A && (8.3.15)\\
&= \mathbb{I}_A && (8.3.16)
\end{aligned}
$$

(c $\implies$ d): rank(K_i)=1: $K_i = |\mathcal{X}_i\rangle_B\langle\omega_i|_A$ for some vectors $|\mathcal{X}_i\rangle_B$, $|\omega_i\rangle_A$, $\langle\mathcal{X}_i|\mathcal{X}_i\rangle = 1$.

$$
\begin{aligned}
\mathcal{N}(\rho) &= \sum_i K_i \rho K_i^\dagger && (8.3.17)\\
&= \sum_i \langle\omega_i|\rho|\omega_i\rangle\,|\mathcal{X}_i\rangle\langle\mathcal{X}_i|_B && (8.3.18)
\end{aligned}
$$

POVM: $\omega = \{\omega_i\}$, states: $\mathcal{X}_i$

$$\mathbb{I} = \sum_i K_i^\dagger K_i \tag{8.3.19}$$
$$= \sum_i |\omega_i\rangle \langle \mathcal{X}_i|\mathcal{X}_i\rangle \langle\omega_i| \tag{8.3.20}$$
$$= \sum_i |\omega_i\rangle \langle\omega_i| \tag{8.3.21}$$

(d $\implies$ a): Let ρ_{RA} be arbitrary:

$$(\mathbb{I}_R \otimes \mathcal{N})(\rho_{RA}) = \sum_i \mathrm{tr}_B\left[(\mathbb{I}_A \otimes E_i)\rho_{RA}\right] \otimes \sigma_i \tag{8.3.22}$$
$$= \sum_i \mathrm{tr}_B\left[(\mathbb{I}_R \otimes \sqrt{E_i})\rho_{RA}(\mathbb{I}_R \otimes \sqrt{E_i})\right] \otimes \sigma_i \tag{8.3.23}$$
$$= \sum_i p_i\omega_i \otimes \sigma_i \tag{8.3.24}$$

where $p_i = \mathrm{tr}(E_i\rho_{RA}), \omega_i = \frac{1}{p_i}\mathrm{tr}_B\left[(\mathbb{I}_R \otimes \sqrt{E_i})\rho_{RA}(\mathbb{I}_A \otimes \sqrt{E_i})\right] \geq 0$. □

We note that there are *some* channel capacities of entanglement-breaking channels that are understood.

- Quantum information transmission is equivalent to generating entanglement. Because entanglement-breaking channels cannot do that, their quantum capacity is zero.
- $C(\mathcal{N}) \geq \mathcal{N}, C(\mathcal{N}) = \sup_{n\in\mathbb{N}}\frac{1}{n}\mathcal{X}(\mathcal{N}^{\otimes n})$. Entanglement-breaking channels destroy entanglement between different inputs. This implies that

$$C(\mathcal{N}) = \mathcal{X}(\mathcal{N}) \tag{8.3.25}$$

BUT: $\mathcal{X}(\mathcal{N})$ is NP-hard to compute, so this relationship doesn't actually get us much.

8.4 PPT-channels

Checking separability is NP-hard. So, is there some easier criterion? The standard relaxed criterion is following.

Definition 8.4.1. *Peres-Horodecki criterion: if ρ_{AB} is separable if ρ_{AB} has a positive partial transpose with respect to either party.*

Let us see why this is a reasonable criterion for separability. If ρ_{AB} is separable,

$$\rho_{AB} = \sum_i p_i \omega_A^i \otimes \sigma_B^i \tag{8.4.1}$$

$$\implies \rho_{AB}^{T_B} = \sum_i p_i \omega_A^i \otimes (\sigma_B^i)^T \geq 0 \tag{8.4.2}$$

- if ρ_{AB} is NPT $\implies$ $\rho_{AB} \notin$SEP.
- if $|A| \cdot |B| \leq 6$, then $\rho_{AB} \in$ SEP $\Leftrightarrow \rho_{AB} \in$ PPT.

Definition 8.4.2. *A channel $\mathcal{N} : A \to B$ is called PPT if $(\mathcal{I}_R \otimes \mathcal{N})(\rho_{RA})$ is PPT for all ρ_{RA}.*

Proposition 8.4.1. *The following are all equivalent*

- $\mathcal{N} : A \to B$ *is PPT*
- $\tau_{AB}^{\mathcal{N}}$ *is PPT*
- $\vartheta \circ \mathcal{N}$ *is CP (Recall that when $\vartheta : X \mapsto X^T : (\mathbb{I}_R \otimes \vartheta)(\gamma) = \mathbb{F} \not\geq 0$)*

Proof. (a $\Rightarrow$ b): True by definition. We know

$$(\mathbb{I}_A \otimes \mathcal{N})(\rho_{AA'}) \tag{8.4.3}$$

is PPT, in particular for $\rho = \gamma_{AA'}$.
(b $\Rightarrow$ c):

$$(\mathbb{I}_A \otimes \vartheta \circ \mathcal{N})(\gamma) = (\mathbb{I} \otimes \vartheta)(\mathbb{I} \otimes \mathcal{N})(\gamma_{AA'}) \tag{8.4.4}$$

$$= (\mathbb{I}_A \otimes \vartheta)(\tau_{AB}^{\mathcal{N}}) \tag{8.4.5}$$

$$= (\tau_{AB}^{\mathcal{N}})^{T_B} \geq 0 \tag{8.4.6}$$

(c $\Rightarrow$ a) $\vartheta \circ \mathcal{N}$ is CP:

$$(\mathbb{I}_R \otimes \vartheta \circ \mathcal{N})(\rho_{RA}) \geq 0 \quad \forall \rho_{RA} \geq 0 \tag{8.4.7}$$

This implies $\mathcal{N}$ is PPT. □

Remarks: In general, $\vartheta \circ \mathcal{N}$ is *not* CP. However, for every CP map $\mathcal{N}$, the map $\vartheta \circ \mathcal{N} \circ \vartheta$ is CP (see exercises).

What about the capacities of PPT channels?

- Horodeckis: PPT states are undistillable. Entanglement distillation: given iid copies of a state ρ_{AB}, the goal is to convert these copies into a smaller number of maximally entangled states. If there is an entanglement distillation protocol (LOCC) such that

1. error of the protocol tends to 0 as $n \Rightarrow \infty$
2. rate c:= $\frac{1}{n} \log m_n \to c > 0$ as n goes to infinity

then ρ_{AB} is *distillable*. We conclude that the quantum capacity for PPT channels is zero. This result even holds if two-way classical communication is allowed!

We note that the set of protocols allowing only one-way communication is a strict subset of the set of protocols allowing two-way communication. Even still, the Horodecki's showed that the quantum capacity of all PPT channels is zero in both cases. Also note that the classical capacity is generally unknown.

8.5 Anti-degradable channels

Let $\mathcal{N}$ be a quantum channel from A to B with an isometry $V : \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E$ such that $\mathcal{N}(\rho) = \mathrm{tr}_E V \rho V^\dagger$. Recall that the completmentary channel is $\mathcal{N}^c(\rho) = \mathrm{tr}_B V \rho V^\dagger$.

Definition 8.5.1. $\mathcal{N}$ *as above is called antidegradable if there exists a channel* $\mathcal{A} : E \to B$ *such that*

$$\mathcal{N} = \mathcal{A} \circ \mathcal{N}^c \tag{8.5.1}$$

Intuition: Eve (environment) can locally obtain Bob's output via the channel $\mathcal{A}$. Antidegradable channels cannot transmit quantum information and thus have zero quantum capacity, $Q(\mathcal{N}) = 0$.

A "proof" of this is as follows. Assume that this channel has non-zero quantum capacity. This means that Alice can faithfully send qubits to Bob at a positive rate. But there is a protocol based on the channel $\mathcal{A}$ that lets Eve implement the same protocol that Alice and Bob use. This violates no-cloning and is thus prohibited.

Examples of anti-degradable channels:

1. erasure channel $\mathcal{E}_p : \rho \mapsto (1-p)\rho + p\mathrm{tr}(\rho) |e\rangle \langle e|$ for $p \geq \frac{1}{2}$.

   Let $\mathcal{H}_1 = \mathbb{C}^2$ be the input space. Let the $\mathcal{H}_2 = \mathbb{C}$ be an erasure flag. Let $\mathcal{E}_p : B(\mathcal{H}_1) \to B(\mathcal{H}_1 \oplus \mathcal{H}_2)$. That is, an input state maps as

$$\rho \mapsto (1-p)\tilde{\rho} + p\mathrm{tr}(\rho) |e\rangle \langle e| \tag{8.5.2}$$

So, how do we embed our input state in the larger space? We use a pretty trivial embedding that is represented as:

$$\hat{\rho} = \begin{pmatrix} \rho_{00} & \rho_{01} & 0 \\ \rho_{10} & \rho_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{8.5.3}$$

and the erasure flag is simply

$$|e\rangle \langle e| = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{8.5.4}$$

Note that the complementary channel of the erasure channel has the same form but with the probabilities flipped:

$$\mathcal{E}_p^c : B(\mathcal{H}_1) \to B(\mathcal{H}_1 \oplus \mathcal{H}_2), \rho \mapsto p\tilde{\rho} + (1-p)\mathrm{tr}(\rho) |e\rangle \langle e| . \tag{8.5.5}$$

Now consider $p \geq \frac{1}{2}$ and define $q = \frac{2p-1}{p}$. the idea is that we erase $\tilde{\rho}$ with probability q and do nothing with $|e\rangle \langle e|$.

$$\begin{aligned} \mathcal{E}_q(p\rho) &= p(1-q)\tilde{\rho} + pq\mathrm{tr}(\rho) |e\rangle \langle e| & (8.5.6) \\ &= (1-p)\tilde{\rho} + (2p-1) |e\rangle \langle e| & (8.5.7) \\ &= (1-p) |e\rangle \langle e| & (8.5.8) \end{aligned}$$

almost certainly messed up p's and q's here.

We need to extend the action of $\mathcal{E}_q$ to $B(\mathcal{H}_1 \otimes \mathcal{H}_2)$. The solution is to define $\mathcal{A}$ in the Kraus representation. The Kraus operators are

$$K_0 = \sqrt{1-q} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{8.5.9}$$

$$K_1 = \sqrt{q} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{8.5.10}$$

$$K_2 = \sqrt{q} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{8.5.11}$$

$$K_3 = \sqrt{q} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{8.5.12}$$

where the first three operators correspond to erasure on $B(\mathcal{H}_1 \oplus 0)$ and the last to doing nothing on $B(0 \oplus \mathcal{H}_2)$. This definition ensures that $\mathcal{E}_p = \mathcal{A} \circ \mathcal{E}_p^c$ for $p \geq \frac{1}{2}$. We have $B(\mathcal{H}_1 \oplus 0), B(0 \oplus \mathcal{H}_2) : \mathcal{A}_p = \mathcal{E}_p \oplus \mathbb{I}$.

For $\rho \in B(\mathcal{H}_1)$, $\sigma = (1-\lambda)\tilde{\rho} + \lambda |e\rangle \langle e|$. This has the block matrix:

$$\begin{pmatrix} (1-\lambda)\rho & 0 \\ 0 & \lambda \end{pmatrix} \tag{8.5.13}$$

So

$$(\mathcal{E}_p \oplus \mathbb{I})(\sigma) = \begin{pmatrix} \mathcal{E}_p((1-\lambda)\rho) & 0 \\ 0 & \mathbb{I}(\lambda) \end{pmatrix} \tag{8.5.14}$$

Then for $\omega \in B(\mathcal{H}_1 \oplus \mathcal{H}_2)$ we have

$$\omega = \begin{pmatrix} \omega_{\mathcal{H}_1} & * \\ * & \omega_{\mathcal{H}_2} \end{pmatrix} \tag{8.5.15}$$

which has off-diagonal elements. We can get rid of these by measuring with respect to $\mathcal{H}_1 \oplus \mathcal{H}_2 = \mathcal{H}$. Let $P_2 = |e\rangle \langle e|, P_1 = \mathbb{I} - |e\rangle \langle e|$. Then $P_1 \omega P_1 + P_2 \omega P_2$ is diagonal. Finally, $\mathcal{A}_M = P_1 \cdot P_1 + P_2 \cdot P_2$ which implies $\mathcal{A} - \mathcal{A}_p \circ \mathcal{A}_M$ where $\mathcal{A}_p = \mathcal{E}_p \oplus \mathbb{I}$.

2. amplitude damping channel $\mathcal{A}_\gamma$ for $\gamma \geq \frac{1}{2}$.

3. depolarizing channel $\mathcal{D}_p$ for $p \geq \frac{1}{4}$

Definition 8.5.2. *Let $\mathcal{N} : A \to B$ be a quantum channel and let G be a group with unitary representations U_g on $\mathcal{H}_A$ and V_g on $\mathcal{H}_B$. Then $\mathcal{N}$ is called covariant with respect to (G, U_g, V_g) if*

$$V_g \mathcal{N}(\cdot) V_g^\dagger = \mathcal{N}(U_g \cdot U_g^\dagger) \tag{8.5.16}$$

for all $g \in G$.

Representation theory basics:

- (φ, V) is a representation of G: $\varphi : G \to GL(V), \varphi(gh) = \varphi(g)\varphi(h)$. Subspace $W \leq V$ is called G-invariant, if $\varphi(g)w \in W \quad \forall w \in W, \forall g \in G$.
- $\{0\}, V$ are always G-invariant
- (φ, V) is called irreducible if $\{0\}, V$ are the only G-inv subspaces.
- Schur's lemma: $(\varphi, V), (\psi, W)$ representations of G. G-linear map $f : f \circ \varphi(g) = \psi(g) \circ f$. If φ, ψ are irreducible representations, then either $V \ncong W$ and $f = 0$ or $V \cong W$ and $f = \lambda \mathbb{I}_{V \to W}$ for some $\lambda \in \mathbb{C}$.

Some examples:

- Pauli channels: $\rho \mapsto p_0\rho + p_1 X\rho X + p_2 Y\rho Y + p_3 Z\rho Z$

  Covariance group: Pauli group, $P = \{\pm 1, \pm i\} \bigcup \{\mathbb{I}, X, Y, Z\}$

  For $\theta_1, \vartheta_2 \in P$: $\vartheta_1\theta_2 \cdot \theta_2^\dagger \vartheta_1^\dagger = \vartheta_2\theta_1 \cdot \theta_1^\dagger \vartheta_2^\dagger$

- depolarizing channel: $\rho \mapsto (1-p)\rho + \frac{p}{3}(X\rho X + Y\rho Y + Z\rho Z)$. The covariance group is $\mathcal{U}(2)$. This is easy to see in the "q-representation": $\rho \mapsto (1-q)\rho + q\mathrm{tr}\rho\frac{1}{2}\mathbb{I}$

$$\mathcal{D}_q(U\rho U^\dagger) = (1-q)U\rho U^\dagger + q\mathrm{tr}(U\rho U^\dagger)\frac{1}{2}\mathbb{I} = U\mathcal{D}_p(\rho)U^\dagger \tag{8.5.17}$$

  In $d \geq$ dimensions: $\rho \mapsto (1-q)\rho + q\mathrm{tr}\rho\frac{1}{d}\mathbb{I}$. The covariance group is $\mathcal{U}(d)$.

- Amplitude damping channel $A_\gamma = \{K_0, K_1\}$ as normal. Covariance group: $\{\mathbb{I}, Z\} \cong \mathbb{Z}_2$.

- Erasure channel $\mathcal{E}_p(\rho) = (1-p)\rho + p\mathrm{tr}\rho\,|e\rangle\langle e|$. The covariance group is again $\mathcal{U}(2)$. Similarly, in $d \geq 2$ dimensions, the d-dimensional erasure channel with covariance group $\mathcal{U}(d)$.

Proposition 8.5.1. *A channel $\mathcal{N} : A \to B$ is (G, U_g, V_g)-covariant iff $\tau^{\mathcal{N}_{AB}} = (\bar{U}_g \otimes V_g)\tau_{AB}^{\mathcal{N}}(\bar{U}_g \otimes V_g)^\dagger$ for all $g \in G$.*

Proof. ($\Rightarrow$): $V_g\mathcal{N}(\cdot)V_g^\dagger = \mathcal{N}(U_g \cdot U_g^\dagger)$ for all $g \in G$. This is equivalent to $\mathcal{N}(\cdot) = V_g^\dagger\mathcal{N}(U_g \cdot U_g^\dagger)V_g$ for all $g \in G$.

$$\begin{aligned}
\tau_{AB}^{\mathcal{N}} &= (\mathbb{I} \otimes \mathcal{N})(\gamma) && (8.5.18)\\
&= (\mathbb{I}_1 \otimes V_g^\dagger)(\mathbb{I} \otimes \mathcal{N})\left((\mathbb{I}_1 \otimes U_g)\gamma(\mathbb{I} \otimes U_g^\dagger)\right)(\mathbb{I}_1 \otimes V_g) && (8.5.19)\\
&= (\mathbb{I} \otimes V_g)(\mathbb{I} \otimes \mathcal{N})\left((U_g^T \otimes \mathbb{I}_2)\gamma(U_g^T \otimes \mathbb{I}_2)^\dagger\right)(\mathbb{I} \otimes V_g) && (8.5.20)\\
&= (U_g^T \otimes V_g^\dagger)(\mathbb{I} \otimes \mathcal{N})(\gamma)(\bar{U}_g \otimes V_g) && (8.5.21)\\
\Longrightarrow \tau_{AB}^{\mathcal{N}} &= (U_g^T \otimes V_g^\dagger)\tau_{AB}^{\mathcal{N}}(\bar{U}_g \otimes V_g) \quad \text{for all} \quad g \in G && (8.5.22)
\end{aligned}$$

($\Leftarrow$): $\tau_{AB}^{\mathcal{N}} = (\bar{U}_g \otimes V_g)\tau_{AB}^{\mathcal{N}}(\bar{U}_g \otimes V_g)^\dagger$ for all $g \in G$. This implies $(U_g^T \otimes \mathbb{I})\tau_{AB}^{\mathcal{N}}(U_g^T \otimes \mathbb{I})^\dagger = (\mathbb{I} \otimes V_g)\tau_{AB}^{\mathcal{N}}(\mathbb{I} \otimes V_g)^\dagger$.

Recall the choi isomorphism: $\mathcal{N}(X) = \mathrm{tr}_1\tau_{AB}^{\mathcal{N}}(X^T \otimes \mathbb{I})$.

$$\mathcal{N}(U_g X U_g^\dagger) = \mathrm{tr}_1\left[\tau_{AB}^{\mathcal{N}}((U_g X U_g^\dagger)^T \otimes \mathbb{I})\right] \tag{8.5.23}$$

$$= \mathrm{tr}_1\left[(U_g^T \otimes \mathbb{I})\tau_{AB}(U_g^T \otimes \mathbb{I})^\dagger (X^T \otimes \mathbb{I})\right] \tag{8.5.24}$$

$$= \mathrm{tr}_1\left[(\mathbb{I} \otimes V_g)\tau_{AB}(\mathbb{I} \otimes V_g)^\dagger (X^T \otimes \mathbb{I})\right] \tag{8.5.25}$$

$$= V_g \mathrm{tr}_1\left[\tau^{\mathcal{N}_{AB}}(X^T \otimes \mathbb{I})\right] V_g^\dagger \tag{8.5.26}$$

$$= V_g \mathcal{N}(X) V_g^\dagger \tag{8.5.27}$$

□

Problem: with the above proposition is that it is basis-dependent (via the Choi operator).

Solution: is that there is also a basis independent version based on the Jamiolkowski operator $J_{AB}^{\mathcal{N}} = (\mathbb{I} \otimes \mathcal{N})(\mathbb{F}_{AA'})$ where $\mathbb{F}_{AB} = |\gamma\rangle\langle\gamma|^{T_A}$. In terms of the Jamialkowski operator, (G, U_g, V_g)-covariance of $\mathbb{N}$ is equivalent to $(U_g \otimes V_g)J_{AB}^{\mathcal{N}}(U_g \otimes V_g)^\dagger = J_{AB}^{\mathcal{N}}$ for all $g \in G$.

We note that a d-dimensional depolarizing channel; $\rho \mapsto (1-q)\rho + q\mathrm{tr}\rho\frac{1}{d}\mathbb{I}_d$ has $\mathcal{U}(d)$ as its covariance group.

Proposition 8.5.2. *Let $\mathcal{N} : A \to B$ be a channel with input and output spaces of equal dimension d. If $U\mathcal{N}(\cdot)U^\dagger = \mathcal{N}(U \cdot U^\dagger)$ for all $U \in mathcalU(d)$, then*

$$\mathcal{N} = (1-q)\cdot + qtr(\cdot)\frac{1}{d}\mathbb{I}_d \tag{8.5.28}$$

with $q = (1-f)/(1-d)$ where $f = \langle\gamma|\tau^{\mathcal{N}}|\gamma\rangle$.

Note Zanqiu Shen University of Toronto

Chapter 9

Introduction to Quantum Optics

9.1 Introduction

- classicaal: classical atom and light
- semiclassical: quantized atom and classical light
- quantum mechanical: quantized atom and light

Light-Atom Interaction Hamiltonian

- classical dipole in eletric field: dipole moment $\overrightarrow{d} = q\overrightarrow{r}$, $U_I = -\overrightarrow{d} \cdot \overrightarrow{E}$. We have

$$\hat{H}_I = -\hat{d} \cdot \overrightarrow{E}(\overrightarrow{v_0}, t), \tag{9.1.1}$$

  where $\hat{d} = q\hat{v}$ is the dipole operator.

- induced atomic dipole

9.2 Light Atom Quantum Evolution

Time Evolution We have the Schrodinger equation (both sides) as

$$i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = (\hat{H}_0 + \hat{H}_I(t))|\Psi(t)\rangle, \tag{9.2.1}$$

where the general ansatz (assumption) is

$$|\Psi(t)\rangle = \sum_n c_n(t)e^{-iE_nt/\hbar}|n\rangle, \tag{9.2.2}$$

and

$$\hat{H}_0|n\rangle = E_n|n\rangle \tag{9.2.3}$$

is the atomic eigenstates. Inserting $|\Psi(t)\rangle$ and $\hat{H}_0|n\rangle$ into Schrodinger equation, we get

$$i\hbar\sum_n\left\{\dot{c}_n e^{-iE_nt/\hbar}|n\rangle - \frac{iE_n}{\hbar}c_n e^{-iE_nt/\hbar}|n\rangle\right\} = \sum_n\left\{c_n e^{-iE_nt/\hbar}|n\rangle + c_n e^{-iE_nt/\hbar}\hat{H}_I|n\rangle\right\} \tag{9.2.4}$$

$$\Longrightarrow i\hbar\sum_n \dot{c_n}e^{-iE_nt/\hbar}|n\rangle = \sum_n c_n e^{-iE_nt/\hbar}\hat{H}_I|n\rangle \tag{9.2.5}$$

$$\Longrightarrow i\hbar\dot{c_n}e^{-iE_kt/\hbar}| = \sum_n c_n(t)e^{-iE_nt/\hbar}\langle k\hat{H}_I(t)|n\rangle \tag{9.2.6}$$

$$\Longrightarrow i\hbar\dot{c}_k = \sum_n c_n(t)e^{-iE_{n,k}t/\hbar}\langle k|\hat{H}_I(t)|n\rangle, \tag{9.2.7}$$

where we use

$$\langle k|n\rangle = \delta_{kn}, \tag{9.2.8}$$

$$E_{n,k} = E_n - E_k, \tag{9.2.9}$$

$$\omega_{nk} = (E_n - E_k)/\hbar. \tag{9.2.10}$$

and $\langle k|\hat{H}_I(t)|n\rangle$ is the matrix element.

9.3 Time Dependent Perturbation Theory

Recall the time evolution:

$$i\hbar\dot{c}_k = \sum_n c_n(t)e^{-i\omega_{nk}t}\langle k|\hat{H}_I(t)|n\rangle, \tag{9.3.1}$$

and

$$\omega_{nk} = (E_n - E_k)/\hbar. \tag{9.3.2}$$

Consider the Simplification (Perturbation Theory)

- System only in state $|1\rangle$ at $t = 0 \Longrightarrow c_1|0\rangle = 1$ (only the ground state $|1\rangle$),
- Perturbative treatment of interaction term: weak perturbation $\forall|c_k(t)|^2 << 1$.

We then have

$$i\hbar\dot{c}_k = e^{i\omega_{1k}t}\langle k|\hat{H}_I(t)|1\rangle, \tag{9.3.3}$$

with $c_k(0) = 0$, we obtain:

$$c_k(t) = \frac{1}{i\hbar}\int_0^t e^{-i\omega_{1k}t}\langle k|\hat{H}_I(t')|1\rangle \mathrm{d}t'. \tag{9.3.4}$$

Example 9.3.1 (Sinusoidal perturbation)**.** *Define*

$$\hat{H}(t) = \hat{H}_I e^{-i\omega t}. \tag{9.3.5}$$

Given the figure in the video, we have

$$c_k(T) = \frac{1}{i\hbar}\int_0^T e^{i\Delta\omega t}\langle k|\hat{H}_I|1\rangle \mathrm{d}t \tag{9.3.6}$$

$$\implies \textit{Transition probability } P_{k1}(T) = |c_k(T)|^2 = \frac{1}{\hbar^2}|\langle k|\hat{H}_I|1\rangle|^2 Y(\Delta\omega, T), \tag{9.3.7}$$

with

$$Y(\Delta\omega, T) = \frac{\sin^2(\Delta\omega T/2)}{(\Delta\omega/2)^2} \tag{9.3.8}$$

$$\sim \mathrm{sinc}^2 x, \tag{9.3.9}$$

where $\Delta\omega = \omega - \omega_{1k}$ *is the detwining.*

Let's take a look at the sinc function $Y(\Delta\omega, T) = \mathrm{sinc}^2 x$. Transition for $\Delta\omega \leq \frac{2\pi}{T}$, we have $\Delta\omega \cdot T \leq 2\pi$, which implies

$$\Delta E \cdot T \leq h, \tag{9.3.10}$$

which is the time-frequency uncertainty. (The expression in the video seems wrong, so I make corrections abrove.) We have the following case

$$\frac{1}{2\pi T} Y(\Delta\omega, T) \overset{T\to\infty}{\to} \delta(\Delta\omega), \tag{9.3.11}$$

then we have

$$P_{k1}(T \to \infty) = \frac{2\pi}{\hbar^2}|\langle k|\hat{H}_I|i\rangle|^2 \delta(\Delta\omega) T. \tag{9.3.12}$$

Fermi's Golden Rule $|k\rangle$ Quasi continuum of final states. We have the transition probability

$$P_{k1} = \Gamma_{k1} T, \tag{9.3.13}$$

where

$$\Gamma_{k1} = \frac{2\pi}{\hbar}|\langle k|\hat{H}_I|1\rangle|^2 \rho(E_k = E_1 + \hbar\omega) \tag{9.3.14}$$

is called the Femi's Golden Rule,

$$|\langle k|\hat{H}_I|1\rangle|^2 \tag{9.3.15}$$

is the coupling strength $\propto E_0^2$ and $\propto I$,

$$\rho(E_k = E_1 + \hbar\omega) \tag{9.3.16}$$

is the density states which is number of availble final states to the system,

$$\Gamma_{k1} \hat{=} Transition\ Rate = \frac{\mathrm{d}P_{k1}}{\mathrm{d}T}, \tag{9.3.17}$$

and density states

$$\rho(E) = \frac{\mathrm{d}N}{\mathrm{d}E}, \tag{9.3.18}$$

where ΔN is the number of states in an energy interval ΔE around energy E_k and we let ΔE approaches 0.

9.4 Two Level Atom (TLA)

Given by the figure, in state $|1\rangle$, we have $E_1 = \hbar\omega_1$ and in state $|2\rangle$, we have $E_2 = \hbar\omega_2$ and $E_2 - E_1 = \hbar(\omega_2 - \omega_1) = \omega_{21}$. We have the Hamiltonian

$$\hat{H} = \hat{H}_0 - \hat{d} \cdot E(t), \tag{9.4.1}$$

where

$$E(t) = \varepsilon E_0 \cos(\omega t), \tag{9.4.2}$$

where ε is the polarization vector, E_0 is the field amplitude, and ω is the frequency of the light field.

Ansatz for Solving TLA We have

$$|\Psi(t)\rangle = c_1(t)e^{-i\omega_1 t}|1\rangle + c_2(t)e^{-i\omega_2 t}|2\rangle. \tag{9.4.3}$$

Time Evolution Amplitude We have

$$\dot{c_1}(t) = i\frac{d_{12}^{\varepsilon} E_0}{\hbar} e^{-\omega_{21}} \cos(\omega t) c_2(t) \tag{9.4.4}$$

$$\dot{c_2}(t) = i\frac{d_{12}^{\varepsilon} E_0}{\hbar} e^{+\omega_{21}} \cos(\omega t) c_1(t), \tag{9.4.5}$$

where

$$\begin{aligned} d_{12}^{\varepsilon} &= \langle 1|\hat{d} \cdot \varepsilon|2\rangle & (9.4.6)\\ &= \langle 1|\hat{d}|2\rangle \cdot \varepsilon & (9.4.7)\\ &= \langle 1|\hat{d}_x|2\rangle \cdot \varepsilon_x + \langle 1|\hat{d}_y|2\rangle \cdot \varepsilon_y + \langle 1|\hat{d}_z|2\rangle \cdot \varepsilon_z. & (9.4.8) \end{aligned}$$

is the Dipole Matrix Element, which is the atomic property and we assume it's real. We also define

$$\Omega_0 = \frac{d_{12}^{\varepsilon} E_0}{\hbar} \tag{9.4.9}$$

as the Rubi frequency.

Time Evolution Using Euler' form, we have

$$\dot{c_1}(t) = i\frac{\Omega_0}{2}e^{-\omega_{21}}(e^{i\omega t} + e^{-i\omega t})c_2(t) \tag{9.4.10}$$

$$\dot{c_2}(t) = i\frac{\Omega_0}{2}e^{+\omega_{21}}(e^{i\omega t} + e^{-i\omega t})c_1(t) \tag{9.4.11}$$

by

$$\cos\alpha = \frac{1}{2}(e^{i\alpha} + e^{-i\alpha}) \tag{9.4.12}$$

and

$$e^{i\alpha} = \cos\alpha + i\sin\alpha. \tag{9.4.13}$$

Rotating Wave Approximation We have

$$\dot{c_1}(t) = i\frac{\Omega_0}{2}(e^{+i(\omega-\omega_{21})t} + e^{-i(\omega+\omega_{21})t})c_2(t) \tag{9.4.14}$$

$$\dot{c_2}(t) = i\frac{\Omega_0}{2}(e^{-i(\omega-\omega_{21})t} + e^{+i(\omega+\omega_{21})t})c_1(t), \tag{9.4.15}$$

and we ignore the sum frequency term and get

$$\dot{c_1}(t) = i\frac{\Omega_0}{2}e^{+i(\omega-\omega_{21})t}c_2(t) \tag{9.4.16}$$

$$\dot{c_2}(t) = i\frac{\Omega_0}{2}e^{-i(\omega-\omega_{21})t}c_1(t), \tag{9.4.17}$$

which is a good approcimation for detwining $\delta = \omega - \omega_{21} \approx 0$. We introduce

$$\tilde{c_1}(t) = c_1(t)e^{-i\frac{\delta}{2}t} \tag{9.4.18}$$

$$\tilde{c_2}(t) = c_2(t)e^{+i\frac{\delta}{2}t}. \tag{9.4.19}$$

$$\tag{9.4.20}$$

Ansatz Wavefunctions for TLA Whole time evolution in state amplitudes

$$|\Psi(t)\rangle = c_1'(t)|1\rangle + c_2'(t)|2\rangle. \tag{9.4.21}$$

Time evolution when field is off

$$|\Psi(t)\rangle = c_1'(0)e^{-i\omega_1 t}|1\rangle + c_2'(0)e^{-i\omega_2}|2\rangle. \tag{9.4.22}$$

However, this is boring. We chose different ansatz as

$$|\Psi(t)\rangle = c_1(t)e^{-i\omega_1 t}|1\rangle + c_2(t)e^{-i\omega_2 t}|2\rangle \tag{9.4.23}$$

$$\Longleftrightarrow |\Psi(t)\rangle = c_1(t)|1\rangle + c_2(t)e^{-i\omega_{21} t}|2\rangle, \tag{9.4.24}$$

where $c_1(t)$ and $c_2(t)$ capture time evolution on top of eigenstate evolution! We now have

$$|\Psi(t)\rangle = c_1(t)|1\rangle + c_2(t)e^{-i\omega_{21} t}|2\rangle, \tag{9.4.25}$$

which is called the rotating frame of atom. We also have Rotating frame of light field as

$$|\Psi(t)\rangle = \tilde{c_1}(t)|1\rangle + \tilde{c_2}(t)e^{-i\omega t}|2\rangle, \tag{9.4.26}$$

where ω is the light frequency, $\tilde{c_1}$ and $\tilde{c_2}$ describe time evolution on top of fast light field oscilation.

Solving the TLA Dynamics We have the following equations:

$$\frac{d}{dt}\begin{pmatrix}\tilde{c_1}(t)\\ \tilde{c_2}(t)\end{pmatrix} = \frac{i}{2}\begin{pmatrix}-\delta & \Omega_0\\ \Omega_0 & +\delta\end{pmatrix}\begin{pmatrix}\tilde{c_1}(t)\\ \tilde{c_2}(t)\end{pmatrix}. \tag{9.4.27}$$

Considering the simplest case $\delta = 0$

$$\frac{d}{dt}\tilde{c_1}(t) = \frac{i}{2}\Omega_0\tilde{c_2}(t) \tag{9.4.28}$$

$$\frac{d}{dt}\tilde{c_2}(t) = \frac{i}{2}\Omega_0\tilde{c_1}(t). \tag{9.4.29}$$

Take time dirivative of the first equation, then we have

$$\ddot{\tilde{c_1}}(t) = -\frac{\Omega_0^2}{4}\tilde{c_1}(t), \tag{9.4.30}$$

the solutions of which are

$$\tilde{c_1}(t) = \cos(\Omega_0 t/2) \tag{9.4.31}$$

$$\tilde{c_2}(t) = i\sin(\Omega_0 t/2) \tag{9.4.32}$$

for $\tilde{c_1}(0) = 1$ and $\tilde{c_2}(0) = 0$. Also we can obtain the excited state probability as

$$P_2(t) = |c_2(t)|^2 \tag{9.4.33}$$

$$= |\tilde{c_2}(t)|^2. \tag{9.4.34}$$

Rabi Oscillations (Resonant Case) Nonlinear Response can be seen from the figure.

General Rabi Oscillations (with detuning) Given the figurem.

$$|\tilde{c_2}(t)|^2 = \frac{\Omega_0^2}{\Omega}\sin^2\left(\frac{1}{2}\Omega t\right) \tag{9.4.35}$$

$$= \frac{\Omega_0^2}{2\Omega^2}\left\{1-\cos(\Omega t)\right\}, \tag{9.4.36}$$

where $\Omega = \sqrt{\Omega_0^2+\delta^2}$ is the effective Rabi frequency.

Interesting Special Cases a) Pi-Puls $\Omega_0\tau = \pi$: swap population

$$|1\rangle \to i|2\rangle \tag{9.4.37}$$

$$|2\rangle \to i|1\rangle. \tag{9.4.38}$$

b) 2Pi-Puls $\Omega_0\tau = 2\pi$: flip the sign

c) Pi/2-Puls $\Omega_0\tau = \pi/2$: superposition state

9.5 Oscillating Dipoles

Atomic Eigenstates

$$|\Psi_{nlm}(t)\rangle = e^{-iE_{nlm}t/\hbar}|\Psi_{nlm}(0)\rangle, \tag{9.5.1}$$

$$\hat{H}_0|\Psi_{nlm}(0)\rangle = E_{nlm}|\Psi_{nlm}\rangle, \tag{9.5.2}$$

and the electron density is

$$\rho(r,\theta,\phi) = |\Psi(r,\theta,\phi,t=0)^2|. \tag{9.5.3}$$

Atomic Dipole Calculate (Oscillating) Dipole Moment for Atomic Eigenstate. We denote $|1\rangle = |\Psi_{nlm}\rangle$. We have

$$d(t) = \langle 1(t)|\hat{d}|1(t)\rangle \tag{9.5.4}$$

$$= \langle\hat{d}|1\rangle \tag{9.5.5}$$

$$= -e\langle 1|\hat{r}|1\rangle. \tag{9.5.6}$$

Then,

$$-e\langle 1|\hat{r}|1\rangle = -e\langle 1|\hat{P}\hat{P}^{-1}\hat{r}\hat{P}\hat{P}^{-1}|1\rangle \tag{9.5.7}$$

$$= +e\langle 1|\hat{r}|1\rangle, \tag{9.5.8}$$

which implies

$$\langle 1|\hat{r}|1\rangle = 0. \tag{9.5.9}$$

Atomic Dipole - Superposition States Calculate (Oscillating) Dipole Moment for Atomic Superposition State

$$|\Psi(0)\rangle = \frac{1}{\sqrt{2}}(|1\rangle + i|2\rangle). \tag{9.5.10}$$

Evolution

$$|\Psi(t)\rangle = \frac{1}{\sqrt{2}}(|1\rangle + ie^{-i\omega_{21}t}|2\rangle). \tag{9.5.11}$$

We have

$$\begin{aligned}
d(t) &= \langle\Psi(t)|\hat{d}|\Psi(t)\rangle && (9.5.12)\\
&= \frac{1}{2}\left\{\langle 1|\hat{d}|1\rangle + \langle 2|\hat{d}|2\rangle + ie^{-i\omega_{21}t}\langle 1|\hat{d}|2\rangle - ie^{-i\omega_{21}t}\langle 2|\hat{d}|1\rangle\right\} && (9.5.13)\\
&= d_{12}i\frac{1}{2}\left\{e^{-i\omega_{21}t} - e^{i\omega_{21}t}\right\} && (9.5.14)\\
&= d_{12}\sin(\omega_{21}t), && (9.5.15)
\end{aligned}$$

where d_{12} is the dipole moment amplitude, ω_{21} is the natural resonance frequency.

Electron Density - Superposition States Calculate Electron Probability Density for Superposition State. The superposition state is

$$\Psi(r,t) = \frac{1}{\sqrt{2}}\left(\Psi_1(r) + ie^{-i\omega_{21}t}\Psi_2 D(t)\right). \tag{9.5.16}$$

The Electron Probability Density is

$$\begin{aligned}
\rho(r,t) &= |\Psi(r,t)|^2 && (9.5.17)\\
&= \Psi^*\Psi && (9.5.18)\\
&= \frac{1}{2}\left\{|\Psi_1(r)|^2 + |\Psi_2(r)|^2 + 2\mathrm{Re}\left(ie^{-i\omega_{21}t}\Psi_1^*(r)\Psi_2(r)\right)\right\}, && (9.5.19)
\end{aligned}$$

where $2\mathrm{Re}\left(ie^{-i\omega_{21}t}\Psi_1^*(r)\Psi_2(r)\right)$ is the interference term.

Examples This is shown by animation and figure in the video.

9.6 The Bloch Sphere

General Two-Level State

- General State Description

$$\begin{aligned}
|\Psi\rangle &= c_1'|1\rangle + c_2'|2\rangle && (9.6.1)\\
&\text{Up to a global phase} && (9.6.2)\\
&= |c_1'||1\rangle + e^{i\phi}|c_2'|2\rangle && (9.6.3)
\end{aligned}$$

satisfying $|c_1'|^2 + |c_2'|^2 = 1$.

- Alternative way

$$|\Psi\rangle = \cos(\theta/2)|1\rangle + e^{i\phi}\sin(\theta/2)|2\rangle, \tag{9.6.4}$$

since $\cos(\theta/2)^2 + \sin(\theta/2)^2 = 1$.

Geometric Description - Bloch Sphere We then have

$$|\Psi\rangle = \cos(\theta/2)|1\rangle + e^{i\phi}\sin(\theta/2)|2\rangle \tag{9.6.5}$$

with $0 \leq \theta \leq \pi$ as the latitude and $0 \leq \phi \leq 2\pi$ as the longitude. This is the Bloch Sphere representation. The definition of θ and θ and their ranges are different from my familiar coordinate system.

Special States on Bloch Sphere

Analogy to Spin -1/2 States Is is shown in the figure.

9.7 Density Operator and Density Matrix

The Problem How do we describe "imperfect state preparation" in an experiment? For example, $50\%|1\rangle$ and $50\%|2\rangle$. We may think of

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle + |2\rangle\right).??? \tag{9.7.1}$$

This is $100\%|\Psi\rangle$ pure state. We need stable relative phase between the two states!

Optical Analogy - Controlled Phase The double slit problem is shown in the video.

Intensity on Detection Screen:

$$I \propto |E|^2 = |E_1 + e^{i\phi}E_2|^2 \tag{9.7.2}$$

$$= |E_1|^2 + |E_2|^2 + 2\mathrm{Re}\left(E_1 E_2 e^{i\phi}\right). \tag{9.7.3}$$

As ϕ varies, Interference pattern "washed out"!

We need new formalism to describe mixed states!(imperfect state preparation, spontaneous emission,...)

Density Operator and Matrix The description of mixed states can be handled by the density operator (matrix) formalism!

- Density operator (hermitian)

$$\hat{\rho} = \sum p_i |\Psi_i\rangle\langle\Psi_i| \tag{9.7.4}$$

$$\hat{\rho} = I\hat{\rho}I \tag{9.7.5}$$

$$= \sum_{i,j} |i\rangle\langle i|\hat{\rho}|j\rangle\langle j| \tag{9.7.6}$$

$$= \rho_{11}|1\rangle\langle 1| + \rho_{12}|1\rangle\langle 2| + \rho_{21}|2\rangle\langle 1| + \rho_{22}|2\rangle\langle 2|, \tag{9.7.7}$$

where $I = \sum_i |i\rangle\langle i|$.

- Density matrix

$$\rho = \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix}, \tag{9.7.8}$$

where ρ_{11} and ρ_{22} are the populations, ρ_{12} and ρ_{21} are the coherence. Since ρ is hermitian, we have

$$\rho_{12} = \rho_{21}^*. \tag{9.7.9}$$

Example 9.7.1 (Example: Density Matrix of Pure State)**.** *We have*

$$|\Psi\rangle = |c_1||1\rangle + e^{i\phi}|c_2||2\rangle. \tag{9.7.10}$$

The corresponding density operator of the ***pure state*** *is* $\hat{\rho} = |\Psi\rangle\langle\Psi|$*. Then the corresponding density matrix is*

$$\rho = \begin{bmatrix} |c_1|^2 & |c_1||c_2|e^{-i\phi} \\ |c_1||c_2|e^{i\phi} & |c_2|^2 \end{bmatrix}, \tag{9.7.11}$$

where $|c_1||c_2|e^{-i\phi}$ *and* $|c_1||c_2|e^{i\phi}$ *are relative phase between states* $|1\rangle$ *and* $|2\rangle$*.*

specific example:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle + |2\rangle\right), \tag{9.7.12}$$

so

$$\rho = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}. \tag{9.7.13}$$

Example 9.7.2 (Example: Fully Incoherent Mixture)**.**

$$\hat{\rho} = \frac{1}{2}|1\rangle\langle 1| + \frac{1}{2}|2\rangle\langle 2| \tag{9.7.14}$$

with

$$\rho = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}, \tag{9.7.15}$$

where vanishingly coherence and the phase varies from 0 *to* 2π*. It means that we did not control phase.*

Useful Facts

- Expectation values: $\langle \hat{A} \rangle = \mathrm{Tr}(\hat{\rho}\hat{A}) = \mathrm{Tr}(\rho A)$
- Time evolution (von Neumann equation)

$$i\hbar \frac{\partial \hat{\rho}}{\partial t} = [\hat{H}, \hat{\rho}] \tag{9.7.16}$$

- Pure state: $\mathrm{Tr}(\rho^2) = 1$
- Mixed states: $\mathrm{Tr}(\rho^2) < 1$

9.8 Optical Bloch Equations

Time Evolution of Density Matrix How to calculate time evolution of density matrix?

$$i\hat{\hbar} \frac{\partial \hat{\rho}}{\partial t} = [\hat{H}, \hat{\rho}]. \tag{9.8.1}$$

Assume pure state

$$\begin{aligned} \frac{d}{dt}\rho_{11} &= \frac{d}{dt}(c_1 c_1^*) \qquad (9.8.2)\\ &= \dot{c}_1 c_1^* + c_1 \dot{c}_1^* \qquad (9.8.3)\\ &= i\frac{\Omega_0}{2}\left(e^{i\delta t}\rho_{21} - e^{-i\delta t}\rho_{12}\right) \qquad (9.8.4) \end{aligned}$$

Transformation to rotality frame of light

$$= i\frac{\Omega_0}{2}\left(\tilde{\rho}_{21} - \tilde{\rho}_{12}\right), \tag{9.8.5}$$

where

$$\dot{c}_1(t) = i\frac{\Omega_0}{2}e^{+i\delta t}c_2(t) \tag{9.8.6}$$

$$\dot{c}_2(t) = i\frac{\Omega_0}{2}e^{-i\delta t}c_1(t) \tag{9.8.7}$$

$$\tilde{\rho}_{12} = e^{-i\delta t}\rho_{12} \tag{9.8.8}$$

$$\tilde{\rho}_{21} = e^{+i\delta t}\rho_{21}. \tag{9.8.9}$$

Other elements obtained in analogy!

$$\frac{d}{dt}\rho_{11} = i\frac{\Omega_0}{2}\left(\tilde{\rho}_{21} - \tilde{\rho}_{12}\right) \tag{9.8.10}$$

$$\frac{d}{dt}\rho_{22} = i\frac{\Omega_0}{2}\left(\tilde{\rho}_{12} - \tilde{\rho}_{21}\right) \tag{9.8.11}$$

$$\frac{d}{dt}\tilde{\rho}_{12} = -i\delta\tilde{\rho}_{12} + i\frac{\Omega_0}{2}\left(\rho_{22} - \rho_{11}\right) \tag{9.8.12}$$

$$\frac{d}{dt}\tilde{\rho}_{21} = +i\delta\tilde{\rho}_{21} + i\frac{\Omega_0}{2}\left(\rho_{11} - \rho_{22}\right). \tag{9.8.13}$$

Noting that $\tilde{\rho}_{12} = \tilde{\rho}_{21}$ due to hermitian matrix, the third and the forth equations are the same. So we have

$$\frac{d}{dt}\rho_{11} = i\frac{\Omega_0}{2}\left(\tilde{\rho}_{21} - \tilde{\rho}_{12}\right) \tag{9.8.14}$$

$$\frac{d}{dt}\rho_{22} = i\frac{\Omega_0}{2}\left(\tilde{\rho}_{12} - \tilde{\rho}_{21}\right) \tag{9.8.15}$$

$$\frac{d}{dt}\tilde{\rho}_{12} = -i\delta\tilde{\rho}_{12} + i\frac{\Omega_0}{2}\left(\rho_{22} - \rho_{11}\right). \tag{9.8.16}$$

Optical Bloch Equations with Damping Phenomenological damping and spontaneous emission in the figure. Combine the decay, we have

$$\frac{d}{dt}\rho_{11} = i\frac{\Omega_0}{2}\left(\tilde{\rho}_{21} - \tilde{\rho}_{12}\right) + \gamma\rho_{22} \tag{9.8.17}$$

$$\frac{d}{dt}\rho_{22} = i\frac{\Omega_0}{2}\left(\tilde{\rho}_{12} - \tilde{\rho}_{21}\right) - \gamma\rho_{22} \tag{9.8.18}$$

$$\frac{d}{dt}\tilde{\rho}_{12} = -i\delta\tilde{\rho}_{12} + i\frac{\Omega_0}{2}\left(\rho_{22} - \rho_{11}\right) - (\gamma/2)\tilde{\rho}_{12}. \tag{9.8.19}$$

We now define the inversion $w = \rho_{22} - \rho_{11}$. We have Optical Bloch Equations with Damping

$$\frac{d}{dt}\tilde{\rho}_{21} = -\left(\gamma/2 - i\delta\right)\tilde{\rho}_{21} - \frac{i\omega\Omega_0}{2} \tag{9.8.20}$$

$$\frac{d}{dt}\omega = -\gamma\left(\omega + 1\right) - i\Omega_0\left(\tilde{\rho}_{21} - \tilde{\rho}_{12}\right) \tag{9.8.21}$$

in the Density Matrix Form.

9.9 Optical Bloch Equations - Dynamics and Steady State

Dynamical Evolution of System Shown in the figure in the picture.

Steady State Solution Conditions: $\frac{d}{dt}\tilde{\rho}_{21} = 0$ and $\frac{d}{dt}\omega = 0$. Then we have the solutions

$$\omega = -\frac{1}{1+S} \tag{9.9.1}$$

$$\tilde{\rho}_{21} = \frac{2\Omega_0}{2\left(\gamma/2 - -\delta\right)\left(1+S\right)} \tag{9.9.2}$$

$$S = \frac{\Omega_0^2/2}{\delta^2 + \gamma^2/4} = \frac{S_0}{1+4\delta^2/\gamma^2} \tag{9.9.3}$$

$$S_0 = \frac{2\Omega_0^2}{\gamma^2} = \frac{I}{O_{sat}}, \tag{9.9.4}$$

where S is called the saturation parameter, S_0 is called resonant saturation parameter.

Limiting Cases:

- $S \leq\leq 1$: $w \to -1$ where $w = \rho_{22} - \rho_{11}$. Atom is mainly in ground state.
- $S \geq\geq 1$: $S \to \infty$, $w \to 0$.
- Excited State Population:

$$\rho_{22} \tag{9.9.5}$$

Combine with $\rho_{22} + \rho_{11} = 1$

$$= \frac{1}{2}(1+w) \tag{9.9.6}$$

$$= \frac{S}{2(1+S)} \tag{9.9.7}$$

$$= \frac{S_0/2}{1+S_0+4\delta^2/\gamma^2} \tag{9.9.8}$$

$$\xrightarrow{S_0\to\infty,\delta=0} \frac{1}{2}. \tag{9.9.9}$$

- Photon Scattering Rate: $\Gamma_{ph} = \gamma\rho_{22} = \frac{\gamma}{2}\frac{S_0}{1+S_0+4\delta^2/\gamma^2}$. $\Gamma_{ph} \to \gamma/2$ for $S_0 \to \infty$ and $\delta = 0$. We rewrite it as

$$\Gamma_{ph} = \left(\frac{S_0}{1+S_0}\right)\left(\frac{\gamma/2}{1+4\delta^2/\gamma'^2}\right) \tag{9.9.10}$$

$$\gamma' = \gamma\sqrt{1+S_0}. \tag{9.9.11}$$

It has a figure in the video. The saturation brodening is shown in the figure.

9.10 Lambert-Beer Law

Attenuation of Light It is shown in the figure.

Scattered Light from Slab of Atoms scattered light power by slab of length dz

$$dP_{sc} = \Gamma_{ph} \times nAdz \times \hbar\omega, \tag{9.10.1}$$

where Γ_{ph} is the single atom photon scattering rate, $\hbar\omega$ is the energy of single atom, $nAdz$ is the number of atoms. Then we have

$$\frac{dP_{sc}}{dz} = \Gamma_{ph} \times nA \times \hbar\omega. \tag{9.10.2}$$

Scattered Light from Slab of Atoms Energy conservation requires

$$\frac{dP}{dz} = -\frac{dP_{sc}}{dz} \tag{9.10.3}$$

$$\frac{dP}{dz} = \frac{dI}{dI}A. \tag{9.10.4}$$

Put every thing together:

$$\frac{dI}{dz} = -\Gamma n\hbar\omega. \tag{9.10.5}$$

We have

$$\frac{dI(z)}{dz} = -n\sigma I(z), \tag{9.10.6}$$

where σ is the atomic scattering cross section.

Lambert-Beer Law (no saturation) We compute the solutions

$$I(z) = I(0)e^{-n\sigma z}, \tag{9.10.7}$$

which is the Lambert-Beer Law of Absorption.

Laser induced Fluorescence Shown in a video.

9.11 Bloch Vector

Density Matrix Revisited Density Matrix of TLA

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \tag{9.11.1}$$

Density Matrix hermitian

$$\rho = \rho^\dagger = (\rho^T)^*, \tag{9.11.2}$$

so we have

$$\rho = \begin{pmatrix} \rho_{11} & \mathrm{Re}\rho_{12} + i\mathrm{Im}\rho_{12} \\ \mathrm{Re}\rho_{12} - i\mathrm{Im}\rho_{12} & \rho_{22} \end{pmatrix}. \tag{9.11.3}$$

Pauli matrices are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \sigma_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{9.11.4}$$

The decomposition of Density matrix into Pauli matrices

$$\rho = \frac{1}{2}\left(I + b_x\sigma_x + b_y\sigma_y + b_z\sigma_z\right), \tag{9.11.5}$$

where $b_x, b_y, b_y \in \mathbb{R}$.

Bloch Vector We have the density matrix in rotating frame of light

$$\tilde{\rho} = \begin{pmatrix} \rho_{11} & \tilde{\rho}_{12} \\ \tilde{\rho}_{21} & \rho_{22} \end{pmatrix}, \tag{9.11.6}$$

where $\tilde{\rho}_{12} = \rho_{12}e^{-i\omega t}$. We use following sign convention and have

$$\tilde{\rho} = \frac{1}{2}\left(I + u\sigma_x - v\sigma_y - w\sigma_z\right), \tag{9.11.7}$$

and the bloch vector is defined as

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}. \tag{9.11.8}$$

It can be easily shown that

$$u = 2\mathrm{Re}(\tilde{\rho}_{12}) = \tilde{\rho}_{12} + \tilde{\rho}_{12}^* \tag{9.11.9}$$

$$v = 2\mathrm{Im}(\tilde{\rho}_{12}) = i(\tilde{\rho}_{12}^* + \tilde{\rho}_{12}) \tag{9.11.10}$$

$$w = \rho_{22} - \rho_{11}, \tag{9.11.11}$$

$$\tag{9.11.12}$$

where u is the dispersive component, v is the absorption component and w is the inversion.

Bloch vector can be used to describe any state of TLA density matrix!

Properties of Bloch Vector

- Mixed State: $u^2 + v^2 + w^2 < 1$
- Pure State: $u^2 + v^2 + w^2 = 1$

9.12 Understanding Bloch Vector

What physical behaviour do the components stand for?

- $w = -1$ atom in ground state. $w = +1$ atom in excited state.
- What about u, v?

$$\langle \hat{d}_i(t) = \mathrm{Tr}(\hat{\rho}\hat{d}) \tag{9.12.1}$$

$$= \mathrm{Tr}\left[\begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{12}^* & \rho_{22} \end{pmatrix}\begin{pmatrix} 0 & d_{12}^i \\ d_{12}^i & 0 \end{pmatrix}\right], \tag{9.12.2}$$

where $d_{12}^x = \langle 1| - q\hat{x}|2\rangle$.

Written in the vector form, we have

$$\langle \hat{d}\rangle(t) = d_{12}\left(\rho_{12} + \rho_{12}^*\right) \tag{9.12.3}$$

$$= d_{12}\left(\tilde{\rho}_{12}e^{i\omega t} + \tilde{\rho}_{12}^* e^{-i\omega t}\right) \tag{9.12.4}$$

$$= d_{12}\left[u\cos(\omega t) - v\sin(\omega t)\right], \tag{9.12.5}$$

where we use $\rho_{12} = \tilde{\rho}_{12}e^{i\omega t}$, u denotes in phase and v denotes $90°$ out of phase component.

Reminder: $E(t) = \epsilon E_0 \cos(\omega t)$.

- Which component responsible for absorption/emission? We have a figure in the video to show the classical picture.

Average absorbed power per atom (classical ensemble average)

$$\langle \frac{dW}{dt} = \epsilon E_0 \cos(\omega t)\langle -q\frac{dr}{dt}\rangle \tag{9.12.6}$$

$$= \epsilon E_0 \cos(\omega t)\langle \dot{d}\rangle. \tag{9.12.7}$$

Quantum mechanical analogue (Ehrenfest)

$$\langle \frac{dW}{dt}\rangle = \epsilon E_0 \cos(\omega t)\langle \dot{d}\rangle \tag{9.12.8}$$

$$\langle \hat{d}\rangle(t) = d_{12}[u\cos(\omega t) - v\sin(\omega t)]. \tag{9.12.9}$$

$$\langle \frac{dW}{dt}\rangle = -d_{12}\cdot \epsilon E_0 \omega(u\cos(\omega t)\sin(\omega t) + v\sin(\omega t)^2) \tag{9.12.10}$$

$$\overline{\langle \frac{dW}{dt}\rangle} = \frac{1}{T}\int dt \langle \frac{dW}{dt}\rangle \tag{9.12.11}$$

$$= -\frac{d_{12}\cdot \epsilon E_0 \omega v}{2} \tag{9.12.12}$$

$$= -\hbar \frac{d_{12}\epsilon E_0}{\hbar}\omega \frac{v}{2} \tag{9.12.13}$$

$$= -\hbar \Omega_0 \omega \frac{v}{2}, \tag{9.12.14}$$

which is the absorption.

9.13 Optical Bloch Equations using Bloch Vector

9.14 Interlude: The Mach-Zehnder Interferometer

9.15 Ramsey Interferometer

9.16 Review: QM of the Harmonic Oscillator

[SZQ: 2023.04.20: I have understandard the content in this video.]

9.17 Wave equation and energy density of classical radiation field

This section is also known as the review of Maxwell equations vector potentials.

Fundamentaals Maxwell equations in free space

$$\nabla \cdot E = 0, \nabla \times E = -\frac{\partial B}{\partial t} \tag{9.17.1}$$

$$\nabla \cdot B = 0, \nabla \times B = \frac{1}{c^2}\frac{\partial E}{\partial t}. \tag{9.17.2}$$

Lemma 9.17.1 (Coulomb Gauge). *Considering Coulomb Gauge, we have*

$$\nabla \cdot A = 0. \tag{9.17.3}$$

Then we can express the eletric field and magnetic field in terms of the vector potential

$$B(r,t) = \nabla \times A(r,t) \tag{9.17.4}$$

$$E(r,t) = -\frac{\partial A(r,t)}{\partial t}. \tag{9.17.5}$$

Lemma 9.17.2 (Wave equation). *Considering Coulomb Gauge, the wave equation is*

$$\nabla^2 A - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}A = 0. \tag{9.17.6}$$

Proof. Using **??** and the forth equation in **??**, we have

$$\nabla \times B = \nabla \times (\nabla \times A(r,t)), \tag{9.17.7}$$

and

$$\nabla \times B = \frac{1}{c^2}\frac{\partial E}{\partial t} \tag{9.17.8}$$
$$= -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}A(r,t). \tag{9.17.9}$$

So we have

$$\nabla \times (\nabla \times A) = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}. \tag{9.17.10}$$

Then use the rule in vector Calculus

$$\nabla \times (\nabla \times A) = \nabla(\nabla \cdot A) - \Delta A. \tag{9.17.11}$$

Use lemma **??**, we then have

$$-\Delta A = -\frac{1}{c^2}\frac{\partial^2}{}\partial t^2 A \tag{9.17.12}$$
$$= -\nabla^2 A. \tag{9.17.13}$$

So we have

$$\nabla^2 A - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = 0. \tag{9.17.14}$$

□

Solutions of Wave Equation

Lemma 9.17.3 (Solutions of Wave Equation)**.** *Plane waves:*

$$\boldsymbol{A}_{\boldsymbol{k},\alpha} = \epsilon_{\boldsymbol{k},\alpha} A_{\boldsymbol{k},\alpha} \exp\left[i(\boldsymbol{k}\boldsymbol{r} - \omega_k t)\right], \tag{9.17.15}$$

where $\epsilon_{\boldsymbol{k},\alpha}$ is polarization, $A_{\boldsymbol{k},\alpha}$ is complex amplitude, $|k| = \frac{2\pi}{\lambda}$ is wavenumber i.e., the magnitude of the wave vector, $\boldsymbol{k}$ is the wave vector, $\omega_k = ck$.

Which wave vectors are possible? (a). in finite space, $\mathbf{k}$ distributed continuous; (b). finite box of length L, $\mathbf{k}$ distributed discretely (periodic boundary conditions)

$$k_x = \frac{2\pi}{L}n_x, k_y = \frac{2\pi}{L}n_y, k_z = \frac{2\pi}{L}n_z \tag{9.17.16}$$
$$\mathbf{A}(\mathbf{r},t) = \sum_{\mathbf{k},\alpha} \epsilon_{\mathbf{k},\alpha}\left(A_{\mathbf{k},\alpha}\exp[i(\mathbf{kr}-\omega_k t)] + A^*_{\mathbf{k},\alpha}\exp[-i(\mathbf{kr}-\omega_k t)]\right) \tag{9.17.17}$$
$$\mathbf{E}(\mathbf{r},t) = -\frac{\partial}{\partial t}\mathbf{A}(r,t) = \sum_{\mathbf{k},\alpha} \epsilon_{\mathbf{k},\alpha} i\omega_k \left[A_{\mathbf{k},\alpha}\exp[i(\mathbf{kr}-\omega_k t)] - A^*_{\mathbf{k},\alpha}\exp[-i(\mathbf{kr}-\omega_k t)]\right] \tag{9.17.18}$$
$$\mathbf{B}(\mathbf{r},t) = \sum_{\mathbf{k},\alpha} i(\mathbf{k}\times\epsilon_{\mathbf{k},\alpha})\left[A_{\mathbf{k},\alpha}\exp[i(\mathbf{kr}-\omega_k t)] - A^*_{\mathbf{k},\alpha}\exp[-i(\mathbf{kr}-\omega_k t)]\right]. \tag{9.17.19}$$

[SZQ: 2023.04.20: The complex conjugate term is used to eliminate the imaginary part.]

Total Energy of Radiation Field Total energy of radiation field in volume $V = L^3$. [SZQ: 2023.04.20: The total energy is the integration of the electric density and magenetic density over the volume.]

The electric density is

$$\frac{1}{2}\varepsilon_0 E(r,t)^2. \tag{9.17.20}$$

The magenetic density is

$$\frac{1}{2\mu_0}B(r,t)^2. \tag{9.17.21}$$

Then the total energy of radiation field in volume $V = L^3$ is

$$H = \frac{1}{2}\int_V dV \left[\varepsilon_0 E(r,t)^2 + \frac{1}{\mu_0}B(r,t)^2\right] \tag{9.17.22}$$

$$= \sum_{\mathbf{k},\alpha} \varepsilon_0 V \omega_k^2 \left[A_{\mathbf{k},\alpha}A^*_{\mathbf{k},\alpha} + A^*_{\mathbf{k},\alpha}A_{\mathbf{k},\alpha}\right] \tag{9.17.23}$$

$$= \sum_{\mathbf{k},\alpha} E_{\mathbf{k},\alpha}, \tag{9.17.24}$$

where

$$E_{\mathbf{k},\alpha} = \varepsilon_0 V \omega_k^2 \left[A_{\mathbf{k},\alpha}A^*_{\mathbf{k},\alpha} + A^*_{\mathbf{k},\alpha}A_{\mathbf{k},\alpha}\right]. \tag{9.17.25}$$

[SZQ: 2023.04.20: This expression is similar to the quantum harmonic oscillators.] [SZQ: 2023.04.20: I ignore the bold. So you should understand where you should use the bold.]

9.18 Quantization of the e.m. field

Fundamental Idea RadiationMode (k, α)

- **To every radiation mode, we associate a harmonic oscillator!** Creation and annihilation operators can change the degree of excitation of mode (occupation with photons)
- **A photon is an excitation quantum of the harmonic oscillator associated with a mode!**

Creation and Annihilation Operators $\hat{a}_k|n_k\rangle = \sqrt{n_k}|n_k - 1\rangle$: decrease photon number by one photon.

$\hat{a}_k^\dagger|n_k\rangle = \sqrt{n_k+1}|n_k+1\rangle$: increase photon number by one photon.

Number operator: $\hat{n}_k|n_k\rangle = n_k|n_k\rangle$. $\hat{a}_k^\dagger\hat{a}_k = \hat{n}_k$.

Fock state: $|n_k\rangle$. Fock state is the eigenstate of quantum harmonic oscillator.

Hamitonian of Radiation Field The Hamitonian of Radiation Field is the sum of the hamitonian of harmonic oscillator of each mode as

$$\hat{H}_R = \sum_k \hat{H}_k, \tag{9.18.1}$$

where

$$\hat{H}_k = \frac{1}{2}\hbar\omega_k \left(\hat{a}_k\hat{a}_k^\dagger + \hat{a}_k^\dagger\hat{a}_k\right). \tag{9.18.2}$$

We can compare it with classicall expression

$$E_{k,\alpha} = \epsilon_0 V\omega_k^2 \left(A_{k,\alpha}A_{k,\alpha}^* + A_{k,\alpha}^* A_{k,\alpha}\right). \tag{9.18.3}$$

If we replace A_k with

$$A_k = \sqrt{\frac{\hbar}{2\epsilon_0 V\omega_k}}\hat{a}_k, \tag{9.18.4}$$

and replace A_k^* with

$$A_k^* = \sqrt{\frac{\hbar}{2\epsilon_0 V\omega_k}}\hat{a}_k^\dagger. \tag{9.18.5}$$

We will arrive at $\hat{H}_k$. Also we can obtain the quantum version of vector potential operator. The classical vector potential operator is

$$A_k(r,t) = \epsilon_k \left[A_k \exp[i(kr - \omega_k t)] + A_k^* \exp[-i(kr-\omega_k t)]\right]. \tag{9.18.6}$$

The quantum version will be

$$\hat{A}_k(r,t) = \epsilon_k\sqrt{\frac{\hbar}{2\epsilon_0 V\omega_k}}\left[\hat{a}_k \exp[i(kr-\omega_k t)] + \hat{a}_k^\dagger \exp[-i(kr-\omega_k t)]\right]. \tag{9.18.7}$$

Use the quantum vector potential, we can derive the quantum electric field operator as

$$\hat{E}_k(r,t) = -\frac{\partial}{\partial t}\hat{A}_k(r,t) \tag{9.18.8}$$

$$= -\epsilon_k\sqrt{\frac{\hbar}{2\epsilon_0\omega_k V}}(-\omega_k)\left[i\hat{a}_k \exp[i(kr-\omega_k t)] - i\hat{a}_k^\dagger \exp[-i(kr-\omega t)]\right]. \tag{9.18.9}$$

Recall that $i = \exp[i\pi/2]$ and define

$$\chi_k(r,t) = -kr + \omega_k t - \pi/2. \tag{9.18.10}$$

We then have the compact form

$$\hat{E}(r,t) = \sum_k \epsilon_k\sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}}\left[\hat{a}_k \exp[-i\chi_k(r,t)] + \hat{a}_k^\dagger \exp[i\chi_k(r,t)]\right] \tag{9.18.11}$$

$$= \sum_k \hat{E}_k(r,t) \tag{9.18.12}$$

$$:= \hat{E}^+(r,t) + \hat{E}^-(r,t). \tag{9.18.13}$$

Hamiltonian of Radiation Field The Hamiltonian of Radiation Field is

$$\hat{H}_R = \frac{1}{2}\int_V dV \left[\epsilon_0 \hat{E}\cdot\hat{E} + \frac{1}{\mu_0}\hat{B}\cdot\hat{B}\right] \tag{9.18.14}$$

$\hat{B}$, $\hat{E}$ are the quantum operator of B, E

$$= \sum_k \frac{\hbar\omega_k}{2}\left[\hat{a}_k^\dagger \hat{a}_k + \hat{a}_k \hat{a}_k^\dagger\right] \tag{9.18.15}$$

Use the commutation relation

$$= \sum_k \left(\hat{a_k}^\dagger \hat{a}_k + 1/2\right). \tag{9.18.16}$$

Use this hamiltonian, we derive the energy of multi-mode Fock states as

$$\hat{H}_R|n_{k_1}, n_{k_2}, \dots\rangle = \sum_k \hbar\omega_k \left(n_k + \frac{1}{2}\right)|n_{k_1}, n_{k_2}, \dots\rangle \tag{9.18.17}$$

using the fact that $\hat{a}_k^\dagger \hat{a}_k$ is the number operator $\hat{n}_k$.

Also the vacuum state energy will be

$$E_0 = \sum_k \frac{1}{2}\hbar\omega_k \tag{9.18.18}$$

corresponds to

$$|0\rangle = |0\rangle \otimes \cdots \otimes |0\rangle. \tag{9.18.19}$$

This is divergent, but do not worry. When we calculate the difference, this term will be canceled.

9.19 Field state of single radiation field mode: Fock States

We focus discussion on a **single mode of the radiation field (wave vector k)** We define the phase factor

$$\chi = \chi_k(r,t) = \omega_k t - \mathbf{kr} - \pi/2. \tag{9.19.1}$$

Then we have

$$\hat{E}(\chi) = \hat{E}^+(\chi) + \hat{E}^-(-\chi) \tag{9.19.2}$$

$$= \left(\frac{\hbar\omega}{2\varepsilon_0 V}\right)^{1/2} (\hat{a}\exp[-i\chi] + \hat{a}^\dagger \exp[i\chi]). \tag{9.19.3}$$

We write the field operator in natural units $2\left(\frac{\hbar\omega}{2\varepsilon_0 V}\right)^{1/2}$, which is also called vacuum field strength. We then have

$$\hat{E}(\chi) = \frac{1}{2}\left(\hat{a}\exp[-i\chi] - \hat{a}^\dagger \exp[i\chi]\right). \tag{9.19.4}$$

Fock states: $|n\rangle$ means n photons in radiation mode, also means eigenstate of number operator $\hat{n}$.

$$\hat{H}|n\rangle = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})|n\rangle \tag{9.19.5}$$
$$= \hbar\omega(\hat{n} + \frac{1}{2})|n\rangle \tag{9.19.6}$$
$$= \hbar\omega(n + \frac{1}{2})|n\rangle. \tag{9.19.7}$$

Lemma 9.19.1. *Fluctuations in n is*

$$(\Delta n)^2 = 0. \tag{9.19.8}$$

Proof.

$$(\Delta n)^2 = \langle n|(\hat{n} - \langle\hat{n}\rangle)^2|n\rangle \tag{9.19.9}$$
$$= \langle n|(\hat{n}^2 - 2\langle\hat{n}\rangle\hat{n} + \langle\hat{n}\rangle^2)|n\rangle \tag{9.19.10}$$
$$= \langle n|\hat{n}^2|n\rangle - 2\langle\hat{n}\rangle^2 + \langle\hat{n}\rangle^2 \tag{9.19.11}$$
$$= \langle n|\hat{n}^2|n\rangle - \langle n|\hat{n}|n\rangle^2 \tag{9.19.12}$$
$$= \langle n|n^2|n\rangle - \langle n|n|n\rangle^2 \tag{9.19.13}$$
$$= n^2 - n^2 \tag{9.19.14}$$
$$= 0, \tag{9.19.15}$$

where we use

$$\langle n|\hat{n}|n\rangle = \langle\hat{n}\rangle. \tag{9.19.16}$$

□

Lemma 9.19.2. *The expectation value of the field is*

$$E = \langle n|\hat{E}(\chi)|n\rangle = 0. \tag{9.19.17}$$

Proof.

$$E = \langle n|\hat{E}(\chi)|n\rangle \tag{9.19.18}$$
$$= \frac{1}{2}\langle n|\hat{a}\exp[-i\chi] + \hat{a}^\dagger\exp[i\chi]|n\rangle \tag{9.19.19}$$
$$= \frac{1}{2}\langle n|\hat{a}\exp[-i\chi]|n\rangle + \frac{1}{2}\langle n|\hat{a}^\dagger\exp[i\chi]|n\rangle \tag{9.19.20}$$
$$= 0 + 0 \tag{9.19.21}$$
$$= 0. \tag{9.19.22}$$

□

Lemma 9.19.3. *Field fluctuations is*

$$(\Delta E(\chi))^2 = \frac{1}{2}(n+\frac{1}{2}). \tag{9.19.23}$$

Proof.

$$\begin{aligned}
(\Delta E(\chi))^2 &= \langle n|\hat{E}(\chi)^2|n\rangle - \langle n|\hat{E}(\chi)|n\rangle^2 && (9.19.24)\\
&= \langle n|\hat{E}(\chi)^2|n\rangle - 0 && (9.19.25)\\
&= \frac{1}{4}\langle n|(\hat{a}\exp[-i\chi] + \hat{a}^\dagger \exp[i\chi])^2|n\rangle && (9.19.26)\\
&= \frac{1}{4}\langle n|\hat{a}^2 \exp[-2i\chi] + \hat{a}^{\dagger 2}\exp[2i\chi] + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}|n\rangle && (9.19.27)\\
&= \frac{1}{4}\langle n|2\hat{n}+2|n\rangle && (9.19.28)\\
&= \frac{1}{4}\langle n|2n+1|n\rangle && (9.19.29)\\
&= \frac{1}{2}(n+\frac{1}{2}). && (9.19.30)
\end{aligned}$$

□

When $n = 0$, which is vacuum state, we have the standard deviation as $1/2$, which is half of the unit, i.e., vacuum field strength.

9.20 Field state of single radiation field mode: Coherent States

How to reproduce classical motion Superposition of Fock states reproduces oscillating wavepacket motion!

$$\begin{aligned}
|\alpha\rangle &\propto \exp[-i\frac{1}{2}\omega t]|0\rangle + \exp[-i\frac{3}{2}\omega t]\alpha|1\rangle + \exp[-i\frac{5}{2}\omega t]\frac{\alpha^2}{\sqrt{2}}|2\rangle + \dots && (9.20.1)\\
&= \sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\exp\left[-iE_n t/\hbar\right]. && (9.20.2)
\end{aligned}$$

What we want: states of light, whose expectation value corresponds to classical e.m. waves!

Solotion: Coherent States

Definition 9.20.1. *Coherent States:*

$$|\alpha\rangle = \exp\left[-\frac{1}{2}|\alpha|^2\right]\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle, \tag{9.20.3}$$

where α is complex number, amplitude of coherent state. $\alpha = |\alpha|\exp[i\theta]$, $|n\rangle$ is the fock state.

Lemma 9.20.1. *Coherent state is normalized:* $\langle\alpha|\alpha\rangle = 1$.

Proof.

$$\langle\alpha|\alpha\rangle = \exp\left[-|\alpha|^2\right]\sum_{n,n'}\frac{(\alpha^*)^n\alpha^{n'}}{\sqrt{n!}\sqrt{n'!}}\langle n|n'\rangle \tag{9.20.4}$$

$$= \exp\left[-|\alpha|^2\right]\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!} \tag{9.20.5}$$

$$= \exp\left[-|\alpha|^2\right]\exp\left[+|\alpha|^2\right] \tag{9.20.6}$$

$$= 1. \tag{9.20.7}$$

□

Lemma 9.20.2. *Coherent state is quasi orthogonal:* $|\alpha-\beta| >> 1 \longrightarrow |\langle\alpha|\alpha\rangle| \to 0$.

Proof.

$$\langle\alpha|\beta\rangle = \exp\left[-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2 + \alpha^*\beta\right]. \tag{9.20.8}$$

$$|\langle\alpha|\beta\rangle|^2 = \langle\alpha|\beta\rangle^*\langle\alpha|\beta\rangle \tag{9.20.9}$$

$$= \exp\left[-|\alpha|^2 - |\beta|^2 + \alpha^*\beta + \beta^*\alpha\right] \tag{9.20.10}$$

$$= \exp\left[-|\alpha-\beta|^2\right]. \tag{9.20.11}$$

We have $|\alpha-\beta| >> 1 \longrightarrow |\langle\alpha|\alpha\rangle| \to 0$, which is called quasi-orthogonal. □

Lemma 9.20.3. *Coherent states are eigenstates of destruction operator* $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$.

Proof.

$$\hat{a}|\alpha\rangle = \exp\left[-\frac{1}{2}|\alpha|^2\right]\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle \tag{9.20.12}$$

$$= \exp\left[-\frac{1}{2}|\alpha|^2\right]\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\hat{a}|n\rangle \tag{9.20.13}$$

$$= \exp\left[-\frac{1}{2}|\alpha|^2\right]\sum_{n=1}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\sqrt{n}|n-1\rangle \tag{9.20.14}$$

$$= \alpha\exp\left[-\frac{1}{2}|\alpha|^2\right]\sum_{n=1}^{\infty}\frac{\alpha^{n-1}}{\sqrt{(n-1)!}}|n-1\rangle \tag{9.20.15}$$

k:= n-1

$$= \alpha\exp\left[-\frac{1}{2}|\alpha|^2\right]\sum_{k=0}^{\infty}\frac{\alpha^k}{\sqrt{k!}}|k\rangle \tag{9.20.16}$$

$$= \alpha|\alpha\rangle. \tag{9.20.17}$$

□

Lemma 9.20.4. *The average photon number of coherent states: $\overline{n} = |\alpha|^2$.*

Proof.

$$\begin{aligned}
\overline{n} &= \langle\alpha|\hat{n}|\alpha\rangle && (9.20.18)\\
&= \langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle && (9.20.19)\\
&\text{Use ??}\\
&= \langle\alpha^*\alpha|\alpha\rangle && (9.20.20)\\
&= |\alpha|^2\langle\alpha|\alpha\rangle && (9.20.21)\\
&= |\alpha|^2. && (9.20.22)
\end{aligned}$$

□

[SZQ: 2023.04.20: coherent state is robust!]

Lemma 9.20.5. *Photon number variance of coherent states is*

$$(\Delta n)^2 = \overline{n}. \quad (9.20.23)$$

Proof.

$$\begin{aligned}
(\Delta n)^2 &= \langle\alpha|\hat{n}^2|\alpha\rangle - \langle\alpha|\hat{n}|\alpha\rangle^2 && (9.20.24)\\
&= \langle\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}|\alpha\rangle - |\alpha|^4 && (9.20.25)\\
&\hat{a}\hat{a}^\dagger = \hat{a}^\dagger\hat{a} + 1\\
&= \langle\alpha|\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}|\alpha\rangle + \langle\hat{a}^\dagger\hat{a}|\alpha\rangle - |\alpha|^4 && (9.20.26)\\
&= |\alpha|^2 && (9.20.27)\\
&= \overline{n}. && (9.20.28)
\end{aligned}$$

□

Lemma 9.20.6. *The photon number distribution is*

$$
\begin{aligned}
P(n) &= |\langle n|\alpha\rangle|^2 && (9.20.29)\\
&= |\langle n| \exp\left[-\frac{|\alpha|^2}{2}\right] \sum_{n'=0}^{\infty} \frac{\alpha^{n'}}{\sqrt{n!}} |n'\rangle|^2 && (9.20.30)\\
&= |\exp\left[-\frac{|\alpha|^2}{2}\right] \sum_{n'=0}^{\infty} \frac{\alpha^{n'}}{\sqrt{n!}} \langle n|n'\rangle|^2 && (9.20.31)\\
&= |\exp\left[-\frac{|\alpha|^2}{2}\right] \frac{\alpha^{n}}{\sqrt{n!}}|^2 && (9.20.32)\\
&= \exp\left[-|\alpha|^2\right] \frac{|\alpha|^{2n}}{n!} && (9.20.33)
\end{aligned}
$$

Use lemma ??

$$
= \exp\left[-\overline{n}\right] \frac{\overline{n}^n}{n!}. \qquad (9.20.34)
$$

This is known as **Poisson distribution**. The standard deviation is

$$
\Delta n = \sqrt{\overline{n}}. \qquad (9.20.35)
$$

The standard deviation relative to the mean is

$$
\frac{\Delta n}{\overline{n}} = \frac{1}{\sqrt{\overline{n}}}, \qquad (9.20.36)
$$

which shows that the fluctuations is smaller and smaller when the mean becomes larger and larger.

For large $\overline{n}$, we have

$$
P(n) \simeq \frac{1}{\sqrt{2\pi\overline{n}}} \exp\left[-\frac{1}{2}\frac{(n-\overline{n})^2}{\overline{n}}\right]. \qquad (9.20.37)
$$

[SZQ: $\simeq$ means asymptoticallly equal to.]

Lemma 9.20.7. *The expectation value of the field operator*

$$
\begin{aligned}
\langle \alpha \hat{E}(\chi)|\alpha\rangle &= \frac{1}{2}(\langle\alpha|\hat{a}\exp\left[-i\chi\right] + \hat{a}^\dagger \exp\left[i\chi\right]|\alpha\rangle) && (9.20.38)\\
&= \frac{1}{2}\Big(\langle\alpha|\hat{a}\exp\left[-i\chi\right]|\alpha\rangle + \langle\alpha|\hat{a}^\dagger \exp\left[i\chi\right]|\alpha\rangle\Big) && (9.20.39)\\
&= \frac{1}{2}\left(\alpha\exp\left[-i\chi\right] + \alpha^* \exp\left[i\chi\right]\right) && (9.20.40)\\
&= |\alpha|\cos\left(\chi-\theta\right), && (9.20.41)
\end{aligned}
$$

where α is the complex amplitude and $\alpha = |\alpha|\exp\left[i\theta\right]$.

We can plot $\langle\alpha|\hat{E}(\chi)|\alpha\rangle$ in Phasor Diagram in terms of $|\alpha|$ and θ.

Lemma 9.20.8. *The fluctuations (variance) of the E-field*

$$(\Delta\hat{E}(\chi))^2 = \langle\alpha|\hat{E}(\chi)^2|\alpha\rangle - \langle\alpha|\hat{E}(\chi)|\alpha\rangle^2 = \frac{1}{4}. \tag{9.20.42}$$

Proof.

$$\begin{aligned}
(\Delta\hat{E}(\chi))^2 &= \langle\alpha|\hat{E}(\chi)^2|\alpha\rangle - \langle\alpha|\hat{E}(\chi)|\alpha\rangle^2 && (9.20.43)\\
&= \frac{1}{4}\langle\alpha|\left[\hat{a}\exp[-i\chi] + \hat{a}^\dagger\exp[i\chi]\right]^2|\alpha\rangle - |\alpha|^2\cos^2(\chi-\theta) && (9.20.44)\\
&= \frac{1}{4}\langle\alpha|\hat{a}^2\exp[-2i\chi] + \hat{a}^{\dagger 2}\exp[2i\chi] + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}|\alpha\rangle - |\alpha|^2\cos^2(\chi-\theta) && (9.20.45)\\
&= \frac{1}{4}\left(\alpha^2\exp[-2i\chi] + \alpha^{*2}\exp[2i\chi] + 2|\alpha|^2 + 1\right) - |\alpha|^2\cos^2(\chi-\theta) && (9.20.46)\\
&= \frac{1}{4}\left(|\alpha\exp[-i\chi] + \alpha^*\exp[i\chi]|^2 + 1\right) - |\alpha|^2\cos^2(\chi-\theta) && (9.20.47)\\
&= \frac{1}{4}\left(4|\alpha|^2\cos^2(\chi-\theta) + 1\right) - |\alpha|^2\cos^2(\chi-\theta) && (9.20.48)\\
&= |\alpha|^2\cos^2(\chi-\theta) + \frac{1}{4} - |\alpha|^2\cos^2(\chi-\theta) && (9.20.49)\\
&= \frac{1}{4}. && (9.20.50)
\end{aligned}$$

□

Lemma 9.20.9. *The expectation value of the Energy of Coherent States*

$$\langle\alpha|\hat{H}|\alpha\rangle = \hbar\omega(\overline{n} + 1/2). \tag{9.20.51}$$

Proof.

$$\begin{aligned}
\langle\alpha|\hat{H}|\alpha\rangle &= \hbar\omega\langle\alpha|\hat{n} + \frac{1}{2}|\alpha\rangle && (9.20.52)\\
&= \hbar\omega\left(|\alpha|^2 + 1/2\right) && (9.20.53)\\
&= \hbar\omega\left(\overline{n} + \frac{1}{2}\right). && (9.20.54)
\end{aligned}$$

□

Lemma 9.20.10. *The fluctuations of the energy of coherent States*

$$\Delta H = \hbar\omega|\alpha|. \tag{9.20.55}$$

Proof.

$$(\Delta H)^2 = \langle\alpha|\hat{H}^2|\alpha\rangle - \langle\alpha|\hat{H}|\alpha\rangle^2, \tag{9.20.56}$$

where

$$\langle\alpha|\hat{H}^2|\alpha\rangle = \langle\alpha|(\hbar\omega(\hat{n}+\frac{1}{2}))^2|\alpha\rangle \tag{9.20.57}$$

$$= \hbar^2\omega^2\langle\alpha|\hat{n}^2+\hat{n}+\frac{1}{4}|\alpha\rangle \tag{9.20.58}$$

$$= \hbar^2\omega^2\left(|\alpha|^4+|\alpha|^2+\frac{1}{4}\right). \tag{9.20.59}$$

Then

$$(\Delta H)^2 = \hbar^2\omega^2\left(|\alpha|^4+|\alpha|^2+\frac{1}{4}\right) - \hbar^2\omega^2\left(|\alpha|^2+\frac{1}{2}\right)^2 \tag{9.20.60}$$

$$= 0 \tag{9.20.61}$$

□

[SZQ: 2023.04.21: I derive wrong answer. Where is the mistake?]

9.21 Quadrature Operators and Phase Space of Field States

Definition 9.21.1. *Classical eletromagnetic field is*

$$E(t) = E_0\cos(\omega t+\theta) \tag{9.21.1}$$

$$= E_0\cos\theta\cos\omega t - E_0\sin\theta\sin\omega t \tag{9.21.2}$$

$$= X_1\cos\omega t + X_2\sin\omega t, \tag{9.21.3}$$

where X_1, X_2 *are quadrature variables defined as*

$$X_1 = E_0\cos\theta \tag{9.21.4}$$

$$X_2 = -E_0\sin\theta. \tag{9.21.5}$$

Definition 9.21.2. *Phasor representation of field*

$$a(t) = E_0\exp[-i\theta]\exp[-i\omega t] = a\exp[-i\omega t], \tag{9.21.6}$$

where a *is defined as* $E_0\exp[-i\theta]$. $a(t)$ ***is called the phaor****.*

Lemma 9.21.1 (Relations between the Phasor representation and quadrature variables)**.**

$$a = X_1 + iX_2. \tag{9.21.7}$$

Proof. By definition **??**,

$$\begin{aligned} a &= E_0 \exp\left[-i\theta\right] && (9.21.8)\\ &= E_0 \left(\cos\theta - i\sin\theta\right) && (9.21.9)\\ &= E_0\cos\theta - iE_0\sin\theta && (9.21.10)\\ &\overset{\text{By definition ??}}{=} X_1 + iX_2. && (9.21.11)\end{aligned}$$

□

[SZQ: 2023.04.21: a is determined by E_0 and θ.]

Corollary 9.21.1.

$$X_1 = Re(a) = \frac{1}{2}\left(a + a^*\right) \tag{9.21.12}$$

$$X_2 = Im(a) = \frac{1}{2i}\left(a - a^*\right). \tag{9.21.13}$$

Proof. By lemma **??**, we can prove this corollary. □

Quantum-Quadrature Operators

Definition 9.21.3. *Quadrature operators:*

$$\hat{x} \hat{=} \hat{X}_1 = \frac{1}{2}\left(\hat{a} + \hat{a}^\dagger\right) \tag{9.21.14}$$

$$\hat{p} \hat{=} \hat{X}_2 = \frac{1}{2i}\left(\hat{a} - \hat{a}^\dagger\right). \tag{9.21.15}$$

[SZQ: $\hat{=}$ means "define"] [SZQ: Quadrature operators are **Hermitian operators**, i.e., observables.]

Lemma 9.21.2. *The commutation and uncertainty relations:*

$$\left[\hat{X}_1, \hat{X}_2\right] = \frac{i}{2} \tag{9.21.16}$$

$$\Delta X_1 \Delta X_2 \geq \frac{1}{4}. \tag{9.21.17}$$

Proof. □

Definition 9.21.4. *Generalized quadrature operators have the same commutation relations with quadrature operators:*

$$\hat{X}_\phi = \frac{1}{2}\left(\hat{a}\exp\left[-i\phi\right] + \hat{a}^\dagger \exp\left[i\phi\right]\right) \tag{9.21.18}$$

$$\hat{X}_{\phi+\frac{\pi}{2}} = \frac{1}{2i}\left(\hat{a}\exp\left[-i\phi\right] - \hat{a}^\dagger \exp\left[i\phi\right]\right). \tag{9.21.19}$$

Phase space distribution of field states

Lemma 9.21.3. *In vacuum state* $|0\rangle$*, we have*

$$P^{(0)}(X_1) = |\langle X_1|0\rangle|^2 = \sqrt{\frac{2}{\pi}} \exp\left[-2X_1^2\right] \tag{9.21.20}$$

$$P^{(0)}(X_2) = |\langle X_2|0\rangle|^2 = \sqrt{\frac{2}{\pi}} \exp\left[-2X_2^2\right]. \tag{9.21.21}$$

$$\textit{Fluctuations:}\Delta X_1 = \sqrt{\langle 0|\hat{X}_1^2|0\rangle - \langle 0|\hat{X}_1|0\rangle^2} = \frac{1}{2}. \tag{9.21.22}$$

Lemma 9.21.4. *In fock state* $|n\rangle$*, we have*

$$\begin{aligned} P^{(n)}(X_1) &= |\langle X_1|n\rangle|^2 && (9.21.23)\\ &= \sqrt{\frac{2}{\pi}} \frac{1}{2^n n!} \exp\left[-2X_1^2\right] \left(H_n(\sqrt{2}X_1)\right)^2 && (9.21.24)\\ \Delta X_1 &= \frac{1}{2}\sqrt{2n+1}. && (9.21.25) \end{aligned}$$

Coherent state - A displaced vacuum

Definition 9.21.5. *Displacement operator (shifts any coherent state by* α*)*

$$\hat{D}(\alpha) = \exp\left[\alpha \hat{a}^\dagger - \alpha^* \hat{a}\right]. \tag{9.21.26}$$

Corollary 9.21.2. *Coherent state from vacuum state*

$$|\alpha\rangle = \hat{D}(\alpha)|0\rangle. \tag{9.21.27}$$

Squeezed states of light We have nice pictures in the video.

We have phase squeezed state and amplitude squeezed state.

9.22 The Classical Beamsplitter

Assume all beams have same polarization and frequency. We input E_1, E_2 and output E_3, E_4. There is a nice picture to illustrate it. Then we have

$$E_3 = RE_1 + TE_2 \tag{9.22.1}$$

$$E_4 = T'E_1 + R'E_2, \tag{9.22.2}$$

which is

$$\begin{pmatrix} E_3 \\ E_4 \end{pmatrix} = \begin{pmatrix} R & T \\ T' & R' \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} \tag{9.22.3}$$

Chapter 10

A Nutshell of Quantum Mechanics

一、量子论基础

1. Plank 黑体辐射假说

能量量子化 $E_\nu = h\nu$

2. 光电效应的实验现象 (为什么不能从波动的观点进行解释)

光强大小正比于光子数目；䧳止频率 $\nu = A/h$ ；光子吸收、发射时间很短

3. Einstein 的光量子假说

频率 ν 的电磁波辐射场由宏观多个光量子组成，$E_\nu = h\nu = \hbar\omega$, 逸出功 $A = h\nu - \frac{1}{2}mv^2$.

4. Bohr 的氢原子理论

定态假设；量子化条件即角动量量子化 $L = n\hbar$ ；跃迁条件 $\Delta E = h\left(\nu_1 - \nu_2\right)$ ；给出了能级公式 $E_n \propto -1/n^2$.

5. 德布罗意物质波

$$\lambda = \frac{h}{p}, \quad \boldsymbol{p} = \hbar \boldsymbol{k}$$

二、量子力学的基本概念

1. 量子力学的基本假设 (sy 的 3 条版)

1. 量子坬立系统由量子态描述，量子态是希尔伯特空间中是态矢 $|\psi\rangle$. 其随时间的演化满足薛定谔方程

$$i\hbar\frac{\partial}{\partial t}\Psi = \hat{H}\Psi$$

其中，$\hat{H}$ 是哈密顿量

II. 每一个可观测量 A, 与希尔伯特空间中的一个厄米算符 $\hat{A}$ 相关联。测量结果是它的本征值之一，概率幅是原量子态在该本征值相应的本征态上的分量 (即展开系数)

$$\hat{A}|i\rangle = a_i|i\rangle; |\psi(t)\rangle = \sum_i C_i(t)|i\rangle; p_i(t) = |\langle i \mid \psi(t)\rangle|^2 = |C_i(t)|^2$$

III. 对于由态矢 $|\psi\rangle$ 描述的系统，如测量可观测量 A ，得到结果 a_n ，那么测量刚结束时，系统的状态是

$$\frac{P_n|\psi\rangle}{\sqrt{\langle\psi\,|P_n|\,\psi\rangle}}$$

P_n 为投影到相应于 a_n 对应的 A 的本征矢量张成的子空间。

2. 定态薛定谔方程的得到

$$i\hbar\frac{\partial}{\partial t}\Psi = \hat{H}\Psi = \left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\Psi$$

当 H 不含时间时，可分离变量 $\Psi(\boldsymbol{r}, t) = \psi(\boldsymbol{r})\phi(t)$

$$\begin{aligned} i\hbar\tfrac{1}{\phi}\tfrac{\partial\phi}{\partial t} &= -\tfrac{\hbar^2}{2m}\tfrac{1}{\psi}\nabla^2\psi + V = E \\ \left(-\tfrac{\hbar^2}{2m}\nabla^2 + V\right)\psi &= E\psi, \phi(t) = \exp[-iEt/\hbar] \end{aligned}$$

3. 波函数的统计诠释

$\int_a^b |\Psi(x,t)|^2 dx$ 表示 t 时刻发现粒子处于 a 和 b 之间的几率.

4. 波函数的归一化条件及其意义

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = 1$$

全空间的总几率为 1 。

5. 量子力学中概率流密度的连续性方程

$$\begin{aligned} \tfrac{\partial}{\partial t}\rho + \nabla\cdot j(\boldsymbol{r}, t) &= 0 \\ \rho = |\Psi|^2, j &= \tfrac{i\hbar}{2m}\left[\Psi\nabla\Psi^* - \Psi^*\nabla\Psi\right] \end{aligned}$$

6. 可观测量 A 的期待值

$$\langle\hat{A}\rangle = \int \Psi^*(r,t)\hat{A}\Psi(r,t)\mathrm{d}^3 r$$

7. 为什么可观测量对应厄米算符

$$\langle\hat{Q}\rangle = \langle\hat{Q}\rangle^* \to \langle\psi \mid \hat{Q}\psi\rangle = \langle\hat{Q}\psi \mid \psi\rangle \to \hat{Q} = \hat{Q}^\dagger$$

8. 证明不同本征值的本征函数正交

$$\hat{Q}\psi_i = q_i\psi_i; \quad q_2\left\langle\psi_1 \mid \psi_2\right\rangle = \left\langle\psi_1 \mid \hat{Q}\psi_2\right\rangle = \left\langle\hat{Q}\psi_1 \mid \psi_2\right\rangle = q_1\left\langle\psi_1 \mid \psi_2\right\rangle$$

9. 同时对角化定理 ($\hat{A}, \hat{B}$ 具有共同本征态的条件)

$$[\hat{A}, \hat{B}] = 0$$

10. Gram-Schmidt 正交化法则

$$\psi_i' = \psi_i - \sum_{k=1}^{i-1} \frac{\langle \psi_k' \mid \psi_i \rangle}{\langle \psi_k' \mid \psi_k' \rangle} \psi_k'$$

11. 不确定原理

$$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}; \quad \sigma_x \sigma_p \geq \tfrac{\hbar}{2}; \quad \Delta E \Delta t \geq \tfrac{\hbar}{2}$$
$$\sigma_A^2 \sigma_B^2 \geq \left(\tfrac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2$$

[* 请利用 Schwarz 不等式自行推导]

12. Heisenberg 动力学方程

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \hat{Q} \rangle = \frac{i}{\hbar} \langle [\hat{H}, \hat{Q}] \rangle + \left\langle \frac{\partial \hat{Q}}{\partial t} \right\rangle$$

13. 动量空间的本征函数及其正交与完备条件

$$\hat{p} \psi_p(x) = p \psi_p(x), \quad \psi_p(x) = \tfrac{1}{\sqrt{2\pi\hbar}} \exp[ipx/\hbar]$$
$$\int \psi_{p'}(x) \psi_p(x) \mathrm{d}x = \delta\left(p - p'\right)$$
$$\int \psi_p^*(x) \psi_p(y) \mathrm{d}p = \delta(x - y) \text{ 或 } \int |p\rangle\langle p| = 1$$

14. Dirac 符号下能量本征函数的正交与完备条件

$$\langle m \mid n \rangle = \delta_{mn}, \sum_n |n\rangle\langle n| = 1$$

15. 位置算符与动量算符的对易关系

$$[\hat{x}, \hat{p}_x] = i\hbar, \quad [\hat{x}, \hat{p}_y] = 0$$

三、定态薛定谔方程的解

1. 无限深势阱的解及函数图像

$$\psi_n = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right), \quad E_n = \frac{n^2 \pi^2 \hbar^2}{2ma^2}$$

2. 谐振子的阶梯算符, 对偶关系及其对本征波函数的作用效果

$$\hat{a}_\pm = \tfrac{1}{\sqrt{2\hbar m\omega}} (\mp i\hat{p} + m\omega x), \quad [\hat{a}_-, \hat{a}_+] = 1$$
$$\hat{a}_+ |n\rangle = \sqrt{n+1} |n+1\rangle, \quad \hat{a}_- |n\rangle = \sqrt{n} |n-1\rangle$$

3. 谐振子的本征函数图像及总能

$$\hat{H} = \left(\hat{a}_+ \hat{a}_- + \frac{1}{2}\right) \hbar\omega, \quad E_n = \left(n + \frac{1}{2}\right) \hbar\omega$$

4. 束缚态与散射态的定义

束缚态: 能量 $E<V(-\infty)$ and $V(\infty)$, 波函数可归一化

散射态: 能量 $E>V(-\infty)$ or $V(\infty)$, 波函数不可归一化

5. δ 势阱中波函数满足的条件

波函数连续 $\psi\left(0^{-}\right)=\psi\left(0^{+}\right)$

波函数的一阶导连续 $\int_{0^{-}}^{0^{+}}\left[-\frac{\hbar^{2}}{2m}\frac{\mathrm{d}^{2}\psi}{\mathrm{d}x^{2}}-\alpha\delta(x)\psi\right]\mathrm{d}x=E\int_{0^{-}}^{0^{+}}\psi\mathrm{d}x$

$$\rightarrow \psi'\left(0^{+}\right)-\psi'\left(0^{-}\right)=-\frac{2\alpha m}{\hbar^{2}}\psi(0)$$

6. $V=-\alpha\delta$ 势阱中束缚态的解及函数图像

$$\psi(x)=\frac{\sqrt{m\alpha}}{\hbar}\exp\left[-m\alpha|x|/\hbar^{2}\right],\quad E=-\frac{m\alpha^{2}}{2\hbar^{2}}$$

7. $V=-\alpha\delta$ 势阱中反射率 R 与透射率 T ，其中左侧波函数为 $e^{ikx}+re^{-ikx}$, 右侧波函数为 te^{ikx}

$$\beta=\frac{\alpha m}{\hbar^{2}k},\quad r=\frac{i\beta}{1-i\beta},\quad t=\frac{1}{1-i\beta}$$

$$R=|r|^{2}=\frac{\beta^{2}}{1+\beta^{2}},T=|t|^{2}=\frac{1}{1+\beta^{2}}$$

8. 有限深势阱的束缚态的奇函数解

$$\psi(x)=\begin{cases}Be^{-\kappa x} & x>a\\ D\cos(lx), & 0<x<a\\ \psi(-x) & x<0\end{cases}$$

$$\kappa=\frac{\sqrt{-2mE}}{\hbar},l=\frac{\sqrt{2m\left(E+V_{0}\right)}}{\hbar},\kappa a=la\tan(la)$$

9. 自由粒子解

$$\psi(x,t)=\sqrt{\tfrac{1}{L}}e^{ik_{n}x}e^{-iE_{n}t/\hbar},k_{n}=\tfrac{2n\pi}{L}=\pm\tfrac{\sqrt{2mE_{n}}}{\hbar}$$

$$\Psi(x,t)=\tfrac{1}{\sqrt{2\pi}}\int\phi_{k}\exp\left[ikx-i\tfrac{\hbar k^{2}}{2m}t\right]dk$$

四、三维空间中的量子力学

1. 氢原子薛定谔方程中的哈密顿量

$$\hat{H}=-\frac{\hbar^{2}}{2m}\nabla^{2}-\frac{e^{2}}{4\pi\epsilon_{0}}\frac{1}{r}$$

2. 氢原子薛定谔方程的解

$$\hat{H}\psi_{nlm}=E_{n}\psi_{nlm},\quad \psi_{nlm}=R_{nl}(r)Y_{l}^{m}(\theta,\phi)$$

$$n=1,2,3,\cdots,\quad l=0,1,\cdots,n-1\quad m=-l,-l+1,\cdots,l-1,l$$

$$E_{n}=\tfrac{E_{1}}{n^{2}},E_{1}=\tfrac{m}{2\hbar^{2}}\left(\tfrac{e^{2}}{4\pi\epsilon_{0}}\right)^{2}=-13.6\mathrm{eV}$$

3. (自旋/轨道) 角动量的对易关系

$$\left[\hat{L}_x, \hat{L}_y\right] = i\hbar\hat{L}_z, \quad \left[\hat{L}_y, \hat{L}_z\right] = i\hbar\hat{L}_x, \quad \left[\hat{L}_z, \hat{L}_x\right] = i\hbar\hat{L}_y, \quad \left[\hat{L}^2, \hat{L}\right] = 0$$

$\hat{S}$ 同理

[* 请自行从 $L = r \times p$ 推导得到]

4. 轨道角动量与自旋角动量的本征方程

$$\begin{aligned}\hat{\boldsymbol{L}}^2|lm\rangle = \hbar^2 l(l+1)|lm\rangle, \quad \hat{L}_z|lm\rangle = \hbar m|lm\rangle \\ l = 0, 1, 2, \cdots \quad m = -l, -l+1, \cdots, l-1, l \\ \hat{\boldsymbol{S}}^2|sm\rangle = h^2 s(s+1)|sm\rangle, \quad \hat{S}_z|sm\rangle = hm|sm\rangle \\ s = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \cdots \quad m = -s, -s+1, \cdots, s-1, s\end{aligned}$$

5. 角动量的上升下降算符，及其对本征波函数的作用

$$\hat{L}_\pm = \hat{L}_x \pm iL_y, \quad \hat{L}_\pm|lm\rangle = \hbar\sqrt{l(l+1) - m(m\pm 1)}|l, m\pm 1\rangle$$

$\hat{S}$ 同理

6. 1/2 自旋的角动量在本征波函数空间的矩阵表示

$$\begin{aligned}&\hat{S} = \frac{\hbar}{2}\sigma \\ &\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\end{aligned}$$

7. 自旋在磁场中受到的相互作用及拉莫尔进动频率

$$\boldsymbol{\mu} = \gamma\boldsymbol{S}, \quad \hat{H} = -\gamma\boldsymbol{B}\cdot\hat{\boldsymbol{S}}, \quad \omega = \gamma B_0$$

8. 角动量的叠加，$\hat{S} = \hat{S}_1 + \hat{S}_2$, 则 $\hat{\boldsymbol{S}}^2, \hat{S}_z$?

$$\hat{S}^2 = \hat{S}_1^2 + \hat{\boldsymbol{S}}_2^2 + 2\hat{\boldsymbol{S}}_1\cdot\hat{\boldsymbol{S}}_2, \quad \hat{S}_z = \hat{S}_{1z} + \hat{S}_{2z}$$

9. 2 个自旋 1/2 的相加

$$\text{三重态}(s=1): \begin{cases} |1,1\rangle = |\uparrow\uparrow\rangle \\ |1,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \\ |1,-1\rangle = |\downarrow\downarrow\rangle \end{cases}$$

单态 $(s=0): |0,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$

10. 角动量叠加后，本征态用 Clebsch-Gordan (CG) 系数和原本的态表示

$$|sm\rangle = \sum_{m_1+m_2=m} C^{sm}_{s_1m_1;s_2m_2}\,|s_1m_1; s_2m_2\rangle$$

五、全同粒子

1. 玻色子和费米子的区别

玻色子: 自旋为整数，$\psi_{+}\left(r_{1}, r_{2}\right)=\psi_{+}\left(r_{2}, r_{1}\right)$

费米子: 自旋为半整数，$\psi_{-}\left(r_{1}, r_{2}\right)=-\psi_{-}\left(r_{2}, r_{1}\right)$

2. 由两个全同粒子组成的波函数为 (考虑自旋与否)

$$\psi_{\pm}\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}\right)=A\left[\psi_{a}\left(\boldsymbol{r}_{1}\right) \psi_{b}\left(\boldsymbol{r}_{2}\right) \pm \psi_{b}\left(\boldsymbol{r}_{1}\right) \psi_{a}\left(\boldsymbol{r}_{2}\right)\right]$$

考虑自旋的玻色子: $\psi_{\pm}\left(r_{1}, r_{2}\right)\left[\chi_{a}\left(s_{1}\right) \chi_{b}\left(s_{2}\right) \pm \chi_{b}\left(s_{1}\right) \chi_{a}\left(s_{2}\right)\right]$

考虑自旋的费米子: $\psi_{\pm}\left(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}\right)\left[\chi_{a}\left(\boldsymbol{s}_{1}\right) \chi_{b}\left(\boldsymbol{s}_{2}\right) \mp \chi_{b}\left(\boldsymbol{s}_{1}\right) \chi_{a}\left(\boldsymbol{s}_{2}\right)\right]$

3. 证明费米子满足泡利不相容原理

若 $\psi_{a}=\psi_{b}$ ，则 $\psi_{-}=0$

4. 交换相互作用的表现

在不考虑自旋情况下，空间对称波函数（玻色子）受到牵引力，空间反对称波函数（费米子）受到排斥力

六、定态微扰理论

1. 非简并微扰 (一阶波函数，二阶能量，微扰项为 $\hat{H}^{\prime}$)

$$E_{n}=E_{n}^{0}+\left\langle\psi_{n}^{0}\left|\hat{H}^{\prime}\right| \psi_{n}^{0}\right\rangle+\sum_{m \neq n} \frac{\left|\left\langle\psi_{m}^{0}\left|\hat{H}^{\prime}\right| \psi_{n}^{0}\right\rangle\right|^{2}}{E_{n}^{0}-E_{m}^{0}}$$
$$\psi_{n}=\psi_{n}^{0}+\sum_{m \neq n} \frac{\left\langle\psi_{m}^{0}\left|\hat{H}^{\prime}\right| \psi_{n}^{0}\right\rangle}{E_{n}^{0}-E_{m}^{0}} \psi_{m}^{0}$$

[* 请自行推导]

2. 非简并微扰成立条件

能级非简并，且 $\left|\left\langle\psi_{m}^{0}\left|\hat{H}^{\prime}\right| \psi_{n}^{0}\right\rangle\right| \ll\left|E_{n}^{0}-E_{m}^{0}\right|$

3. 近简并微扰

$$\left(\begin{array}{cccc}H_{11}^{\prime} & H_{12}^{\prime} & \cdot & H_{1 g}^{\prime} \\ \vdots & \vdots & \ddots & \vdots \\ H_{g 1}^{\prime} & H_{g 2}^{\prime} & \cdot & H_{g g}^{\prime}\end{array}\right)\left(\begin{array}{c}a_{1} \\ \vdots \\ a_{g}\end{array}\right)=E^{1}\left(\begin{array}{c}a_{1} \\ \vdots \\ a_{g}\end{array}\right)$$
$$H_{i j}^{\prime}=\left\langle\psi_{n, i}^{0}\left|\hat{H}^{\prime}\right| \psi_{n, j}^{0}\right\rangle, \quad \phi_{n}=\sum_{i} a_{i} \psi_{n, i}$$

4. 氢原子哈密顿量的相对论效应修正

$$T=\sqrt{p^{2} c^{2}+m^{2} c^{4}}-m c^{2} \approx \frac{p^{2}}{2 m}-\frac{p^{4}}{8 m^{3} c^{2}} \bigg(\text{泰勒展开}) \rightarrow H^{\prime}=-\frac{p^{4}}{8 m^{3} c^{2}}$$

5. 氢原子的自旋轨道耦合修正的哈密顿量及对应的好量子数

$$B=\frac{\mu_0 I}{2r}=\frac{1}{4\pi\epsilon_0}\frac{e}{mc^2r^3}\boldsymbol{L},\boldsymbol{\mu}_e=-\frac{e}{m}\boldsymbol{S}$$
$$\hat{H}'=\frac{1}{2}\frac{e^2}{4\pi\epsilon_0}\frac{1}{m^2c^2r^3}\hat{\boldsymbol{S}}\cdot\hat{\boldsymbol{L}}$$
$$\left\{\hat{H}_0,\hat{L}^2,\hat{S}^2,\hat{J}^2,\hat{\boldsymbol{J}}_z\right\},\hat{\boldsymbol{J}}=\hat{\boldsymbol{L}}+\hat{\boldsymbol{S}}$$

6. 外加电场下，氢原子 Stark 效应的微扰哈密顿量

$$\hat{H}'=-qEz=eE\hat{z}$$

7. 氢原子的精细结构能级修正

$$E_n^{(1)}=\frac{E_n^2}{2mc^2}\left(3-\frac{4n}{j+1/2}\right)$$

8. 外加磁场下，氢原子 Zeeman 效应的微扰哈密顿量

$$\hat{H}'=-\left(\boldsymbol{\mu}_L+\boldsymbol{\mu}_S\right)\cdot\boldsymbol{B}=\frac{e}{2m}(\hat{\boldsymbol{L}}+2\hat{\boldsymbol{S}})\cdot\boldsymbol{B}=\frac{eB}{2m}\left(\hat{L}_z+2\hat{S}_z\right)$$

[* 请自行推导，在强弱磁场下该如何选择好量子数及对应的能量修正]

七、变分原理与 WKB 近似

1. 变分原理

$$E_{gs}\leq\langle\psi|\hat{H}|\psi\rangle$$

取到等号时即对应基态能量与波函数。

2. WKB 近似公式

经典区域 $(E>V):\psi(x)=\frac{C}{\sqrt{p(x)}}\exp\left[\pm\frac{i}{\hbar}\int p(x)\mathrm{d}x\right],\quad p(x)=\sqrt{2m(E-V(x))}$

隧穿区域 $(E<V):\psi(x)=\frac{C}{\sqrt{|p(x)|}}\exp\left[\pm\frac{1}{\hbar}\int p(x)\mathrm{d}x\right]$

八、含时微扰理论

1. 含时微扰理论的系数满足的方程

$$i\hbar\frac{d}{dt}c_m(t)=\sum_n c_n(t)\left\langle m\left|\hat{H}'(t)\right|n\right\rangle e^{i\omega_{mn}t},\quad \omega_{mn}=\frac{E_m-E_n}{\hbar}$$

2. 2 能级系统含时微扰理论的一级修正 $(c_m(0)=0,c_n(0)=1)$

$$c_m(t)=\frac{1}{i\hbar}\int_0^t\left\langle m\left|\hat{H}'\left(t'\right)\right|n\right\rangle e^{i\omega_{mn}t'}\mathrm{d}t',\quad c_n(t)=1$$
$$\omega_{mn}=\frac{E_m-E_n}{\hbar},\quad P_{n\to m}=\left|c_m(t)\right|^2$$

3. 正弦微扰 $H'(\boldsymbol{r},t)=V(r)\cos(\omega t)\,(\omega_0+\omega>>|\omega_0-\omega|)$ 下的跃迁几率

$$P_{a\to b}(t)=\frac{\left|V_{ab}\right|^2}{\hbar^2}\frac{\sin^2\left[\left(\omega_0-\omega\right)t/2\right]}{\left(\omega_0-\omega\right)^2}$$

4，光与原子相互作用的三种类型及辐射的物理原因

吸收、受激辐射与自发辐射

受激发射：一个光子进入，两个光子出来，包含引起跃迁的原始光子和另一个来自原子的光子、可以通过计算跃迁几率得到。是激光的理论基础。

$$P_{2\rightarrow 1}(t)=\left(\frac{\left|e\left\langle\psi_1|z|\psi_2\right\rangle\right|E_0}{\hbar}\right)^2\frac{\sin^2\left[\left(\omega_0-\omega\right)t/2\right]}{\left(\omega_0-\omega\right)^2},\quad E_2-E_1=\hbar\omega_0$$

自发辐射其实并不是真正的"自发"，而是受到热涨落或电磁波的量子零点涨落（真空涨落）的刺激。

5. 跃迁的选择定则及物理原因

$$\boldsymbol{P}=e\left\langle\psi_2|\boldsymbol{r}|\psi_1\right\rangle\neq 0\rightarrow\Delta m=0,\pm 1\text{ 且}\Delta l=\pm 1$$

因为光子的自旋为 1 , 角动量 (z 分量) 守恒要求原子失去的等于光子获得的角动量, 角动量的叠加规律只允许 $l'=l+1, l'=l, l'=l-1$, 但对于电偶极辐射，$l'=l$ 不会发生。

九、散射理论

1. 考虑平面波入射的散射情形下的波函数

$$\psi(r,\theta,\phi)=A\left[e^{ikx}+f(\theta,\phi)\frac{e^{ikr}}{r}\right]$$

2. 微分散射截面与散射截面

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}=|f(\theta,\phi)|^2,\sigma=\int\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\mathrm{d}\Omega$$

3. l 分波具有 δ_l 的相移时，对应的散射截面

$$\sigma=\frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\left(\delta_t\right)$$

4. 散射理论中的 Born 近似及球对称下的表述

$$f(\theta,\phi)=-\frac{m}{2\pi\hbar^2}\int V\left(\boldsymbol{r}_0\right)\exp\left[i\left(\boldsymbol{k}'-\boldsymbol{k}\right)\cdot\boldsymbol{r}_0\right]\mathrm{d}^3\boldsymbol{r}_0$$
$$f(\theta,\phi)=-\frac{2m}{\hbar^2\kappa}\int rV(r)\sin(\kappa r)\mathrm{d}r,\quad\kappa=2k\sin\frac{\theta}{2}$$

θ 为 k,k' 之间的夹角

Part V

Quantum Information

Chapter 11

Estimation

11.1 DFE

Chapter 12

Sensing

12.1 Joint measurement of TFE via SFG

12.2 Mathematical preliminaries

12.3 Physics preminaries

12.3.1 SFE

The SFG process in a $\chi^{(2)}$ nonlinear medium could be modeled as the following evolution operator:

$$V = I + \varepsilon \left(\int \mathrm{d}\omega_p \mathrm{d}\omega_s \mathrm{d}\omega_i a_p^\dagger(\omega_p) a_s(\omega_s) a_i(\omega_i) \delta(\omega_p - \omega_s - \omega_i) - H.C. \right), \tag{12.3.1}$$

where photons in the signal mode $a_s(\omega_s)$ and ideler mode $a_i(\omega_i)$ are annihilated to generate photons in the pump mode $a_p(\omega_p)$ and ε characterizes the interaction strength. The SPDC process is the time-reversal process of SFG, which can also be described by the same revolution operator.

12.3.2 SPDC

12.4 Introduction

12.5 Terminology

12.6 Problem formulation

The frequency sum and time difference of two photons could be simultaneously measured through the sum-frequency generation process.

12.7 Protocol

Given the close connection between the spontaneous parametric down-conversion (SPDC) process and time-frequency entanglement (TFE), it's natural to utilize the time-reversal of the SPDC process, i.e., sum frequency generation (SFG) to obtain a TFE joint measurement based protocol.

Definition 12.7.1 (Frequency sum (FS) operator). *The frequency sum operator $P_{\delta_\omega}(\omega)$ that selects states with the frequency sum $\omega_s + \omega_i$ of the signal and idler photon being around ω within uncertatinty δ_ω is defined as:*

$$P_{\delta_\omega}(\omega) = \int\int \mathrm{d}\omega_s \mathrm{d}\omega_i a_s^\dagger(\omega_s) a_i^\dagger(\omega_i) a_s(\omega_s) a_i(\omega_i) \mathrm{Gate}\left(\frac{\omega - \omega_s - \omega_i}{\delta_\omega}\right), \tag{12.7.1}$$

where $\mathrm{Gate}(x) = 1$ *for* $|x| \leq 1/2$ *and* $\mathrm{Gate}(x) = 0$ *otherwise.*

Lemma 12.7.1 (Frequency sum operator is a projection operator). *The frequency sum operator $P_{\delta_\omega}(\omega)$ is a projection operator satisfying*

$$P_{\delta_\omega}(\omega)^2 = P_{\delta_\omega}(\omega). \tag{12.7.2}$$

Definition 12.7.2 (Time difference (TD) operator). *The time difference operator $P_{\delta_t}(t)$ that selects states with the time difference $t_s - t_i$ of the signal and idler photon being around t within uncertatinty δ_t is defined as:*

$$P_{\delta_t}(t) = \int\int \mathrm{d}t_s \mathrm{d}t_i \tilde{a_s}^\dagger(t_s) \tilde{a_i}^\dagger(t_i) \tilde{a_s}(t_s) \tilde{a_i}(t_i) \mathrm{Gate}\left(\frac{t_s - t_i - t}{\delta_t}\right), \tag{12.7.3}$$

where $\mathrm{Gate}(x) = 1$ *for* $|x| \leq 1/2$ *and* $\mathrm{Gate}(x) = 0$ *otherwise and*

$$\tilde{a}_x = \frac{1}{\sqrt{2\pi}} \int \mathrm{d}\omega \exp\left(-i\omega t\right) a_x(\omega) \tag{12.7.4}$$

Lemma 12.7.2 (Time difference operator is a projection operator)**.** *The time difference operator $P_{\delta_t}(t)$ is a projection operator satisfying*

$$P_{\delta_t}(t)^2 = P_{\delta_t}(t). \tag{12.7.5}$$

Definition 12.7.3 (Joint projection operator)**.** *The joint projection operator of the time difference and frequency sum is defined as*

$$P_{\delta_\omega,\delta_t}(\omega, t) = P_{\delta_\omega}(\omega)P_{\delta_t}(t), \tag{12.7.6}$$

which means selecting states of which the time difference between the signal and idler photon $t_s - t_i$ is around t within uncertainty δ_t and frequency sum of the signal and idler photon being around ω within unicertainty δ_ω, simultaneously.

Lemma 12.7.3 (Commutation relationship between frequency-time operators)**.** *We have the commutation relationship*

$$[P_{\delta_\omega}(\omega), P_{\delta_t}(t)] = 0. \tag{12.7.7}$$

Definition 12.7.4 (TD an FS probability density operator (PDF))**.** *We define*

$$P(\omega, t) = \lim_{\delta_t\to 0,\delta_\omega\to 0} \frac{1}{\delta_\omega\delta_t} P_{\delta_\omega}(\omega)P_{\delta_t}(t). \tag{12.7.8}$$

Lemma 12.7.4. *We have*

$$P(\omega, t) = \frac{1}{2\pi} B_p^\dagger B_p, \tag{12.7.9}$$

where

$$B_p = \int\int \mathrm{d}\omega_s \mathrm{d}\omega_i \delta(\omega_s + \omega_i - \omega) \exp\left[i\omega_i t\right] a_s(\omega_s)a_i(\omega_i). \tag{12.7.10}$$

Lemma 12.7.5 (Connection between TD and FS PDF and SFG Process)**.** *We have*

$$B^\dagger B = 2\pi P(\omega, 0). \tag{12.7.11}$$

This lemma did not show the SFG process

Lemma 12.7.6 (Discrete sum of evolution operator of SFG process)**.** *The discrete sum of the evolution operator of SFG process can be obtained by a* ***two-step Schmidt decomposition*** *as:*

$$v = I + \varepsilon \sum_m \left(\sqrt{\lambda_m^{(1)}} A_m^\dagger B_m - H.C.\right), \tag{12.7.12}$$

where

$$B_m = \sum_n \sqrt{\lambda^{(2)}_{m,n}} F_{m,n} G_{m,n}, \tag{12.7.13}$$

$$A_m = \int \mathrm{d}\omega \psi_{A,m}(\omega) a_p(\omega), \tag{12.7.14}$$

$$B_m = \int \mathrm{d}\omega_s \mathrm{d}\omega_i \psi_{B,m}(\omega_s, \omega_i) a_s(\omega_s) a_i(\omega_i), \tag{12.7.15}$$

$$F_{m,n} = \int \mathrm{d}\psi_{F,m,n}(\omega) a_s(\omega), \tag{12.7.16}$$

$$G_{m,n} = \int \mathrm{d}\psi_{G,m,n}(\omega) a_i(\omega). \tag{12.7.17}$$

Lemma 12.7.7 (Non-uniqueness of the first step Schmidt Decomposition)**.** *If the function f_0 can be written in the following form:*

$$\delta(\omega_p - \omega_s - \omega_i) f_0(\omega_p - \omega_s - \omega_i) = \delta(\omega_p - \omega_s - \Omega_i) f(\frac{\omega_s - \Omega_i}{\sqrt{2}}), \tag{12.7.18}$$

then the first step Schmidt decomposition in the main text is not unique.

Lemma 12.7.8 (An useful communtation relationship)**.** *An useful communtation relationship:*

$$[B_{m'}, B^\dagger_{m''}] = \delta_{m'm''} + \int \mathrm{d}\omega'_s \mathrm{d}\omega_i \mathrm{d}\omega''_s \psi^*_{B,m''}(\omega'_s, \omega_i) a^\dagger_s(\omega''_s) a_s(\omega'_s) \tag{12.7.19}$$

$$+ \int \mathrm{d}\omega_s \mathrm{d}\omega'_i \mathrm{d}\omega''_i \psi^*_{B,m''}(\omega_s, \omega''_i) \psi_{B,m'} a^\dagger_i(\omega''_i) a_i(\omega'_i). \tag{12.7.20}$$

Corollary 12.7.1. *By (**??**), we have*

$$[B_{m'}, B^\dagger_{m''}] = \delta_{m'm''} |0\rangle. \tag{12.7.21}$$

Lemma 12.7.9. *We have the commutation relation between time difference projection operator and frequency sum projection operator:*

$$[P_{\delta_\omega}(\omega), P_{\delta_t}(t)] = 0. \tag{12.7.22}$$

Lemma 12.7.10. *The frequency spectrum $S(\omega)$ of the generated pump photon is given by the expectation value of the spectral density operator $a^\dagger_p(\omega_p) a_p(\omega_p)$*

$$S(\omega_p) = \frac{\epsilon^2 \exp\left[\frac{1}{8}\left(-4\Delta t^2 \sigma_-^2 - \frac{\Delta\omega^2}{\sigma_-^2} - \frac{4(\Delta\omega + \omega_0 - \Omega_p)^2}{\sigma_+^2}\right)\right]}{2\sqrt{\pi}\sigma_+}. \tag{12.7.23}$$

Definition 12.7.5. *The discrete mode operator $F^{(}_{m,n}b)$ for the noise photons is defined as*

$$F^{(b)}_{m,n} = \int \mathrm{d}\omega \psi_{F_{m,n}}(\omega) a^{(b)}_s(\omega). \tag{12.7.24}$$

Definition 12.7.6. *The virtual beam-splitter is modeledd as the following unitary transform:*

$$U_{loss} = \Pi_n \exp\left[i\arccos(\eta)(F^{\dagger}_{0,n}F^{(b)}_{0,n} + H.C.)\right]. \tag{12.7.25}$$

Definition 12.7.7. *We use a density matrix ρ_b that satisfies the following conditions to describe the noise photons:*

$$\mathrm{Tr}[F^{(b)\dagger}_{0,n''}F^{(b)}_{0,n'}\rho_b] = \delta_{n',n''}\mu_b, \tag{12.7.26}$$
$$\mathrm{Tr}[F^{(b)}_{0,n'}\rho_b] = 0 \tag{12.7.27}$$

Definition 12.7.8. *The signal and idler photon pair source is described by the biphoton state $|pair\rangle$:*

$$|pair\rangle = B^{\dagger}_0|0\rangle \tag{12.7.28}$$
$$= \sum_n \sqrt{\lambda^{(2)}_{0,n}} F^{\dagger}_{0,n}G_{0,n}|0\rangle. \tag{12.7.29}$$

Definition 12.7.9. *The unitary trasform of the SFG process is given by:*

$$V = I + \epsilon\sum_m \left[\sqrt{\lambda^{(1)}_m} A^{\dagger}_m B_m - H.C.\right]. \tag{12.7.30}$$

Definition 12.7.10. *In the Heisenberg picture, the photon number operator of the generated pump photon in each pump mode A_m after the beam-splitter transform and the SFG process is given by:*

$$U^{\dagger}_{loss}V^{\dagger}A^{\dagger}_m V U_{loss}. \tag{12.7.31}$$

Proposition 12.7.1. *When the transmission of the signal photon is perfect ($\eta = 1$), the pump photon can only generate in mode $A_0(m = 0)$.*

Lemma 12.7.11. *We have*

$$\langle U^{\dagger}_{loss}V^{\dagger}A^{\dagger}_0 A_0 V|U_{loss}\rangle = \epsilon^2\lambda^{(1)}_0(\eta + \mu_b\sum_n \lambda^2_{0,n}). \tag{12.7.32}$$

Definition 12.7.11. *The generated SPDC state is given by:*

$$V = |0\rangle - \epsilon\sqrt{\lambda^{(1)}_0}\alpha B^{\dagger}_0|0\rangle. \tag{12.7.33}$$

Lemma 12.7.12. *The joint density operator of the noise-idler state ρ_j is given by the tensor product of ρ_i and ρ_b:*

$$\rho_j = \rho_i \otimes \rho_b \tag{12.7.34}$$
$$= \mu_b \int\int \mathrm{d}\omega_s \mathrm{d}\omega'_s \int\int \mathrm{d}\omega'_i \mathrm{d}\omega''_i \phi^*_0(\omega'_s,\omega'_i)\phi_0(\omega'_s,\omega''_i)a^{\dagger}_i(\omega'_i)a^{\dagger}_s(\omega_s)|0\rangle\langle 0|a_i(\omega''_i)a_s(\omega_s). \tag{12.7.35}$$

Lemma 12.7.13. *The spectral density $S(\omega)$ of the upconverted photons is*

$$S(\omega) = \frac{\epsilon^2 \mu_b \exp\left[-\frac{(\omega-\omega_0)^2}{8\sigma_-^2 - 2\sigma_+^2}\right]}{\sqrt{\pi}\sqrt{4\sigma_-^2 + \sigma_+^2}}. \quad (12.7.36)$$

Theorem 11. *The error exponent of the classical Chernoff bound of the TFE QI protocol is given by C_{QI} [?]:*

$$C_{QI} = -\log \min_{s\in[0,1]} \left\{ \sum_{b\in 0,1} p_0(b)^s p_1(b)^{(1-s)} \right\}. \quad (12.7.37)$$

12.8 Performance evaluation

Lemma 12.8.1.

$$ \quad (12.8.1)$$

Chapter 13

Imaging

13.1 Quantum and non-local effects offer over 40 dB noise resilience advantage towards quanutm lidar