Chapter 1

Sensing

1.1 Quantum state distrimination with Bosonic channels aand Gaussian states

This PhD thesis is written by Sihui Tan, who is the one that first propose Quantum Illumination. So this thesis is a very valuable material to learn QI.

1.1.1 Mathematical preliminaries

Definition 1.1.1 (Direct sum of matrix)**.**

1.1.2 Physics preliminaries

1.1.3 Introduction

1.1.4 Problem formulation

1.1.5 Protocol

1.1.6 Protocol Details

1.1.7 Performance evaluation

1.2 Joint Measurement of TFE via SFG [LH20]

1.2.1 Mathematical preliminaries

Proposition 1.2.1 (The Fourier Transform of 1)**.**

1.2.2 Physics preliminaries

Definition 1.2.1 (Sum Frequency Generation (SFG))**.** *The SFG process in a $\chi^{(2)}$ non-linear medium could be modeled as the following evolution operator:*

$$V = I + \varepsilon \left(\int \mathrm{d}\omega_p \mathrm{d}\omega_s \mathrm{d}\omega_i a_p^\dagger(\omega_p) a_s(\omega_s) a_i(\omega_i) \delta(\omega_p - \omega_s - \omega_i) - H.C. \right), \tag{1.2.1}$$

where photons in the signal mode $a_s(\omega_s)$ and ideler mode $a_i(\omega_i)$ are annihilated to generate photons in the pump mode $a_p(\omega_p)$ and ε characterizes the interaction strength. The SPDC process is the time-reversal process of SFG, which can also be described by the same revolution operator.

Definition 1.2.2 (SPDC)**.**

Definition 1.2.3 (Evolution operator)**.**

Theorem 1 (Schmidt' decomposition [NC19])**.**

Definition 1.2.4 (Schmidt's number [NC19])**.**

Definition 1.2.5 (Heisenberg picture)**.**

Proposition 1.2.2 (The relationship between annihilation operator and frequency)**.**

Definition 1.2.6 (Time Correlation)**.**

Definition 1.2.7 (Frequency/Spectral/Energy Correlation)**.**

1.2.3 Introduction

SPDC is widely adopted to generate TFE photons. SFG can be used to perform TFE joint-measurement.

[SZQ: 2023.06.03: TFE joint-measurement is just entangled measurement. SPDC is similart to unitary evolution for state preparation, in particular, the time-frequency entangled states. Thus, the reversed process SPDC, i.e., SFG process, is the entangled measurement. This is analogy to bell state preparation and bell measurements.]

1.2.4 Problem formulation

The frequency sum and time difference of two photons could be simultaneously measured through the sum-frequency generation process.

1.2.5 Protocol

Given the close connection between the spontaneous parametric down-conversion (SPDC) process and time-frequency entanglement (TFE), it's natural to utilize the time-reversal of the SPDC process, i.e., sum frequency generation (SFG) to obtain a TFE joint measurement based protocol.

Definition 1.2.8 (Frequency sum (FS) operator)**.** *The frequency sum operator $P_{\delta_\omega}(\omega)$ that selects states with the frequency sum $\omega_s + \omega_i$ of the signal and idler photon being around ω within uncertatinty δ_ω is defined as:*

$$P_{\delta_\omega}(\omega) = \int\int \mathrm{d}\omega_s \mathrm{d}\omega_i a_s^\dagger(\omega_s) a_i^\dagger(\omega_i) a_s(\omega_s) a_i(\omega_i) \mathrm{Gate}\left(\frac{\omega - \omega_s - \omega_i}{\delta_\omega}\right), \tag{1.2.2}$$

where $\mathrm{Gate}(x) = 1$ *for* $|x| \leq 1/2$ *and* $\mathrm{Gate}(x) = 0$ *otherwise.*

Lemma 1.2.1 (Frequency sum operator is a projection operator)**.** *The frequency sum operator $P_{\delta_\omega}(\omega)$ is a projection operator satisfying*

$$P_{\delta_\omega}(\omega)^2 = P_{\delta_\omega}(\omega). \tag{1.2.3}$$

Definition 1.2.9 (Time difference (TD) operator)**.** *The time difference operator $P_{\delta_t}(t)$ that selects states with the time difference $t_s - t_i$ of the signal and idler photon being around t within uncertatinty δ_t is defined as:*

$$P_{\delta_t}(t) = \int\int \mathrm{d}t_s \mathrm{d}t_i \tilde{a_s}^\dagger(t_s) \tilde{a_i}^\dagger(t_i) \tilde{a_s}(t_s) \tilde{a_i}(t_i) \mathrm{Gate}\left(\frac{t_s - t_i - t}{\delta_t}\right), \tag{1.2.4}$$

where $\mathrm{Gate}(x) = 1$ *for* $|x| \leq 1/2$ *and* $\mathrm{Gate}(x) = 0$ *otherwise and*

$$\tilde{a}_x = \frac{1}{\sqrt{2\pi}} \int \mathrm{d}\omega \exp\left(-i\omega t\right) a_x(\omega) \tag{1.2.5}$$

Lemma 1.2.2 (Time difference operator is a projection operator)**.** *The time difference operator $P_{\delta_t}(t)$ is a projection operator satisfying*

$$P_{\delta_t}(t)^2 = P_{\delta_t}(t). \tag{1.2.6}$$

Definition 1.2.10 (Joint projection operator)**.** *The joint projection operator of the time difference and frequency sum is defined as*

$$P_{\delta_\omega,\delta_t}(\omega, t) = P_{\delta_\omega}(\omega)P_{\delta_t}(t), \tag{1.2.7}$$

which means selecting states of which the time difference between the signal and idler photon $t_s - t_i$ is around t within uncertainty δ_t and frequency sum of the signal and idler photon being around ω within unicertainty δ_ω, simultaneously.

Lemma 1.2.3 (Commutation relationship between frequency-time operators)**.** *We have the commutation relationship*

$$[P_{\delta_\omega}(\omega), P_{\delta_t}(t)] = 0. \tag{1.2.8}$$

Definition 1.2.11 (TD an FS probability density operator (PDF))**.** *We define*

$$P(\omega, t) = \lim_{\delta_t \to 0, \delta_\omega \to 0} \frac{1}{\delta_\omega \delta_t} P_{\delta_\omega}(\omega)P_{\delta_t}(t). \tag{1.2.9}$$

Lemma 1.2.4. *We have*

$$P(\omega, t) = \frac{1}{2\pi} B_p^\dagger B_p, \tag{1.2.10}$$

where

$$B_p = \int\int \mathrm{d}\omega_s \mathrm{d}\omega_i \delta(\omega_s + \omega_i - \omega) \exp\left[i\omega_i t\right] a_s(\omega_s) a_i(\omega_i). \tag{1.2.11}$$

Lemma 1.2.5 (Connection between TD and FS PDF and SFG Process)**.** *We have*

$$B^\dagger B = 2\pi P(\omega, 0). \tag{1.2.12}$$

This lemma did not show the SFG process

Lemma 1.2.6 (Discrete sum of evolution operator of SFG process)**.** *The discrete sum of the evolution operator of SFG process can be obtained by a* ***two-step Schmidt decomposition*** *as:*

$$v = I + \varepsilon \sum_m \left(\sqrt{\lambda_m^{(1)}} A_m^\dagger B_m - H.C. \right), \tag{1.2.13}$$

where

$$B_m = \sum_n \sqrt{\lambda^{(2)}_{m,n}} F_{m,n} G_{m,n}, \tag{1.2.14}$$

$$A_m = \int \mathrm{d}\omega \psi_{A,m}(\omega) a_p(\omega), \tag{1.2.15}$$

$$B_m = \int \mathrm{d}\omega_s \mathrm{d}\omega_i \psi_{B,m}(\omega_s, \omega_i) a_s(\omega_s) a_i(\omega_i), \tag{1.2.16}$$

$$F_{m,n} = \int \mathrm{d}\psi_{F,m,n}(\omega) a_s(\omega), \tag{1.2.17}$$

$$G_{m,n} = \int \mathrm{d}\psi_{G,m,n}(\omega) a_i(\omega). \tag{1.2.18}$$

Lemma 1.2.7 (Non-uniqueness of the first step Schmidt Decomposition)**.** *If the function f_0 can be written in the following form:*

$$\delta(\omega_p - \omega_s - \omega_i) f_0(\omega_p - \omega_s - \omega_i) = \delta(\omega_p - \omega_s - \Omega_i) f(\frac{\omega_s - \Omega_i}{\sqrt{2}}), \tag{1.2.19}$$

then the first step Schmidt decomposition in the main text is not unique.

Lemma 1.2.8 (An useful communtation relationship)**.** *An useful communtation relationship:*

$$[B_{m'}, B^\dagger_{m''}] = \delta_{m'm''} + \int \mathrm{d}\omega'_s \mathrm{d}\omega_i \mathrm{d}\omega''_s \psi^*_{B,m''}(\omega'_s, \omega_i) a^\dagger_s(\omega''_s) a_s(\omega'_s) \tag{1.2.20}$$

$$+ \int \mathrm{d}\omega_s \mathrm{d}\omega'_i \mathrm{d}\omega''_i \psi^*_{B,m''}(\omega_s, \omega''_i) \psi_{B,m'} a^\dagger_i(\omega''_i) a_i(\omega'_i). \tag{1.2.21}$$

Corollary 1.2.1. *By (1.2.8), we have*

$$[B_{m'}, B^\dagger_{m''}] = \delta_{m'm''} |0\rangle. \tag{1.2.22}$$

Lemma 1.2.9. *We have the commutation relation between time difference projection operator and frequency sum projection operator:*

$$[P_{\delta_\omega}(\omega), P_{\delta_t}(t)] = 0. \tag{1.2.23}$$

Lemma 1.2.10. *The frequency spectrum $S(\omega)$ of the generated pump photon is given by the expectation value of the spectral density operator $a^\dagger_p(\omega_p) a_p(\omega_p)$*

$$S(\omega_p) = \frac{\epsilon^2 \exp\left[\frac{1}{8}\left(-4\Delta t^2 \sigma_-^2 - \frac{\Delta\omega^2}{\sigma_-^2} - \frac{4(\Delta\omega + \omega_0 - \Omega_p)^2}{\sigma_+^2}\right)\right]}{2\sqrt{\pi}\sigma_+}. \tag{1.2.24}$$

Definition 1.2.12. *The discrete mode operator $F^{(}_{m,n}b)$ for the noise photons is defined as*

$$F^{(b)}_{m,n} = \int \mathrm{d}\omega \psi_{F_{m,n}}(\omega) a^{(b)}_s(\omega). \tag{1.2.25}$$

Definition 1.2.13. *The virtual beam-splitter is modeledd as the following unitary transform:*

$$U_{loss} = \Pi_n \exp\left[i\arccos(\eta)(F^\dagger_{0,n}F^{(b)}_{0,n} + H.C.)\right]. \tag{1.2.26}$$

Definition 1.2.14. *We use a density matrix ρ_b that satisfies the following conditions to describe the noise photons:*

$$\mathrm{Tr}[F^{(b)\dagger}_{0,n''}F^{(b)}_{0,n'}\rho_b] = \delta_{n',n''}\mu_b, \tag{1.2.27}$$

$$\mathrm{Tr}[F^{(b)}_{0,n'}\rho_b] = 0 \tag{1.2.28}$$

Definition 1.2.15. *The signal and idler photon pair source is described by the biphoton state $|pair\rangle$:*

$$|pair\rangle = B^\dagger_0|0\rangle \tag{1.2.29}$$

$$= \sum_n \sqrt{\lambda^{(2)}_{0,n}} F^\dagger_{0,n}G_{0,n}|0\rangle. \tag{1.2.30}$$

Definition 1.2.16. *The unitary trasform of the SFG process is given by:*

$$V = I + \epsilon\sum_m \left[\sqrt{\lambda^{(1)}_m}A^\dagger_m B_m - H.C.\right]. \tag{1.2.31}$$

Definition 1.2.17. *In the Heisenberg picture, the photon number operator of the generated pump photon in each pump mode A_m after the beam-splitter transform and the SFG process is given by:*

$$U^\dagger_{loss}V^\dagger A^\dagger_m V U_{loss}. \tag{1.2.32}$$

Proposition 1.2.3. *When the transmission of the signal photon is perfect ($\eta = 1$), the pump photon can only generate in mode $A_0(m = 0)$.*

Lemma 1.2.11. *We have*

$$\langle U^\dagger_{loss}V^\dagger A^\dagger_0 A_0 V|U_{loss}\rangle = \epsilon^2\lambda^{(1)}_0(\eta + \mu_b\sum_n \lambda^2_{0,n}). \tag{1.2.33}$$

Definition 1.2.18. *The generated SPDC state is given by:*

$$V = |0\rangle - \epsilon\sqrt{\lambda^{(1)}_0}\alpha B^\dagger_0|0\rangle. \tag{1.2.34}$$

Lemma 1.2.12. *The joint density operator of the noise-idler state ρ_j is given by the tensor product of ρ_i and ρ_b:*

$$\rho_j = \rho_i \otimes \rho_b \tag{1.2.35}$$

$$= \mu_b \int\int \mathrm{d}\omega_s \mathrm{d}\omega'_s \int\int \mathrm{d}\omega'_i \mathrm{d}\omega''_i \phi^*_0(\omega'_s,\omega'_i)\phi_0(\omega'_s,\omega''_i)a^\dagger_i(\omega'_i)a^\dagger_s(\omega_s)|0\rangle\langle 0|a_i(\omega''_i)a_s(\omega_s). \tag{1.2.36}$$

Lemma 1.2.13. *The spectral density $S(\omega)$ of the upconverted photons is*

$$S(\omega) = \frac{\epsilon^2 \mu_b \exp\left[-\frac{(\omega-\omega_0)^2}{8\sigma_-^2 - 2\sigma_+^2}\right]}{\sqrt{\pi}\sqrt{4\sigma_-^2 + \sigma_+^2}}. \tag{1.2.37}$$

Theorem 2. *The error exponent of the classical Chernoff bound of the TFE QI protocol is given by C_{QI} [NS09]:*

$$C_{QI} = -\log \min_{s\in[0,1]} \left\{ \sum_{b\in 0,1} p_0(b)^s p_1(b)^{(1-s)} \right\}. \tag{1.2.38}$$

1.2.6 Performance evaluation

Lemma 1.2.14.

$$\tag{1.2.39}$$

1.2.7 Ideas

1.3 Optimum Mixed-State Discrimination for Noisy Entanglement-Enhanced Sensing [ZZS17]

1.3.1 Mathematical preliminaries

1.3.2 Physics preliminaries

1.3.3 Introduction

1.3.4 Problem formulation

1.3.5 Protocol

1.3.6 Protocol Details

1.3.7 Performance evaluation

Lemma 1.3.1.

$$\tag{1.3.1}$$

1.3.8 Ideas

1.4 Quantum Estimation Methods for Quantum Illumination [SLHGR+17]

1.4.1 Mathematical preliminaries

Definition 1.4.1 (Quantum Fisher Information (QFI)). *Define a quantum state ρ_η parameterized by η. Then the QFI for ρ_η is*

$$H := 2\sum_{mn} \frac{|\langle \phi_m|(\partial_\eta \rho_\eta|_{\eta=0}|\phi_n\rangle)|^2}{\lambda_m + \lambda_n}, \tag{1.4.1}$$

where λ_n is the eigenvalue of $\rho_{\eta=0}$ corresponding to the eigenstate $|\phi_n\rangle$, and the derivative is evaluated as $\eta = 0$.

Definition 1.4.2 (Crameŕ-Rao Bound (CROB) in Eq. (2) [SLHGR+17]). *The CROB asserts that the limits on the achievable precision of an unbiased estimator $\tilde{\eta}$ is*

$$\Delta\tilde{\eta}^2 \geq \frac{1}{MH}, \tag{1.4.2}$$

*where H is the QFI (**??**) of ρ_η and M is the number of copies.*

Theorem 3 (Cramér-Chernoff theorem [?]).

1.4.2 Physics preliminaries

1.4.3 Introduction

1.4.4 Problem formulation

The quantum illumination problem is modeled as a **Reflectivity Estimation Problem** as the following.

Let us consider a general bipartite pure state representation of sinal-ideler system written in the Schmidt decomposition form

$$|\psi\rangle_{SI} = \sum_\alpha \sqrt{p_\alpha}|w_\alpha\rangle_S|v_\alpha\rangle_I. \tag{1.4.3}$$

In the Quantum Illumimation (QI) protocol, the signal modes of M copies of $|\psi\rangle_{SI}$ are sent to the target region embedded in a bright thermal noise, in which there could possibly be an object. We receive M copies of the state

$$\rho_\eta = \mathrm{Tr}\left[U_\eta \psi \bigotimes \rho_B U_\eta^\dagger\right], \tag{1.4.4}$$

where

$$U_\eta := \exp[\sin^{-1}(\eta) - s^\dagger b - sb^\dagger] \simeq \exp[\eta(s^\dagger b - sb^\dagger)] \tag{1.4.5}$$

is the signal-object interaction, modeled as a beamsplitter with amplitude reflectivity $\eta << 1$, and

$$\rho_B := \sum_n \frac{N_B^n}{(1+N_B)^{1+n}} |n\rangle\langle n| \tag{1.4.6}$$

is the thermal state with mean photon number N_B.

In this framework, at the receiver side, $\eta = 0$ corresponds to the absence of the object in the target region. We note η is unknown parameter. So we measurement ρ_η many times and get the estimation $\tilde{\eta}$. We then use $\hat{\eta}$ to determine whether is target is absent or present. This lie in the Estimation problem and state discrimination problem!

1.4.5 Protocol

The quantum illumination protocl has beed presented the last subsection. Here, we focus on the estimation protocol.

The main idea of the estimation protocol is to use QFI (**??**) and CROB (**??**) for estimating η.

Lemma 1.4.1 (The QFI for the returned state ρ_η in Eq. (3) [SLHGR+17])**.** *The QFI for ρ_η is*

$$H := \frac{4}{1+N_B} \sum_{\alpha\alpha'} \frac{p_\alpha p_{\alpha'}}{p_{\alpha'} + p_\alpha \frac{N_B}{1+N_B}} |\langle w_{\alpha'}|s|w_\alpha\rangle|^2. \tag{1.4.7}$$

Proof. □

We choose the observable that optimize the QFI of the returned state (**??**) and measure ρ_η, the outcome is o_i. Then by the strong law of large numbers, we perform the sample mean as

$$\tilde{\eta} = \frac{1}{M} \sum_{i=1}^{M} o_i. \tag{1.4.8}$$

Algorithm 1 QI estimation protocol

Input: $|\psi_{SI}\rangle$: the signal-idler state,
η: the reflectivity parameter,
ρ_B: the thermal state
M: the number of copies of $|\psi_{SI}\rangle$

Output: Estimation of the reflectivity $\tilde{\eta}$.

for $i = 1, \cdots, M$ **do**
  Sent the signal mode of $\psi_{SI}^{(1)}$ to the target region,
  Rceive the state $\rho_{\eta}^{(1)}$,
  Choose the observable that optimize the QFI,
  Measure the observable and get the outcome o_i.
end for
Calculate $\tilde{\eta}$(1.4.8).
Output the estimated reflectivity parameter $\tilde{\eta}$.

After that, we use the estimation $\tilde{\eta}$ for state discrimination.

Algorithm 2 QI discrimination protocol

Input: $\tilde{\eta}$: the estimation reflectivity,
ξ: the scaling parameter.

Output: Absent or present.

if $\tilde{\eta} > \xi\eta$ **then**
  Output present.
else if $\tilde{\eta} \leq \xi\eta$ **then**
  Output absent.
else
  Output 'Wrong input!'
end if

1.4.6 Protocol Details

1.4.7 Performance evaluation

Theorem 4 (Type I-II error probabilities Theorem in [SLHGR+17]). *Let* $|\psi\rangle_{SI} = \sqrt{p_\alpha}|n\rangle_S|v_n\rangle_I$ *be the Schmidt decomposition of the signal-idler state, and denote* ρ_S *the state of the signal. Then* $P_{I,II} \sim \exp\left(-\frac{\eta_{I,II}^2 HM}{2}\right)$ *provided that* $\exists C > 0$ *s.t.* $\langle s^k s^{\dagger k}\rangle_{\rho_S} \leq k!C^k, \forall k \in \mathcal{N}$.

Proof. □

1.4.8 Exapmles

Example 1.4.1 (Gaussian states)**.**

Example 1.4.2 (Schrödinger's cat state)**.**

1.4.9 Main ideas summarized

This subsection summarizes the main ideas in this paper.

Choose the observable that optimize the QFI.

1.4.10 Ideas

How many copies of ψ_{SI} is needed to achieve a given confidence interval? This is a problem about the quantum resources?

Bibliography

[LH20] Han Liu and Amr S Helmy. Joint measurement of time–frequency entanglement via sum frequency generation. *npj Quantum Information*, 6(1):66, 2020.

[NC19] Michael A. Nielsen and Isaac L. Chuang. *Quantum computation and quantum information*. Cambridge University Press, 2019.

[NS09] Michael Nussbaum and Arleta Szkoła. The chernoff lower bound for symmetric quantum hypothesis testing. 2009.

[SLHGR+17] Mikel Sanz, U Las Heras, Juan José García-Ripoll, Enrique Solano, and R Di Candia. Quantum estimation methods for quantum illumination. *Physical review letters*, 118(7):070803, 2017.

[ZZS17] Quntao Zhuang, Zheshen Zhang, and Jeffrey H Shapiro. Optimum mixed-state discrimination for noisy entanglement-enhanced sensing. *Physical review letters*, 118(4):040801, 2017.