Chapter 1

Advent of Mathematical Symbols

- Kronecker delta:

$$\delta_{ij} := \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \tag{1.1}$$

- Levi-Civita symbol:

$$\varepsilon_{ijk} := \begin{cases} 1, & (i,j,k) = (1,2,3) \text{ or } (2,3,1) \text{ or } (3,1,2) \\ -1, & (i,j,k) = (3,2,1) \text{ or } (2,1,3) \text{ or } (1,3,2) \\ 0, & \text{else.} \end{cases} \tag{1.2}$$

  Example 1.0.1.

$$(a \times b)_i = \sum_{j,k=1}^{3} \varepsilon_{ijk} a_j b_k, \tag{1.3}$$

  where a, b are three dimensional vectors and " $\times$ " denotes cross product.

- Nabla symbol:

$$\nabla := \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{pmatrix}. \tag{1.4}$$

- Factorial: $n! := n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$.
  Recursive definition: $0! := 1$, $n! := n \cdot (n-1)!,\ n \in \mathbb{N}$.

- Gamma function:

$$\Gamma(z) := \int_0^{\infty} x^{z-1} \cdot \mathrm{e}^{-x} \mathrm{d}x,\ \mathrm{Re}(z) \geq 0. \tag{1.5}$$

  Property 1.0.1.

$$\Gamma(n) = (n-1)!,\ n \in \mathbb{N};\ \Gamma(z+1) = z \cdot \Gamma(z). \tag{1.6}$$

- Composition: $(g \circ f)(x) := g(f(x))$.

- Sum symbol: $\sum_{k=1}^{n} a_k := a_1 + a_2 + \cdots + a_n$.

  Recursive defintion: $\sum_{k=1}^{0} a_k := 0$, $\sum_{k=1}^{n} a_k := \left(\sum_{k=1}^{n-1} a_k\right) + a_n$.

- Product: $\Pi_{k=1} n a_k := a_1 \cdot a_2 \cdot \cdots \cdot a_n$.

  Recursive defintion: $\Pi_{k=1}^{0} a_k := 1$, $\Pi_{k=1}^{n} a_k := \left(\Pi_{k=1}^{n-1} a_k\right) \cdot a_n$

- Restriction: $f|_A : A \to Y$. For $f : X \to Y$ and $A \subseteq X$, we define $f|_A(x) = f(x)$ for all $x \in A$.

- Pauli matrices:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} . \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} . \sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} . \tag{1.7}$$

  Property 1.0.2. *We have $\sigma_k^2 = I$ and $\sigma_j \sigma_k - \sigma_k \sigma_j = 2i\varepsilon_{jkl}\sigma_l$.*

- Set brackets: $\{f(x)|x \in A\}$.

  Example 1.0.2.

$$\{2x + 1 | x \in \{0, 1, 2, 3\}\} = \{1, 3, 5, 7\} . \tag{1.8}$$

- Big O: $f(x) = O(g(x))$, $(x \to a)$, which means that $|f(x)| \leq M \cdot |g(x)|$, i.e., $\limsup_{x \to a} \frac{f(x)}{g(x)} < \infty$.

$$x^2 + x + 2 = O(x^2), \ (x \to \infty) \tag{1.9}$$

$$x^2 + x + 2 = O(x^3), \ (x \to \infty). \tag{1.10}$$

- Binomial coefficient:

$$\binom{n}{k} = \frac{n \cdot (n-1) \cdots (n-k-1)}{k!} \tag{1.11}$$

$$= \frac{n!}{k!(n-k)!}. \tag{1.12}$$

- Modulo: $x \bmod n := r \in [0, n)$ with $x = n \cdot q + r$ where q is the integer.

  Example 1.0.3.

$$5 \bmod 3 = 2 \tag{1.13}$$

$$6 \bmod 3 = 0 \tag{1.14}$$

$$7.1 \bmod 3 = 1.1 \tag{1.15}$$

$$9.7 \bmod 2.1 = 1.3. \tag{1.16}$$

- Beta function:

$$\beta(x, y) := \int_0^1 t^{x-1}(1-t) \ \mathrm{d}t, \tag{1.17}$$

  where $x, y \in \mathbb{C}$, $\mathrm{Re}(x) > 0$ and $\mathrm{Re}(y) > 0$.

Lemma 1.0.1 (Identity between β func. and Γ func.)**.**

$$\beta(x, y) = \frac{\Gamma(x) \cdot \Gamma(y)}{\Gamma(x+y)}, \tag{1.18}$$

where $\Gamma(\cdot)$ *is related to factorial and* $\beta(x, y)$ *is related to binomial coefficient.*

- Map arrows: $f : \ X \to Y$ where X is the domain and Y is the codomain. This map can also be denoted as elementwise-mapping as $x \longmapsto f(x)$.

  Example 1.0.4.

$$f := \mathbb{R} \to \mathbb{R} \tag{1.19}$$
$$x \longmapsto x^2. \tag{1.20}$$

- Little o: $f(x) = o(g(x))$, $(x \to a)$, which means $\lim_{x \to a} |\frac{f(x)}{g(x)}| = 0$.

  Example 1.0.5.

$$8 \cdot x^2 \neq o(x^2), \ (x \to \infty) \tag{1.21}$$
$$8 \cdot x^2 \neq o(x^3), \ (x \to \infty). \tag{1.22}$$

- Outer product (Kronecker product for vectors):

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \otimes \begin{pmatrix} w_1 & w_2 & w_3 \end{pmatrix} = \begin{pmatrix} v_1 w_1 & v_1 w_2 & v_1 w_3 \\ v_2 w_1 & v_2 w_2 & v_2 w_3 \end{pmatrix}, \tag{1.23}$$

  i.e. matrix entries $(V \otimes W)_{ij} = v_i \cdot w_j$.

- Euler's phi function: $\phi : \mathbb{N} \to \mathbb{N}$ defined as

$$\phi(n) = \text{count numbers } a \in \mathbb{N} \text{ with} \tag{1.24}$$
$$(1)\ a \leq n \tag{1.25}$$
$$(2)\ \gcd(a, n) = 1(\text{mutually prime}). \tag{1.26}$$

  Example 1.0.6.

$$\phi(4) = 2 \tag{1.27}$$
$$\phi(5) = 4 \tag{1.28}$$
$$\phi(p) = p - 1 \ for \ p \ prime. \tag{1.29}$$

- Laplace operator (Laplacian):

$$\Delta f(x) = \frac{\partial^2 f}{\partial x_1^2}(x) + \frac{\partial^2 f}{\partial x_2^2}(x) + \frac{\partial^2 f}{\partial x_3^2}(x), \tag{1.30}$$

  where $f : \mathbb{R}^3 \to \mathbb{R}$.

- Convolution: $(f * g)(x) := \int_{-\infty}^{\infty} f(\tau) \cdot g(x - \tau) \mathrm{d}\tau$, where $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ and $f * g : \mathbb{R} \to \mathbb{R}$.

- Heaviside function:

$$H(x) := \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases} \tag{1.31}$$

Property 1.0.3.

$$H' = \delta. \tag{1.32}$$

- Quaternions:

$$\mathbb{H} \supseteq \mathbb{C}, \tag{1.33}$$

where $a, b, c, d \in \mathbb{R}$, the element in $\mathbb{H}$ is $a + i \cdot b + j \cdot c + k \cdot d$ with $i^2 = -1, j^2 = -1, k^2 = -1, ijk = -1$. $\mathbb{H}$ is not commutative in multiplication, i.e., $i \cdot j = -j \cdot i$.

- Infinity: ∞.

Example 1.0.7. *In measure theory:* $[0, \infty]$. *We have*

$$a + \infty = \infty + a = \infty \ for \ a \in [a, \infty] \tag{1.34}$$

$$\tag{1.35}$$

- means equivalence reltation. For example, x y means x is equivalent to y for some conditions.