

# Note

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# Chapter 1

## Introduction to Quantum Optics

### 1.1 Introduction

- classical: classical atom and light
- semiclassical: quantized atom and classical light
- quantum mechanical: quantized atom and light

#### Light-Atom Interaction Hamiltonian

- classical dipole in electric field: dipole moment  $\vec{d} = q\vec{r}$ ,  $U_I = -\vec{d} \cdot \vec{E}$ . We have

$$\hat{H}_I = -\hat{d} \cdot \vec{E}(\vec{r}_0, t), \quad (1.1.1)$$

where  $\hat{d} = q\hat{v}$  is the dipole operator.

- induced atomic dipole

### 1.2 Light Atom Quantum Evolution

**Time Evolution** We have the Schrodinger equation (both sides) as

$$i\hbar \frac{\partial}{\partial t} |\Psi(t)\rangle = (\hat{H}_0 + \hat{H}_I(t)) |\Psi(t)\rangle, \quad (1.2.1)$$

where the general ansatz (assumption) is

$$|\Psi(t)\rangle = \sum_n c_n(t) e^{-iE_n t/\hbar} |n\rangle, \quad (1.2.2)$$

and

$$\hat{H}_0|n\rangle = E_n|n\rangle \quad (1.2.3)$$

is the atomic eigenstates. Inserting  $|\Psi(t)\rangle$  and  $\hat{H}_0|n\rangle$  into Schrodinger equation, we get

$$i\hbar \sum_n \left\{ \dot{c}_n e^{-iE_n t/\hbar} |n\rangle - \frac{iE_n}{\hbar} c_n e^{-iE_n t/\hbar} |n\rangle \right\} = \sum_n \left\{ c_n e^{-iE_n t/\hbar} |n\rangle + c_n e^{-iE_n t/\hbar} \hat{H}_I |n\rangle \right\} \quad (1.2.4)$$

$$\implies i\hbar \sum_n \dot{c}_n e^{-iE_n t/\hbar} |n\rangle = \sum_n c_n e^{-iE_n t/\hbar} \hat{H}_I |n\rangle \quad (1.2.5)$$

$$\implies i\hbar \dot{c}_n e^{-iE_n t/\hbar} = \sum_n c_n(t) e^{-iE_n t/\hbar} \langle n | \hat{H}_I(t) | n \rangle \quad (1.2.6)$$

$$\implies i\hbar \dot{c}_k = \sum_n c_n(t) e^{-iE_{n,k} t/\hbar} \langle k | \hat{H}_I(t) | n \rangle, \quad (1.2.7)$$

where we use

$$\langle k | n \rangle = \delta_{kn}, \quad (1.2.8)$$

$$E_{n,k} = E_n - E_k, \quad (1.2.9)$$

$$\omega_{nk} = (E_n - E_k)/\hbar. \quad (1.2.10)$$

and  $\langle k | \hat{H}_I(t) | n \rangle$  is the matrix element.

### 1.3 Time Dependent Perturbation Theory

Recall the time evolution:

$$i\hbar \dot{c}_k = \sum_n c_n(t) e^{-i\omega_{nk} t} \langle k | \hat{H}_I(t) | n \rangle, \quad (1.3.1)$$

and

$$\omega_{nk} = (E_n - E_k)/\hbar. \quad (1.3.2)$$

Consider the Simplification (Perturbation Theory)

- System only in state  $|1\rangle$  at  $t = 0 \implies c_1|0\rangle = 1$  (only the ground state  $|1\rangle$ ),
- Perturbative treatment of interaction term: weak perturbation  $\forall |c_k(t)|^2 < 1$ .

We then have

$$i\hbar \dot{c}_k = e^{i\omega_{1k} t} \langle k | \hat{H}_I(t) | 1 \rangle, \quad (1.3.3)$$

with  $c_k(0) = 0$ , we obtain:

$$c_k(t) = \frac{1}{i\hbar} \int_0^t e^{-i\omega_{1k} t'} \langle k | \hat{H}_I(t') | 1 \rangle dt'. \quad (1.3.4)$$

**Example 1.3.1** (Sinusoidal perturbation). *Define*

$$\hat{H}(t) = \hat{H}_I e^{-i\omega t}. \quad (1.3.5)$$

*Given the figure in the video, we have*

$$c_k(T) = \frac{1}{i\hbar} \int_0^T e^{i\Delta\omega t} \langle k | \hat{H}_I | 1 \rangle dt \quad (1.3.6)$$

$$\implies \text{Transition probability } P_{k1}(T) = |c_k(T)|^2 = \frac{1}{\hbar^2} |\langle k | \hat{H}_I | 1 \rangle|^2 Y(\Delta\omega, T), \quad (1.3.7)$$

*with*

$$Y(\Delta\omega, T) = \frac{\sin^2(\Delta\omega T/2)}{(\Delta\omega/2)^2} \quad (1.3.8)$$

$$\sim \text{sinc}^2 x, \quad (1.3.9)$$

*where  $\Delta\omega = \omega - \omega_{1k}$  is the detuning.*

Let's take a look at the sinc function  $Y(\Delta\omega, T) = \text{sinc}^2 x$ . Transition for  $\Delta\omega \leq \frac{2\pi}{T}$ , we have  $\Delta\omega \cdot T \leq 2\pi$ , which implies

$$\Delta E \cdot T \leq h, \quad (1.3.10)$$

which is the time-frequency uncertainty. (The expression in the video seems wrong, so I make corrections above.) We have the following case

$$\frac{1}{2\pi T} Y(\Delta\omega, T) \xrightarrow{T \rightarrow \infty} \delta(\Delta\omega), \quad (1.3.11)$$

then we have

$$P_{k1}(T \rightarrow \infty) = \frac{2\pi}{\hbar^2} |\langle k | \hat{H}_I | i \rangle|^2 \delta(\Delta\omega) T. \quad (1.3.12)$$

**Fermi's Golden Rule**  $|k\rangle$  Quasi continuum of final states. We have the transition probability

$$P_{k1} = \Gamma_{k1} T, \quad (1.3.13)$$

where

$$\Gamma_{k1} = \frac{2\pi}{\hbar} |\langle k | \hat{H}_I | 1 \rangle|^2 \rho(E_k = E_1 + \hbar\omega) \quad (1.3.14)$$

is called the Femi's Golden Rule,

$$|\langle k | \hat{H}_I | 1 \rangle|^2 \quad (1.3.15)$$

is the coupling strength  $\propto E_0^2$  and  $\propto I$ ,

$$\rho(E_k = E_1 + \hbar\omega) \quad (1.3.16)$$

is the density states which is number of available final states to the system,

$$\Gamma_{k1} \hat{=} \text{Transition Rate} = \frac{dP_{k1}}{dT}, \quad (1.3.17)$$

and density states

$$\rho(E) = \frac{dN}{dE}, \quad (1.3.18)$$

where  $\Delta N$  is the number of states in an energy interval  $\Delta E$  around energy  $E_k$  and we let  $\Delta E$  approaches 0.

## 1.4 Two Level Atom (TLA)

Given by the figure, in state  $|1\rangle$ , we have  $E_1 = \hbar\omega_1$  and in state  $|2\rangle$ , we have  $E_2 = \hbar\omega_2$  and  $E_2 - E_1 = \hbar(\omega_2 - \omega_1) = \omega_{21}$ . We have the Hamiltonian

$$\hat{H} = \hat{H}_0 - \hat{d} \cdot E(t), \quad (1.4.1)$$

where

$$E(t) = \varepsilon E_0 \cos(\omega t), \quad (1.4.2)$$

where  $\varepsilon$  is the polarization vector,  $E_0$  is the field amplitude, and  $\omega$  is the frequency of the light field.

**Ansatz for Solving TLA** We have

$$|\Psi(t)\rangle = c_1(t)e^{-i\omega_1 t}|1\rangle + c_2(t)e^{-i\omega_2 t}|2\rangle. \quad (1.4.3)$$

**Time Evolution Amplitude** We have

$$\dot{c}_1(t) = i \frac{d_{12}^\varepsilon E_0}{\hbar} e^{-\omega_{21} t} \cos(\omega t) c_2(t) \quad (1.4.4)$$

$$\dot{c}_2(t) = i \frac{d_{12}^\varepsilon E_0}{\hbar} e^{+\omega_{21} t} \cos(\omega t) c_1(t), \quad (1.4.5)$$

where

$$d_{12}^\varepsilon = \langle 1 | \hat{d} \cdot \varepsilon | 2 \rangle \quad (1.4.6)$$

$$= \langle 1 | \hat{d} | 2 \rangle \cdot \varepsilon \quad (1.4.7)$$

$$= \langle 1 | \hat{d}_x | 2 \rangle \cdot \varepsilon_x + \langle 1 | \hat{d}_y | 2 \rangle \cdot \varepsilon_y + \langle 1 | \hat{d}_z | 2 \rangle \cdot \varepsilon_z. \quad (1.4.8)$$

is the Dipole Matrix Element, which is the atomic property and we assume it's real. We also define

$$\Omega_0 = \frac{d_{12}^\varepsilon E_0}{\hbar} \quad (1.4.9)$$

as the Rabi frequency.

**Time Evolution** Using Euler' form, we have

$$\dot{c}_1(t) = i \frac{\Omega_0}{2} e^{-\omega_{21}t} (e^{i\omega t} + e^{-i\omega t}) c_2(t) \quad (1.4.10)$$

$$\dot{c}_2(t) = i \frac{\Omega_0}{2} e^{+\omega_{21}t} (e^{i\omega t} + e^{-i\omega t}) c_1(t) \quad (1.4.11)$$

by

$$\cos \alpha = \frac{1}{2}(e^{i\alpha} + e^{-i\alpha}) \quad (1.4.12)$$

and

$$e^{i\alpha} = \cos \alpha + i \sin \alpha. \quad (1.4.13)$$

**Rotating Wave Approximation** We have

$$\dot{c}_1(t) = i \frac{\Omega_0}{2} (e^{+i(\omega-\omega_{21})t} + e^{-i(\omega+\omega_{21})t}) c_2(t) \quad (1.4.14)$$

$$\dot{c}_2(t) = i \frac{\Omega_0}{2} (e^{-i(\omega-\omega_{21})t} + e^{+i(\omega+\omega_{21})t}) c_1(t), \quad (1.4.15)$$

and we ignore the sum frequency term and get

$$\dot{c}_1(t) = i \frac{\Omega_0}{2} e^{+i(\omega-\omega_{21})t} c_2(t) \quad (1.4.16)$$

$$\dot{c}_2(t) = i \frac{\Omega_0}{2} e^{-i(\omega-\omega_{21})t} c_1(t), \quad (1.4.17)$$

which is a good approximation for detuning  $\delta = \omega - \omega_{21} \approx 0$ . We introduce

$$\tilde{c}_1(t) = c_1(t) e^{-i\frac{\delta}{2}t} \quad (1.4.18)$$

$$\tilde{c}_2(t) = c_2(t) e^{+i\frac{\delta}{2}t}. \quad (1.4.19)$$

$$(1.4.20)$$

**Ansatz Wavefunctions for TLA** Whole time evolution in state amplitudes

$$|\Psi(t)\rangle = c'_1(t)|1\rangle + c'_2(t)|2\rangle. \quad (1.4.21)$$

Time evolution when field is off

$$|\Psi(t)\rangle = c'_1(0)e^{-i\omega_1 t}|1\rangle + c'_2(0)e^{-i\omega_2 t}|2\rangle. \quad (1.4.22)$$

However, this is boring. We chose different ansatz as

$$|\Psi(t)\rangle = c_1(t)e^{-i\omega_1 t}|1\rangle + c_2(t)e^{-i\omega_2 t}|2\rangle \quad (1.4.23)$$

$$\Longleftrightarrow |\Psi(t)\rangle = c_1(t)|1\rangle + c_2(t)e^{-i\omega_{21} t}|2\rangle, \quad (1.4.24)$$

where  $c_1(t)$  and  $c_2(t)$  capture time evolution on top of eigenstate evolution! We now have

$$|\Psi(t)\rangle = c_1(t)|1\rangle + c_2(t)e^{-i\omega_{21} t}|2\rangle, \quad (1.4.25)$$

which is called the rotating frame of atom. We also have Rotating frame of light field as

$$|\Psi(t)\rangle = \tilde{c}_1(t)|1\rangle + \tilde{c}_2(t)e^{-i\omega t}|2\rangle, \quad (1.4.26)$$

where  $\omega$  is the light frequency,  $\tilde{c}_1$  and  $\tilde{c}_2$  describe time evolution on top of fast light field oscillation.

**Solving the TLA Dynamics** We have the following equations:

$$\frac{d}{dt} \begin{pmatrix} \tilde{c}_1(t) \\ \tilde{c}_2(t) \end{pmatrix} = \frac{i}{2} \begin{pmatrix} -\delta & \Omega_0 \\ \Omega_0 & +\delta \end{pmatrix} \begin{pmatrix} \tilde{c}_1(t) \\ \tilde{c}_2(t) \end{pmatrix}. \quad (1.4.27)$$

Considering the simplest case  $\delta = 0$

$$\frac{d}{dt} \tilde{c}_1(t) = \frac{i}{2} \Omega_0 \tilde{c}_2(t) \quad (1.4.28)$$

$$\frac{d}{dt} \tilde{c}_2(t) = \frac{i}{2} \Omega_0 \tilde{c}_1(t). \quad (1.4.29)$$

Take time derivative of the first equation, then we have

$$\ddot{\tilde{c}}_1(t) = -\frac{\Omega_0^2}{4} \tilde{c}_1(t), \quad (1.4.30)$$

the solutions of which are

$$\tilde{c}_1(t) = \cos(\Omega_0 t/2) \quad (1.4.31)$$

$$\tilde{c}_2(t) = i \sin(\Omega_0 t/2) \quad (1.4.32)$$

for  $\tilde{c}_1(0) = 1$  and  $\tilde{c}_2(0) = 0$ . Also we can obtain the excited state probability as

$$P_2(t) = |c_2(t)|^2 \quad (1.4.33)$$

$$= |\tilde{c}_2(t)|^2. \quad (1.4.34)$$

**Rabi Oscillations (Resonant Case)** Nonlinear Response can be seen from the figure.

**General Rabi Oscillations (with detuning)** Given the figurem.

$$|\tilde{c}_2(t)|^2 = \frac{\Omega_0^2}{\Omega} \sin^2 \left( \frac{1}{2} \Omega t \right) \quad (1.4.35)$$

$$= \frac{\Omega_0^2}{2\Omega^2} \{1 - \cos(\Omega t)\}, \quad (1.4.36)$$

where  $\Omega = \sqrt{\Omega_0^2 + \delta^2}$  is the effective Rabi frequency.

**Interesting Special Cases** a) Pi-Puls  $\Omega_0 \tau = \pi$ : swap population

$$|1\rangle \rightarrow i|2\rangle \quad (1.4.37)$$

$$|2\rangle \rightarrow i|1\rangle. \quad (1.4.38)$$

b) 2Pi-Puls  $\Omega_0 \tau = 2\pi$ : flip the sign

c) Pi/2-Puls  $\Omega_0 \tau = \pi/2$ : superposition state

## 1.5 Oscillating Dipoles

**Atomic Eigenstates**

$$|\Psi_{nlm}(t)\rangle = e^{-iE_{nlm}t/\hbar} |\Psi_{nlm}(0)\rangle, \quad (1.5.1)$$

$$\hat{H}_0 |\Psi_{nlm}(0)\rangle = E_{nlm} |\Psi_{nlm}\rangle, \quad (1.5.2)$$

and the electron density is

$$\rho(r, \theta, \phi) = |\Psi(r, \theta, \phi, t = 0)|^2. \quad (1.5.3)$$

**Atomic Dipole** Calculate (Oscillating) Dipole Moment for Atomic Eigenstate. We denote  $|1\rangle = |\Psi_{nlm}\rangle$ . We have

$$d(t) = \langle 1(t) | \hat{d} | 1(t) \rangle \quad (1.5.4)$$

$$= \langle \hat{d} | 1 \rangle \quad (1.5.5)$$

$$= -e \langle 1 | \hat{r} | 1 \rangle. \quad (1.5.6)$$

Then,

$$-e \langle 1 | \hat{r} | 1 \rangle = -e \langle 1 | \hat{P} \hat{P}^{-1} \hat{r} \hat{P} \hat{P}^{-1} | 1 \rangle \quad (1.5.7)$$

$$= +e \langle 1 | \hat{r} | 1 \rangle, \quad (1.5.8)$$

which implies

$$\langle 1 | \hat{r} | 1 \rangle = 0. \quad (1.5.9)$$



**Atomic Dipole - Superposition States** Calculate (Oscillating) Dipole Moment for Atomic Superposition State

$$|\Psi(0)\rangle = \frac{1}{\sqrt{2}}(|1\rangle + i|2\rangle). \quad (1.5.10)$$

Evolution

$$|\Psi(t)\rangle = \frac{1}{\sqrt{2}}(|1\rangle + ie^{-i\omega_{21}t}|2\rangle). \quad (1.5.11)$$

We have

$$d(t) = \langle \Psi(t) | \hat{d} | \Psi(t) \rangle \quad (1.5.12)$$

$$= \frac{1}{2} \left\{ \langle 1 | \hat{d} | 1 \rangle + \langle 2 | \hat{d} | 2 \rangle + ie^{-i\omega_{21}t} \langle 1 | \hat{d} | 2 \rangle - ie^{-i\omega_{21}t} \langle 2 | \hat{d} | 1 \rangle \right\} \quad (1.5.13)$$

$$= d_{12}i \frac{1}{2} \{ e^{-i\omega_{21}t} - e^{i\omega_{21}t} \} \quad (1.5.14)$$

$$= d_{12} \sin(\omega_{21}t), \quad (1.5.15)$$

where  $d_{12}$  is the dipole moment amplitude,  $\omega_{21}$  is the natural resonance frequency.

**Electron Density - Superposition States** Calculate Electron Probability Density for Superposition State. The superposition state is

$$\Psi(r, t) = \frac{1}{\sqrt{2}} (\Psi_1(r) + ie^{-i\omega_{21}t}\Psi_2(r)). \quad (1.5.16)$$

The Electron Probability Density is

$$\rho(r, t) = |\Psi(r, t)|^2 \quad (1.5.17)$$

$$= \Psi^* \Psi \quad (1.5.18)$$

$$= \frac{1}{2} \{ |\Psi_1(r)|^2 + |\Psi_2(r)|^2 + 2\text{Re} (ie^{-i\omega_{21}t}\Psi_1^*(r)\Psi_2(r)) \}, \quad (1.5.19)$$

where  $2\text{Re} (ie^{-i\omega_{21}t}\Psi_1^*(r)\Psi_2(r))$  is the interference term.

**Examples** This is shown by animation and figure in the video.

## 1.6 The Bloch Sphere

### General Two-Level State

- General State Description

$$|\Psi\rangle = c'_1|1\rangle + c'_2|2\rangle \quad (1.6.1)$$

$$\text{Up to a global phase} \quad (1.6.2)$$

$$= |c'_1||1\rangle + e^{i\phi}|c'_2||2\rangle \quad (1.6.3)$$

satisfying  $|c'_1|^2 + |c'_2|^2 = 1$ .

- Alternative way

$$|\Psi\rangle = \cos(\theta/2)|1\rangle + e^{i\phi} \sin(\theta/2)|2\rangle, \quad (1.6.4)$$

since  $\cos(\theta/2)^2 + \sin(\theta/2)^2 = 1$ .

**Geometric Description - Bloch Sphere** We then have

$$|\Psi\rangle = \cos(\theta/2)|1\rangle + e^{i\phi} \sin(\theta/2)|2\rangle \quad (1.6.5)$$

with  $0 \leq \theta \leq \pi$  as the latitude and  $0 \leq \phi \leq 2\pi$  as the longitude. This is the Bloch Sphere representation. The definition of  $\theta$  and  $\phi$  and their ranges are different from my familiar coordinate system.

**Special States on Bloch Sphere**

**Analogy to Spin -1/2 States** Is shown in the figure.

## 1.7 Density Operator and Density Matrix

**The Problem** How do we describe "imperfect state preparation" in an experiment? For example, 50% $|1\rangle$  and 50% $|2\rangle$ . We may think of

$$|\Psi\rangle = \frac{1}{\sqrt{2}} (|1\rangle + |2\rangle) . ??? \quad (1.7.1)$$

This is 100% $|\Psi\rangle$  pure state. We need stable relative phase between the two states!

**Optical Analogy - Controlled Phase** The double slit problem is shown in the video.

Intensity on Detection Screen:

$$I \propto |E|^2 = |E_1 + e^{i\phi} E_2|^2 \quad (1.7.2)$$

$$= |E_1|^2 + |E_2|^2 + 2\text{Re} \left( E_1 E_2 e^{i\phi} \right) . \quad (1.7.3)$$

As  $\phi$  varies, Interference pattern "washed out"!

We need new formalism to describe mixed states!(imperfect state preparation, spontaneous emission,...)

**Density Operator and Matrix** The description of mixed states can be handled by the density operator (matrix) formalism!

- Density operator (hermitian)

$$\hat{\rho} = \sum p_i |\Psi_i\rangle \langle \Psi_i| \quad (1.7.4)$$

$$\hat{\rho} = I \hat{\rho} I \quad (1.7.5)$$

$$= \sum_{i,j} |i\rangle \langle i| \hat{\rho} |j\rangle \langle j| \quad (1.7.6)$$

$$= \rho_{11}|1\rangle\langle 1| + \rho_{12}|1\rangle\langle 2| + \rho_{21}|2\rangle\langle 1| + \rho_{22}|2\rangle\langle 2|, \quad (1.7.7)$$

where  $I = \sum_i |i\rangle \langle i|$ .

- Density matrix

$$\rho = \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix}, \quad (1.7.8)$$

where  $\rho_{11}$  and  $\rho_{22}$  are the populations,  $\rho_{12}$  and  $\rho_{21}$  are the coherence. Since  $\rho$  is hermitian, we have

$$\rho_{12} = \rho_{21}^*. \quad (1.7.9)$$

**Example 1.7.1** (Example: Density Matrix of Pure State). *We have*

$$|\Psi\rangle = |c_1||1\rangle + e^{i\phi}|c_2||2\rangle. \quad (1.7.10)$$

*The corresponding density operator of the **pure state** is  $\hat{\rho} = |\Psi\rangle \langle \Psi|$ . Then the corresponding density matrix is*

$$\rho = \begin{bmatrix} |c_1|^2 & |c_1||c_2|e^{-i\phi} \\ |c_1||c_2|e^{i\phi} & |c_2|^2 \end{bmatrix}, \quad (1.7.11)$$

*where  $|c_1||c_2|e^{-i\phi}$  and  $|c_1||c_2|e^{i\phi}$  are relative phase between states  $|1\rangle$  and  $|2\rangle$ .*

*specific example:*

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle), \quad (1.7.12)$$

*so*

$$\rho = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}. \quad (1.7.13)$$

**Example 1.7.2** (Example: Fully Incoherent Mixture).

$$\hat{\rho} = \frac{1}{2}|1\rangle\langle 1| + \frac{1}{2}|2\rangle\langle 2| \quad (1.7.14)$$

with

$$\rho = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}, \quad (1.7.15)$$

where vanishingly coherence and the phase varies from 0 to  $2\pi$ . It means that we did not control phase.

### Useful Facts

- Expectation values:  $\langle \hat{A} \rangle = \text{Tr}(\hat{\rho}\hat{A}) = \text{Tr}(\rho A)$
- Time evolution (von Neumann equation)

$$i\hbar \frac{\partial \hat{\rho}}{\partial t} = [\hat{H}, \hat{\rho}] \quad (1.7.16)$$

- Pure state:  $\text{Tr}(\rho^2) = 1$
- Mixed states:  $\text{Tr}(\rho^2) < 1$

## 1.8 Optical Bloch Equations

**Time Evolution of Density Matrix** How to calculate time evolution of density matrix?

$$i\hbar \frac{\partial \hat{\rho}}{\partial t} = [\hat{H}, \hat{\rho}]. \quad (1.8.1)$$

Assume pure state

$$\frac{d}{dt}\rho_{11} = \frac{d}{dt}(c_1 c_1^*) \quad (1.8.2)$$

$$= \dot{c}_1 c_1^* + c_1 \dot{c}_1^* \quad (1.8.3)$$

$$= i\frac{\Omega_0}{2} \left( e^{i\delta t} \rho_{21} - e^{-i\delta t} \rho_{12} \right) \quad (1.8.4)$$

Transformation to rotality frame of light

$$= i\frac{\Omega_0}{2} (\tilde{\rho}_{21} - \tilde{\rho}_{12}), \quad (1.8.5)$$

where

$$\dot{c}_1(t) = i\frac{\Omega_0}{2}e^{+i\delta t}c_2(t) \quad (1.8.6)$$

$$\dot{c}_2(t) = i\frac{\Omega_0}{2}e^{-i\delta t}c_1(t) \quad (1.8.7)$$

$$\tilde{\rho}_{12} = e^{-i\delta t}\rho_{12} \quad (1.8.8)$$

$$\tilde{\rho}_{21} = e^{+i\delta t}\rho_{21}. \quad (1.8.9)$$

Other elements obtained in analogy!

$$\frac{d}{dt}\rho_{11} = i\frac{\Omega_0}{2}(\tilde{\rho}_{21} - \tilde{\rho}_{12}) \quad (1.8.10)$$

$$\frac{d}{dt}\rho_{22} = i\frac{\Omega_0}{2}(\tilde{\rho}_{12} - \tilde{\rho}_{21}) \quad (1.8.11)$$

$$\frac{d}{dt}\tilde{\rho}_{12} = -i\delta\tilde{\rho}_{12} + i\frac{\Omega_0}{2}(\rho_{22} - \rho_{11}) \quad (1.8.12)$$

$$\frac{d}{dt}\tilde{\rho}_{21} = +i\delta\tilde{\rho}_{21} + i\frac{\Omega_0}{2}(\rho_{11} - \rho_{22}). \quad (1.8.13)$$

Noting that  $\tilde{\rho}_{12} = \tilde{\rho}_{21}$  due to hermitian matrix, the third and the forth equations are the same. So we have

$$\frac{d}{dt}\rho_{11} = i\frac{\Omega_0}{2}(\tilde{\rho}_{21} - \tilde{\rho}_{12}) \quad (1.8.14)$$

$$\frac{d}{dt}\rho_{22} = i\frac{\Omega_0}{2}(\tilde{\rho}_{12} - \tilde{\rho}_{21}) \quad (1.8.15)$$

$$\frac{d}{dt}\tilde{\rho}_{12} = -i\delta\tilde{\rho}_{12} + i\frac{\Omega_0}{2}(\rho_{22} - \rho_{11}). \quad (1.8.16)$$

**Optical Bloch Equations with Damping** Phenomenological damping and spontaneous emission in the figure. Combine the decay, we have

$$\frac{d}{dt}\rho_{11} = i\frac{\Omega_0}{2}(\tilde{\rho}_{21} - \tilde{\rho}_{12}) + \gamma\rho_{22} \quad (1.8.17)$$

$$\frac{d}{dt}\rho_{22} = i\frac{\Omega_0}{2}(\tilde{\rho}_{12} - \tilde{\rho}_{21}) - \gamma\rho_{22} \quad (1.8.18)$$

$$\frac{d}{dt}\tilde{\rho}_{12} = -i\delta\tilde{\rho}_{12} + i\frac{\Omega_0}{2}(\rho_{22} - \rho_{11}) - (\gamma/2)\tilde{\rho}_{12}. \quad (1.8.19)$$

We now define the inversion  $w = \rho_{22} - \rho_{11}$ . We have Optical Bloch Equations with Damping

$$\frac{d}{dt}\tilde{\rho}_{21} = -(\gamma/2 - i\delta)\tilde{\rho}_{21} - \frac{i\omega\Omega_0}{2} \quad (1.8.20)$$

$$\frac{d}{dt}w = -\gamma(w + 1) - i\Omega_0(\tilde{\rho}_{21} - \tilde{\rho}_{12}) \quad (1.8.21)$$

in the Density Matrix Form.

## 1.9 Optical Bloch Equations - Dynamics and Steady State

**Dynamical Evolution of System** Shown in the figure in the picture.

**Steady State Solution** Conditions:  $\frac{d}{dt}\tilde{\rho}_{21} = 0$  and  $\frac{d}{dt}\omega = 0$ . Then we have the solutions

$$\omega = -\frac{1}{1+S} \quad (1.9.1)$$

$$\tilde{\rho}_{21} = \frac{2\Omega_0}{2(\gamma/2 - \delta)(1+S)} \quad (1.9.2)$$

$$S = \frac{\Omega_0^2/2}{\delta^2 + \gamma^2/4} = \frac{S_0}{1 + 4\delta^2/\gamma^2} \quad (1.9.3)$$

$$S_0 = \frac{2\Omega_0^2}{\gamma^2} = \frac{I}{O_{sat}}, \quad (1.9.4)$$

where  $S$  is called the saturation parameter,  $S_0$  is called resonant saturation parameter.

Limiting Cases:

- $S \leq 1$ :  $w \rightarrow -1$  where  $w = \rho_{22} - \rho_{11}$ . Atom is mainly in ground state.
- $S \geq 1$ :  $S \rightarrow \infty$ ,  $w \rightarrow 0$ .
- Excited State Population:

$$\rho_{22} \quad (1.9.5)$$

Combine with  $\rho_{22} + \rho_{11} = 1$

$$= \frac{1}{2}(1+w) \quad (1.9.6)$$

$$= \frac{S}{2(1+S)} \quad (1.9.7)$$

$$= \frac{S_0/2}{1 + S_0 + 4\delta^2/\gamma^2} \quad (1.9.8)$$

$$\xrightarrow{S_0 \rightarrow \infty, \delta=0} \frac{1}{2}. \quad (1.9.9)$$

- Photon Scattering Rate:  $\Gamma_{ph} = \gamma\rho_{22} = \frac{\gamma}{2} \frac{S_0}{1+S_0+4\delta^2/\gamma^2}$ .  $\Gamma_{ph} \rightarrow \gamma/2$  for  $S_0 \rightarrow \infty$  and  $\delta = 0$ . We rewrite it as

$$\Gamma_{ph} = \left( \frac{S_0}{1+S_0} \right) \left( \frac{\gamma/2}{1+4\delta^2/\gamma'^2} \right) \quad (1.9.10)$$

$$\gamma' = \gamma\sqrt{1+S_0}. \quad (1.9.11)$$

It has a figure in the video. The saturation broadening is shown in the figure.

## 1.10 Lambert-Beer Law

**Attenuation of Light** It is shown in the figure.

**Scattered Light from Slab of Atoms** scattered light power by slab of length  $dz$

$$dP_{sc} = \Gamma_{ph} \times nAdz \times \hbar\omega, \quad (1.10.1)$$

where  $\Gamma_{ph}$  is the single atom photon scattering rate,  $\hbar\omega$  is the energy of single atom,  $nAdz$  is the number of atoms. Then we have

$$\frac{dP_{sc}}{dz} = \Gamma_{ph} \times nA \times \hbar\omega. \quad (1.10.2)$$

**Scattered Light from Slab of Atoms** Energy conservation requires

$$\frac{dP}{dz} = -\frac{dP_{sc}}{dz} \quad (1.10.3)$$

$$\frac{dP}{dz} = \frac{dI}{dI} A. \quad (1.10.4)$$

Put every thing together:

$$\frac{dI}{dz} = -\Gamma n \hbar\omega. \quad (1.10.5)$$

We have

$$\frac{dI(z)}{dz} = -n\sigma I(z), \quad (1.10.6)$$

where  $\sigma$  is the atomic scattering cross section.

**Lambert-Beer Law (no saturation)** We compute the solutions

$$I(z) = I(0)e^{-n\sigma z}, \quad (1.10.7)$$

which is the Lambert-Beer Law of Absorption.

**Laser induced Fluorescence** Shown in a video.

## 1.11 Bloch Vector

**Density Matrix Revisited** Density Matrix of TLA

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \quad (1.11.1)$$

Density Matrix hermitian

$$\rho = \rho^\dagger = (\rho^T)^*, \quad (1.11.2)$$

so we have

$$\rho = \begin{pmatrix} \rho_{11} & \text{Re}\rho_{12} + i\text{Im}\rho_{12} \\ \text{Re}\rho_{12} - i\text{Im}\rho_{12} & \rho_{22} \end{pmatrix}. \quad (1.11.3)$$

Pauli matrices are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (1.11.4)$$

The decomposition of Density matrix into Pauli matrices

$$\rho = \frac{1}{2} (I + b_x \sigma_x + b_y \sigma_y + b_z \sigma_z), \quad (1.11.5)$$

where  $b_x, b_y, b_z \in \mathbb{R}$ .

**Bloch Vector** We have the density matrix in rotating frame of light

$$\tilde{\rho} = \begin{pmatrix} \rho_{11} & \tilde{\rho}_{12} \\ \tilde{\rho}_{21} & \rho_{22} \end{pmatrix}, \quad (1.11.6)$$

where  $\tilde{\rho}_{12} = \rho_{12} e^{-i\omega t}$ . We use following sign convention and have

$$\tilde{\rho} = \frac{1}{2} (I + u \sigma_x - v \sigma_y - w \sigma_z), \quad (1.11.7)$$

and the bloch vector is defined as

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}. \quad (1.11.8)$$

It can be easily shown that

$$u = 2\text{Re}(\tilde{\rho}_{12}) = \tilde{\rho}_{12} + \tilde{\rho}_{12}^* \quad (1.11.9)$$

$$v = 2\text{Im}(\tilde{\rho}_{12}) = i(\tilde{\rho}_{12}^* - \tilde{\rho}_{12}) \quad (1.11.10)$$

$$w = \rho_{22} - \rho_{11}, \quad (1.11.11)$$

$$(1.11.12)$$

where  $u$  is the dispersive component,  $v$  is the absorption component and  $w$  is the inversion.

Bloch vector can be used to describe any state of TLA density matrix!

Properties of Bloch Vector

- Mixed State:  $u^2 + v^2 + w^2 < 1$
- Pure State:  $u^2 + v^2 + w^2 = 1$



## 1.12 Understanding Bloch Vector

What physical behaviour do the components stand for?

- $w = -1$  atom in ground state.  $w = +1$  atom in excited state.
- What about  $u, v$ ?

$$\langle \hat{d}_i(t) \rangle = \text{Tr}(\hat{\rho} \hat{d}) \quad (1.12.1)$$

$$= \text{Tr} \left[ \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{12}^* & \rho_{22} \end{pmatrix} \begin{pmatrix} 0 & d_{12}^i \\ d_{12}^i & 0 \end{pmatrix} \right], \quad (1.12.2)$$

where  $d_{12}^x = \langle 1 | -q\hat{x} | 2 \rangle$ .

Written in the vector form, we have

$$\langle \hat{d} \rangle(t) = d_{12} (\rho_{12} + \rho_{12}^*) \quad (1.12.3)$$

$$= d_{12} (\tilde{\rho}_{12} e^{i\omega t} + \tilde{\rho}_{12}^* e^{-i\omega t}) \quad (1.12.4)$$

$$= d_{12} [u \cos(\omega t) - v \sin(\omega t)], \quad (1.12.5)$$

where we use  $\rho_{12} = \tilde{\rho}_{12} e^{i\omega t}$ ,  $u$  denotes in phase and  $v$  denotes  $90^\circ$  out of phase component.

Reminder:  $E(t) = \epsilon E_0 \cos(\omega t)$ .

- Which component responsible for absorption/emission? We have a figure in the video to show the classical picture.

Average absorbed power per atom (classical ensemble average)

$$\left\langle \frac{dW}{dt} \right\rangle = \epsilon E_0 \cos(\omega t) \left\langle -q \frac{dr}{dt} \right\rangle \quad (1.12.6)$$

$$= \epsilon E_0 \cos(\omega t) \langle \dot{d} \rangle. \quad (1.12.7)$$

Quantum mechanical analogue (Ehrenfest)

$$\left\langle \frac{dW}{dt} \right\rangle = \epsilon E_0 \cos(\omega t) \langle \dot{d} \rangle \quad (1.12.8)$$

$$\langle \dot{d} \rangle(t) = d_{12} [u \cos(\omega t) - v \sin(\omega t)]. \quad (1.12.9)$$

$$\left\langle \frac{dW}{dt} \right\rangle = -d_{12} \cdot \epsilon E_0 \omega (u \cos(\omega t) \sin(\omega t) + v \sin(\omega t)^2) \quad (1.12.10)$$

$$\overline{\left\langle \frac{dW}{dt} \right\rangle} = \frac{1}{T} \int dt \left\langle \frac{dW}{dt} \right\rangle \quad (1.12.11)$$

$$= -\frac{d_{12} \cdot \epsilon E_0 \omega v}{2} \quad (1.12.12)$$

$$= -\hbar \frac{d_{12} \epsilon E_0}{\hbar} \omega \frac{v}{2} \quad (1.12.13)$$

$$= -\hbar \Omega_0 \omega \frac{v}{2}, \quad (1.12.14)$$

which is the absorption.

### 1.13 Optical Bloch Equations using Bloch Vector

### 1.14 Interlude: The Mach-Zehnder Interferometer

### 1.15 Ramsey Interferometer

### 1.16 Review: QM of the Harmonic Oscillator

[SZQ: 2023.04.20: I have understandard the content in this video.]

### 1.17 Wave equation and energy density of classical radiation field

This section is also known as the review of Maxwell equations vector potentials.

**Fundamentaals** Maxwell equations in free space

$$\nabla \cdot E = 0, \nabla \times E = -\frac{\partial B}{\partial t} \quad (1.17.1)$$

$$\nabla \cdot B = 0, \nabla \times B = \frac{1}{c^2} \frac{\partial E}{\partial t}. \quad (1.17.2)$$

**Lemma 1.17.1** (Coulomb Gauge). *Considering Coulomb Gauge, we have*

$$\nabla \cdot A = 0. \quad (1.17.3)$$

*Then we can express the electric field and magnetic field in terms of the vector potential*

$$B(r, t) = \nabla \times A(r, t) \quad (1.17.4)$$

$$E(r, t) = -\frac{\partial A(r, t)}{\partial t}. \quad (1.17.5)$$

**Lemma 1.17.2** (Wave equation). *Considering Coulomb Gauge, the wave equation is*

$$\nabla^2 A - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} A = 0. \quad (1.17.6)$$

*Proof.* Using 1.17.1 and the forth equation in 1.17.1, we have

$$\nabla \times B = \nabla \times (\nabla \times A(r, t)), \quad (1.17.7)$$

and

$$\nabla \times B = \frac{1}{c^2} \frac{\partial E}{\partial t} \quad (1.17.8)$$

$$= -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} A(r, t). \quad (1.17.9)$$

So we have

$$\nabla \times (\nabla \times A) = -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} A. \quad (1.17.10)$$

Then use the rule in vector Calculus

$$\nabla \times (\nabla \times A) = \nabla(\nabla \cdot A) - \Delta A. \quad (1.17.11)$$

Use lemma 1.17.1, we then have

$$-\Delta A = -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} A \quad (1.17.12)$$

$$= -\nabla^2 A. \quad (1.17.13)$$

So we have

$$\nabla^2 A - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} A = 0. \quad (1.17.14)$$

□

### Solutions of Wave Equation

**Lemma 1.17.3** (Solutions of Wave Equation). *Plane waves:*

$$\mathbf{A}_{\mathbf{k}, \alpha} = \epsilon_{\mathbf{k}, \alpha} A_{\mathbf{k}, \alpha} \exp[i(\mathbf{k}\mathbf{r} - \omega_k t)], \quad (1.17.15)$$

where  $\epsilon_{\mathbf{k}, \alpha}$  is polarization,  $A_{\mathbf{k}, \alpha}$  is complex amplitude,  $|k| = \frac{2\pi}{\lambda}$  is wavenumber i.e., the magnitude of the wave vector,  $\mathbf{k}$  is the wave vector,  $\omega_k = ck$ .

Which wave vectors are possible? (a). in finite space,  $\mathbf{k}$  distributed continuous; (b). finite box of length  $L$ ,  $\mathbf{k}$  distributed discretely (periodic boundary conditions)

$$k_x = \frac{2\pi}{L} n_x, k_y = \frac{2\pi}{L} n_y, k_z = \frac{2\pi}{L} n_z \quad (1.17.16)$$

$$\mathbf{A}(\mathbf{r}, t) = \sum_{\mathbf{k}, \alpha} \epsilon_{\mathbf{k}, \alpha} (A_{\mathbf{k}, \alpha} \exp[i(\mathbf{k}\mathbf{r} - \omega_k t)] + A_{\mathbf{k}, \alpha}^* \exp[-i(\mathbf{k}\mathbf{r} - \omega_k t)]) \quad (1.17.17)$$

$$\mathbf{E}(\mathbf{r}, t) = -\frac{\partial}{\partial t} \mathbf{A}(\mathbf{r}, t) = \sum_{\mathbf{k}, \alpha} \epsilon_{\mathbf{k}, \alpha} i\omega_k [A_{\mathbf{k}, \alpha} \exp[i(\mathbf{k}\mathbf{r} - \omega_k t)] - A_{\mathbf{k}, \alpha}^* \exp[-i(\mathbf{k}\mathbf{r} - \omega_k t)]] \quad (1.17.18)$$

$$\mathbf{B}(\mathbf{r}, t) = \sum_{\mathbf{k}, \alpha} i(\mathbf{k} \times \epsilon_{\mathbf{k}, \alpha}) [A_{\mathbf{k}, \alpha} \exp[i(\mathbf{k}\mathbf{r} - \omega_k t)] - A_{\mathbf{k}, \alpha}^* \exp[-i(\mathbf{k}\mathbf{r} - \omega_k t)]] . \quad (1.17.19)$$

[SZQ: 2023.04.20: The complex conjugate term is used to eliminate the imaginary part.]

**Total Energy of Radiation Field** Total energy of radiation field in volume  $V = L^3$ .  
 [SZQ: 2023.04.20: The total energy is the integration of the electric density and magnetic density over the volume.]

The electric density is

$$\frac{1}{2}\varepsilon_0 E(r, t)^2. \quad (1.17.20)$$

The magnetic density is

$$\frac{1}{2\mu_0} B(r, t)^2. \quad (1.17.21)$$

Then the total energy of radiation field in volume  $V = L^3$  is

$$H = \frac{1}{2} \int_V dV \left[ \varepsilon_0 E(r, t)^2 + \frac{1}{\mu_0} B(r, t)^2 \right] \quad (1.17.22)$$

$$= \sum_{\mathbf{k}, \alpha} \varepsilon_0 V \omega_k^2 [A_{\mathbf{k}, \alpha} A_{\mathbf{k}, \alpha}^* + A_{\mathbf{k}, \alpha}^* A_{\mathbf{k}, \alpha}] \quad (1.17.23)$$

$$= \sum_{\mathbf{k}, \alpha} E_{\mathbf{k}, \alpha}, \quad (1.17.24)$$

where

$$E_{\mathbf{k}, \alpha} = \varepsilon_0 V \omega_k^2 [A_{\mathbf{k}, \alpha} A_{\mathbf{k}, \alpha}^* + A_{\mathbf{k}, \alpha}^* A_{\mathbf{k}, \alpha}]. \quad (1.17.25)$$

[SZQ: 2023.04.20: This expression is similar to the quantum harmonic oscillators.] [SZQ: 2023.04.20: I ignore the bold. So you should understand where you should use the bold.]

## 1.18 Quantization of the e.m. field

**Fundamental Idea** RadiationMode ( $k, \alpha$ )

- **To every radiation mode, we associate a harmonic oscillator!** Creation and annihilation operators can change the degree of excitation of mode (occupation with photons)
- **A photon is an excitation quantum of the harmonic oscillator associated with a mode!**

**Creation and Annihilation Operators**  $\hat{a}_k |n_k\rangle = \sqrt{n_k} |n_k - 1\rangle$ : decrease photon number by one photon.

$\hat{a}_k^\dagger |n_k\rangle = \sqrt{n_k + 1} |n_k + 1\rangle$ : increase photon number by one photon.

**Number operator:**  $\hat{n}_k |n_k\rangle = n_k |n_k\rangle$ .  $\hat{a}_k^\dagger \hat{a}_k = \hat{n}_k$ .

**Fock state:**  $|n_k\rangle$ . Fock state is the eigenstate of quantum harmonic oscillator.

**Hamitonian of Radiation Field** The Hamitonian of Radiation Field is the sum of the hamitonian of harmonic oscillator of each mode as

$$\hat{H}_R = \sum_k \hat{H}_k, \quad (1.18.1)$$

where

$$\hat{H}_k = \frac{1}{2} \hbar \omega_k \left( \hat{a}_k \hat{a}_k^\dagger + \hat{a}_k^\dagger \hat{a}_k \right). \quad (1.18.2)$$

We can compare it with classically expression

$$E_{k,\alpha} = \epsilon_0 V \omega_k^2 \left( A_{k,\alpha} A_{k,\alpha}^* + A_{k,\alpha}^* A_{k,\alpha} \right). \quad (1.18.3)$$

If we replace  $A_k$  with

$$A_k = \sqrt{\frac{\hbar}{2\epsilon_0 V \omega_k}} \hat{a}_k, \quad (1.18.4)$$

and replace  $A_k^*$  with

$$A_k^* = \sqrt{\frac{\hbar}{2\epsilon_0 V \omega_k}} \hat{a}_k^\dagger. \quad (1.18.5)$$

We will arrive at  $\hat{H}_k$ . Also we can obtain the quantum version of vector potential operator. The classical vector potential operator is

$$A_k(r, t) = \epsilon_k \left[ A_k \exp[i(kr - \omega_k t)] + A_k^* \exp[-i(kr - \omega_k t)] \right]. \quad (1.18.6)$$

The quantum version will be

$$\hat{A}_k(r, t) = \epsilon_k \sqrt{\frac{\hbar}{2\epsilon_0 V \omega_k}} \left[ \hat{a}_k \exp[i(kr - \omega_k t)] + \hat{a}_k^\dagger \exp[-i(kr - \omega_k t)] \right]. \quad (1.18.7)$$

Use the quantum vector potential, we can derive the quantum electric field operator as

$$\hat{E}_k(r, t) = -\frac{\partial}{\partial t} \hat{A}_k(r, t) \quad (1.18.8)$$

$$= -\epsilon_k \sqrt{\frac{\hbar}{2\epsilon_0 \omega_k V}} (-\omega_k) \left[ i \hat{a}_k \exp[i(kr - \omega_k t)] - i \hat{a}_k^\dagger \exp[-i(kr - \omega_k t)] \right]. \quad (1.18.9)$$

Recall that  $i = \exp[i\pi/2]$  and define

$$\chi_k(r, t) = -kr + \omega_k t - \pi/2. \quad (1.18.10)$$

We then have the compact form

$$\hat{E}(r, t) = \sum_k \epsilon_k \sqrt{\frac{\hbar \omega_k}{2\epsilon_0 V}} \left[ \hat{a}_k \exp[-i\chi_k(r, t)] + \hat{a}_k^\dagger \exp[i\chi_k(r, t)] \right] \quad (1.18.11)$$

$$= \sum_k \hat{E}_k(r, t) \quad (1.18.12)$$

$$:= \hat{E}^+(r, t) + \hat{E}^-(r, t). \quad (1.18.13)$$

**Hamiltonian of Radiation Field** The Hamiltonian of Radiation Field is

$$\hat{H}_R = \frac{1}{2} \int_V dV \left[ \epsilon_0 \hat{E} \cdot \hat{E} + \frac{1}{\mu_0} \hat{B} \cdot \hat{B} \right] \quad (1.18.14)$$

*$\hat{B}, \hat{E}$  are the quantum operator of  $B, E$*

$$= \sum_k \frac{\hbar \omega_k}{2} \left[ \hat{a}_k^\dagger \hat{a}_k + \hat{a}_k \hat{a}_k^\dagger \right] \quad (1.18.15)$$

*Use the commutation relation*

$$= \sum_k \left( \hat{a}_k^\dagger \hat{a}_k + 1/2 \right). \quad (1.18.16)$$

Use this hamiltonian, we derive the energy of multi-mode Fock states as

$$\hat{H}_R |n_{k_1}, n_{k_2}, \dots\rangle = \sum_k \hbar \omega_k \left( n_k + \frac{1}{2} \right) |n_{k_1}, n_{k_2}, \dots\rangle \quad (1.18.17)$$

using the fact that  $\hat{a}_k^\dagger \hat{a}_k$  is the number operator  $\hat{n}_k$ .

Also the vacuum state energy will be

$$E_0 = \sum_k \frac{1}{2} \hbar \omega_k \quad (1.18.18)$$

corresponds to

$$|0\rangle = |0\rangle \otimes \dots \otimes |0\rangle. \quad (1.18.19)$$

This is divergent, but do not worry. When we calculate the difference, this term will be canceled.

## 1.19 Field state of single radiation field mode: Fock States

We focus discussion on a **single mode of the radiation field (wave vector  $k$ )**. We define the phase factor

$$\chi = \chi_k(r, t) = \omega_k t - \mathbf{k} \cdot \mathbf{r} - \pi/2. \quad (1.19.1)$$

Then we have

$$\hat{E}(\chi) = \hat{E}^+(\chi) + \hat{E}^-(\chi) \quad (1.19.2)$$

$$= \left( \frac{\hbar \omega}{2 \epsilon_0 V} \right)^{1/2} (\hat{a} \exp[-i\chi] + \hat{a}^\dagger \exp[i\chi]). \quad (1.19.3)$$

We write the field operator in natural units  $2 \left( \frac{\hbar \omega}{2 \epsilon_0 V} \right)^{1/2}$ , which is also called vacuum field strength. We then have

$$\hat{E}(\chi) = \frac{1}{2} \left( \hat{a} \exp[-i\chi] - \hat{a}^\dagger \exp[i\chi] \right). \quad (1.19.4)$$

**Fock states:**  $|n\rangle$  means  $n$  photons in radiation mode, also means eigenstate of number operator  $\hat{n}$ .

$$\hat{H}|n\rangle = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})|n\rangle \quad (1.19.5)$$

$$= \hbar\omega(\hat{n} + \frac{1}{2})|n\rangle \quad (1.19.6)$$

$$= \hbar\omega(n + \frac{1}{2})|n\rangle. \quad (1.19.7)$$

**Lemma 1.19.1.** *Fluctuations in  $n$  is*

$$(\Delta n)^2 = 0. \quad (1.19.8)$$

*Proof.*

$$(\Delta n)^2 = \langle n | (\hat{n} - \langle \hat{n} \rangle)^2 | n \rangle \quad (1.19.9)$$

$$= \langle n | (\hat{n}^2 - 2\langle \hat{n} \rangle \hat{n} + \langle \hat{n} \rangle^2) | n \rangle \quad (1.19.10)$$

$$= \langle n | \hat{n}^2 | n \rangle - 2\langle \hat{n} \rangle^2 + \langle \hat{n} \rangle^2 \quad (1.19.11)$$

$$= \langle n | \hat{n}^2 | n \rangle - \langle n | \hat{n} | n \rangle^2 \quad (1.19.12)$$

$$= \langle n | n^2 | n \rangle - \langle n | n | n \rangle^2 \quad (1.19.13)$$

$$= n^2 - n^2 \quad (1.19.14)$$

$$= 0, \quad (1.19.15)$$

where we use

$$\langle n | \hat{n} | n \rangle = \langle \hat{n} \rangle. \quad (1.19.16)$$

□

**Lemma 1.19.2.** *The expectation value of the field is*

$$E = \langle n | \hat{E}(\chi) | n \rangle = 0. \quad (1.19.17)$$

*Proof.*

$$E = \langle n | \hat{E}(\chi) | n \rangle \quad (1.19.18)$$

$$= \frac{1}{2} \langle n | \hat{a} \exp[-i\chi] + \hat{a}^\dagger \exp[i\chi] | n \rangle \quad (1.19.19)$$

$$= \frac{1}{2} \langle n | \hat{a} \exp[-i\chi] | n \rangle + \frac{1}{2} \langle n | \hat{a}^\dagger \exp[i\chi] | n \rangle \quad (1.19.20)$$

$$= 0 + 0 \quad (1.19.21)$$

$$= 0. \quad (1.19.22)$$

□

**Lemma 1.19.3.** *Field fluctuations is*

$$(\Delta E(\chi))^2 = \frac{1}{2}\left(n + \frac{1}{2}\right). \quad (1.19.23)$$

*Proof.*

$$(\Delta E(\chi))^2 = \langle n | \hat{E}(\chi)^2 | n \rangle - \langle n | \hat{E}(\chi) | n \rangle^2 \quad (1.19.24)$$

$$= \langle n | \hat{E}(\chi)^2 | n \rangle - 0 \quad (1.19.25)$$

$$= \frac{1}{4} \langle n | (\hat{a} \exp[-i\chi] + \hat{a}^\dagger \exp[i\chi])^2 | n \rangle \quad (1.19.26)$$

$$= \frac{1}{4} \langle n | \hat{a}^2 \exp[-2i\chi] + \hat{a}^{\dagger 2} \exp[2i\chi] + \hat{a} \hat{a}^\dagger + \hat{a}^\dagger \hat{a} | n \rangle \quad (1.19.27)$$

$$= \frac{1}{4} \langle n | 2\hat{n} + 2 | n \rangle \quad (1.19.28)$$

$$= \frac{1}{4} \langle n | 2n + 1 | n \rangle \quad (1.19.29)$$

$$= \frac{1}{2}\left(n + \frac{1}{2}\right). \quad (1.19.30)$$

□

When  $n = 0$ , which is vacuum state, we have the standard deviation as  $1/2$ , which is half of the unit, i.e., vacuum field strength.

## 1.20 Field state of single radiation field mode: Coherent States

**How to reproduce classical motion** Superposition of Fock states reproduces oscillating wavepacket motion!

$$|\alpha\rangle \propto \exp[-i\frac{1}{2}\omega t]|0\rangle + \exp[-i\frac{3}{2}\omega t]\alpha|1\rangle + \exp[-i\frac{5}{2}\omega t]\frac{\alpha^2}{\sqrt{2}}|2\rangle + \dots \quad (1.20.1)$$

$$= \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \exp[-iE_n t/\hbar]. \quad (1.20.2)$$

**What we want:** states of light, whose expectation value corresponds to classical e.m. waves!

**Solution:** Coherent States

**Definition 1.20.1.** *Coherent States:*

$$|\alpha\rangle = \exp\left[-\frac{1}{2}|\alpha|^2\right] \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \quad (1.20.3)$$

where  $\alpha$  is complex number, amplitude of coherent state.  $\alpha = |\alpha| \exp[i\theta]$ ,  $|n\rangle$  is the fock state.



**Lemma 1.20.1.** *Coherent state is normalized:  $\langle \alpha | \alpha \rangle = 1$ .*

*Proof.*

$$\langle \alpha | \alpha \rangle = \exp[-|\alpha|^2] \sum_{n,n'} \frac{(\alpha^*)^n \alpha^{n'}}{\sqrt{n!} \sqrt{n'!}} \langle n | n' \rangle \quad (1.20.4)$$

$$= \exp[-|\alpha|^2] \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} \quad (1.20.5)$$

$$= \exp[-|\alpha|^2] \exp[+|\alpha|^2] \quad (1.20.6)$$

$$= 1. \quad (1.20.7)$$

□

**Lemma 1.20.2.** *Coherent state is quasi orthogonal:  $|\alpha - \beta| \gg 1 \longrightarrow |\langle \alpha | \alpha \rangle| \rightarrow 0$ .*

*Proof.*

$$\langle \alpha | \beta \rangle = \exp \left[ -\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2 + \alpha^* \beta \right]. \quad (1.20.8)$$

$$|\langle \alpha | \beta \rangle|^2 = \langle \alpha | \beta \rangle^* \langle \alpha | \beta \rangle \quad (1.20.9)$$

$$= \exp[-|\alpha|^2 - |\beta|^2 + \alpha^* \beta + \beta^* \alpha] \quad (1.20.10)$$

$$= \exp[-|\alpha - \beta|^2]. \quad (1.20.11)$$

We have  $|\alpha - \beta| \gg 1 \longrightarrow |\langle \alpha | \alpha \rangle| \rightarrow 0$ , which is called quasi-orthogonal. □

**Lemma 1.20.3.** *Coherent states are eigenstates of destruction operator  $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ .*

*Proof.*

$$\hat{a}|\alpha\rangle = \exp \left[ -\frac{1}{2}|\alpha|^2 \right] \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \quad (1.20.12)$$

$$= \exp \left[ -\frac{1}{2}|\alpha|^2 \right] \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \hat{a} |n\rangle \quad (1.20.13)$$

$$= \exp \left[ -\frac{1}{2}|\alpha|^2 \right] \sum_{n=1}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \sqrt{n} |n-1\rangle \quad (1.20.14)$$

$$= \alpha \exp \left[ -\frac{1}{2}|\alpha|^2 \right] \sum_{n=1}^{\infty} \frac{\alpha^{n-1}}{\sqrt{(n-1)!}} |n-1\rangle \quad (1.20.15)$$

$k := n-1$

$$= \alpha \exp \left[ -\frac{1}{2}|\alpha|^2 \right] \sum_{k=0}^{\infty} \frac{\alpha^k}{\sqrt{k!}} |k\rangle \quad (1.20.16)$$

$$= \alpha |\alpha\rangle. \quad (1.20.17)$$

□

**Lemma 1.20.4.** *The average photon number of coherent states:  $\bar{n} = |\alpha|^2$ .*

*Proof.*

$$\bar{n} = \langle \alpha | \hat{n} | \alpha \rangle \quad (1.20.18)$$

$$= \langle \alpha | \hat{a}^\dagger \hat{a} | \alpha \rangle \quad (1.20.19)$$

Use 1.20.3

$$= \langle \alpha^* \alpha | \alpha \rangle \quad (1.20.20)$$

$$= |\alpha|^2 \langle \alpha | \alpha \rangle \quad (1.20.21)$$

$$= |\alpha|^2. \quad (1.20.22)$$

□

[SZQ: 2023.04.20: coherent state is robust!]

**Lemma 1.20.5.** *Photon number variance of coherent states is*

$$(\Delta n)^2 = \bar{n}. \quad (1.20.23)$$

*Proof.*

$$(\Delta n)^2 = \langle \alpha | \hat{n}^2 | \alpha \rangle - \langle \alpha | \hat{n} | \alpha \rangle^2 \quad (1.20.24)$$

$$= \langle \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a} | \alpha \rangle - |\alpha|^4 \quad (1.20.25)$$

$$\hat{a} \hat{a}^\dagger = \hat{a}^\dagger \hat{a} + 1$$

$$= \langle \alpha | \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} | \alpha \rangle + \langle \hat{a}^\dagger \hat{a} | \alpha \rangle - |\alpha|^4 \quad (1.20.26)$$

$$= |\alpha|^2 \quad (1.20.27)$$

$$= \bar{n}. \quad (1.20.28)$$

□

**Lemma 1.20.6.** *The photon number distribution is*

$$P(n) = |\langle n|\alpha \rangle|^2 \quad (1.20.29)$$

$$= |\langle n| \exp \left[ -\frac{|\alpha|^2}{2} \right] \sum_{n'=0}^{\infty} \frac{\alpha^{n'}}{\sqrt{n'!}} |n'\rangle|^2 \quad (1.20.30)$$

$$= |\exp \left[ -\frac{|\alpha|^2}{2} \right] \sum_{n'=0}^{\infty} \frac{\alpha^{n'}}{\sqrt{n'!}} \langle n|n'\rangle|^2 \quad (1.20.31)$$

$$= |\exp \left[ -\frac{|\alpha|^2}{2} \right] \frac{\alpha^n}{\sqrt{n!}}|^2 \quad (1.20.32)$$

$$= \exp \left[ -|\alpha|^2 \right] \frac{|\alpha|^{2n}}{n!} \quad (1.20.33)$$

*Use lemma 1.20.4*

$$= \exp \left[ -\bar{n} \right] \frac{\bar{n}^n}{n!}. \quad (1.20.34)$$

This is known as **Poisson distribution**. The standard deviation is

$$\Delta n = \sqrt{\bar{n}}. \quad (1.20.35)$$

The standard deviation relative to the mean is

$$\frac{\Delta n}{\bar{n}} = \frac{1}{\sqrt{\bar{n}}}, \quad (1.20.36)$$

which shows that the fluctuations is smaller and smaller when the mean becomes larger and larger.

For large  $\bar{n}$ , we have

$$P(n) \simeq \frac{1}{\sqrt{2\pi\bar{n}}} \exp \left[ -\frac{1}{2} \frac{(n - \bar{n})^2}{\bar{n}} \right]. \quad (1.20.37)$$

[SZQ:  $\simeq$  means asymptotically equal to.]

**Lemma 1.20.7.** *The expectation value of the field operator*

$$\langle \alpha | \hat{E}(\chi) | \alpha \rangle = \frac{1}{2} (\langle \alpha | \hat{a} \exp[-i\chi] + \hat{a}^\dagger \exp[i\chi] | \alpha \rangle) \quad (1.20.38)$$

$$= \frac{1}{2} \left( \langle \alpha | \hat{a} \exp[-i\chi] | \alpha \rangle + \langle \alpha | \hat{a}^\dagger \exp[i\chi] | \alpha \rangle \right) \quad (1.20.39)$$

$$= \frac{1}{2} (\alpha \exp[-i\chi] + \alpha^* \exp[i\chi]) \quad (1.20.40)$$

$$= |\alpha| \cos(\chi - \theta), \quad (1.20.41)$$

where  $\alpha$  is the complex amplitude and  $\alpha = |\alpha| \exp[i\theta]$ .

We can plot  $\langle \alpha | \hat{E}(\chi) | \alpha \rangle$  in Phasor Diagram in terms of  $|\alpha|$  and  $\theta$ .

**Lemma 1.20.8.** *The fluctuations (variance) of the E-field*

$$(\Delta \hat{E}(\chi))^2 = \langle \alpha | \hat{E}(\chi)^2 | \alpha \rangle - \langle \alpha | \hat{E}(\chi) | \alpha \rangle^2 = \frac{1}{4}. \quad (1.20.42)$$

*Proof.*

$$(\Delta \hat{E}(\chi))^2 = \langle \alpha | \hat{E}(\chi)^2 | \alpha \rangle - \langle \alpha | \hat{E}(\chi) | \alpha \rangle^2 \quad (1.20.43)$$

$$= \frac{1}{4} \langle \alpha | \left[ \hat{a} \exp[-i\chi] + \hat{a}^\dagger \exp[i\chi] \right]^2 | \alpha \rangle - |\alpha|^2 \cos^2(\chi - \theta) \quad (1.20.44)$$

$$= \frac{1}{4} \langle \alpha | \hat{a}^2 \exp[-2i\chi] + \hat{a}^{\dagger 2} \exp[2i\chi] + \hat{a} \hat{a}^\dagger + \hat{a}^\dagger \hat{a} | \alpha \rangle - |\alpha|^2 \cos^2(\chi - \theta) \quad (1.20.45)$$

$$= \frac{1}{4} (\alpha^2 \exp[-2i\chi] + \alpha^{*2} \exp[2i\chi] + 2|\alpha|^2 + 1) - |\alpha|^2 \cos^2(\chi - \theta) \quad (1.20.46)$$

$$= \frac{1}{4} (|\alpha \exp[-i\chi] + \alpha^* \exp[i\chi]|^2 + 1) - |\alpha|^2 \cos^2(\chi - \theta) \quad (1.20.47)$$

$$= \frac{1}{4} (4|\alpha|^2 \cos^2(\chi - \theta) + 1) - |\alpha|^2 \cos^2(\chi - \theta) \quad (1.20.48)$$

$$= |\alpha|^2 \cos^2(\chi - \theta) + \frac{1}{4} - |\alpha|^2 \cos^2(\chi - \theta) \quad (1.20.49)$$

$$= \frac{1}{4}. \quad (1.20.50)$$

□

**Lemma 1.20.9.** *The expectation value of the Energy of Coherent States*

$$\langle \alpha | \hat{H} | \alpha \rangle = \hbar\omega(\bar{n} + 1/2). \quad (1.20.51)$$

*Proof.*

$$\langle \alpha | \hat{H} | \alpha \rangle = \hbar\omega \langle \alpha | \hat{n} + \frac{1}{2} | \alpha \rangle \quad (1.20.52)$$

$$= \hbar\omega (|\alpha|^2 + 1/2) \quad (1.20.53)$$

$$= \hbar\omega \left( \bar{n} + \frac{1}{2} \right). \quad (1.20.54)$$

□

**Lemma 1.20.10.** *The fluctuations of the energy of coherent States*

$$\Delta H = \hbar\omega |\alpha|. \quad (1.20.55)$$

*Proof.*

$$(\Delta H)^2 = \langle \alpha | \hat{H}^2 | \alpha \rangle - \langle \alpha | \hat{H} | \alpha \rangle^2, \quad (1.20.56)$$

where

$$\langle \alpha | \hat{H}^2 | \alpha \rangle = \langle \alpha | (\hbar\omega(\hat{n} + \frac{1}{2}))^2 | \alpha \rangle \quad (1.20.57)$$

$$= \hbar^2\omega^2 \langle \alpha | \hat{n}^2 + \hat{n} + \frac{1}{4} | \alpha \rangle \quad (1.20.58)$$

$$= \hbar^2\omega^2 \left( |\alpha|^4 + |\alpha|^2 + \frac{1}{4} \right). \quad (1.20.59)$$

Then

$$(\Delta H)^2 = \hbar^2\omega^2 \left( |\alpha|^4 + |\alpha|^2 + \frac{1}{4} \right) - \hbar^2\omega^2 \left( |\alpha|^2 + \frac{1}{2} \right)^2 \quad (1.20.60)$$

$$= 0 \quad (1.20.61)$$

□

[SZQ: 2023.04.21: I derive wrong answer. Where is the mistake?]

## 1.21 Quadrature Operators and Phase Space of Field States

**Definition 1.21.1.** *Classical eletromagnetic field is*

$$E(t) = E_0 \cos(\omega t + \theta) \quad (1.21.1)$$

$$= E_0 \cos \theta \cos \omega t - E_0 \sin \theta \sin \omega t \quad (1.21.2)$$

$$= X_1 \cos \omega t + X_2 \sin \omega t, \quad (1.21.3)$$

where  $X_1, X_2$  are quadrature variables defined as

$$X_1 = E_0 \cos \theta \quad (1.21.4)$$

$$X_2 = -E_0 \sin \theta. \quad (1.21.5)$$

**Definition 1.21.2.** *Phasor representation of field*

$$a(t) = E_0 \exp[-i\theta] \exp[-i\omega t] = a \exp[-i\omega t], \quad (1.21.6)$$

where  $a$  is defined as  $E_0 \exp[-i\theta]$ .  $a(t)$  **is called the phaor.**

**Lemma 1.21.1** (Relations between the Phasor representation and quadrature variables).

$$a = X_1 + iX_2. \quad (1.21.7)$$

*Proof.* By definition 1.21.2,

$$a = E_0 \exp[-i\theta] \quad (1.21.8)$$

$$= E_0 (\cos \theta - i \sin \theta) \quad (1.21.9)$$

$$= E_0 \cos \theta - i E_0 \sin \theta \quad (1.21.10)$$

By definition 1.21.1

$$= X_1 + i X_2. \quad (1.21.11)$$

□

[SZQ: 2023.04.21:  $a$  is determined by  $E_0$  and  $\theta$ .]

**Corollary 1.21.1.**

$$X_1 = \text{Re}(a) = \frac{1}{2} (a + a^*) \quad (1.21.12)$$

$$X_2 = \text{Im}(a) = \frac{1}{2i} (a - a^*). \quad (1.21.13)$$

*Proof.* By lemma 1.21.1, we can prove this corollary. □

## Quantum-Quadrature Operators

**Definition 1.21.3.** *Quadrature operators:*

$$\hat{x} \hat{=} \hat{X}_1 = \frac{1}{2} (\hat{a} + \hat{a}^\dagger) \quad (1.21.14)$$

$$\hat{p} \hat{=} \hat{X}_2 = \frac{1}{2i} (\hat{a} - \hat{a}^\dagger). \quad (1.21.15)$$

[SZQ:  $\hat{=}$  means "define"] [SZQ: Quadrature operators are **Hermitian operators**, i.e., observables.]

**Lemma 1.21.2.** *The commutation and uncertainty relations:*

$$[\hat{X}_1, \hat{X}_2] = \frac{i}{2} \quad (1.21.16)$$

$$\Delta X_1 \Delta X_2 \geq \frac{1}{4}. \quad (1.21.17)$$

*Proof.* □

**Definition 1.21.4.** *Generalized quadrature operators have the same commutation relations with quadrature operators:*

$$\hat{X}_\phi = \frac{1}{2} (\hat{a} \exp[-i\phi] + \hat{a}^\dagger \exp[i\phi]) \quad (1.21.18)$$

$$\hat{X}_{\phi+\frac{\pi}{2}} = \frac{1}{2i} (\hat{a} \exp[-i\phi] - \hat{a}^\dagger \exp[i\phi]). \quad (1.21.19)$$

**Phase space distribution of field states**

**Lemma 1.21.3.** *In vacuum state  $|0\rangle$ , we have*

$$P^{(0)}(X_1) = |\langle X_1|0\rangle|^2 = \sqrt{\frac{2}{\pi}} \exp[-2X_1^2] \quad (1.21.20)$$

$$P^{(0)}(X_2) = |\langle X_2|0\rangle|^2 = \sqrt{\frac{2}{\pi}} \exp[-2X_2^2]. \quad (1.21.21)$$

$$\text{Fluctuations: } \Delta X_1 = \sqrt{\langle 0|\hat{X}_1^2|0\rangle - \langle 0|\hat{X}_1|0\rangle^2} = \frac{1}{2}. \quad (1.21.22)$$

**Lemma 1.21.4.** *In fock state  $|n\rangle$ , we have*

$$P^{(n)}(X_1) = |\langle X_1|n\rangle|^2 \quad (1.21.23)$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{2^n n!} \exp[-2X_1^2] \left( H_n(\sqrt{2}X_1) \right)^2 \quad (1.21.24)$$

$$\Delta X_1 = \frac{1}{2} \sqrt{2n+1}. \quad (1.21.25)$$

**Coherent state - A displaced vacuum**

**Definition 1.21.5.** *Displacement operator (shifts any coherent state by  $\alpha$ )*

$$\hat{D}(\alpha) = \exp[\alpha \hat{a}^\dagger - \alpha^* \hat{a}]. \quad (1.21.26)$$

**Corollary 1.21.2.** *Coherent state from vacuum state*

$$|\alpha\rangle = \hat{D}(\alpha)|0\rangle. \quad (1.21.27)$$

**Squeezed states of light** We have nice pictures in the video.

We have phase squeezed state and amplitude squeezed state.

**1.22 The Classical Beamsplitter**

Assume all beams have same polarization and frequency. We input  $E_1, E_2$  and output  $E_3, E_4$ . There is a nice picture to illustrate it. Then we have

$$E_3 = RE_1 + TE_2 \quad (1.22.1)$$

$$E_4 = T'E_1 + R'E_2, \quad (1.22.2)$$

which is

$$\begin{pmatrix} E_3 \\ E_4 \end{pmatrix} = \begin{pmatrix} R & T \\ T' & R' \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}, \quad (1.22.3)$$

where  $R, R'$  denote reflection coefficients, which are complex coefficients,  $T, T'$  denote transmission coefficients, which are complex coefficients.

The simplified case: symmetric Beamsplitter

$$\begin{pmatrix} E_3 \\ E_4 \end{pmatrix} = \begin{pmatrix} R & T \\ T & R \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}. \quad (1.22.4)$$

**Lemma 1.22.1.** *The energy conservation*

$$|E_1|^2 + |E_2|^2 = |E_3|^2 + |E_4|^2 \quad (1.22.5)$$

leads to

$$|R|^2 + |T|^2 = 1 \quad (1.22.6)$$

$$RT^* + TR^* = 0. \quad (1.22.7)$$

*Proof.*

$$|E_3|^2 + |E_4|^2 = (RE_1 + TE_2)(RE_1 + TE_2)^* + (TE_1 + RE_2)(TE_1 + RE_2)^* \quad (1.22.8)$$

$$= |R|^2|E_1|^2 + |T|^2|E_2|^2 + RT^*E_1E_2^* + TR^*E_2E_1^* \quad (1.22.9)$$

$$+ |T|^2|E_1|^2 + |R|^2|E_2|^2 + TR^*E_1E_2^* + RT^*E_2E_1^* \quad (1.22.10)$$

$$= (|R|^2 + |T|^2)|E_1|^2 + (|R|^2 + |T|^2)|E_2|^2 \quad (1.22.11)$$

$$+ (RT^* + TR^*)E_1E_2^* + (RT^* + TR^*)E_2E_1^* \quad (1.22.12)$$

$$= |E_1|^2 + |E_2|^2. \quad (1.22.13)$$

Therefore,

$$|R|^2 + |T|^2 = 1 \quad (1.22.14)$$

$$RT^* + TR^* = 0. \quad (1.22.15)$$

□

**Lemma 1.22.2.** *Define  $R = |R|e^{i\phi_R}$  and  $T = |T|e^{i\phi_T}$ . We have  $\phi_R - \phi_T = \frac{\pi}{2}$ .*

*Proof.* By the second requirement in 1.22.1, we have

$$|R||T|e^{i(\phi_R - \phi_T)} + |R||T|e^{-i(\phi_R - \phi_T)} = 0. \quad (1.22.16)$$

So

$$2 \cos(\phi_R - \phi_T) = 0, \quad (1.22.17)$$

leads to

$$\phi_R - \phi_T = \frac{\pi}{2}. \quad (1.22.18)$$

□



Set  $\phi_T = 0$ , then  $\phi_R = \frac{\pi}{2}$ . We then have

$$E_3 = i|R|E_1 + |T|E_2 \quad (1.22.19)$$

$$E_4 = |T|E_1 + i|R|E_2. \quad (1.22.20)$$

**50/50 beamsplitter input-output** Symmetrized beamsplitter input-output relations

$$\begin{pmatrix} E_3 \\ E_4 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}. \quad (1.22.21)$$

In this case, light is always equally split in port 3 and port 4.

## 1.23 The Quantum Beamsplitter

In the quantum beamsplitter,  $E_1, E_2, E_3, E_4$  are replaced by  $\hat{a}_1, \hat{a}_2, \hat{a}_3, \hat{a}_4$ . There is a nice picture. We have

$$\hat{a}_3 = R\hat{a}_1 + T\hat{a}_2 \quad (1.23.1)$$

$$\hat{a}_4 = T\hat{a}_1 + R\hat{a}_2, \quad (1.23.2)$$

which is

$$\begin{pmatrix} \hat{a}_3 \\ \hat{a}_4 \end{pmatrix} = \begin{pmatrix} R & T \\ T & R \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix}, \quad (1.23.3)$$

where  $R$  denotes reflection coefficient, which is complex coefficient,  $T$  denotes transmission coefficient, which is complex coefficient.

**What about  $\hat{a}_1$  and  $\hat{a}_2$ ?**

**Lemma 1.23.1.**

$$\hat{a}_1 = R^*\hat{a}_3 + T^*\hat{a}_4 \quad (1.23.4)$$

$$\hat{a}_2 = T^*\hat{a}_3 + R^*\hat{a}_4. \quad (1.23.5)$$

*Proof.* Recall that

$$\hat{a}_3 = R\hat{a}_1 + T\hat{a}_2 \quad (1.23.6)$$

$$\hat{a}_4 = T\hat{a}_1 + R\hat{a}_2. \quad (1.23.7)$$

We have

$$R^*\hat{a}_3 = |R|^2\hat{a}_1 + R^*T\hat{a}_2 \quad (1.23.8)$$

$$T^*\hat{a}_4 = |T|^2\hat{a}_1 + T^*R\hat{a}_2. \quad (1.23.9)$$

Sum two equations, we have

$$(|R|^2 + |T|^2)\hat{a}_1 + R^*T\hat{a}_2 + T^*R\hat{a}_2 = R^*\hat{a}_3 + T^*\hat{a}_4. \quad (1.23.10)$$

Use lemma 1.22.1, we have

$$\hat{a}_1 = R^*\hat{a}_3 + T^*\hat{a}_4. \quad (1.23.11)$$

Similarly, we can get

$$\hat{a}_2 = T^*\hat{a}_3 + R^*\hat{a}_4. \quad (1.23.12)$$

□

**Lemma 1.23.2.** *We have the creation operators*

$$\hat{a}_1^\dagger = R\hat{a}_3^\dagger + T\hat{a}_4^\dagger \quad (1.23.13)$$

$$\hat{a}_2^\dagger = T\hat{a}_3^\dagger + R\hat{a}_4^\dagger \quad (1.23.14)$$

$$\hat{a}_3^\dagger = R^*\hat{a}_1^\dagger + T^*\hat{a}_2^\dagger \quad (1.23.15)$$

$$\hat{a}_4^\dagger = T^*\hat{a}_1^\dagger + R^*\hat{a}_2^\dagger. \quad (1.23.16)$$

**Single photon on BS Input state** We have a nice picture in the video.

$$|1\rangle_1|0\rangle_2 \quad (1.23.17)$$

$$|1\rangle_1|0\rangle_2 = \hat{a}_1^\dagger|0\rangle_1|0\rangle_2 \quad (1.23.18)$$

**Output state**

$$\hat{a}_1^\dagger = R\hat{a}_3^\dagger + T\hat{a}_4^\dagger. \quad (1.23.19)$$

This shows how  $\hat{a}_1^\dagger$  is splitted in port 3 and port 4. We have the output state as

$$\left(R\hat{a}_3^\dagger + T\hat{a}_4^\dagger\right)|0\rangle_3|0\rangle_4 = R|1\rangle_3|0\rangle_4 + T|0\rangle_3|1\rangle_4, \quad (1.23.20)$$

which is an entangled state of photon between field modes.

**Single photon on 50/50 BS** We have

$$\frac{1}{\sqrt{2}}(i|1\rangle_3|0\rangle_4 + |0\rangle_3|1\rangle_4). \quad (1.23.21)$$

**Lemma 1.23.3.** *Average output photon number at port 3*

$$\langle \hat{n}_3 \rangle = \frac{1}{2}, \quad (1.23.22)$$

and at port 4

$$\langle \hat{n}_4 \rangle = \frac{1}{2}, \quad (1.23.23)$$

*Proof.*

$$\langle \hat{n}_3 \rangle = \langle \hat{a}_3^\dagger \hat{a}_3 \rangle \quad (1.23.24)$$

$$= {}_2 \langle 0|_1 \langle 1| \hat{a}_3^\dagger \hat{a}_3 |1\rangle_1 |0\rangle_2 \quad (1.23.25)$$

$$= {}_2 \langle 0|_1 \langle 1| (R^* \hat{a}_1^\dagger + T^* \hat{a}_2^\dagger) (R \hat{a}_1 + T \hat{a}_2) |1\rangle_1 |0\rangle_2 \quad (1.23.26)$$

number operator  $\hat{n} = \hat{a}^\dagger \hat{a}$

$$= |R|_2^2 \langle 0|_1 \langle 1| 1|1\rangle_1 |0\rangle_2 \quad (1.23.27)$$

$$= \frac{1}{2}. \quad (1.23.28)$$

Similiarly, we have

$$\langle \hat{n}_4 \rangle = \frac{1}{2}. \quad (1.23.29)$$

□

**Lemma 1.23.4.** *Correlations*

$$\langle \hat{n}_3 \hat{n}_4 \rangle = {}_2 \langle 0|_1 \langle 1| \hat{n}_3^\dagger \hat{n}_4 |1\rangle_1 |0\rangle_2 = 0. \quad (1.23.30)$$

**Remark 1.23.1.** • *non-classical Correlations.*

- *due to entangled single photon state between field modes.*

**What about coherent states?**

## 1.24 Balanced Homodyne Detection