# Chapter 1

# Measure Theory

## 1.1 Sigma algebra

**Definition 1.1.1** (Power set). *We define $\mathcal{P}(X)$ as the power set of set $X$. Assume that set $X = \{a, b\}$, the power set $P(X)$ would be $\{\emptyset, X, \{a\}, \{b\}\}$*

**Definition 1.1.2** (Sigma algebra). *$\mathcal{A} \subseteq \mathcal{P}(X)$ is called a $\sigma -$ algebra:*

$$(a)\ \emptyset, X \in \mathcal{A} \tag{1.1.1}$$

$$(b)\ A \in \mathcal{A} \Longrightarrow A^c := X \setminus A \in \mathcal{A} \tag{1.1.2}$$

$$(c)\ A_i \in \mathcal{A},\ i \in \mathcal{N} \Longrightarrow \cup_{i=1}^{\infty} A_i \in \mathcal{A}. \tag{1.1.3}$$

**Definition 1.1.3** (Measurable sets). *$A \in \mathcal{A}$ is called a $\mathcal{A}$-measurable set.*

**Example 1.1.1.**

$$(1)\ \mathcal{A} = \{\emptyset, X\} \tag{1.1.4}$$

$$(2)\ \mathcal{A} = \{\mathcal{P}(X)\}. \tag{1.1.5}$$

**Lemma 1.1.1.** *Assume $\mathcal{A}_i$ is $\sigma$-algebra on $X$, $i \in I$ (index set). Then, we have $\cap_{i \in I} \mathcal{A}_i$ is also a $\sigma$-algebra on $X$.*

**Definition 1.1.4** (Sigma algebra generated by $\mathcal{M}$). *For $\mathcal{M} \subseteq \mathcal{P}(X)$, there is a smallest $\sigma$-algebra that contains $\mathcal{M}$:*

$$\sigma(\mathcal{M}) := \cap_{\mathcal{A} \supseteq \mathcal{M},\ a\ \sigma-algebra} \mathcal{A}. \tag{1.1.6}$$

**Example 1.1.2.** *We define $X = \{a, b, c, d\}$ and $\mathcal{M} = \{\{a\}, \{b\}\}$. Then we have*

$$\sigma(\mathcal{M}) = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c, d\}, \{a, c, d\}, \{c, d\}\}. \tag{1.1.7}$$

Here is a very nice picture which shows the connections between the topological space, metric space and vector space. It is from this video:

**Definition 1.1.5** (Topological spaxe)**.** *Topological space $(X, \tau)$, where $X$ is a set and $\tau$ is a collection of subsets of $X$. This $\tau$ is called the togology on $X$. The pair $(X, \tau)$ is called the topological space. In order to be a topological space, the collection of subsets must satisfy three properties:*

- *Empth set $\emptyset, X \in \tau$.*
- *Unions must be in $\tau$, i.e., $\cup_{i=1}^{\infty} \tau_i \in \tau$.*
- *Intersections must be in $\tau$, i.e., $\cap_{i=1}^{n} \tau_i \in \tau$.*

**Definition 1.1.6** (Indiscrete topology)**.** *Indiscrete topology is defined as $\tau = \{\emptyset, X\}$.*

*Proof.* This can proved by the definition of 1.1.5. □

**Definition 1.1.7** (Discrete topology)**.** *Discrete topology is the power set 1.1.1 of $X$.*

**Proposition 1.1.1.** *Any topology $\tau$ on $X$ satisfy the following relation: $\{\emptyset, X\} \subset \tau \subset \mathcal{P}(X)$, where $\{\emptyset, X\}$ is the indiscrete topology 1.1.6 and $\mathcal{P}(X)$ is the discrete topology 1.1.7.*

**Definition 1.1.8** (Borel sigma algebra)**.** *Let $(X, \mathcal{T})$ be a topological space 1.1.5 (Let $X$ be a metric space/Let $X$ be a subset of $\mathbb{R}^n$; We need "open sets".). We then define $\mathcal{B}(X)$ is the borel $\sigma$-algebra on $X$ as*

$$\mathcal{B}(X) := \sigma(\mathcal{T}), \tag{1.1.8}$$

*which is the $\sigma$-algebra generated by the open sets $\mathcal{T}$.*

**Definition 1.1.9** (Borel sets)**.** *Any set contained in Borel $\sigma$-algebra is called Borel set. If $A \in \mathcal{B}(X)$, then $A$ is borel set.*

**Proposition 1.1.2.** *Let $\Omega = [0, 1)$ and $b \in \Omega$, then the singleton $\{b\}$ is a Borel set.*

*Proof.*

$$\{b\} = \cap_{n=1}^{\infty} \left[ (b - \frac{1}{n}, b + \frac{1}{n}) \cap \Omega \right]. \tag{1.1.9}$$

□

[SZQ: 2023.04.08: This is a standard trick to prove.]

**Proposition 1.1.3.** *Let $\Omega = [0,1)$ and $b \in \Omega$, then $(a,b]$, $[a,b]$ and $[a,b)$ are Borel sets.*

*Proof.* We write

$$(a,b] = \cap_{n=1}^{\infty}(a, b+\frac{1}{n}) \cap \Omega. \tag{1.1.10}$$

Then we can prove $(a,b]$ is a borel set.

We can also write $(a,b]$ as the union of singletons and there use 1.1.2 and the fact that the union of borel sets is also a borel set. □

**Definition 1.1.10** (Borel measure)**.** *A borel measure on $\mathbb{R}$ is a measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.*

**Definition 1.1.11** (Cumulative distribution function)**.** *A CDF(cumulative distribution function) is a function $F : \mathbb{R} \to \mathbb{R}$ such that*

- *$F$ is nondecreasing ($x \leq y \implies F(x) \leq F(y)$)*
- *$F$ is right continuous ($\lim_{a \to a^+} = F(a)$)*
- $\lim_{x\to\infty} F(x) = 1$
- $\lim_{x\to-\infty} F(x) = 0$

**Theorem 1.**
- *If $F$ is a CDF then there si a unique Borel probability measure on $\mathbb{R}$ such that $P((-\infty, x]) = F(x), \forall x \in \mathbb{R}$.*
- *If $P$ is a Borel probability measure on $\mathbb{R}$ then there is a unique CDF $F$ such that $F(x) = P((-\infty, x]), \forall x \in \mathbb{R}$.*

*That is, there is an equivalence between CDFs and Borel probability measure.*

## 1.2 What is a measure?

**Definition 1.2.1** (Measure)**.** *$(X, \mathcal{A})$ is called a measurable space, where $X$ is a set and $\mathcal{A}$ is a $\sigma$-algebra on $X$. A map $\mu : \mathcal{A} \to [0, \infty] := [0, \infty) + \{\infty\}$ is called a measure if it satisfies:*

$$(a)\ \mu(\emptyset) = 0 \tag{1.2.1}$$

$$(b)\ \mu(\cup_{i=1}^{\infty}\mathcal{A}_i) = \sum_{i=1}^{\infty} \mu(\mathcal{A}_i)\ with\ \mathcal{A}_i \cap \mathcal{A}_j = \emptyset,\ i \neq j\ for\ all\ \mathcal{A}_i \in \mathcal{A}.(\sigma - additive) \tag{1.2.2}$$

**Definition 1.2.2.** *$(X, \mathcal{A}, \mu)$ is called a measure space.*

**Example 1.2.1.** *Given $X$ and $\mathcal{A} = \mathcal{P}(X)$.*

- *Counting measure ($A \in \mathcal{A}$) is defined as*

$$\mu(A) := \begin{cases} \#A, & A \text{ has finitely many elements} \\ \infty & \text{else} \end{cases} \tag{1.2.3}$$

*where $\#A$ means the number of elements in $A$.*

*Calculation rules in $[0, \infty]$:*

$$x + \infty := \infty \ for\ all\ x \in [0, \infty] \tag{1.2.4}$$

$$x \cdot \infty := \infty \ for\ all\ x \in (0, \infty] \tag{1.2.5}$$

$$0 \cdot \infty := 0 \ (\text{only true in most cases in measure theory!}) \tag{1.2.6}$$

- *Dirac measure for $p \in X$ is defined as*

$$\delta_p(A) := \begin{cases} 1, & p \in A \\ 0, & \text{else} \end{cases} \tag{1.2.7}$$

- *We search a measure on $X \in \mathcal{R}^n$ satisfying:*

$$(1)\ \mu([0,1]^n) = 1 \tag{1.2.8}$$

$$(2)\ \mu(x + A) = \mu(A)\ for\ all\ x \in \mathcal{R}^n, \tag{1.2.9}$$

*which is known as Lebesgue measure where the $\sigma$-algebra is not equal to power set.*

## 1.3 Not everything is lebesgue measurable

**Measure problem:** search measure $\mu$ on $\mathcal{P}(\mathbb{R})$ with:

- (1) $\mu([a, b]) = b - a,\ b > a,$
- (2) $\mu(x + A) = \mu(A),\ A \in \mathcal{P}(\mathbb{R}),\ x \in \mathbb{R}.$

$\Longrightarrow \mu$ does not exist.

**Claim:** Let $\mu$ be a measure on $\mathcal{P}(\mathbb{R})$ with $\mu((0, 1]) < \infty$ and (2). $\Longrightarrow \mu = 0$.

*Proof.* (a) Definitions: $I \in (0, 1]$ with equivalence relation on $I$: $x{\sim}y \iff x - y \in \mathbb{Q}$ i.e., $[x] := \{x + r | r \in \mathbb{Q},\ x + r \in I\}$. Following this definition, we have a disjoint decomposition of $I$ into boxes, possibly uncontable many of them! We then pick one element $a_n$ from each box $[x_n]$ and form a set $A \in I$, i.e., $\{a_1, a_2, \cdots\} = A$. We have $A \in I$ with prperty:

- (1) For each $[x]$, there is an $a \in A$ with $a \in [x]$.
- (2) For all $a, b \in A: \; a, b \in [x] \Longrightarrow a = b$.

In uncountable case, the existence of $A \in I$ with the above property is guaranted by the axiom of choice of set theory.

We define $A_n := r_n + A$, where $(r_n)_{n\in\mathbb{N}}$ enumeration of $\mathbb{Q}_n(-1, 1]$.

(b) We then claim that $A_n \cap A_m = \emptyset \Longleftarrow n \neq m$. The proof is as follows: $x \in A_n \cap A_m \Longrightarrow x = r_n + a_n,\ a_n \in A$ and $x = r_m + a_m,\ a_m \in A. \Longrightarrow r_n + a_n = r_m + a_m \Longrightarrow a_n - a_m = r_n - r_m \in \mathbb{Q} \Longrightarrow a_n \sim a_m \Longrightarrow a_m, a_n \in [a_m] \Longrightarrow a_n = a_m \Longrightarrow r_n = r_m \Longrightarrow n = m$.

(c) We claim that $(0, 1] \subseteq \cup_{n\in\mathbb{N}} A_n \subseteq (-1, 2]$. The proof is as follows:

Assume now: $\mu$ measure on $\mathcal{P}(\mathbb{R})$ with $\mu((0, 1]) < \infty$ and (2).

By (2): $\mu(1 + A) = \mu(A)$ for all $n \in \mathbb{N}$.

By (c): we have

$$\mu((0, 1]) \leq \mu(\cup_{n\in\mathbb{N}} A_n) \leq \mu((-1, 2]) \tag{1.3.1}$$

We know: $\mu((0, 1]) =: C < \infty$. By using (2) and $\sigma$-additivity, we get $\mu((-1, 2]) = \mu\left((-1, 0] \cup (0, 1] \cup (1, 2] = 3C\right)$. $\Longrightarrow_{1.3.1,(b)} C \leq \sum_{n=1}^{\infty} \mu(A_n) \leq 3C \Longrightarrow C \leq \sum_{n=1}^{\infty} \mu(A) \leq 3C \Longrightarrow \mu(A) = 0 \Longrightarrow C = 0$(henceL $\mu\left((0, 1]\right) = 0) \Longrightarrow \mu(\mathbb{R}) = \mu\left(\cup_{n\in\mathbb{Z}} (m, m + 1]\right) = 0 \Longrightarrow \mu = 0$. □

## 1.4 Measurable maps

**Definition 1.4.1** (Measurable maps). *$(\Omega_1, \mathcal{A}_1)$ and $(\Omega_2, \mathcal{A}_2)$ are measurable spaces. $f : \Omega_1 \to \Omega_2$ is a measurable map w.r.t. $\mathcal{A}_1$ and $\mathcal{A}_2$ if $f^{-1}(A_2) \in \mathcal{A}_1$ for all $A_2 \in \mathcal{A}_2$.*

**Example 1.4.1.**

- *$(\Omega, \mathcal{A})$ and $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ are two measurable spaces. We define characteristic fucntion (aksi indicator function) as $\chi_A : \Omega \to \mathbb{R}$, where*

$$\chi_A(w) := \begin{cases} 1, & w \in A \\ 0, & w \notin A \end{cases} \tag{1.4.1}$$

*For all measurable $A \in \mathcal{A}$, $\chi_A$ is a measurable map. We have*

$$\chi_A^{-1}(\emptyset) = \emptyset \in \mathcal{A},\ \chi_A^{-1}(\mathbb{R}) = \Omega \in \mathcal{A} \tag{1.4.2}$$

$$\chi_A^{-1}(\{A\}) = A,\ \chi_A^{-1}(\{0\}) = A^c \in \mathcal{A}. \tag{1.4.3}$$

- *Composition of measurable maps.*

**Lemma 1.4.1.** *$(\Omega_1, \mathcal{A}_1)$, $(\Omega_2, \mathcal{A}_2)$, $(\Omega_3, \mathcal{A}_3)$ are measurable space. We define $\Omega_1 \xrightarrow{f} \Omega_2 \xrightarrow{g} \Omega_3$. Then $f, g$ are measurable implies $g \circ f$ is measurable.*

*Proof.*

$$(g \circ f)^{-1}(A_3) = f^{-1}(g^{-1}(A_3)) \tag{1.4.4}$$

$$\in \mathcal{A}_1 \tag{1.4.5}$$

□

**Important measurable maps**

**Lemma 1.4.2.** *$(\Omega, \mathcal{A})$ and $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ are measurable spaces. $f, g : \Omega \to \mathbb{R}$ are measurable maps indicates that $f + g$, $f - g$, $f \cdot g$, $|f|$ are measurable maps.*

## 1.5 Lebesgue integral

**Example 1.5.1.** *Define Characteristic function $\chi_A : X \to \mathbb{R}$, $A \in \mathcal{A}$. We define $I(A) := \mu(A)$. Surprisingly, $I(A)$ is nothing but the integral of $\chi_A$ over $A$.*

**Definition 1.5.1** (Simple/Step/Staircsae functions,...)**.** *For $A_1, A_2, \ldots, A_n \in \mathcal{A}$, and $c_1, c_2, \cdots, c_n \in \mathbb{R}$. We define*

$$f(x) := \sum_{i=1}^{n} c_i \cdot \chi_{A_i}(x). \tag{1.5.1}$$

*We then have $f(x)$ is measurable and the integraal of $f$ is defined as $I(f) := \sum_{i=1}^{n} c_i \mu(A_i)$.*

**Remark 1.5.1.** *The problem of the integral $I(f)$ is that it is undefined when $\mu(A_i) = \infty$. The problem can be solved by exclude $\infty$ by defintion or the following way.*

**Definition 1.5.2** (Lebesgue integral)**.** *Define $S^+ := \{f : X \to \mathbb{R} | f \text{ simple function}, \ f \geq 0\}$. $f \in S^+$ and choose representation $f(x) = \sum_{i=1}^{n} c_i \chi_{A_i}(x)$, $c_i \geq 0$. The lebesgue integral of $f$ w.r.t. $\mu$ is defined as*

$$\int_X f(x) \ \mathrm{d}\mu(x) = \int_X f \ \mathrm{d}\mu \tag{1.5.2}$$

$$= I(f) \tag{1.5.3}$$

$$= \sum_{i=1}^{n} c_i \cdot \mu(A_i) \tag{1.5.4}$$

$$= [0, \infty]. \tag{1.5.5}$$

**Property 1.5.1.**
- $I(\alpha f + \beta g) = \alpha I(f) + \beta I(g)$, $\alpha, \beta \geq 0$.

- $f \le g \Longrightarrow I(f) \le I(g)$ *(monotomicity)*

**Definition 1.5.3.** *Define a measurable map $f : X \to [0, \infty)$. $h = \sum_{i=1}^{n} c_i \cdot \chi_{A_i}$. The lebesgue integral of $f$ w.r.t. $\mu$ is defined as*

$$\int_X f \, \mathrm{d}\mu := \sup \left\{ I(h) | h \in S^+, \; h \le f \right\} \tag{1.5.6}$$

$$\in [0, \infty]. \tag{1.5.7}$$

*$f$ is called $\mu$-integrable if $\int_X f \, \mathrm{d}\mu < \infty$.*

**Property 1.5.2.** *Define measurable maps $f, g : X \to [0, \infty)$, we have*

- *1. $f = g$ for $\mu$-almost everywhere(a.e.), which satisfies $\mu\left(\{x \in X | f(x) \neq g(x)\}\right) = \Longrightarrow \int_X f \, \mathrm{d}\mu = \int_X g \, \mathrm{d}\mu$.*
- *2. $f \le g$ for $\mu$ a.e. $\Longrightarrow \int_X f \, \mathrm{d}\mu \le \int_X g \, \mathrm{d}\mu$*
- *3. $f = 0$ for $\mu$-a.e. $\Longleftrightarrow \int_X f \, \mathrm{d}\mu = 0$.*

*Proof of 2.: monotonicity.* Let $h := X \to [0, \infty)$ be a simple function, i.e.,

$$h(x) = \sum_{i=1}^{n} c_i \chi_{A_i}(x) \tag{1.5.8}$$

$$= \sum_{t \in h(X)} t \cdot \chi_{\{x \in X | h(x) = t\}}. \tag{1.5.9}$$

Let $X = \tilde{X}^c \cup \tilde{X}$ with $\mu(\tilde{X}^c) = 0$,

$$\tilde{h}(x) := \begin{cases} h(x), & x \in \tilde{X} \\ a, & x \in \tilde{X}^c \end{cases} \tag{1.5.10}$$

$$\tilde{h}(x) = \sum_{t \in h(X)} t \cdot \chi_{\left\{x \in \tilde{X} | h(x) = t\right\}} + a \cdot \chi_{\tilde{X}^c} \tag{1.5.11}$$

$$I(\tilde{h}) = \sum_{t \in h(X)} t \cdot \mu(\left\{x \in \tilde{X} | h(x) = t\right\}) + a \cdot \mu(\tilde{X}^c) \tag{1.5.12}$$

$$= \sum_{t \in h(X)} t \left[ \mu\left( \left\{ x \in \tilde{X} | h(x) = t \right\} \right) + \mu\left( \left\{ x \in \tilde{X}^c | h(x) = t \right\} \right) \right] \tag{1.5.13}$$

$$= \sum_{t \in h(X)} t \left[ \mu\left( \left\{ x \in \tilde{X} | h(x) = t \right\} \cup \left\{ x \in \tilde{X}^c | h(x) = t \right\} \right) \right] \tag{1.5.14}$$

$$I(h) = \sum_{t \in h(X) \setminus \{0\}} t \cdot \mu\left(\{x \in X | h(x) = t\}\right). \tag{1.5.15}$$

We define

$$
\begin{aligned}
\tilde{X} &:= \{x \in X | f(x) \leq g(x)\}, && (1.5.16)\\
\mu(\tilde{X}^c) &= 0 && (1.5.17)\\
\int_X f \ \mathrm{d}\mu &= \sup\left\{I(h) | h \in S^+, h \leq f\right\} && (1.5.18)\\
&= \sup\{I(\tilde{h}) | \tilde{h} \in S^+, \tilde{h} \leq f \ on \ \tilde{X}\} && (1.5.19)\\
&\leq \sup\{I(\tilde{h}) | \tilde{h} \in S^+, h \leq g \ on \ \tilde{X}\} && (1.5.20)\\
&= \sup\{I(h) | h \in S^+, h \leq g\} && (1.5.21)\\
&= \int_X g \ \mathrm{d}\mu. && (1.5.22)
\end{aligned}
$$

□

**Theorem 2** (Monotone convergence theorem)**.** *$(X, \mathcal{A}, \mu)$ measurable spaces, $f_n : X \to [0, \infty]$, $(f : X \to [0, \infty])$ measurable for all $n \in \mathbb{N}$ with*

$$f_1 \leq f_2 \leq f_3 \leq \cdots \quad \mu - a.e. \tag{1.5.23}$$

$$(\lim_{n\to\infty} \int_X f_n \ \mathrm{d}\mu = \int_X f \ \mathrm{d}\mu \quad \mu - a.e. (x \in X)) \tag{1.5.24}$$

*This implies that*

$$\lim_{n\to\infty} \int_X f_n \ \mathrm{d}\mu = \int_X \lim_{n\to} f_n \ \mathrm{d}\mu. \tag{1.5.25}$$

*Proof.* $\int_X f_1 \ \mathrm{d}\mu \leq \int_X f_2 \ \mathrm{d}\mu \leq \cdots$ and $\int_X f_n \ \mathrm{d}\mu \leq \int_X f \ \mathrm{d}\mu$ for $n \in \mathbb{N}$. Then we have

$$\lim_{n\to\infty} \int_X f_n \ \mathrm{d}\mu \leq \int_X f \ \mathrm{d}\mu, \tag{1.5.26}$$

which is the first part of 1.5.25.

Let $h$ be a simple function $0 \leq h \leq f$ and $\varepsilon > 0$. We define

$$X_n := \{x \in X | f_n(x) \geq (1 - \varepsilon) h(x)\} \tag{1.5.27}$$

with $\cup_{n=1}^{\infty} X_n = \tilde{X}$, and $\mu(\tilde{X}^c) = 0$. We have

$$
\begin{aligned}
\int_X f_n \ \mathrm{d}\mu \geq \int_{X_n} f_n \ \mathrm{d}\mu &\geq \int_{X_n} (1 - \varepsilon) h \ \mathrm{d}\mu && (1.5.28)\\
\lim_{n\to\infty} \int_X f_n \ \mathrm{d}\mu &\geq \lim_{n\to\infty} \int_{X_n} (1 - \varepsilon) h \ \mathrm{d}\mu && (1.5.29)\\
&= \int_{\tilde{X}} (1 - \varepsilon) h \ \mathrm{d}\mu && (1.5.30)\\
&= \int_X (1 - \varepsilon) h \ \mathrm{d}\mu. && (1.5.31)
\end{aligned}
$$

This implies

$$\lim_{n\to\infty}\int_X f_n \ \mathrm{d}\mu \geq \int_X h \ \mathrm{d}\mu, \tag{1.5.32}$$

since $\varepsilon > 0$ arbitrarily. Then we have

$$\lim_{n\to\infty}\int_X f_n \ \mathrm{d}\mu \geq \int_X f \mathrm{d}\mu, \tag{1.5.33}$$

since $h$ is arbitrary and $h \leq f$, which is second part of 1.5.25. □

**Applictions** Given a series $(g_n)_{n\in\mathbb{N}}$, $g_n : X \to [0,\infty]$ measurable for all $n$. Then we have $\sum_{n=1}^{\infty} g_n : X \to [0,\infty]$ measurable and

$$\int_X \sum_{n=1}^{\infty} g_n \ \mathrm{d}\mu = \sum_{n=1}^{\infty}\int_X g_n \ \mathrm{d}\mu, \tag{1.5.34}$$

which means the integral and sum can exchange.

## 1.6 Fatou' lemma

**Lemma 1.6.1** (Fatou' lemma)**.** *Given $(X,\mathcal{A},\mu)$ measurable space, $f_n : X \to [0,\infty]$ measurable for all $n \in \mathbb{N}$. Then we have*

$$\int_X \liminf_{n\to\infty} f_n \ \mathrm{d}\mu \leq \liminf_{n\to\infty}\int_X f_n \ \mathrm{d}\mu. \tag{1.6.1}$$

**Remark 1.6.1.** $\liminf_{n\to\infty} f_n : X \to [0,\infty]$ *is a function. This is*

$$\begin{aligned} g(x) &:= \left(\liminf_{n\to\infty} f_n\right)(x) && (1.6.2)\\ &:= \lim_{n\to\infty}\left(\inf_{k\geq n} f_k(x)\right) && (1.6.3)\\ &\in [0,\infty] && (1.6.4)\\ g_n(x) &:= \inf_{k\geq n} f_k(x). && (1.6.5)\end{aligned}$$

*We have*

$$g_1 \leq g_2 \leq g_3 \leq \cdots, \tag{1.6.6}$$

*which is monotonically increasing. All these functions are measurable.*

*Proof.*

Since (2),

$$\int_X \lim_{n\to\infty} g_n \,\mathrm{d}\mu = \lim_{n\to\infty} \int_X g_n \,\mathrm{d}\mu \tag{1.6.7}$$

$$= \liminf_{n\to\infty} \int_X g_n \,\mathrm{d}\mu. \tag{1.6.8}$$

We know that $g_n \leq f_n$ for all $n \in \mathbb{N}$. By (1.5.2), we have

$$\int_X g_n \,\mathrm{d}\mu \leq \int_X f_n \,\mathrm{d}\mu, \tag{1.6.9}$$

for all $n \in \mathbb{N}$. Then we have

$$\int_X \liminf_{n\to\infty} f_n \,\mathrm{d}\mu = \liminf_{n\to\infty} \int_X g_n \,\mathrm{d}\mu \tag{1.6.10}$$

$$\leq \liminf_{n\to\infty} \int_X f_n \,\mathrm{d}\mu. \tag{1.6.11}$$

□

## 1.7 Lebesgue's dominated convergence theorem

$(X, \mathcal{A}, \mu)$, $\mathcal{L}^1 := \left\{f : X \to \mathbb{R} \textit{ measurable} \middle| \int_X |f|^1 \,\mathrm{d}\mu < \infty\right\}$. For $f \in \mathcal{L}^1(\mu)$, write $f = f^+ - f^-$, where $f^+, f^- \geq 0$. Define $\int_X f \,\mathrm{d}\mu := \int_X f^+ \,\mathrm{d}\mu - \int_X f^- \,\mathrm{d}\mu$.

**Theorem 3** (Lebesgue's dominated convergence theorem)**.** *$f_n : X \to \mathbb{R}$ measurable for all $n \in \mathbb{N}$. $f : X \to \mathbb{R}$ with $\overset{n\to\infty}{f(x)}$ for $x \in X$ ($\mu$-a.e.) and $|f_n| \leq g$ with $g \in \mathcal{L}^1(\mu)$ for all $n \in \mathbb{N}$, where $g$ is called integral majorant. Then: we have $f_1, f_2, \cdots \in \mathcal{L}^1(\mu)$, $f \in \mathcal{L}^1(\mu)$ and*

$$\lim_{n\to\infty} \int_X f_n \,\mathrm{d}\mu = \int_X f \,\mathrm{d}\mu. \tag{1.7.1}$$

*Proof.*

$$|f_n| \leq g \overset{\textit{monotonicity}}{\Longrightarrow} \int_X g \,\mathrm{d}\mu < \infty \tag{1.7.2}$$

$$\Longrightarrow f_1, f_2, \cdots \in \mathcal{L}^1(\mu) \tag{1.7.3}$$

$$|f| \leq g \ for\ \mu - \text{a.e.} \Longrightarrow f \in \mathcal{L}^1(\mu) \tag{1.7.4}$$

We will show $\int_X |f_n - f| \,\mathrm{d}\mu \overset{n\to\infty}{\Longrightarrow} 0$.

$$|f_n - f| \leq |f_n| + |f| \leq 2g \tag{1.7.5}$$

$$\Longrightarrow h_n := 2g - |f_n - f| \geq 0 \tag{1.7.6}$$

Hence: $h_n : X \to [0, \infty]$ measurable for all $n \in \mathbb{N}$. Then by (1.6.1),

$$\implies \int_X \liminf_{n\to\infty} h_n \; \mathrm{d}\mu \le \liminf_{n\to\infty} \int_X h_n \; \mathrm{d}\mu \tag{1.7.7}$$

$$\implies \int_X 2g \; \mathrm{d}\mu \le \int_X 2g \; \mathrm{d}\mu - \limsup_{n\to\infty} \int_X |f_n - f| \; \mathrm{d}\mu \tag{1.7.8}$$

$$\implies 0 \le \liminf_{n\to\infty} \int_X |f_n - f| \; \mathrm{d}\mu \le \limsup_{n\to\infty} \int_X |f_n - f| \; \mathrm{d}\mu \le 0 \tag{1.7.9}$$

$$\implies \tag{1.7.10}$$

Limits exists and $\lim_{n\to\infty} |f_n - f| \; \mathrm{d}\mu = 0$. We conclude that

$$\tag{1.7.11}$$

$$0 \le |\int_X f_n \; \mathrm{d}\mu - \int_X f \; \mathrm{d}\mu| = |\int_X (f_n - f) \; \mathrm{d}\mu| \le \int_X |f_n - f| \; \mathrm{d}\mu \overset{n\to\infty}{\longrightarrow} 0,$$

$$\tag{1.7.12}$$

where the third inequality is due to the integral's triangle inequality.

$$\tag{1.7.13}$$

$$\implies \lim_{n\to\infty} \int_X f_n \; \mathrm{d}\mu = \int_X f \; \mathrm{d}\mu. \tag{1.7.14}$$

□

## 1.8 Caratheodory's extension theorem

**Theorem 4** (Caratheodory's extension theorem)**.** *$X$ set, $\mathcal{A} \in \mathcal{P}(X)$ semiring of sets. A map $\mu : \mathcal{A} \to [0, \infty]$. Note that $\mu$ is not a measure, it is called A pre-measure.*

- *Then $\mu$ has an extension $\tilde{\mu} : \sigma(\mathcal{A}) \to [0, \infty]$, where $\tilde{\mu}$ is a measure and $\sigma(\mathcal{A})$ is a $\sigma$-algebra generated by $\mathcal{A}$, i.e., $\mu(A) = \tilde{\mu}(A)$.*
- *If there is sequence $(S_j)$ with $S_j \in \mathcal{A}$, $\cup_{j=1}^{\infty} S_j = X$, then the extension $\tilde{\mu}$ from (a) is unique. ($\tilde{\mu}$ is also $\sigma$-finite)*

**Definition 1.8.1** (Semiring set)**.** *Semiring of sets $\mathcal{A} \subseteq \mathcal{P}(X)$:*

- *$\emptyset \in \mathcal{A}$ (as for $\sigma$-algebra)*
- *$A, B \in \mathcal{A} \implies A \cap B \in \mathcal{A}$*
- *For $A, B \in \mathcal{A}$, there are pairwise disjoint sets $S_1, S_2, \ldots, S_n \in \mathcal{A} : \cup_{j=1}^{n} S_j = A \backslash B$*

**Example 1.8.1.** *$A := \{[a, b)|a, b \in \mathbb{R}, a \le b\}$ not a $\sigma$-algebra because $\mathbb{R} \notin \mathcal{A}$. But $\sigma(\mathcal{A}) = \mathcal{B}(\mathbb{R})$ (Borel $\sigma$-algebra). Check that $\mathcal{A}$ is semiring set:*

- *$\emptyset \in \mathcal{A}$*

- $$[a,b)\cap[c,d)=\begin{cases}\emptyset, & b\le c, d\le a\\ [c,b), & c\in[a,b), d\notin[a,b)\\ \dots\end{cases}\tag{1.8.1}$$

- $$[a,b)\backslash[c,d)=\begin{cases}[a,b), & d\le a, b\le c\\ [a,c), & c\in[a,b), d\notin[a,b)\\ [a,c)\cup[d,b), & c>a, d<b\\ \dots\end{cases}\tag{1.8.2}$$

**Definition 1.8.2** (Pre-measure). $\mu:\mathcal{A}\to[0,\infty]$ *with* $\mathcal{A}$ *semiring os sets:*

- $\mu(\emptyset)=0$
- $\mu(\cup_{j=1}^{\infty})=\sum_{j=1}^{\infty}\mu(A_j)$, *for* $A_j\in\mathcal{A}$, $A_i\cap A_j=\emptyset$ *for* $i\neq j$ *and* $\cup_{j=1}^{\infty}A_j\in\mathcal{A}$.

**Application:** $A:=\{[a,b)|a,b\in\mathbb{R},a\le b\}$, $\mu:\mathcal{A}\to[0,\infty]$, $\mu([a,b))=b-a$ is a pre-measure (We can check by the definition of pre-measure). Then by (4), there is a unique extension to $\mathcal{B}(\mathbb{R})\implies$ lebesgue measure.

## 1.9 Lebesgue-Stieltjes measures

$F:\mathbb{R}\to\mathbb{R}$ monotonically increasing (non-decreasing).$[a,b)$ is the length of the interval. Now we consider new kinds of intervals:

$$F(b^-)-F(a^-)=:\mu_F([a,b)),\tag{1.9.1}$$

where $F(a^-):=\lim_{\varepsilon\to0^+}F(a-\varepsilon)$. Alternatively, we also have

$$F(b^+)-F(a^+)=:\mu_F((a,b]),\tag{1.9.2}$$

where $F(a^+):=\lim_{\varepsilon\to0^+}F(a+\varepsilon)$. We consider the previous one hereafter.

**Definition 1.9.1.** $\mathcal{A}:=\{[a,b):a,b\in\mathbb{R},a\le b\}$ *semiring of sets. Then by Caratheodory' theorem, we have that there exists exactly one measure*

$$\mu_F:\mathcal{B}(\mathbb{R})\to[0,\infty]\tag{1.9.3}$$

*with* $\mu_F([a,b))$

$$.\tag{1.9.4}$$

**Example 1.9.1.**

- *$F(x) = x$, $\mu_F([a,b)) = b - a \to$ Lebesgue measure.*
- *$F(x) = 1$, $\mu_F([a,b)) = 0 \to$ zero measure.*
-
$$F(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases} \tag{1.9.5}$$
*$\mu_F([-\varepsilon, \varepsilon)) = 1 \to$ Dirac measure $\delta_0$.*
- *$F : \mathbb{R} \to \mathbb{R}$ monotonically increasing + continuously differentiable. Then we have*
$$F' : \mathbb{R} \to [0, \infty) \tag{1.9.6}$$
*and*
$$\mu_F([a,b)) = F(b) - F(a) \tag{1.9.7}$$
$$= \int_a^b F'(x)\ \mathrm{d}x, \tag{1.9.8}$$
*which implies*
$$\mu_F : A \longmapsto \int_A F'(x)\ \mathrm{d}x, \tag{1.9.9}$$
*where $F'(x)$ is called the density function.*

## 1.10 Radon-Nikodym theorem and Lebesgue's decomposition theorem

$(X, \mathcal{A}, \lambda)$ measure space. Special case: $X = \mathcal{R}$, $\mathcal{A} = \mathcal{B}(\mathbb{R})$, and $\lambda$ is lebesgue measure. Recall that $\lambda([a,b)) = b - a$. Another measure $\mu : \mathcal{B}(\mathbb{R}) \to [0, \infty]$. We will look how $\mu$ acts w.r.t. the given reference measure: lebesgue measure.

**Definition 1.10.1.**

- *$\mu$ is called absolutely continuous (w.r.t. $\lambda$) if $\lambda(A) = 0 \Longrightarrow \mu(A) = 0$ for all $A \in \mathcal{B}(\mathbb{R})$. One writes: $\mu << \lambda$.*
- *$\mu$ is called singular (w.r.t. $\lambda$) if there is $N \in \mathcal{B}(\mathbb{R})$ with $\lambda(N) = 0$ and $\mu(N^c) = 0$. One writes: $\mu \perp \lambda$.*

**Example 1.10.1.** *$\delta_0$ Dirac measure ($\delta_0(\{0\}) = 1$) $\Longrightarrow \delta_0 \perp \lambda$ (Choose $N = \{0\}$).*

**Theorem 5** (Lebesgue's decomposition theorem)**.** *$\mu : \mathcal{B}(\mathbb{R}) \to [0, \infty]$ ($\sigma$-finite)*

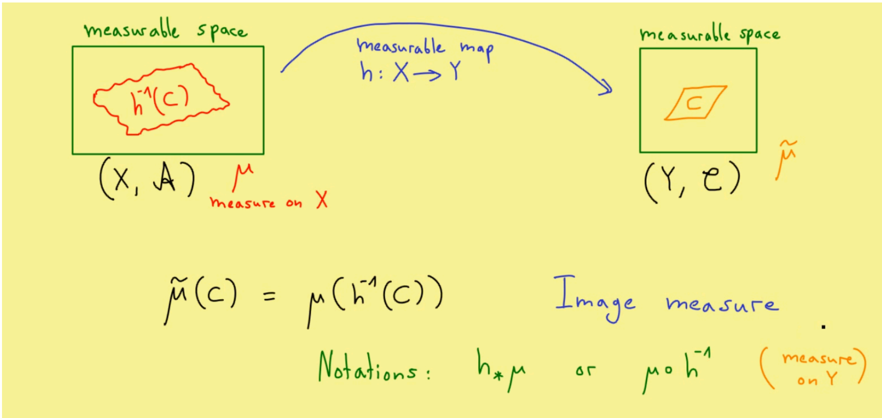
- *There are measures (uniquely determined) $\mu_{ac}, \mu_s : \mathcal{B}(\mathbb{R}) \to [0, \infty]$ with $\mu = \mu_{ac} + \mu_s$, $\mu_{ac} << \lambda, \mu_s \perp \lambda$.*

**Theorem 6** (Radon-Nikodym theorem)**.** $\mu : \mathcal{B}(\mathbb{R}) \to [0, \infty]$ *($\sigma$-finite)*

- *There is a measurable map* $h : \mathbb{R} \to [0, \infty)$ *with* $\mu_{ac} = \int_A d \; \mathrm{d}\lambda$ *for all* $A \in \mathcal{B}(\mathbb{R})$*, where* $h$ *is called the density function.*

## 1.11 Image measure and substitution formula

Image measure is also called pushforward measure. Substitution formula is also called change of variable.




**Definition 1.11.1** (Image measure)**.** *Measure space* $(X, \mathcal{A})$*,* $\mu$ *is a measure on* $X$*. Measure space* $(Y, \mathcal{E})$*,* $\tilde{\mu}$ *is a measure on* $Y$*. Define a measure map* $h : X \to Y$*. See the above figure. We then define the image measure as*

$$\tilde{\mu}(c) = \mu(h^{-1}(c)). \tag{1.11.1}$$

*The notations:* $h * \mu$ *or* $\mu \circ h^{-1}$*.* $h * \mu$ *means pushforward and* $\mu \circ h^{-1}$ *is readble. Remember that* $\tilde{\mu}$ *is aa measure on* $Y$*.*

**Lemma 1.11.1** (Substitution formula)**.** *A integrable function* $g : Y \to \mathbb{R}$*. We have*

$$\int_Y g \; \mathrm{d}(h * \mu) = \int_X g \circ h \; \mathrm{d}\mu, \tag{1.11.2}$$

*which can also be written as*

$$\int_Y g(y) \; \mathrm{d}(\mu \circ h^{-1})(y) = \int_X g(h(x)) \; \mathrm{d}\mu(x), \tag{1.11.3}$$

*which is called the change of variables:* $y = h(x)$*.*

**Example 1.11.1.** $F$ *is a strictly monotonically increasing and continuously differentiable and surjective function from* $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ *with* $\mu_F$ *as* $\mu_F(A) = \int_A F'(x) \; \mathrm{d}x$ *to*

$(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. *We have*

$$\begin{aligned}
(F * \mu_F)([a,b)) &= \mu_F(F^{-1}([a,b))) && (1.11.4)\\
&= \mu_F([F^{-1}(a), F^{-1}(b)]) && (1.11.5)\\
&= \int_{F^{-1}(a)}^{F^{-1}(b)} F'(x)\ \mathrm{d}x && (1.11.6)\\
&= \int_a^b \mathrm{d}y && (1.11.7)\\
&= \lambda([a,b)) && (1.11.8)\\
&\Longrightarrow F_x \mu_F = \lambda, && (1.11.9)
\end{aligned}$$

*Substitution formula:*

$$\begin{aligned}
\int_Y g\ \mathrm{d}(F * \mu_F) &= \int_X g \circ F\ \mathrm{d}\mu_F && (1.11.10)\\
\Longrightarrow \int_{\mathbb{R}} g(y)\ \mathrm{d}y &= \int_{\mathbb{R}} g(F(x))F'(x)\ \mathrm{d}x. && (1.11.11)
\end{aligned}$$

*Proof.* (1) Let $g = \chi_c$ with $C \subseteq Y$ measurable. For the left hand side, we have

$$\begin{aligned}
\int_Y \chi_c\ \mathrm{d}(h * \mu) &= (h * \mu)(c) && (1.11.12)\\
&= \mu(h^{-1}(c)). && (1.11.13)
\end{aligned}$$

For the right hand side, we have

$$\begin{aligned}
\int_X \chi_c \circ h\ \mathrm{d}\mu &= \int_X \chi_c \circ h\ \mathrm{d}\mu && (1.11.14)\\
&= \int_X \chi_c(h(x))\ \mathrm{d}\mu(x) && (1.11.15)\\
&= \int_X \chi_{h^{-1}(c)}\ \mathrm{d}\mu && (1.11.16)\\
&= \mu(h^{-1}(c)), && (1.11.17)
\end{aligned}$$

where

$$\chi_c(h(x)) = \begin{cases} 1, & x \in h^{-1}(c) \\ 0, & x \notin h^{-1}(c) \end{cases} \qquad (1.11.18)$$

(2) Let $g$ be a simple function, i.e.,$g = \sum_{i=1}^{n} \chi_{c_i}$. We then obtatin

$$\int_Y \sum_{i=1}^{n} \lambda_i \chi_{c_i} \, \mathrm{d}(h * \mu) = \sum_{i=1}^{n} \lambda_i \int_Y \chi_{c_i} \, \mathrm{d}(h * \mu) \tag{1.11.19}$$

By (1)

$$= \sum_{i=1}^{n} \lambda_i \int_X \chi_{c_i}(h(x)) \, \mathrm{d}\mu(x) \tag{1.11.20}$$

$$= \int_X (\sum_{i=1}^{n} \lambda_i \chi_{c_i})(h(x)) \, \mathrm{d}\mu(x) \tag{1.11.21}$$

$$= \int_X g \circ h \, \mathrm{d}\mu. \tag{1.11.22}$$

(3) Let $g : Y \to [0, \infty)$ measurable. We have

$$\int_Y g \, \mathrm{d}(h * \mu) = \sup \left\{ \int_Y \tilde{s} \, \mathrm{d}(h * \mu) | \tilde{s} : Y \to [0, \infty) \ simple, \ \tilde{s} \leq g \right\}. \tag{1.11.23}$$

We have the following equivalence relation:

$$\forall y \in h(x) : \tilde{s}(y) \leq g(y) \tag{1.11.24}$$

$$\iff \forall x \in X : \tilde{s}(h(x)) \leq g(h(x)) \tag{1.11.25}$$

$$[i.e., \tilde{s} \circ h \leq (g \circ h)(x)]. \tag{1.11.26}$$

Then we have

$$\int_Y g \, \mathrm{d}(h * \mu) = \sup \left\{ \int_X \tilde{s} \circ \ \mathrm{d}\mu | \tilde{s} : Y \to [0, \infty) \ simple, \tilde{s} \circ h \leq g \circ h \right\} \tag{1.11.27}$$

Left as exercise

$$= \sup \left\{ \int_X s \circ \ \mathrm{d}\mu | s : X \to [0, \infty) \ simple, s \circ h \leq g \circ h \right\} \tag{1.11.28}$$

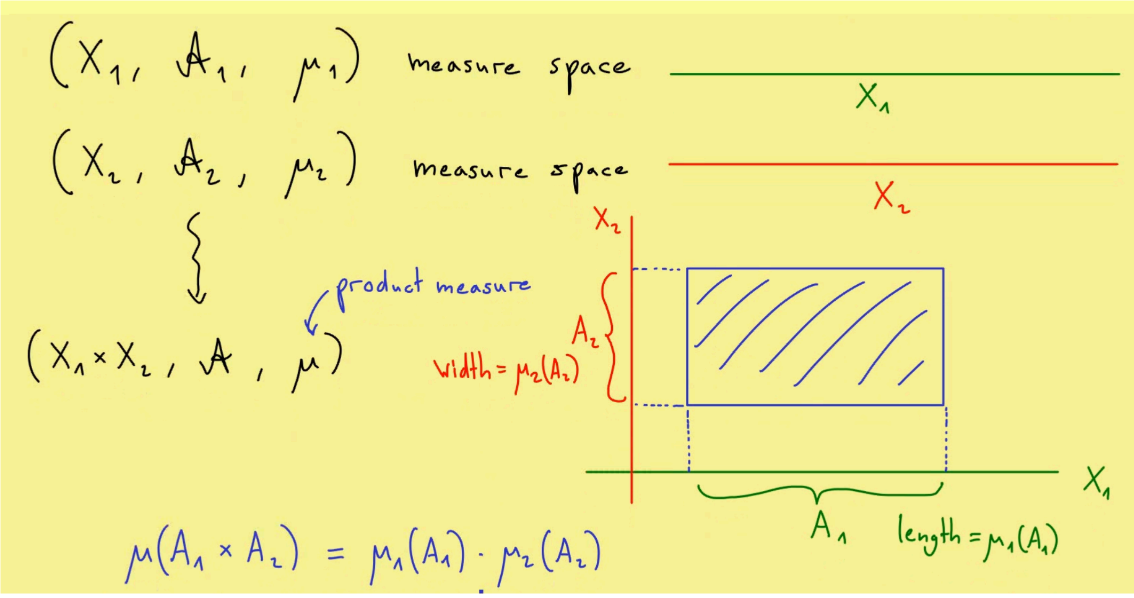
$$= \int_X g \circ h \, \mathrm{d}\mu. \tag{1.11.29}$$

□

## 1.12 Product measure and Cavalieri's principle

$(X_1, \mathcal{A}_1, \mu_1)$ measure space and $(X_2, \mathcal{A}_2, \mu_2)$ measure space,

$$\implies (X_1 \times X_2, \mathcal{A}, \mu), \ where \ \mu \ is \ the \ product \ measure. \tag{1.12.1}$$




We have

$$\mu(A_1 \times A_2) = \mu_1(A_1) \cdot \mu_2(A_2). \tag{1.12.2}$$

**Definition 1.12.1** (Product $\sigma$-algebra)**.**

$$\mathcal{A} = \sigma(\mathcal{A}_1 \times \mathcal{A}_2). \tag{1.12.3}$$

**Remark 1.12.1.** *Set of rectangles (=$\mathcal{A}_1 \times \mathcal{A}_2$) are not a $\sigma$-algebra (but a semiring of sets)*

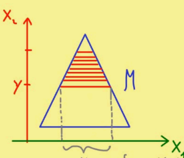
**Definition 1.12.2.** *Define product measure $\mu$ as $\mu(A_1 \times A_2) = \mu_1(A_1) \times \mu_2(A_2)$ for all $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$, and use (4).*

**Remark 1.12.2.** *Product measure in general not unique.*

Proposition: If $\mu_1, \mu_2$ are $\sigma$-finite, then there is exactly one measure $\mu$ with $\mu(A_1 \times A_2) = \mu_1(A_1) \cdot \mu_2(A_2)$.

It satisfies:

$$\mu(M) = \int_{X_2} \mu_1(M_y)\, d\mu_2(y)$$




$M_y := \{x_1 \in X_1 \mid (x_1, y) \in M\}$

$$= \int_{X_1} \mu_2(M_x)\, d\mu_1(x)$$

$M_x := \{x_2 \in X_2 \mid (x, x_2) \in M\}$

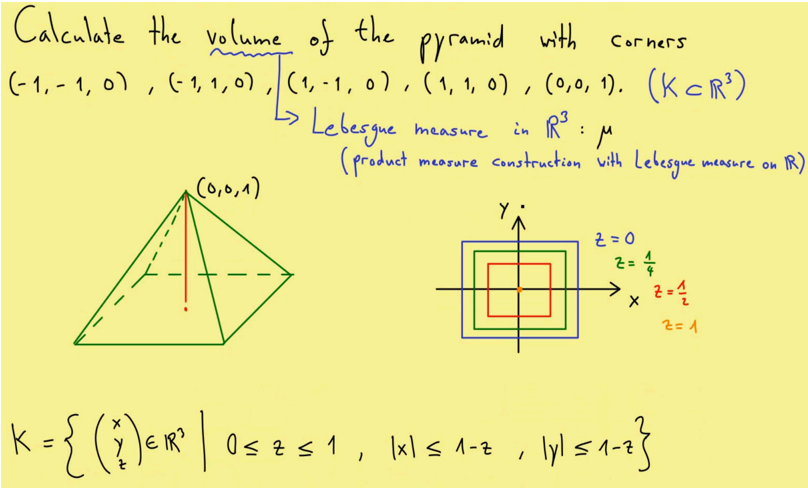
[Cavalieri's principle]

**Proposition 1.12.1** (Cavalieri's principle)**.** *If $\mu_1$, $\mu_2$ are $\sigma$-finite, then there is exactly one measure $\mu$ with $\mu(A_1 \times A_2) = \mu_1(A_1) \cdot \mu_2(A_2)$. Is satisfies:*

$$\mu(M) = \int_{X_2} \mu_1(M_y)\ \mathrm{d}\mu_2(y) \tag{1.12.4}$$

$$= \int_{X_1} \mu_2(M_x)\ \mathrm{d}\mu_1(x). \tag{1.12.5}$$

**Example 1.12.1** (An example for Cavalieri's principle)**.** *Calculate the volume of the pyramid with corners (-1,-1,0), (-1,1,0),(1,-1,0),(1,1,0),(0,0,1), $K \subset \mathbb{R}^3$, where the volume if the lebesgue measure in $\mathbb{R}^3$ : $\mu$(Recall product measure construction with lebesgus measure on $\mathbb{R}$).*




*Proof.* Set

$$K = \left\{(x, y, z)^T \in \mathbb{R}^3 | 0 \le z \le 1, |x| \le 1 - z, |y| \le 1 - z\right\}. \tag{1.12.6}$$

Define $\mu$ as a product measure of $\mu_1$ and $\mu_2$, where $\mu_1$ is the lebesgue measure in $\mathbb{R}$(z-coordinate) and $\mu_2$ is the lebesgue measure on $\mathbb{R}^2$(x- and y-coordinate). Following the definition of product measure, we have the volume of $K$ as

$$\mu(k) = \int_{\mathbb{R}} \mu_2(M_{z_0})\ \mathrm{d}\mu_1(z_0) \tag{1.12.7}$$

$$= \int_{[0,1]} 4 \cdot (1 - z_0)^2 \mathrm{d}\mu_1(z_0) \tag{1.12.8}$$

$$= \frac{4}{3}, \tag{1.12.9}$$

where

$$M_{z_0} := \left\{(x, y)^T \in \mathbb{R}^2 | |x| \le 1 - z_0, |y| \le 1 - z_0\right\}, \tag{1.12.10}$$

and $\mu_2(M_{z_0})$ is the area of the square only for $z_0 \in [0, 1]$. ☐

## 1.13 Fubini's theorem

**Theorem 7** (Fubini's theorem)**.** *Let $\mu_1$ and $\mu_2$ be $\sigma$-finite, $\mu$ be the product measure and*

$$f : X_1 \times X_2 \to [0, \infty] \textit{ measurable } [\textit{or } f \in \mathcal{L}^1(\mu)], \tag{1.13.1}$$

*then:*

$$\int_{X_1\times X_2} f \, \mathrm{d}\mu = \int_{X_2} \left( \int_{X_1} f(x, y) \, \mathrm{d}\mu_1(x) \right) \, \mathrm{d}\mu_2(x) \tag{1.13.2}$$

$$= \int_{X_1} \left( \int_{X_2} f(x, y) \, \mathrm{d}\mu_2(x) \right) \, \mathrm{d}\mu_1(x). \tag{1.13.3}$$

**Example 1.13.1.** *$\mu$ lebesgue measure for $\mathbb{R}^2$. Calculate $\int_A f \, \mathrm{d}\mu = ?$, where*

$$A = \left\{(x, y) \in [0, 1] \times [0, 1] | x \geq y \geq x^2\right\}, \tag{1.13.4}$$

$$f(x, y) = 2xy. \tag{1.13.5}$$

*We have*

$$\int_A f \, \mathrm{d}\mu = \int_{\mathbb{R}^2} f \cdot \chi_A \, \mathrm{d}\mu \tag{1.13.6}$$

$$= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} f(x, y)\chi_A(x, y) \, \mathrm{d}y \right) \, \mathrm{d}x \tag{1.13.7}$$

$$= \int_0^1 \left( \int_x^{x^2} 2xy \, \mathrm{d}y \right) \, \mathrm{d}x \tag{1.13.8}$$

$$= \frac{1}{12}. \tag{1.13.9}$$

## 1.14 Outer measure

- tools for the proof of (4)
- "outer measure" is a new notion. "Outer measure" is not an attribute for "measure"! "Outer mesure" do not have to be measures!

**Definition 1.14.1** (Outer measure)**.** *A map $\phi : \mathcal{P}(X) \to [0, \infty]$ is called an outer measure if:*

- *(a)* $\phi(\emptyset) = 0$
- *(b)* $A \subseteq B \Longrightarrow \phi(A) \leq \phi(B)$. *(monotonicity)*
- *(c)* $A_1, A_2, \ldots, \in \mathcal{P}(X) \Longrightarrow \phi\left(\cup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \phi(A_n)$. *($\sigma$-subadditivity)*

**Question:** $\phi : \mathcal{P}(X) \to [0, \infty]$ outer measure $\xrightarrow{?}$ $\mu$ measrue?

**Definition 1.14.2** ($\phi$-measurable)**.** *Let $\phi$ be an outer measure. $A \in \mathcal{P}(X)$ is called $\phi$-measurable if for all $Q \in \mathcal{P}(X)$ we have:*

$$\phi(Q) \geq \phi(Q \cap A) + \phi(Q \cap A^c). \tag{1.14.1}$$

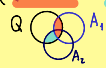
**Proposition 1.14.1.** *If $\phi : \mathcal{P}(X) \to [0, \infty]$ is an outer measure, then:*

- *$\mathcal{A}_\phi := \{A \subseteq X | A\ \phi\ \text{measurable}\}$ is a $\sigma$-algebra.*
- *$\mu : \mathcal{A}_\phi \to [0, \infty]$, $\mu(A) := \phi(A)$, is a measrue.*

• union with two sets: $A_1, A_2 \in \mathcal{A}_\varphi$

$$\varphi(Q) = \varphi(Q \cap A_1) + \varphi(\overbrace{Q \cap A_1^c}^{\tilde{Q}}) = \varphi(Q \cap A_1) + \varphi(\tilde{Q} \cap A_2) + \varphi(\tilde{Q} \cap A_2^c)$$

$$\geq \varphi\big((Q \cap A_1) \cup (\tilde{Q} \cap A_2)\big) + \varphi(\tilde{Q} \cap A_2^c)$$




$Q \cap (A_1^c \cap A_2^c)$

*Proof.* • $\phi \in \mathcal{A}_\phi$? Is $\emptyset$ $\phi$ -measurable?

$$\phi(Q) = \phi(Q \cap \emptyset) + \phi(Q \cup \emptyset^c) \tag{1.14.2}$$

$$= 0 + \phi(Q) \tag{1.14.3}$$

- $X \in \mathcal{A}_\phi$? Is $X$ $\phi$-measurable?

$$\phi(Q) = \phi(Q \cap X) + \phi(Q \cap X^c) \tag{1.14.4}$$

$$= \phi(Q) + \phi(\emptyset). \tag{1.14.5}$$

- $A \in \mathcal{A}_\phi \implies$

$$\phi(Q) = \phi(Q \cap A) + \phi(Q \cap A^c) \tag{1.14.6}$$

$$= \phi(Q \cap A^c) + \phi(Q \cap (A^c)^c) \tag{1.14.7}$$

$$\implies A^c \in \mathcal{A}_\phi. \tag{1.14.8}$$

- union with two sets: $A_1, A_2 \in \mathcal{A}$

$$\phi(Q) = \phi(Q \cap A_1) + \phi(Q \cap A_1^c) \tag{1.14.9}$$

Define $\tilde{Q} := Q \cap A_1^c$

$$= \phi(Q \cap A_1) + \phi(\tilde{Q} \cap A_2) + \phi(\tilde{Q} \cap A_2^c) \tag{1.14.10}$$

$$\geq \phi\Big((Q \cap A_1) \cup (\tilde{Q} \cap A_2)\Big) + \phi(\tilde{Q} \cap A_2^c) \tag{1.14.11}$$

$$= \phi(Q \cap (A_1 \cup A_2)) + \phi(Q \cap (A_1 \cup A_2)^c), \tag{1.14.12}$$

$$\implies \phi(Q) \geq \phi(Q \cap (A_1 \cup A_2)) + \phi(Q \cap (A_1 \cup A_2)^c) \tag{1.14.13}$$

$$\implies A_1 \cup A_2 \in \mathcal{A}_\phi, \tag{1.14.14}$$

where the fouth equation is obtain by the above figure.

- countable union: $A_1, A_2, \dots \in \mathcal{A}_\phi$, $A := \cup_{j=1}^{\infty} A_j \in \mathcal{A}_\phi$?

$$\phi(Q) = \phi(Q \cap A_1) + \phi(Q \cap A_1^c) \tag{1.14.15}$$

Set $Q = \hat{Q} \cap (A_1 \cup A_2)$

$$= \phi(\hat{Q} \cap A_1) + \phi(\hat{Q} \cap A_2). \tag{1.14.16}$$

Induction: $\phi(\hat{Q} \cap \cup_{j=1}^{n} A_j) = \sum_{j=1}^{n} \phi(\hat{Q} \cap A_j)$. We have:

$$\phi(\hat{Q}) = \phi(\hat{Q} \cap \cup_{j=1}^{n}) + \phi(\hat{Q} \cap (\cup_{j=1}^{n} A_j)^c) \tag{1.14.17}$$

$$\geq \sum_{j=1}^{n} \phi(\hat{Q} \cap A_j) + \phi(\hat{Q} \cap A^c) \tag{1.14.18}$$

$$\Longrightarrow \phi(\hat{Q}) \geq \sum_{j=1}^{n} \phi(\hat{Q} \cap A_j) + \phi(\hat{Q} \cap A^c) \tag{1.14.19}$$

$$\geq \phi(\hat{Q} \cap A) + \phi(\hat{Q} \cap A^c) \tag{1.14.20}$$

$$\geq \phi(\hat{Q}) \tag{1.14.21}$$

$$\Longrightarrow A \in \mathcal{A}_\phi. \tag{1.14.22}$$

□

**Example 1.14.1.** *(1)* $\phi : \mathcal{P}(\mathbb{R}) \to [0, \infty]$,

$$\phi(A) = \begin{cases} 0, & A = \emptyset \\ 1, & A \neq \emptyset. \end{cases} \qquad \Longrightarrow \textit{outer measure but not a measure!} \tag{1.14.23}$$

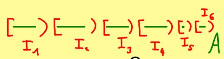
**Example 1.14.2.** $\phi : \mathcal{P}(\mathbb{N}) \to [0, \infty]$,

$$\phi(A) = \begin{Bmatrix} |A|, & A \textit{ finite} \\ \infty, & A \textit{ not finite.} \end{Bmatrix} \tag{1.14.24}$$

$$\Longrightarrow \textit{outer measure but a measure!(counting measure)} \tag{1.14.25}$$

(3) $\mathcal{I} = \{ [a,b) \mid a,b \in \mathbb{R},\ a \leq b \}$, $\mu([a,b)) = b - a$ ("length")

Define $\varphi : \mathcal{P}(\mathbb{R}) \longrightarrow [0,\infty)$ by :




$$\varphi(A) := \inf \left\{ \sum_{j=1}^{\infty} \mu(I_j) \;\middle|\; I_j \in \mathcal{I},\ A \subseteq \bigcup_{j=1}^{\infty} I_j \right\}$$

$\rightsquigarrow$ $\varphi$ is an outer measure!

**Example 1.14.3.** *$\mathcal{I} = \{[a,b)|a,b \in \mathbb{R},\ a \leq b\}$, $\mu([a,b)) = b - a$("length").*

*Define $\phi : \mathcal{P}(\mathbb{R}) \to [0,\infty)$ by:*

$$\phi(A) := \inf \left\{ \sum_{j=1}^{\infty} \mu(I_j) | I_j \in \mathcal{I},\ A \subseteq \cup_{j=1}^{\infty} I_j \right. \tag{1.14.26}$$

$$\Longrightarrow \phi \textit{ is an outer measure!} \tag{1.14.27}$$

*Proof.* check (a) of (1.14.1): $\phi(\emptyset) = 0$.

check (b) of (1.14.1): monotonicity,

$$A \subseteq B \Longrightarrow \phi(B) \tag{1.14.28}$$

$$= \inf \left\{ \sum_{j=1}^{\infty} \mu(I_j) | I_j \in \mathcal{I},\ B \subseteq U_{j=1}^{\infty} I_j \right\} \tag{1.14.29}$$

$$\geq \inf \left\{ \sum_{j=1}^{\infty} \mu(I_j) | I_j \in \mathcal{I},\ A \subseteq U_{j=1}^{\infty} I_j \right\}, \tag{1.14.30}$$

since $A \subseteq B$.

check (c) of (1.14.1): show that $\phi(\cup_{n\in\mathbb{N}} A_n) \leq \sum_{n\in\mathbb{N}} \phi(A_n)$. Let $\varepsilon > 0$. Choose $\varepsilon_n > 0$ with $\sum_{n\in\mathbb{N}} \varepsilon_n = \varepsilon$. Then there are intervals $I_{j,n}$ with:

$$\phi(A_n) \geq \sum_{j=1}^{\infty} \mu(I_{j,n}) - \varepsilon_n, \tag{1.14.31}$$

and

$$A_n \subseteq \cup_{j=1}^{\infty} I_{j,n}. \tag{1.14.32}$$

Then: $\cup_{n\in\mathbb{N}} \subseteq \cup_{n\in\mathbb{N}} \cup_{j\in\mathbb{N}} I_{j,n} = \cup_{j,n} I_{j,n}$.

$$\Longrightarrow \phi(\cup_{n\in\mathbb{N}}) \overset{(b)}{\leq} \phi(\cup_{j,n} I_{j,n}) \tag{1.14.33}$$

$$\leq \sum_{j,n} \mu(I_{j,n}) \tag{1.14.34}$$

$$= \sum_{n\in\mathbb{N}} \left\{ \sum_{j\in\mathbb{N}} \mu(I_{j,n}) \right\} \tag{1.14.35}$$

$$\leq \sum_{n\in\mathbb{N}} (\phi(A_n) + \varepsilon_n) \tag{1.14.36}$$

$$= \sum_{n\in\mathbb{N}} \phi(A_n) + \varepsilon. \tag{1.14.37}$$

□